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Magic p-dimensional cubes

by

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A *magic p-dimensional cube* of order n is a p-dimensional matrix

$$\mathbf{M}_n^p = |\mathbf{m}(i_1, \ldots, i_p) : 1 \leq i_1, \ldots, i_p \leq n|,$$

containing natural numbers $1, \ldots, n^p$ such that the sum of the numbers along every row and every diagonal is the same.

By a *row* of $\mathbf{M}_n^p$ we mean an n-tuple of elements $\mathbf{m}(i_1, \ldots, i_p)$ which have identical coordinates at $p-1$ places. A *diagonal* of $\mathbf{M}_n^p$ is an n-tuple $\{\mathbf{m}(x, i_2, \ldots, i_p) : x = 1, \ldots, n,\ i_j = x \text{ or } i_j = 2^p + 1 - x \text{ for all } 2 \leq j \leq p\}$. A magic p-dimensional cube $\mathbf{M}_n^p$ has pn^{p-1} rows and 2^{p-1} diagonals. The symbol $\lfloor x \rfloor$ denotes the integer part of x. A magic 1-dimensional cube $\mathbf{M}_n^1$ of order n is given by an arbitrary permutation of integers $1, \ldots, n$. Evidently, a magic p-dimensional cube of order 2 for $p \geq 2$ does not exist.

In [5] there is a construction of $\mathbf{M}_n^3$ for every $n \neq 2$ and in [6] it is proved that a magic p-dimensional cube $\mathbf{M}_n^p$ of order n exists for every integer p and $n \not\equiv 2 \pmod 4$. (The reader can find more information in [2, 3, 5, 6].) These results are improved in

THEOREM. *A magic p-dimensional cube $\mathbf{M}_n^p$ of order n exists if and only if $p \geq 2$ and $n \neq 2$ or $p = 1$.*

Before we begin the proof, we demonstrate a construction of a magic square $\mathbf{M}_6^2$. The construction starts from four copies of a Latin square $\mathbf{U} = |\mathbf{u}(i_1, i_2) : 1 \leq i_1, i_2 \leq 3|$ of order 3 defined by the relation $\mathbf{u}(i_1, i_2) \equiv (i_1 - i_2) \pmod 3$. We insert these squares into a 6×6 table, so that Latin squares are symmetric about the lines going through the centres of two opposite sides. All elements of Latin squares are replaced by $0, 1, 2, 3$ as shown in Figure A. On the left hand side in every cell there is an element of the Latin square, and its substitution on the right hand side.

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$0 \to 3$	$2 \to 0$	$1 \to 1$	$1 \to 3$	$2 \to 1$	$0 \to 1$
$1 \to 1$	$0 \to 3$	$2 \to 0$	$2 \to 1$	$0 \to 1$	$1 \to 3$
$2 \to 0$	$1 \to 1$	$0 \to 3$	$0 \to 1$	$1 \to 3$	$2 \to 1$
$2 \to 3$	$1 \to 0$	$0 \to 2$	$0 \to 0$	$1 \to 2$	$2 \to 2$
$1 \to 0$	$0 \to 2$	$2 \to 3$	$2 \to 2$	$0 \to 0$	$1 \to 2$
$0 \to 2$	$2 \to 3$	$1 \to 0$	$1 \to 2$	$2 \to 2$	$0 \to 0$

Fig. A

$27+6$	7	$9+2$	$27+2$	$9+7$	$9+6$
$9+1$	$27+5$	9	$9+9$	$9+5$	$27+1$
8	$9+3$	$27+4$	$9+4$	$27+3$	$9+8$
$27+8$	3	$18+4$	4	$18+3$	$18+8$
1	$18+5$	$27+9$	$18+9$	5	$18+1$
$18+6$	$27+7$	2	$18+2$	$18+7$	6

Fig. B

By multiplying all elements by 9 and adding elements of four copies of a magic square $\mathbf{M}_3^2$ we obtain the magic square $\mathbf{M}_6^2$ of order 6 which is shown in Figure B.

Proof of the Theorem. For $n \not\equiv 2 \pmod 4$ the proof is in [6]. That paper gives the construction of $\mathbf{M}_n^p$ for $n \equiv 1 \pmod 2$ or $n \equiv 0 \pmod 4$ and $p \geq 2$.

Let $n \equiv 2 \pmod 4$, $n \neq 2$ and $p \geq 2$ be two fixed natural numbers and let $m = n/2$. The construction of $\mathbf{M}_n^p$ is described in 6 steps.

1. Let $\mathbf{D} = |\mathbf{d}(j,x) : 1 \leq j \leq m, 1 \leq x \leq 2^p|$ be a matrix defined by the following relations:

$$\mathbf{d}(1,x) = 2^{p-1} \cdot 2^{x \,(\mathrm{mod}\, 2)} - \left\lfloor \frac{x+1}{2} \right\rfloor,$$

$$\mathbf{d}(2,x) = 2^{p-1} \cdot 2^{(x+1) \,\mathrm{mod}\, 2)} - \left\lfloor \frac{x+1}{2} \right\rfloor,$$

$$\mathbf{d}(3,x) = x + (-1)^x \left\lfloor \frac{x-1}{2^{p-1}} \right\rfloor [(p+1) \pmod 2],$$

$$\mathbf{d}(j,x) = \begin{cases} x-1, & j = 4,6,8,\ldots,m-1, \\ 2^p - x, & j = 5,7,9,\ldots,m. \end{cases}$$

This definition yields the following facts which are crucial in our construction:

(a) for every $1 \leq x \leq 2^p$,

$$\sum_{j=1}^{m} \mathbf{d}(j,x) = \frac{n}{4}(2^p - 1) - \frac{1}{2} + \left[x + \left\lfloor \frac{x-1}{2^{p-1}} \right\rfloor (p+1)\right] \pmod 2,$$

(b) $\{\mathbf{d}(j,1),\ldots,\mathbf{d}(j,2^p)\} = \{1,\ldots,2^p\}$ for all $1 \leq j \leq m$,

(c) $\mathbf{d}(1,x) + \mathbf{d}(1,2^p - x + 1) = 2^p - 1$ for all $1 \leq x \leq 2^{p-1}$ (this is important only for $p \equiv 0 \pmod 2$).

2. Let σ be a permutation of the set $\{1,\ldots,2^p\}$ which satisfies:

(i) if the number of ones in the binary representation of the number $k-1$ is even (odd) then $\sigma(k)$ is an even (odd) number for every $k = 1,\ldots,2^{p-1}$,

(ii) if $k \leq 2^{p-1}$ then $\sigma(k) \leq 2^{p-1}$,

(iii) if $2^{p-1} < k \leq 2^p$ then $\sigma(k) = 2^p - \sigma(2^p - k + 1)$.

3. Let $\mathbf{U} = |\mathbf{u}(i_1, \ldots, i_p) : 1 \leq i_1, \ldots, i_p \leq m|$ be a p-dimensional matrix defined by

$$\mathbf{u}(i_1, \ldots, i_p) = \Big(\sum_{x=1}^{p} (-1)^{x+1} i_x \Big) \pmod m.$$

Every row of $\mathbf{U}$ is the set $\{0, 1, \ldots, m-1\}$. If $p \equiv 1 \pmod 2$ then the diagonal $\{\mathbf{u}(i, \ldots, i) : i = 1, \ldots, m\}$ of $\mathbf{U}$ is the set $\{0, 1, \ldots, m-1\}$. If $p \equiv 0 \pmod 2$ then it is $\{0, 0, \ldots, 0\}$.

4. Let $\mathbf{V}_{(k)} = |\mathbf{v}_{(k)}(i_1, \ldots, i_p) : 1 \leq i_1, \ldots, i_p \leq m|$, $1 \leq k \leq 2^p$, be p-dimensional matrices defined by

$$\text{if} \quad \mathbf{u}(i_1, \ldots, i_p) = q \quad \text{then} \quad \mathbf{v}_{(k)}(i_1, \ldots, i_p) = \mathbf{d}(q, \sigma(k)).$$

5. Let $\mathbf{M}_{(k)} = |\mathbf{m}_{(k)}(i_1, \ldots, i_p) : 1 \leq i_1, \ldots, i_p \leq m|$, $1 \leq k \leq 2^p$, be p-dimensional matrices defined by

$$\mathbf{m}_{(k)}(i_1, \ldots, i_p) = \mathbf{v}_{(k)}(i_1, \ldots, i_p) m^p + \mathbf{m}(i_1, \ldots, i_p),$$

where $\mathbf{m}(i_1, \ldots, i_p)$ is the element of $\mathbf{M}_m^p$ which is constructed in [6].

Because $\mathbf{v}_{(j)}(i_1, \ldots, i_p) \neq \mathbf{v}_{(k)}(i_1, \ldots, i_p)$ for all $j \neq k$ and from the previous relation it follows that:

(a) no two elements $\mathbf{m}_{(k)}(i_1, \ldots, i_p)$ with different coordinates or indices are equal,

(b) the row sum of $\mathbf{M}_{(k)}$ for fixed k is the same, i.e.

$$\left[\frac{m}{2}(2^p - 1) + \frac{(-1)^{\omega}}{2} \right] m^p + \frac{m(m^p + 1)}{2}, \qquad \text{where } \omega = 1 \text{ or } 2,$$

(c) if $p \equiv 1 \pmod 2$ then $\sum_{i=1}^{m} \mathbf{m}_{(k)}(i, \ldots, i)$ is equal to the row sum of $\mathbf{M}_{(k)}$, if $p \equiv 0 \pmod 2$ then

$$\sum_{i=1}^{m} \mathbf{m}_{(k)}(i, \ldots, i) = \mathbf{d}(1, \sigma(k)) m^{p+1} + m(m^p + 1)/2.$$

6. We define a magic p-dimensional cube $\mathbf{M}_n^p = |\mathbf{m}(i_1, \ldots, i_p) : 1 \leq i_1, \ldots, i_p \leq n|$ of order $n \equiv 2 \pmod 4$ by

$$\mathbf{m}(i_1, \ldots, i_p) = \mathbf{m}_{(k)}(i_1^*, \ldots, i_p^*),$$

where $i_j^* = \min\{i_j, n + 1 - i_j\}$ and $k = \sum_{x=1}^{p} \lfloor \frac{i_x - 1}{m} \rfloor 2^{x-1} + 1$.

From the definition of $\mathbf{M}_n^p$ we get:

(a) every row of $\mathbf{M}_n^p$ consists of one row of $\mathbf{M}_{(j)}$ and one row of $\mathbf{M}_{(k)}$ which have different row sums,

(b) every diagonal of $\mathbf{M}_n^p$ consists of $\mathbf{m}_{(k)}(i, \ldots, i)$, $\mathbf{m}_{(2^p+1-k)}(i, \ldots, i)$, $i = 1, \ldots, m$. If $p \equiv 1 \pmod 2$ then $\mathbf{M}_{(k)}$ and $\mathbf{M}_{(2^p-k+1)}$ have different row sums. If $p \equiv 0 \pmod 2$ then the row sums of $\mathbf{M}_{(k)}$ and $\mathbf{M}_{(2^p-k+1)}$ are the same.

It is easy to see that $\mathbf{M}_n^p$, which is a union of 2^p matrices $\mathbf{M}_{(k)}$, satisfies the conditions for a magic p-dimensional cube.

REMARK 1. Magic squares have fascinated people for centuries. Mathematicians have studied many properties of magic squares and formulated problems which have not been solved. (See [1].) We can formulate similar problems for magic cubes, too.

REMARK 2. Another "magic" p-dimensional cube was studied by J. Ivančo. In [4] it is proved that if $4 \leq p \equiv 0 \pmod 2$ then the edges of a p-dimensional cube can be labelled by integers $1, 2, \ldots, 2^{p-1}p$ in such a way that the sum of the labels of edges incident to each vertex is the same.

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