MAGIC CUBES

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Since antiquity mathematicians (and not only them) have taken an interest in constructing magic squares. Probably the first magic square ever created is the one shown in Fig. 1. Its origin is shrouded in the mystical legends of ancient China. It became to be known as Luo Shu (Luo river writing). There was no clear connection between this configuration and mathematical study until the time of Yang Hui, even though it was described in the sixth century. Another well-known magic square (Fig. 2) is in the painting Melancholy ([3], p.6) made by the famous renaissance artist Albrecht Dürer in 1514 (the year is formed in the middle of the lowest row).

$$\begin{array}{ccc}
 4 & 9 & 2 \\
 3 & 5 & 7 \\
 8 & 1 & 6 \\
\end{array}$$

$$\begin{array}{cccc}
 16 & 3 & 2 & 13 \\
 5 & 10 & 11 & 8 \\
 9 & 6 & 7 & 12 \\
 4 & 15 & 14 & 1 \\
\end{array}$$

Figure 1

Figure 2

A magic square of order $n$ is an $n \times n$ matrix (square table) containing the natural numbers $1, 2, 3, \ldots, n^2$ in some order, and such that the sum of the numbers along any row, column, or main diagonal is a fixed constant. It is easy to see that this constant in a square of order $n$ must be $\frac{n(n^2+1)}{2}$. In [3] and elsewhere we can find constructions of magic squares of order $n$ for all natural numbers $n \neq 2$. There is no magic square of order 2, as the reader may easily verify.

A generalization of magic squares is magic cubes. In Fig. 3 a magic cube of order 4 is depicted, layer by layer.

**Definition.** A magic cube of order $n$ is a 3-dimensional $n \times n \times n$ matrix (cubical table)

$$Q_n = [q(i, j, k); 1 \leq i, j, k \leq n]$$

containing the natural numbers $1, 2, 3, \ldots, n^3$ in some order, and such that

$$\sum_{x=1}^{n} q(x, j, k) = \sum_{x=1}^{n} q(i, x, k) = \sum_{x=1}^{n} q(i, j, x) = \frac{n(n^3+1)}{2} \quad \text{for all} \quad i, j, k = 1, \ldots, n$$

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(Note that in a magic cube we make no requirement about the sums of elements on any diagonal.)

The triple of numbers \((i, j, k)\) called the coordinates of the element \(q(i, j, k)\).

\[
\begin{array}{ccc}
16 & 19 & 34 & 61 \\
18 & 13 & 64 & 35 \\
45 & 50 & 3 & 32 \\
51 & 48 & 29 & 2 \\
\end{array}
\begin{array}{ccc}
37 & 58 & 11 & 24 \\
59 & 40 & 21 & 10 \\
8 & 27 & 42 & 53 \\
26 & 5 & 56 & 43 \\
\end{array}
\begin{array}{ccc}
57 & 38 & 23 & 12 \\
9 & 60 & 22 & 14 \\
28 & 7 & 54 & 41 \\
6 & 25 & 44 & 55 \\
\end{array}
\begin{array}{ccc}
20 & 15 & 62 & 33 \\
14 & 17 & 36 & 63 \\
49 & 6 & 31 & 4 \\
47 & 52 & 1 & 30 \\
\end{array}
\]

**Figure 3**

A magic square of order 1 is a magic cube of order 1. Just as a magic square of order 2 does not exist a magic cube of order 2 does not exist either.

In this paper we prove the following theorem:

**Theorem.** For every natural number \(n \neq 2\) there exists a magic cube of order \(n\).

Before we prove this theorem we will consider Latin squares, which will be used in the proof. A Latin square \(R_n = [r(i, j); 1 \leq i, j \leq n]\) of order \(n\) is an \(n \times n\) matrix such that every row and every column is a permutation of the set of natural numbers \(\{1, 2, \ldots, n\}\). Two Latin squares \(R_n = [r(i, j)]\) and \(S_n = [s(i, j)]\) of order \(n\) are called orthogonal, if whenever \(i, i', j, j' \in \{1, 2, \ldots, n\}\) are such that \(r(i, j) = r(i', j')\) and \(s(i, j) = s(i', j')\), then we must have \(i = i'\) and \(j = j'\).

Two orthogonal Latin squares of order 4 are depicted in Fig. 4.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3 \\
3 & 4 & 1 & 2 \\
4 & 3 & 2 & 1 \\
\end{array}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
3 & 4 & 1 & 2 \\
4 & 3 & 2 & 1 \\
2 & 1 & 4 & 3 \\
\end{array}
\]

**Figure 4**

In 1960 R. C. Bose, S.S. Shrikhande and E. T. Parker [1] proved that two orthogonal Latin squares of order \(n\) exist if and only if \(n \neq 2, 6\). We use this statement to prove our theorem.

**Proof of the Theorem.** Let \(R_n = [r(i, j); 1 \leq i, j \leq n]\) and \(S_n = [s(i, j); 1 \leq i, j \leq n]\) be two orthogonal Latin squares of order \(n\), and let \(M_n = [m(i, j); 1 \leq i, j \leq n]\) be a magic square of order \(n\).

Let us define a magic cube \(Q_n = [q(i, j, k); 1 \leq i, j, k \leq n]\) in the following way:

\[
q(i, j, k) = [s(i, r(j, k)) - 1]n^2 + m(i, s(j, k)) \quad \text{for all } 1 \leq i, j, k \leq n.
\]
We shall prove, in four steps, that $Q_n$ is a magic cube.

1. All elements of the squares $R_n$ and $S_n$ are from the set $\{1, 2, \ldots, n\}$ and all elements of $M_n$ are from the set $\{1, \ldots, n^2\}$. It follows immediately that for all elements of $Q_n$ we have $1 \leq q(i, j, k) \leq n^3$.

2. In this part we prove that two elements of $Q_n$ with different coordinates can not be equal. Let us suppose $q(i, j, k) = q(i', j', k')$. We show that this implies that $i = i'$, $j = j'$ and $k = k'$.

From the definition of $Q_n$ we have

$$[s(i, r(j, k)) - 1]n^2 + m(i, s(j, k)) = [s(i', r(j', k')) - 1]n^2 + m(i', s(j', k'))].$$

By rearranging this equation we get

$$-[s(i, r(j, k)) - s(i', r(j', k'))]n^2 = m(i, s(j, k)) - m(i', s(j', k'))). \tag{1}$$

Because all the elements of $M_n$ are from the set $\{1, \ldots, n^2\}$, the right hand side of (1) is not a non-zero multiple of $n^2$. On the left is a whole multiple of $n^2$. This equality in (1) can only occur if $s(i, r(j, k)) - s(i', r(j', k')) = 0$. Hence

$$s(i, r(j, k)) = s(i', r(j', k')) \tag{2}$$

and

$$m(i, s(j, k)) = m(i', s(j', k')). \tag{3}$$

In a magic square no two elements are identical. If two elements of $M_n$ are equal, then their coordinates are the same, so from (3) we have

$$i = i' \quad \text{and} \quad s(j, k) = s(j', k').$$

By substitution of $i' = i$ in (2) we get $s(i, r(j, k)) = s(i, r(j', k'))$. Because $S_n$ is a Latin square, from the equality of the first coordinate the equality of the second coordinate follows, so that $r(j, k) = r(j', k')$. From the assumption that $S_n, R_n$ are orthogonal Latin squares we get $j = j'$ and $k = k'$.

3. Next we prove that the sum of numbers in every row is the same. We have

$$\sum_{x=1}^{n} q(x, j, k) = \sum_{x=1}^{n} [s(x, r(j, k)) - 1]n^2 + \sum_{x=1}^{n} m(x, s(j, k)) =$$

$$= \left[\frac{n(n+1)}{2} - n\right]n^2 + \frac{n(n^2+1)}{2} = \frac{n(n^3+1)}{2}$$

Then, because

$$\sum_{x=1}^{n} s(x, r(j, k)) = \sum_{x=1}^{n} s(i, r(x, k)) = \sum_{x=1}^{n} s(i, r(j, x))$$
and 
\[ \sum_{x=1}^{n} m(x, s(j, k)) = \sum_{x=1}^{n} m(i, s(x, k)) = \sum_{x=1}^{n} m(i, s(j, x)) \]
similarly we get 
\[ \sum_{x=1}^{n} q(i, x, k) = \sum_{x=1}^{n} q(i, j, x) = \frac{n(n^2+1)}{2} \]

4. The above construction of magic squares is based on the use of two orthogonal Latin squares and therefore is not valid for \( n = 6 \). So to complete the proof, we exhibit the magic cube \( Q_6 \), in Figure 5.

\[
\begin{array}{cccccccc}
6 & 192 & 193 & 199 & 30 & 31 \\
72 & 174 & 168 & 49 & 151 & 37 \\
103 & 138 & 85 & 96 & 115 & 114 \\
139 & 79 & 126 & 127 & 102 & 78 \\
150 & 61 & 55 & 162 & 43 & 180 \\
181 & 7 & 24 & 18 & 210 & 211 \\
214 & 28 & 16 & 21 & 189 & 183 \\
148 & 45 & 57 & 160 & 64 & 177 \\
76 & 117 & 87 & 94 & 135 & 142 \\
111 & 100 & 124 & 129 & 82 & 105 \\
69 & 154 & 165 & 52 & 171 & 40 \\
33 & 207 & 202 & 195 & 10 & 4 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
212 & 26 & 23 & 14 & 191 & 185 \\
146 & 47 & 53 & 167 & 62 & 176 \\
113 & 83 & 128 & 125 & 98 & 104 \\
74 & 137 & 95 & 86 & 116 & 143 \\
71 & 152 & 158 & 56 & 173 & 41 \\
35 & 206 & 194 & 203 & 11 & 2 \\
213 & 9 & 15 & 22 & 208 & 184 \\
39 & 172 & 166 & 51 & 153 & 70 \\
75 & 136 & 88 & 93 & 118 & 141 \\
112 & 81 & 123 & 130 & 99 & 106 \\
178 & 63 & 58 & 159 & 46 & 147 \\
34 & 190 & 201 & 196 & 27 & 3 \\
\end{array}
\]

\[
\begin{array}{cccc}
5 & 209 & 200 & 197 & 8 & 32 \\
179 & 44 & 50 & 164 & 65 & 149 \\
140 & 80 & 131 & 122 & 101 & 77 \\
107 & 134 & 92 & 89 & 119 & 110 \\
38 & 155 & 161 & 59 & 170 & 68 \\
182 & 29 & 17 & 20 & 188 & 215 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 187 & 204 & 198 & 25 & 36 \\
67 & 169 & 157 & 60 & 156 & 42 \\
144 & 97 & 132 & 121 & 84 & 73 \\
108 & 120 & 91 & 90 & 133 & 109 \\
145 & 66 & 54 & 163 & 48 & 175 \\
186 & 12 & 13 & 19 & 205 & 216 \\
\end{array}
\]

**Figure 5**

**Remark 1.** The construction of the magic cube of order 4, shown in Figure 3, was based on the magic square shown in Fig. 2 and from the two orthogonal Latin squares shown in Fig. 4.

For example,

\[
q(2, 3, 4) = (s(2, r(3, 4)) - 1) \times 16 + m(2, s(3, 4))
\]
\[
= (s(2, 2) - 1) \times 16 + m(2, 1) = 3 \times 16 + 5 = 53
\]

and one sees in Fig. 2 that for \( i = 2 \) (that is at level \( B \)), the \((3, 4)\) entry is indeed 53.

**Remark 2.** Two orthogonal Latin squares of order \( n = 5 \) are depicted in Fig. 6, and the pattern used here generalises to arbitrary odd \( n \), as the reader may easily verify.
Using this pattern and the following, it is easy to make a computer program which constructs magic cubes for every odd \( n \). Instead of the magic square \( M_n \) we can use the square \( M^*_n = [m^*(i, j)] \), where \( m^*(i, j) = [r(i, j) - 1]n^2 + s(i, j) \) for all \( 1 \leq i, j \leq n \). In this square the sums of the rows and columns are the same, but the sums of the diagonals need not be the same, so it is not a magic square in the usual sense. This does not matter, since the diagonal sums were not used in the proof of our theorem.

**Remark 3.** Magic cubes generalize to magic hypercubes in \( p \)-dimensional Euclidian space.

A magic \( p \)-dimensional hypercube of order \( n \) is a \( p \)-dimensional \( n \times n \times n \times \cdots \times n \) matrix

\[
Q^p_n = [q^p(i_1, i_2, \ldots, i_p)]
\]

containing the natural numbers \( 1, 2, 3, \ldots, n^p \) in some order, and such that the sum of the numbers along every row is \( \frac{n(n^p+1)}{2} \). (By a row of \( Q^p_n \) we mean an \( n \)-tuple of elements which have at \((p-1)\) places the identical coordinates.)

A Latin \( p \)-dimensional hypercube \( U^p_n \) of order \( n \) is a \( p \)-dimensional matrix \( n \times n \times \cdots \times n \) whose elements in every row are a permutation of the numbers \( 1, 2, 3, \ldots, n \).

By a generalization of the previous proof we can similarly prove:

For all natural numbers \( n \neq 2, 6 \) and \( p \geq 2 \) there exists a magic hypercube \( Q^p_n \).

A magic hypercube \( Q^p_1 \) of order 1 has only one element. Just as a magic square \( M_2 \) of order 2 does not exist a magic hypercube \( Q^p_2 \) does not exist either. Therefore we suppose that \( n \geq 3 \).

We show how to construct a magic hypercube \( Q^p_n \), for all natural numbers \( 3 \leq n, n \neq 6 \) and \( p \geq 3 \). We use mathematical induction on \( p \).

Let us suppose \( 3 \leq n \) and \( n \neq 6 \). If \( p = 2 \), then \( U^2_n \) is a Latin square of order \( n \) and \( Q^2_n \) is a magic square of order \( n \).

Let \( p > 2 \). We suppose that a \((p-1)\)-dimensional Latin hypercube \( U^{p-1}_n = [u^{p-1}(i_1, i_2, \ldots, i_{p-1})] \) and a magic hypercube \( Q^{p-1}_n = [q^{p-1}(i_1, i_2, \ldots, i_{p-1})] \) are already constructed. A \( p \)-dimensional Latin hypercube \( U^p_n = [u^p(i_1, \ldots, i_p)] \) of order \( n \) and a \( p \)-dimensional magic hypercube \( Q^p_n = [q^p(i_1, \ldots, i_p)] \) of order \( n \) can be defined by:

\[
u^p(i_1, i_2, \ldots, i_p) = u^{p-1}(i_1, i_2, \ldots, i_{p-2}, r(i_{p-1}, i_p))
\]

and

\[
q^p(i_1, i_2, \ldots, i_n) = \{u^{p-1}(i_1, i_2, \ldots, i_{p-2}, r(i_{p-1}, i_p)) - 1\}n^{p-1} + q^{p-1}(i_1, i_2, \ldots, i_{p-2}, s(i_{p-1}, i_p)),
\]

for all \( 1 \leq i_1, i_2, \ldots, i_p \leq n \). Details of the proof can safely be left to the reader.
References


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