MAGIC CUBES

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Since antiquity mathematicians (and not only them) have taken an interest in constructing magic squares. Probably the first magic square ever created is the one shown in Fig.1. Its origin is shrouded in the mystical legends of ancient China. It became to be known as *Luo Shu (Luo river writing)*. There was no clear connection between this configuration and mathematical study until the time of *Yang Hui*, even though it was described in the sixth century. Another well-known magic square (Fig. 2) is in the painting *Melancholy* ([3], p.6) made by the famous renaissance artist *Albrecht Dürer* in 1514 (the year is formed in the middle of the lowest row).

4	9	2
3	5	7
8	1	6

Figure 1

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Figure 2

A *magic square* of order n is an $n \times n$ matrix (square table) containing the natural numbers $1, 2, 3, \ldots, n^2$ in some order, and such that the sum of the numbers along any row, column, or main diagonal is a fixed constant. It is easy to see that this constant in a square of order n must be $\frac{n(n^2+1)}{2}$. In [3] and elsewhere we can find constructions of magic squares of order n for all natural numbers $n \neq 2$. There is no magic square of order 2, as the reader may easily verify.

A generalization of magic squares is magic cubes. In Fig. 3 a magic cube of order 4 is depicted, layer by layer.

Definition. A *magic cube* of order n is a 3-dimensional $n \times n \times n$ matrix (cubical table)

$$\mathbf{Q}_n = [\mathbf{q}(i, j, k);\ 1 \leq i, j, k \leq n]$$

containing the natural numbers $1, 2, 3, \ldots, n^3$ in some order, and such that

$$\sum_{x=1}^{n} \mathbf{q}(x, j, k) = \sum_{x=1}^{n} \mathbf{q}(i, x, k) = \sum_{x=1}^{n} \mathbf{q}(i, j, x) = \frac{n(n^3+1)}{2} \qquad \text{for all} \quad i, j, k = 1, \ldots, n$$

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(Note that in a magic cube we make no requirement about the sums of elements on any diagonal.)

The triple of numbers (i, j, k) called the *coordinates* of the element $\mathbf{q}(i, j, k)$.

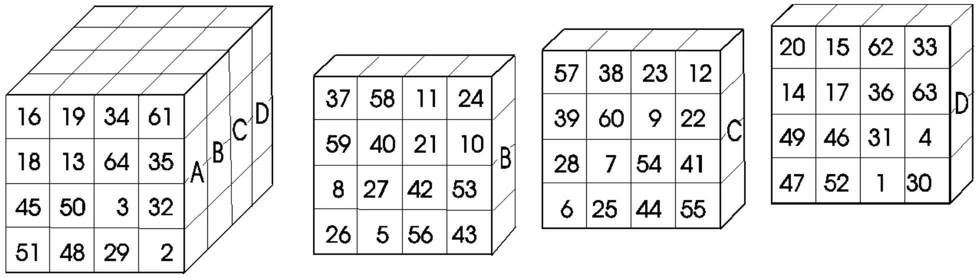

FIGURE 3

A magic square of order 1 is a magic cube of order 1. Just as a magic square of order 2 does not exist a magic cube of order 2 does not exist either.

In this paper we prove the following theorem:

Theorem. *For every natural number $n \neq 2$ there exists a magic cube of order n.*

Before we prove this theorem we will consider Latin squares, which will be used in the proof. A *Latin square* $\mathbf{R}_n = [\mathbf{r}(i, j); 1 \leq i, j \leq n]$ of order n is an $n \times n$ matrix such that every row and every column is a permutation of the set of natural numbers $\{1, 2, \ldots, n\}$. Two Latin squares $\mathbf{R}_n = [\mathbf{r}(i, j)]$ and $\mathbf{S}_n = [\mathbf{s}(i, j)]$ of order n are called *orthogonal*, if whenever $i, i', j, j' \in \{1, 2, \ldots, n\}$ are such that $\mathbf{r}(i, j) = \mathbf{r}(i', j')$ and $\mathbf{s}(i, j) = \mathbf{s}(i', j')$, then we must have $i = i'$ and $j = j'$.

Two orthogonal Latin squares of order 4 are depicted in Fig. 4.

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

FIGURE 4

In 1960 *R.C.Bose, S.S.Shrikhande and E.T.Parker* [1] proved that two orthogonal Latin squares of order n exist if and only if $n \neq 2, 6$. We use this statement to prove our theorem.

Proof of the Theorem. Let $\mathbf{R}_n = [\mathbf{r}(i, j); 1 \leq i, j \leq n]$ and $\mathbf{S}_n = [\mathbf{s}(i, j); 1 \leq i, j \leq n]$ be two orthogonal Latin squares of order n, and let $\mathbf{M}_n = [\mathbf{m}(i, j); 1 \leq i, j \leq n]$ be a magic square of order n.

Let us define a magic cube $\mathbf{Q}_n = [\mathbf{q}(i, j, k); 1 \leq i, j, k \leq n]$ in the following way:

$$\mathbf{q}(i, j, k) = [\mathbf{s}(i, \mathbf{r}(j, k)) - 1]n^2 + \mathbf{m}(i, \mathbf{s}(j, k)) \qquad \text{for all } 1 \leq i, j, k \leq n.$$

We shall prove, in four steps, that $\mathbf{Q}_n$ is a magic cube.

1. All elements of the squares $\mathbf{R}_n$ and $\mathbf{S}_n$ are from the set $\{1, 2, \ldots, n\}$ and all elements of $\mathbf{M}_n$ are from the set $\{1, \ldots, n^2\}$. It follows immediately that for all elements of $\mathbf{Q}_n$ we have $1 \leq \mathbf{q}(i,j,k) \leq n^3$.

2. In this part we prove that two elements of $\mathbf{Q}_n$ with different coordinates can not be equal. Let us suppose $\mathbf{q}(i,j,k) = \mathbf{q}(i',j',k')$. We show that this implies that $i = i'$, $j = j'$ and $k = k'$.
From the definition of $\mathbf{Q}_n$ we have

$$[\mathbf{s}(i, \mathbf{r}(j,k)) - 1]n^2 + \mathbf{m}(i, \mathbf{s}(j,k)) = [\mathbf{s}(i', \mathbf{r}(j',k')) - 1]n^2 + \mathbf{m}(i', \mathbf{s}(j',k')).$$

By rearranging this equation we get

$$-[\mathbf{s}(i, \mathbf{r}(j,k)) - \mathbf{s}(i', \mathbf{r}(j',k'))]n^2 = \mathbf{m}(i, \mathbf{s}(j,k)) - \mathbf{m}(i', \mathbf{s}(j',k')). \tag{1}$$

Because all the elements of $\mathbf{M}_n$ are from the set $\{1, \ldots, n^2\}$, the right hand side of (1) is not a non-zero multiple of n^2. On the left is a whole multiple of n^2. This equality in (1) can only occur if $\mathbf{s}(i, \mathbf{r}(j,k)) - \mathbf{s}(i', \mathbf{r}(j',k')) = 0$. Hence

$$\mathbf{s}(i, \mathbf{r}(j,k)) = \mathbf{s}(i', \mathbf{r}(j',k')) \tag{2}$$

and

$$\mathbf{m}(i, \mathbf{s}(j,k)) = \mathbf{m}(i', \mathbf{s}(j',k')). \tag{3}$$

In a magic square no two elements are identical. If two elements of $\mathbf{M}_n$ are equal, then their coordinates are the same, so from (3) we have

$$i = i' \quad and \quad \mathbf{s}(j,k) = \mathbf{s}(j',k').$$

By substitution of $i' = i$ in (2) we get $\mathbf{s}(i, \mathbf{r}(j,k)) = \mathbf{s}(i, \mathbf{r}(j',k'))$. Because $\mathbf{S}_n$ is a Latin square, from the equality of the first coordinate the equality of the second coordinate follows, so that $\mathbf{r}(j,k) = \mathbf{r}(j',k')$. From the assumption that $\mathbf{S}_n, \mathbf{R}_n$ are orthogonal Latin squares we get $j = j'$ and $k = k'$.

3. Next we prove that the sum of numbers in every row is the same. We have

$$\sum_{x=1}^{n} \mathbf{q}(x,j,k) = \sum_{x=1}^{n} [\mathbf{s}(x, \mathbf{r}(j,k)) - 1]n^2 + \sum_{x=1}^{n} \mathbf{m}(x, \mathbf{s}(j,k)) =$$

$$= [\tfrac{n(n+1)}{2} - n]n^2 + \tfrac{n(n^2+1)}{2} = \tfrac{n(n^3+1)}{2}$$

Then, because

$$\sum_{x=1}^{n} \mathbf{s}(x, \mathbf{r}(j,k)) = \sum_{x=1}^{n} \mathbf{s}(i, \mathbf{r}(x,k)) = \sum_{x=1}^{n} \mathbf{s}(i, \mathbf{r}(j,x))$$

and

$$\sum_{x=1}^{n} \mathbf{m}(x, \mathbf{s}(j,k)) = \sum_{x=1}^{n} \mathbf{m}(i, \mathbf{s}(x,k)) = \sum_{x=1}^{n} \mathbf{m}(i, \mathbf{s}(j,x))$$

similarly we get

$$\sum_{x=1}^{n} \mathbf{q}(i,x,k) = \sum_{x=1}^{n} \mathbf{q}(i,j,x) = \frac{n(n^3+1)}{2}$$

4. The above construction of magic squares is based on the use of two orthogonal Latin squares and therefore is not valid for $n = 6$. So to complete the proof, we exhibit the magic cube $\mathbf{Q}_6$, in Figure 5.

6	192	193	199	30	31
72	174	168	49	151	37
103	138	85	96	115	114
139	79	126	127	102	78
150	61	55	162	43	180
181	7	24	18	210	211

212	26	23	14	191	185
146	47	53	167	62	176
113	83	128	125	98	104
74	137	95	86	116	143
71	152	158	56	173	41
35	206	194	203	11	2

214	28	16	21	189	183
148	45	57	160	64	177
76	117	87	94	135	142
111	100	124	129	82	105
69	154	165	52	171	40
33	207	202	195	10	4

213	9	15	22	208	184
39	172	166	51	153	70
75	136	88	93	118	141
112	81	123	130	99	106
178	63	58	159	46	147
34	190	201	196	27	3

5	209	200	197	8	32
179	44	50	164	65	149
140	80	131	122	101	77
107	134	92	89	119	110
38	155	161	59	170	68
182	29	17	20	188	215

1	187	204	198	25	36
67	169	157	60	156	42
144	97	132	121	84	73
108	120	91	90	133	109
145	66	54	163	48	175
186	12	13	19	205	216

FIGURE 5

Remark 1. The construction of the magic cube of order 4, shown in Figure 3, was based on the magic square shown in Fig. 2 and from the two orthogonal Latin squares shown in Fig. 4.

For example,

$$\begin{aligned}\mathbf{q}(2,3,4) &= (\mathbf{s}(2,\mathbf{r}(3,4)) - 1) * 16 + \mathbf{m}(2,\mathbf{s}(3,4)) \\ &= (\mathbf{s}(2,2) - 1) * 16 + \mathbf{m}(2,1) = 3 * 16 + 5 = 53\end{aligned}$$

and one sees in Fig. 2 that for $i = 2$ (that is at level $\mathbf{B}$), the $(3,4)$ entry is indeed 53.

Remark 2. Two orthogonal Latin squares of order $n = 5$ are depicted in Fig. 6, and the pattern used here generalises to arbitrary odd n, as the reader may easily verify.

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
3	4	5	1	2
2	3	4	5	1

FIGURE 6

Using this pattern and the following, it is easy to make a computer program which constructs magic cubes for every odd n. Instead of the magic square $\mathbf{M}_n$ we can use the square $\mathbf{M}_n^* = [\mathbf{m}^*(i,j)]$, where $\mathbf{m}^*(i,j) = [\mathbf{r}(i,j) - 1]n^2 + \mathbf{s}(i,j)$ for all $1 \le i,j \le n$. In this square the sums of the rows and columns are the same, but the sums of the diagonals need not be the same, so it is not a magic square in the usual sense. This does not matter, since the diagonal sums were not used in the proof of our theorem.

Remark 3. Magic cubes generalize to magic hypercubes in p-dimensional Euclidian space.

A *magic p-dimensional hypercube* of order n is a p-dimensional $n \times n \times n \times \cdots \times n$ matrix

$$\mathbf{Q}_n^p = [\mathbf{q}^p(i_1, i_2, \ldots, i_p)]$$

containing the natural numbers $1, 2, 3, \ldots, n^p$ in some order, and such that the sum of the numbers along every row is $\frac{n(n^p+1)}{2}$. (By a *row* of $\mathbf{Q}_n^p$ we mean an n-tuple of elements which have at $(p-1)$ places the identical coordinates.)

A *Latin p-dimensional hypercube* $\mathbf{U}_n^p$ of order n is a p-dimensional matrix $n \times n \times \cdots \times n$ whose elements in every row are a permutation of the numbers $1, 2, 3, \ldots, n$.

By a generalization of the previous proof we can similarly prove:

For all natural numbers $n \ne 2, 6$ and $p \ge 2$ there exists a magic hypercube $\mathbf{Q}_n^p$.

A magic hypercube $\mathbf{Q}_1^p$ of order 1 has only one element. Just as a magic square $\mathbf{M}_2$ of order 2 does not exist a magic hypercube $\mathbf{Q}_2^p$ does not exist either. Therefore we suppose that $n \ge 3$.

We show how to construct a magic hypercube $\mathbf{Q}_n^p$, for all natural numbers $3 \le n$, $n \ne 6$ and $p \ge 3$. We use mathematical induction on p.

Let us suppose $3 \le n$ and $n \ne 6$. If $p = 2$, then $\mathbf{U}_n^2$ is a Latin square of order n and $\mathbf{Q}_n^2$ is a magic square of order n.

Let $p > 2$. We suppose that a $(p-1)$-dimensional Latin hypercube $\mathbf{U}_n^{p-1} = [\mathbf{u}^{p-1}(i_1, i_2, \ldots, i_{p-1})]$ and a magic hypercube $\mathbf{Q}_n^{p-1} = [\mathbf{q}^{p-1}(i_1, i_2, \ldots, i_{p-1})]$ are already constructed. A p-dimensional Latin hypercube $\mathbf{U}_n^p = [\mathbf{u}^p(i_1, \ldots, i_p)]$ of order n and a p-dimensional magic hypercube $\mathbf{Q}_n^p = [\mathbf{q}^p(i_1, \ldots, i_p)]$ of order n can be defined by:

$$\mathbf{u}^p(i_1, i_2, \ldots, i_p) = \mathbf{u}^{p-1}(i_1, i_2, \ldots, i_{p-2}, \mathbf{r}(i_{p-1}, i_p))$$

and

$$\mathbf{q}^p(i_1, i_2, \ldots, i_n) = \\ = \{\mathbf{u}^{p-1}(i_1, i_2, \ldots, i_{p-2}, \mathbf{r}(i_{p-1}, i_p)) - 1\}n^{p-1} + \mathbf{q}^{p-1}(i_1, i_2, \ldots, i_{p-2}, \mathbf{s}(i_{p-1}, i_p)),$$

for all $1 \le i_1, i_2, \ldots, i_p \le n$. Details of the proof can safely be left to the reader.

References

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