Applications of Modern Mathematical Methods
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CONNECTIONS - MAGIC SQUARES, CUBES AND MATCHINGS

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1. Magic squares and cubes

Magic squares fascinated people throughout centuries. The first references to magic squares can be found in ancient Chinese and Indian literature. The first, well known magic squares is called *Lo Shu*. During the fifteenth century, the Byzantine writer *Manuel Moschopoulos* introduced magic squares in Europe, where, as in other cultures, magic squares were linked with divination, alchemy, and astrology. The first evidence of a magic square appearing in print in Europe was revealed in a famous engraving by the German artist *Albrecht Dürer*. In 1514, Dürer incorporated a magic square into his copperplate engraving *Melancholy* in the upper-right corner (see [3, p.147].) A construction of a magic square of order 3 is introduced in the tragedy *Faust* by *J.W.Göthe*. During the seventeenth century, serious consideration was given to the study of magic squares. In 1687-88, a French aristocrat, *Antoine de la Loubère*, studied the mathematical theory of constructing magic squares. In 1686, *Adamas Kochansky* extended magic squares to three dimensions. Probably, the first mentioned magic cube appeared in a letter of *Pierre de Fermat* from 1640 (see [3, p.314].) During the latter part of the nineteenth century, mathematicians applied the squares to problems in probability and analysis. Today, magic squares are studied in relation to factor analysis, combinatorial mathematics, matrices, modular arithmetic, and geometry. There is a lot of information about magic squares and cubes in cited literature and on many web-pages. (See *http://www.pse.tohoku.ac.jp/˜msuzuki/MagicSquare.html.*)

Figure 1

A magic cube of order n is a cubical array

$$\mathbf{M}_n = |\mathbf{m}_n(i,j,k);\ 1 \le i,j,k \le n|,$$

containing natural numbers $1, 2, \dots, n^3$ such that the sums of the numbers along each row (n-tuple of elements having the same coordinates on two places) and also along each of its four great diagonals are the same, e.i. $\frac{n(n^3+1)}{2}$.

Figure 1 shows the magic cube $\mathbf{M}_3$ which was constructed using the formula (1). The element $\mathbf{m}_3(1,1,1) = 8$ is in three rows containing the triples $\{8, 15, 19\}$,

$\{8,24,10\}$, $\{8,12,22\}$. On the four diagonals there are the triples $\{8,14,20\}$, $\{19,14,9\}$, $\{10,14,18\}$ and $\{6,14,22\}$.

This paper gives the algorithm for making a magic cube of order $n \neq 2$. The proof of the correctness of our formulas follows from [17] and [20]. We use the following denotation:

$x \pmod n$ is a remainder from division of x by n,

$\overline{x} = n + 1 - x$,

$x^* = \min\{x, \overline{x}\}$,

$$\widetilde{x} = \begin{cases} 0 & \text{for} \quad 1 \le x \le \frac{n}{2} \\ 1 & \text{for} \quad \frac{n}{2} < x \le n. \end{cases}$$

We construct a magic cube $\mathbf{M}_n = |\mathbf{m}_n(i,j,k)|$ of order n using the following three formulas:

1. If $n \equiv 1 \pmod 2$ then

$$\mathbf{m}_n(i,j,k) = [(i-j+k-1) \pmod n]\, n^2 + [(i-j-k) \pmod n]\, n + (i+j+k-2) \pmod n + 1 \tag{1}$$

2. If $n \equiv 0 \pmod 4$ then

$$\mathbf{m}_n(i,j,k) = \begin{cases} (i-1)\, n^2 + (j-1)\, n + k & \text{if} \quad \mathbb{F}(i,j,k) = 1 \\ (\overline{i}-1)\, n^2 + (\overline{j}-1)\, n + \overline{k} & \text{if} \quad \mathbb{F}(i,j,k) = 0 \end{cases}$$

where

$$\mathbb{F}(i,j,k) = (i + j + k + \widetilde{i} + \widetilde{j} + \widetilde{k}) \pmod 2$$

3. If $n \equiv 2 \pmod 4$ (in this case $\frac{n}{2}$ is odd) then

$$\mathbf{m}_n(i,j,k) = \mathbf{d}(u,v)\frac{n^3}{8} + \mathbf{m}_{\frac{n}{2}}(i^*, j^*, k^*)$$

where

$u = (i^* - j^* + k^*) \pmod{\frac{n}{2}} + 1$,

$v = 4\widetilde{i} + 2\widetilde{j} + \widetilde{k} + 1$,

$\mathbf{d}(u,v)$ for $1 \le u \le \frac{n}{2}, 1 \le v \le 8$ is defined by the table $(x = 1, 2, \ldots, \frac{n-6}{4})$

	$\mathbf{d}(u,1)$	$\mathbf{d}(u,2)$	$\mathbf{d}(u,3)$	$\mathbf{d}(u,4)$	$\mathbf{d}(u,5)$	$\mathbf{d}(u,6)$	$\mathbf{d}(u,7)$	$\mathbf{d}(u,8)$
$\mathbf{d}(1,v)$	7	3	6	2	5	1	4	0
$\mathbf{d}(2,v)$	3	7	2	6	1	5	0	4
$\mathbf{d}(3,v)$	0	1	3	2	5	4	6	7
$\mathbf{d}(2x+2,v)$	0	1	2	3	4	5	6	7
$\mathbf{d}(2x+3,v)$	7	6	5	4	3	2	1	0

A generalization of a magic square is a magic p-dimensional cube.

A magic p-dimensional cube of order n is a p-dimensional matrix

$$\mathbf{M}_n^p = |\mathbf{m}(i_1, i_2, \dots, i_p);\ 1 \le i_1, i_2, \dots, i_p \le n|,$$

containing natural numbers $1, 2, \dots, n^p$ such that the sum of the numbers along every row and every diagonal is the same, i.e. $\frac{n(n^p+1)}{2}$. (Note. A magic 1-dimensional cube $\mathbf{M}_n^1$ of order n is given by an arbitrary permutation of natural numbers $1, 2, \dots, n$ and for $p = 2$, of a magic p-dimensional cube $\mathbf{M}_n^p$ is a magic square.)

By a *row* of $\mathbf{M}_n^p$ we mean an n-tuple of elements $\mathbf{m}(i_1, i_2, \dots, i_p)$ which have identical coordinates at $p-1$ places. A magic p-dimensional cube $\mathbf{M}_n^p$ contains pn^{p-1} rows. A *diagonal* of $\mathbf{M}_n^p$ is an n-tuple of elements $\{\mathbf{m}(x, i_2, i_3, \dots, i_p),\ x = 1, 2, \dots, n,$ where $i_j = x$ or $i_j = \overline{x}$ for all $2 \le j \le p\}$. Every p-dimensional cube has exactly 2^{p-1} great diagonals.

In Figure 2 are depicted the nine layers of $\mathbf{M}_3^4$. The element $\mathbf{m}(1,1,1,1) = 46$ is in four rows containing the triplets of numbers $\{46, 8, 69\}$, $\{46, 62, 15\}$, $\{46, 17, 60\}$ and $\{46, 59, 18\}$. In the eight diagonals there are the triplets $\{\mathbf{m}(1,1,1,1) = 46, 41, 36\}$, $\{\mathbf{m}(1,1,1,3) = 69, 41, 13\}$, $\{\mathbf{m}(1,1,3,1) = 15, 41, 67\}$, $\{\mathbf{m}(1,1,3,3) = 35, 41, 47\}$,
$\{\mathbf{m}(1,3,1,1) = 60, 41, 22\}$, $\{\mathbf{m}(1,3,1,3) = 26, 41, 56\}$, $\{\mathbf{m}(1,3,3,1) = 53, 41, 29\}$ and
$\{\mathbf{m}(1,3,3,3) = 64, 41, 18\}$. This magic 4-dimensional cube was constructed using the following formula (see [19]).

$$\begin{aligned}\mathbf{m}(i_1, i_2, i_3, i_4) = {} & [(i_1 - i_2 + i_3 - i_4 + \tfrac{n+1}{2} - 1) \pmod n]n^3 \\ & + [(i_1 - i_2 + i_3 + i_4 - \tfrac{n+1}{2} - 1) \pmod n]n^2 \\ & + [(i_1 - i_2 - i_3 - i_4 + 3\tfrac{n+1}{2} - 1) \pmod n]n \\ & + [(i_1 + i_2 + i_3 + i_4 - 3\tfrac{n+1}{2} - 1) \pmod n] + 1.\end{aligned}$$

46	8	69	17	78	28	60	37	26
62	42	19	51	1	71	10	80	33
15	73	35	55	44	24	53	6	64
59	39	25	48	7	68	16	77	30
12	79	32	61	41	21	50	3	70
52	5	66	14	75	34	57	43	23
18	76	29	58	38	27	47	9	67
49	2	72	11	81	31	63	40	20
56	45	22	54	4	65	13	74	36

FIGURE 2 - MAGIC 4-DIMENSIONAL CUBE $\mathbf{M}_3^4$

In [19] and [20] we are concerned with the construction of a magic p-dimensional cube. There is proved the first definitive result for magic hypercubes.

Theorem. *(Trenkler)*

A magic p-dimensional cube $\mathbf{M}_n^p$ *of order n exists if and only if* $n \ne 2$ *and* $p > 1$ *or all n and* $p = 1$.

Various properties of magic squares have been studied, but there are many open problems (see [1]). The similar problems can be formulated and solved for magic p-dimensional cubes.

2. Magic graphs

We shall consider a non-oriented finite graph $\mathbf{G} = [V(\mathbf{G}), E(\mathbf{G})]$ without loops, multiple edges or isolated vertices. If there exists a mapping f from the set of edges $E(\mathbf{G})$ into positive real numbers such that

(i) $f(e_i) \neq f(e_j)$ for all $e_i \neq e_j;\ e_i, e_j \in E(\mathbf{G})$,

(ii) $\sum_{e\in E(G)} \eta(v,e)f(e) = r$ for all $v \in V(\mathbf{G})$,

$$\text{where } \eta(v,e) = \begin{cases} 1 & \text{when vertex } v \text{ and the edge } e \text{ are incident and} \\ 0 & \text{in the opposite case,} \end{cases}$$

then the graph $\mathbf{G}$ is called *magic*. The mapping f is called a *labelling* of $\mathbf{G}$ and the value r is the index of the label f. We say that a magic graph $\mathbf{G}$ is *supermagic* if there exists a mapping f into the set $1, 2, 3, \dots, |E|$.

To study magic graphs was suggested by Czech mathematician *Jiří Sedláček*. He noticed the correspondence between a magic square of order n and a magic complete bipartite graph $K_{n,n}$. Several sufficient conditions for the existence of magic graphs are established by many authors (*e.g. M.Bača, M.Doob, H.Enomoto, N.Hartsfield, J.Ivančo, R.H.Jeurissen, J.Mülbacher, K-W.Lih, S-M.Lee, G.Ringer, J.Sedláček, B.M.Stewart, M.Trenkler, V.Vetchý.*) There were published many papers about magic graphs (see [2] and [9].) Some authors used the term *magic graphs* in another meaning. Two following different characterizations of magic graphs were published in [11] and [12].

First we shall formulate several necessary definitions. We say that a graph $\mathbf{G}$ is of *type* $\mathbb{A}$ if it has two edges e, f such that $\mathbf{G} - e - f$ is a balanced bipartite graph with the partition V_1, V_2, and the edge e joins two vertices of V_1 and f joins two vertices of V_2. A graph $\mathbf{G}$ is of *type* $\mathbb{B}$ if it has two edges e_1, e_2 such that $\mathbf{G} - e_1 - e_2$ is a graph with two components $\mathbf{G}_1$ and $\mathbf{G}_2$ such that $\mathbf{G}_1$ is a balanced bipartite graph with partition V_1, V_2 and $\mathbf{G}_2$ is a non-bipartite graph, and e_i joins a vertex of V_i with a vertex of $V(\mathbf{G}_2)$. As usual, $\Gamma(S)$ denotes the set of vertices adjacent to a vertex in the set S.

Theorem. *(Jeurissen)*

A non-bipartite graph $\mathbf{G}$ *is magic if and only if* $\mathbf{G}$ *is neither of type* $\mathbb{A}$ *nor of type* $\mathbb{B}$*, and* $|\Gamma(S)| > |S|$ *for every independent subset* $S \neq \emptyset$ *of* $V(\mathbf{G})$.

A spanning subgraph $\mathbf{F}$ of the graph $\mathbf{G}$ is called a (1-2)-*factor* of $\mathbf{G}$ if each of its components is an isolated edge or a circuit. We say that a (1-2)-factor *separates the edges* e and f if at least one of them belongs to $\mathbf{F}$ and neither the edge part nor the circuit part contains both of them. In [11] the following theorem is proved. (Note. Later, the same characterization of magic graphs appeared in [7].)

Theorem. *(Jezný, Trenkler)*

A graph $\mathbf{G}$ *is magic if and only if every edge belongs to a (1-2)-factor, and every pair of edges* e*,* f *is separated by a (1-2)-factor.*

From this theorem it follows, then, that a graph is magic with real labels if and only if it is magic with integer labels.

If e and f is an arbitrary couple of edges of a bipartite magic graph $\mathbf{G}$ then it has a 1-factor which contains e and does not contain f. Evidently, every bipartite magic graph is an elementary graph (see [13,p.122]), which has a 1-factor for its arbitrary edge.

We give several results on magic and super-magic graphs.

Theorem. *(Trenkler* [18]*)*
A connected magic graph with n vertices and q edges exists if and only if $n = 2$ and $q = 1$ or $n \geq 5$ and $\frac{5n}{4} < q \leq \frac{n(n-1)}{2}$.

By an *I-graph* we mean a graph $\mathbf{G}$ with a 1-factor $\mathbf{F}$ whose every edge is incident with an *end-vertex* (a vertex of degree 1) of $\mathbf{G}$. The symbol $\mathbf{P}_5$ denotes a path of length 5.

Theorem. *(M.Trenkler, V.Vetchý* [22]*)*
Let a graph $\mathbf{G}$ have order $n \geq 5$. The graph $\mathbf{G}^2$ is magic if and only if $\mathbf{G}$ is not an I-graph and it is different from the path $\mathbf{P}_5$. The graph $\mathbf{G}^i$ is magic for all $i \geq 3$.

From this theorem it follows two unpublished theorems proved by *M.Sekanina*.

Theorem.
If $\mathbf{G}$ is a graph of even-order then $\mathbf{G}^2$ has 1-factor

Theorem.
If $\mathbf{G}$ is a graph of odd-order and v is an arbitrary vertex then $\mathbf{G}^2$ has a factor consisting of isolated edges and one triangle with the vertex v.

B.M.Stewart [16] has proved that for all $n \neq 0 \pmod 4$ and $n > 5$ the complete graph $\mathbf{K}_n$ is super-magic. It is easy to see that the classic concept of a magic square corresponds to the fact that the complete bipartite graph $\mathbf{K}_{n,n}$ is super-magic for all $n \neq 2$. *J.Sedláček* considered the graph $\mathbf{M}_{2n}$ (also called the Möbius ladder) and constructed a super-magic labeling for odd $n > 3$. Super-magic labeling for some classes of regular graphs of degree 4 were described in [4] and [10]. (For more information see [9].)

3. Magic hypergraphs

By a *complete k-partite hypergraph* $\mathbf{H}_n^k$ we mean a hypergraph with kn vertices divided into k independent sets each with n vertices and n^k hyperedges having exactly k vertices. (Note. We obtain $\mathbf{H}_n^k$ from a complete k-partite graph $\mathbf{K}_{n,n,\dots,n}$ by replacing all the edges of its every complete subgraph $\mathbf{K}_k$ by a hyperedge.) A hypergraph $\mathbf{H}_n^k$ is *magic* if the hyperedges can be labeled with different positive integers such that the sum of labels of the hyperedges incident to $(k-1)$ particular vertices is the same for all $(k-1)$-tuples of vertices from $(k-1)$ independent sets. Moreover, if the labels are consecutive integers $1, 2, \dots, n^k$ then $\mathbf{H}_n^k$ is called *super-magic*. In a similar way we can define a *magic* (or *super-magic*) hypergraph and its special case a *magic* (or *super-magic*) graph.

Super-magic complete bipartite graphs $\mathbf{K}_{n,n}$ generalize to super-magic complete k-partite hypergraphs $\mathbf{H}_n^k$. In [18] it is proved the following theorem.

Theorem. *(Trenkler)*

If $n \neq 2,6$ and $k \geq 2$ are positive integers, then the complete k-partite hypergraph $\mathbf{H}_n^k$ is super-magic.

From the correspondence between a supermagic complete k-partite hypergraph and a magic p-dimensional cubes follows that this theorem was improved by [20].

4. Generalized magic graphs and perfect matchings

Let $\mathbf{G} = [V(\mathbf{G}), E(\mathbf{G})]$ be a connected graph (without loops or multiple edges) with n vertices denoted by $v_1, v_2, \ldots, v_n$ and let $\beta = (b_1, b_2, \ldots, b_n)$ be an n-dimensional vector of positive real numbers. The graph $\mathbf{G}$ is called *β-non-negative* or *β-positive* if there exists a non-negative solution f to the system of linear equations

$$\sum_{e \in E(G)} \eta(v_i, e).f(e) = b_i \qquad \text{for} \qquad i = 1, 2, \ldots, n,$$

where $\eta(v_i, e) = 1$ when the vertex v_i and the edge e are incident or 0 otherwise. In other terms, if there exist non-negative or positive edge labels such that the sum of labels incident to v_i is b_i for all $1 \leq i \leq n$.

A β-positive graph is called a *generalized magic graphs* if no two edges have the same label. A characterization of generalized magic graphs was published in [14]. If we consider the vector β and the solution of non-negative integers our problem coincides with the problem known as *perfect b-matching* (see the book [13,p.271]). In the special case when β is a stationary vector of integer, the β-positive graphs has been call a *regularisable graph* on Berge's papers (see [13,p.218]), or a *semimagic graph* in [11] and [14].

In [15] *Ľubica Šándorová-Hudecová and M.Trenkler* proved the following three theorems. (Note. Tutte's characterization of *perfect* 2*-matching graphs* [13,p.216] is a particular case of the following theorem.)

Theorem.

Let $\mathbf{G}$ be a connected graph with n vertices $v_1, v_2, \ldots, v_n$ and let $\beta = (b_1, b_2, \ldots, b_n)$ be a vector of non-negative numbers. The graph $\mathbf{G}$ is β-non-negative if and only if

$$\sum_{v_1 \in S} b_i \leq \sum_{v_j \in \Gamma(S)} b_j \qquad \textit{for all independent} \quad S \neq \emptyset \quad \textit{of} \;\; \mathbf{G}.$$

Theorem.

Let $\mathbf{G}$ be a non-bipartite connected graph with n vertices $v_1, v_2, \ldots, v_n$ and let $\beta = (b_1, b_2, \ldots, b_n)$ be a vector of positive numbers. The graph $\mathbf{G}$ is β-positive if and only if

$$\sum_{v_1 \in S} b_i < \sum_{v_j \in \Gamma(S)} b_j \qquad \textit{for all independent} \quad S \neq \emptyset \quad \textit{of} \quad \mathbf{G}.$$

Theorem.

Let $\mathbf{G}$ be a bipartite graph with a partition V_1, V_2 having n vertices and let $\beta = (b_1, b_2, \ldots, b_n)$ be a vector of positive numbers. The graph $\mathbf{G}$ is β-positive if and only if

$$\sum_{v_i \in V_1} b_i = \sum_{v_j \in V_2} b_j$$

and

$$\sum_{v_1 \in S} b_i < \sum_{v_j \in \Gamma(S)} b_j \quad \textit{for all independent} \quad S \neq \emptyset,\ V_1,\ V_2.$$

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