A THEOREM ON THE STRUCTURE OF
CELL-DECOMPOSITIONS OF ORIENTABLE 2-MANIFOLDS

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1. Introduction. For a cell complex $M$ decomposing the closed orientable 2-
manifold $P^*_g$ of genus $g$ let $p_i(M)$ and $v_j(M)$ denote the number of $i$-gonal cells (faces, 
countries) and $j$-valent vertices of the graph (= 1 - skeleton) of $M$, respectively. It 
will be supposed that $i, j \geq 3$. From Euler’s formula follows

$$
\sum_{k \geq 3} (4 - k)(p_k(M) + v_k(M)) = 8(1 - g),
$$

and

$$
\sum_{k \geq 3} k p_k(M) = \sum_{k \geq 3} k v_k(M) = 2e
$$
is even, where $e$ is the number of edges of $M$. As seen, the relation above does not impose restrictions on the numbers $p_d(M)$, $v_d(M)$. In an attempt to characterize the vectors $\{p_i(M)\}$, $\{v_j(M)\}$ which partially determine the combinatorial structure of the cell-decompositions, the first step could involve answering the question: Given sequences $p = (p_3, p_5, \ldots)$, $v = (v_3, v_5, \ldots)$ of non-negative integers satisfying conditions

$$
\begin{align*}
\sum_{k \geq 3} (4 - k)(p_k + v_k) &= 8(1 - g), \\
\sum_{k \geq 3} k p_k &\equiv 0 \pmod{2},
\end{align*}
$$
does there exist a cell-decomposition $M$ of $P^*_g$, for which $p_i(M) = p_i$, $v_j(M) = v_j$
for all $i, j \neq 4$? (If so, the sequences $p, v$ are called realizable in the sequel. $M$ itself
is a realization of $p, v$. For brevity’s sake we will often use the word map instead of
cell-decomposition in the sequel.)

The problem was formulated and, for $g = 0$, solved in [3] by B. Grünbaum, who
For $g \geq 3$, assuming $v = (0, 0, 0, \ldots)$, E. Jucovič–M. Trenkler [5] gives the answer. 
The aim of the present paper is to settle the entire problem by proving the following

**Main Theorem.** Given sequences of non-negative integers $p = (p_3, p_5, \ldots)$, $v = (v_3, v_5, \ldots)$ satisfying, for some integer $g$, the conditions

$$
\sum_{k \geq 3} (4 - k)(p_k + v_k) = 8(1 - g)
$$

and

$$
\sum_{k \geq 3} k p_k \equiv 0 \pmod{2},
$$

there exists a cell-decomposition $M$ of the closed orientable 2-manifold $P^*_g$ of genus $g$
with $p_i(M) = p_i$, $v_j(M) = v_j$ for all $i, j \neq 4$, except in the cases $g = 1$ and
$p = (1, 1, 0, 0, \ldots)$, $v = (0, 0, \ldots)$ or $p = (0, 0, \ldots)$, $v = (1, 1, 0, 0, \ldots)$.

Because of the results in [1, 3, 4] it suffices to prove the Main theorem for $g \geq 2$.
This will be done in two theorems.
2. THEOREM 1. Every pair of sequences \( p, v \) satisfying (1) with \( g = 2 \) is realizable on \( P_2 \).

The proof consists of constructing, for every pair of sequences \( p, v \) satisfying (1), its realization. Unfortunately, many cases and subcases must be considered. In most of them first a toroidal map is constructed containing all the required \( k \)-gonal faces and \( m \)-valent vertices, \( k, m \neq 4 \), an unspecified number of quadrangles and 4-valent vertices, and two polygons \( O_1, O_2 \) with the same number of vertices which are all 3-valent. These polygons \( O_1, O_2 \), called "openings" in the sequel, are joined by a handle in the concluding stage of the construction; the handle is a cylindrical map containing quadrangles and 4-valent vertices, except those on the boundaries which are 3-valent. In some cases we shall construct two toroidal maps containing one opening each. One of the toroidal maps will realize the required faces (face-part of the map), the second one will realize the required vertices (vertex-part). The last step of the construction consists of unifying the two toroidal maps by identifying vertices and edges of the two openings. From the described strategy of the construction is seen that it is very important to ensure that the openings have equal numbers of 3-valent vertices.

Case 1. \( \sum_{k \geq 5} (k - 4) p_k \geq 10 \). In this case it follows from (1) that \( p_3 \geq 2 \) or \( v_3 \geq 2 \).

On each of two parallel sides of a rectangle \( R \) assign

\[
\left[ \frac{1}{2} \sum_{k \geq 5} (k - 4) p_k \right] + \left[ \frac{1}{2} \sum_{k \geq 5} (k - 4) v_k \right] + 1 = s + 1
\]

points \( K_1, \ldots, K_s, K_{s+1} \), on the mid-line parallel to these sides assign points \( L_1, \ldots, L_s, L_{s+1} \). Decompose one half of \( R \) into two hexagons \( O_1, O_2 \) and 2s - 5 quadrangles and 2s - 8 triangles as depicted by full lines in Fig. 1. In the sequel, this submap of the map constructed will be called the \( a \)-part, and the other submap will be called the \( b \)-part. To form a torus, equally marked vertices are identified, as well as the pairs \( K_1, K_{s+1} \), and \( L_1, L_{s+1} \). The same is done with pairs of points appearing in the course of the following construction on the boundary of \( R \). So not only equal numbers of trivalent vertices on the openings but also equal numbers of vertices on opposite sides of \( R \) must be maintained. The paths \( K_1, L_1 \) .... and \( K_{s+1}, L_{s+1} \) ... will be called "borders" of \( R \) in the sequel.

Before forming the torus the required \( k \)-gons (\( k \geq 5 \)) must be cut off from the polygon \( L_1 \ldots L_s K_s \ldots K_1 \). If \( k = 2m \), join a point \( P_1 \) between \( K_{m-1} \) and \( K_m \) with a point \( P_2 \) between \( L_{m-1} \) and \( L_m \) by an arc in the \( b \)-part. If \( P_2 \) does not belong to an edge of an opening the arc \( P_1, P_2 \) is extended by an edge \( P_2, P_3 \) into the \( a \)-part. If this is not the case one of the openings has increased its number of vertices by two; it will be shown later how balancing up the number of vertices of the second opening is performed.

The faces with odd numbers of vertices are cut off in pairs. Suppose we need a \( k \)-gon, \( k = 2m + 1 \), and a \( h \)-gon, \( h = 2n + 1 \), and these are the first we are constructing. Join, in the \( b \)-part, a point \( Q_1 \) between \( K_{m-1} \) and \( K_m \) with a point \( Q_4 \) between \( L_m \) and \( L_{m+1} \), then join a point \( Q_2 \) between \( K_n \) and \( K_{n+1} \) with a point \( Q_3 \) between \( L_{n-1} \) and \( L_n \). A point \( P_1 \) between \( K_{n+m-2} \) and \( K_{n+m-1} \) must now be joined with a point \( P_2 \) between \( L_{n+m-2} \) and \( L_{n+m-1} \) (dashed lines in Fig. 1, where \( k = 9 \), \( h = 7 \)). Again, as before, it is necessary to balance up the number of vertices on the openings. Another complication arises when an odd number of faces having \( \geq 5 \)
edges is required; from (1) it follows that $p_3$ must be odd in this case. Let the last face we are constructing in the $b$-part have an odd number $k \geq 7$ of edges. We cut off a $(k - 1)$-gon $F$ as before. If $F$ is adjacent to an opening $O_i$ then $O_i$ is surrounded by a belt consisting of 7 quadrangles and one triangle as depicted in Fig. 2a. If $F$ is not adjacent to an opening, i.e. a double triangle meets it, this double triangle is replaced by a triple of triangles and a quadrangle as depicted in Fig. 2b. Both these procedures increase the number of vertices of the face $F$ by one. If $k = 5$ the triangle in the belt mentioned above is put next to an opening, not meeting the $b$-part (Fig. 2c).

The procedure of balancing the number of vertices on the opening $O_1$ or $O_2$, if this is necessary, will now be described. If, for example, on $O_2$ $c = 2d$ ($c$ is always even!) trivalent vertices must be added, join $d$ points between $L_5$ and $L_6$ with $d$ points between $K_3$ and $K_4$ and these $d$ points between $K_3$ and $K_4$ with $d$ points between $K_5$ and $K_6$ by non-intersecting arcs. These form in the $b$-part as well as in the $a$-part quadrangles only (Fig. 3). This procedure of increasing the number of vertices of an opening is made possible since the double triangle can be placed between $O_1$.
and $O_2$ when $p_3 \geq 2$ (Fig. 1). If $p_3 < 2$, and so $v_3 \geq 2$, the double triangle is replaced by the pair of trivalent vertices $K_4, L_4$ as shown in Fig. 5, and the balancing proceeds similarly.

The above argument settles subcase

(a) $v = (0, 0, ...)$.

Now consider the subcase

(b) $\sum_{k \geq 2} (4 - k) v_k = 0$, $\sum_{i \geq 4} v_i \neq 0$, $v \neq (1, 1, 0, ...)$.

From the assumption made in case I it follows that after realizing the face-part a part $R_n$ of the map $R$ is available which does not contain an opening (redrawn in full lines in Fig. 4). In $R_n$ we cut off $v_i$ i-gonal faces for all $i \geq 5$ as before (the submap $R_n$ thus obtained will be called the dual of the vertex-part in the sequel). The numbers of vertices on the openings are balanced, and then $R_n$ is dualized (the dual of the map drawn in full lines in Fig. 4 is indicated by dashed lines). Joining both the face-part and the vertex-part is straightforward. (The pairs of vertices $k_{s+1}$ and $K_1, L_{s+1}$ and $L_1$ are identified, etc.)
(e) $\sum_{k > 3} (4 - k) v_k > 0$, $\sum_{k \geq 4} v_k \neq 0$, $v \neq (1, 1, 0, 0, \ldots)$. In the face-part $\sum_{k > 3} (4 - k) v_k = 2w$ triangles are changed into $w$ pairs of trivalent vertices by removing a pair of edges from each of the $w$ double triangles. (See Fig. 5 where the removed edges are marked by dashed lines.)

(d) $\sum_{k > 3} (4 - k) v_k < 0$, $\sum_{k \geq 4} v_k \neq 0$, $v \neq (1, 1, 0, \ldots)$. In the dual of the vertex-part $\sum_{k > 3} (k - 4) v_k = p_3^* = O (\mod 2)$ triangles are changed into trivalent vertices by the procedure above, and then $R_v$ is dualized.

Remark. The processes described above of changing triangles into trivalent vertices and vice versa will often be used in the sequel without stating it explicitly.

(e) $v = (1, 1, 0, \ldots)$. The vertex-part is depicted in Fig. 6 and is added to the face-part where it is necessary to increase the number of vertices on the border $K_1 L_1$ by two. This is performed as depicted in Fig. 1 by dot-and-dash lines, i.e. two points between $K_1$ and $L_1$ are joined with two points between $K_2$ and $L_2$ by non-intersecting edges. Then the number of vertices of $O_2$ is increased by two as before (Fig. 3).

Case II. $\sum_{k > 5} (k - 4) p_k = 8$ or $9$, $p_3 + 2p_6 \geq 2$, $\sum_{k > 5} (k - 4) v_k < 4.$

In this case the map in Fig. 1 cannot be used because there exist pairs of sequences satisfying (1) and the above conditions with $p_3 < 2$ and $v_5 < 2$. The double triangle between $O_1$ and $O_2$ would be superfluous. Moreover in cutting off the last $k$-gon, $k > 5$, we should be forced to dissect the opening $O_2$, which makes it impossible to continue the construction in the preceding manner. The map $R$ in Fig. 1 is replaced by another one.

(a) $p_3 \geq 2$. In the starting map (see full lines in Fig. 7) there is again a $b$-part, and an $a$-part consisting of two octagonal openings, two pentagons and quadrangles. Cutting off the required $k$-gons, $k > 5$, is slightly different from case I because each opening is surrounded by two belts consisting of quadrangles. The cutting paths join always two points on sides of an opening. As an example in Fig. 7 an octagon is cut off from the $b$-part by dot-and-dash lines.

Balancing the number of vertices on the borders, if needed, and adding the vertex-part to the face-part is performed as in case I. But increasing the number of vertices of an opening, for example $O_1$, is done in another way. A path joining two points of two edges of the opening $O_1$ is depicted around the pentagons and the opening $O_2$ (dashed lines in Fig. 7) forming a new series of quadrangles.

(b) $p_6 \geq 1$. The vertices $M_1, M_2$ as well as the edge $M_1 M_2$ from subcase (a) (Fig. 7) are deleted. The remaining part of the construction is the same as in (a).

Case III. $\sum_{k > 5} (k - 4) p_k = 5, 6, 7, 8$ or $9$, $v_5 + 2v_6 + 2v_7 \geq 2$, $\sum_{k > 5} (k - 4) v_k < 4$.

In fact, there cannot exist a pair of sequences $p, v$ satisfying (1) and $\sum_{k > 5} (k - 4) v_k = 5$ and the other two conditions above. Notice that the last condition implies $v_6 < 1$, $v_7 < 1$. Three subcases must be distinguished.

(a) $v_4 \geq 2$. In the map in Fig. 7 the vertices $M_3$ and $M_4$ are joined by a path in the $a$-part changing the pentagons into quadrangles. (The double triangle occurs there only if $\sum_{k > 5} (k - 4) p_k \geq 8$.) The construction continues as in case II.
(b) \( v_0 = 1 \). The starting map for constructing the face-part is shown in Fig. 8a (full lines) where a vertex \( V \) of degree 6 occurs. (In fact, this graph is a small modification of the graph in Fig. 7.) The same procedure as in subcase II(a) (indicated by dashed lines) is used to increase the number of vertices of an opening. Cutting off the required \( k \)-gons, \( k \geq 5 \), and adding the vertex-part which can contain a 5-valent and a 3-valent vertex (see Fig. 6), is performed as in that subcase too. (Again in Fig. 8a the double triangle occurs only if \( \sum_{k \geq 4} (k - 4) p_k \geq 8 \).)

(c) \( v_7 = 1 \). From the conditions it follows that \( v_3 \geq 1 \), and \( \sum v_i = 0 \) for \( i \neq 7,3 \). The starting map for cutting off all the required \( k \)-gons, \( k \geq 5 \), is depicted in Fig. 8a in full lines supplemented by new edges (dashed lines in Fig. 8b) forming from \( V \) a 7-valent vertex and adding a new 3-valent vertex \( W \). Balancing the number of vertices on the openings is performed as in subcase (b).
Case IV. $\sum_{k \geq 5} (k - 4) p_k = 5, 6, 7, 8$ or $9, p_5 + 2p_6 < 2, v_5 + 2v_6 + 2v_7 < 2,$
$\sum_{k \geq 5} (k - 4) v_k < 4.$

The sequence $\nu$ can be either $\nu^0 = (0, 0, ...)$ or $\nu^1 = (1, 1, 0, ...).$ It thus follows
that there are no pairs $p, v$ of sequences satisfying (1) and the above conditions with
$\sum_{k \geq 5} (k - 4) p_k = 5, 6$ or $7.$

For the sequence $p$ satisfying (1) only one of the following possibilities can occur:

$p^1 \parallel p_{12} = 1, p_i = 0$ for all $i \neq 12$;

$p^2 \parallel p_7 = p_9 = 1, p_i = 0$ for all $i \neq 7, 9$;

$p^3 \parallel p_5 = 2, p_i = 0$ for all $i \neq 8$;

$p^4 \parallel p_5 = 1, p_i = 3, p_i = 0$ for all $i \neq 3, 7$;

$p^5 \parallel p_5 = p_{11} = 1, p_i = 0$ for all $i \neq 5, 11$;

$p^6 \parallel p_5 = p_7 = p_9 = 1, p_i = 0$ for all $i \neq 3, 5, 7, 9$;

$p^7 \parallel p_5 = p_8 = 1, p_i = 2, p_i = 0$ for all $i \neq 3, 5, 8$;

$p^8 \parallel p_5 = p_8 = p_9 = 1, p_i = 0$ for all $i \neq 3, 8, 9$;

$p^9 \parallel p_5 = p_8 = p_{12} = 1, p_i = 0$ for all $i \neq 3, 5, 12$;

$p^{10} \parallel p_5 = p_{13} = 1, p_i = 0$ for all $i \neq 3, 13$;

$p^{11} \parallel p_5 = p_7 = p_8 = 1, p_i = 0$ for all $i \neq 5, 7, 8$;

$p^{12} \parallel p_5 = p_{10} = p_7 = 1, p_i = 0$ for all $i \neq 3, 7, 10.$

Realizations of all pairs $(p^i, v^j), i = 1, \ldots, 12, j = 0, 1,$ will now be demonstrated (see Fig. 9).

In Fig. 9a, 9b, 9c and 9d planar maps $R_1, R_2, R_3, R_4$ are depicted in each of which
the pairs of openings $a_i, b_i$, $i = 1, 2,$ should be joined by two handles. From this,
realization of the pair $(v_0, p^1), (v_0, p^2), (v_0, p^3)$ and $(v_0, p^4),$ respectively, result. All
other realizations $(v^i, p^j), i = 5, \ldots, 12,$ are obtained by adding the dashed arcs $A$
or \(B\) in Fig. 9e or 9f, respectively (where submaps of \(R_1, R_2, R_3, R_4\) are also depicted) as follows:

\((o^0, p^0)\) results from \(R_1\) and \(B\);

\((v_0, p^0)\) results from \(R_2\) and \(A\) and \(B\);

\((v_0, p^1)\) results from \(R_3\) and \(A\) and \(B\); (but instead of the openings \(a_1, a_2\) the openings 
\(b_1, b_2\) are used);

\((v^0, p^0)\) results from \(R_4\) and \(A\) and \(B\);

\((v^0, p^{10})\) results from \(R_1\) and \(A\);

\((v^0, p^{11})\) results from \(R_3\) and \(A\);

\((v^0, p^{12})\) results from \(R_2\) and \(A\).

The realization of the pair \((v^i, p^i), i = 1, \ldots, 12\) is obtained from the realization of 
\((v^0, p^0)\) in adding to it five edges as depicted in Fig. 9g.

Remark. The realizations of the pairs of sequences \((v^0, p^0)\), \((v^0, p^{10})\) and \((v^1, p^1)\) as well as their duals were kindly communicated by J. Zaks to the authors, who were not able to decide the question of their realizability. Zaks's maps enabled the authors to simplify their original proof of case IV.

Case V. \(\sum_{k \geq 5} (k - 4) p_k \geq 4, \sum_{k \geq 5} (k - 4) v_k \geq 4\). Two toroidal maps \(\rho_1, \rho_2\) will be constructed as in case I, but containing only one opening each. On \(\rho_1\), whose \(b\)-part is a \(2\left\lfloor \frac{1}{2} \sum_{k \geq 5} (k - 4) p_k \right\rfloor\)-gon, the \(k\)-gonal faces, \(k \geq 5\), are cut off as before. On \(\rho_2\), whose \(b\)-part is a \(2\left\lfloor \frac{1}{2} \sum_{k \geq 5} (k - 4) v_k \right\rfloor\)-gon, first \(v_k\) \(k\)-gonal faces are cut off and the numbers of vertices of the openings are balanced. This last operation is performed so that the opening is joined with the borders of the appropriate \(\rho_1\) by the suitable number of paths (dashed lines in Fig. 10a, where four vertices were added to \(O\)). Then \(\rho_2\) is dualized to \(\rho_2'\), and associated with the opening \(O_2\) of \(\rho_2\) is a vertex of \(\rho_2'\) incident with triangles only. This vertex is cut off to get an opening \(O_2'\) surrounded by quadrangles (Fig. 10b). The maps \(\rho_1, \rho_2'\) are joined by identifying their openings.

Case VI. \(\sum_{k \geq 5} (k - 4) p_k < 4\). This case is settled by dualization from cases I–V. This completes the proof of Theorem 1.

3. Theorem 2. Every pair of sequences \(p, v\) satisfying condition (1) with \(g \geq 3\) is realizable on \(P_e\).

Proof. Two cases and a number of subcases must be considered separately. The cases are distinguished depending on the sequence \(p\), without explicitly mentioning in all subcases the properties of the vector \(v\). This should be understood so that all vectors \(v\) are considered satisfying (1) and the condition of the appropriate subcase. In that way all pairs \(p, v\) satisfying (1) are indeed investigated.
Case I. \( \sum_{k \geq 5} (k - 4) p_k \geq 8(g - 1) \). Analogously as in the proof of Theorem 1 a toroidal map is first constructed. This consists of a 2s-gon,

\[
s = \left[ \frac{1}{3} \sum_{k \geq 5} (k - 4) p_k \right] + \left[ \frac{1}{3} \sum_{k \geq 5} (k - 4) p_k \right],
\]

2\((g - 1)\) hexagonal openings \(O_1, \ldots, O_{2g-2}\), \(s = 4(g - 1)\) double triangles, and quadrangles. (Fig. 11. Notice that the double triangle between \(O_1\) and \(O_2\) is lacking here.) Cutting off the required \(k\)-gons, \(k \geq 5\), forming the dual of the vertex-part, its dualizing and joining with the face-part are performed as in case 1 of the proof of Theorem 1. However, balancing the number of vertices on the openings and on the borders \(K_1 L_1, K_{s+1} L_{s+1}\) must be done as follows (cf. [4]). In any step when the number of vertices of one opening \(O_i\) is increased by two, the numbers of vertices of all openings \(O_j \neq O_i\) are also increased by two. This is done by joining pairs of openings to the right and to the left from \(O_i\) by two edges (or paths) each. (Fig. 12.) Because on one side of \(O_i\) there is an odd number of openings, the last of them is joined by two edges (or paths) with the border. (It can happen that these paths meet the triangles as in Fig. 12.) In order to finish forming the toroidal map the numbers of vertices on the borders \(K_1 L_1, K_{s+1} L_{s+1}\) must be balanced.
Suppose, for example, that \( r \geq y \), where \( r \) and \( y \) are the number of vertices on the border \( K, L \) and on the border \( K_{s+1}, L_{s+1} \), respectively.

We choose \( r - y \) points \( H_1, ..., H_{r-y} \) between \( K_1 \) and \( L_1 \), and \( r - y \) points \( G_1, ..., G_{r-y} \) between \( K_2 \) and \( L_2 \), and join \( H_i \) and \( G_i \) by non-intersecting edges (or paths) (Fig. 13). As \( g \geq 3 \) there are at least three openings different from \( O_1 \). We choose between the vertices \( K_5 \) and \( L_5 \) (\( K_6 \) and \( L_6 \), \( K_7 \) and \( L_7 \), \( K_8 \) and \( L_8 \)) \( c = \frac{1}{2}(r-y) \) points \( F_1, ..., F_c \) (\( N_1, ..., N_c \), \( P_1, ..., P_c \), and \( S_1, ..., S_c \), respectively), and join \( F_i \) with \( N_i \), and \( P_i \) with \( S_i \), \( i = 1, ..., c \), by non-intersecting paths, increasing the number of 3-valent vertices of the openings \( O_2 \) and \( O_4 \) or \( O_3 \) by \( c \) or \( r - y \), respectively. New quadrangles and possibly 4-valent vertices are also created. At the end of this operation the borders \( K, L \) and \( K_{s+1}, L_{s+1} \), the pairs of openings \( O_1 \) and \( O_3 \), \( O_2 \) and \( O_4 \) have equal numbers of vertices. Of course all other openings \( O_{31}, ..., O_{2s+2} \) have equal numbers of vertices too. The last stage of the construction—forming of the \((g-1)\) handles and adding them to the toroidal map constructed—can now follow.

All the subcases of this case I are precisely identical with those in case I of Theorem I, and are settled in the same manner (except balancing up the numbers of
vertices on the openings and the borders). So, we shall not describe the procedure again.

In case I of Theorem 1, it was assumed that $\sum_{h>5} (k-4)p_h \geq 10(g-1) = 10$, because in the starting map a double triangle was placed between the openings $O_1$ and $O_2$ to enable the balancing of numbers of vertices on the openings. However, in the proof of the Theorem 2 this is done in a different way. Notice that in case I of both Theorem 1 and 2, the face-part still includes all the openings.

Remarks. 1. The described procedure of balancing the numbers of vertices on the borders cannot be employed for $g = 2$. That is why a different approach for $g = 2$ was necessary.

2. Notice that in case I as well as in the following case II the increase in the number of vertices of an opening or a border is always even.

Case II. $\sum_{h>5} (k-4)p_h = 8(u-1) + t$, where $u = 1, 2, 3, \ldots, g-1$.

Four subcases must be considered, according to the value of the number $t$.

(a) $t = 0$ or 4. The face-part is constructed as in case I. The assumptions ensure that the arc cutting off the last $k$-gon, $k \geq 5$, does not meet an opening, and that the face-part does not include all the openings. That is why after forming all $k$-gons, $k \geq 5$, from the polygon $K_1 K_2 \ldots K_5 \ldots L_3 L_4 v_i i$-gons for all $i \geq 5$ are cut off. This will be referred to as the dual of the vertex-part too (although it is not precisely the dual map to that containing the required vertices).

Balancing the number of vertices on the openings and the borders is performed as before. The dual of the vertex-part is separated from the face-part by a series of quadrangles which we include in the dual of the vertex-part, and then this part $R_x$
of the map \( R \) is dualized (Fig. 14) analogously as was done in subcase 1(b) of Theorem 1. Analogously as in case V of the proof of Theorem 1 the vertices of the vertex-part which are associated to the openings of the map \( R \) are cut off to obtain polygons with 3-valent vertices as openings (dot-and-dash lines in Fig. 14) of the vertex-part, and quadrangles. Both the face-part and the vertex-part are then joined as before; handle forming can follow.

(b) \( i = 1 \) or \( 5 \). This condition means that an odd number of \( k \)-gonal faces \( (k \geq 5 \text{ odd}) \) is required. (From (1) it then follows that \( p_3 \equiv 1 \pmod{2} \).) Forming the face-part is started as in subcase (a). The last face formed is one of the \( k \)-gonal
faces, where \( k \equiv 1 \pmod{2} \). This is done in the following way. A \((k - 1)\)-gonal face \( F \) is cut off (the cutting path does not meet an opening). Around the opening, which is adjacent to the face \( F \), a belt consisting of 7 quadrangles and one triangle is put changing \( F \) into a \( k \)-gon (Fig. 2a). Forming the dual of the vertex-part is performed as in subcase (a), but first the number of vertices on the openings and on the borders is balanced.

(c) \( t = 2 \) or 6. If we proceeded in this case as in the preceding ones, the arc cutting off the last face in the face-part would meet an opening, and the rest of the construction would fail. The map employed above for forming the face-part must therefore be changed.

From the assumptions made we see that \( p_3 \) is even.

\( \text{\( p_3 \) is odd} \). If \( p_3 > 1 \) then a double triangle is placed between the border \( K_1 L_1 \) and the opening \( O_1 \) (Fig. 15). If \( v_3 > 1 \) and \( p_3 = 0 \) such a double triangle is changed into a pair of trivalent vertices (cf. Fig. 5); the construction continues as in subcase (a).

If \( p_3 + v_3 = 0 \), then at least one of the following possibilities (c2) or (c3) must occur.

(c2) For some odd numbers \( q, z, v_q \not= 0, v_z \not= 0 \), and \( z + q \equiv 2 \pmod{4} \).

Let \( z = q \). The starting toroidal map is the same as in the preceding subcases II(a), (b). (Fig. 11.) However, before forming the face-part, a \( z \)-valent and a \( q \)-valent vertex will be formed from \( L_1 \) and \( K_1 \), respectively; this will be done in the following way (Fig. 16a): Choose a point \( U_1 \) and \( V_1 \) between the vertices \( L_3 \) and \( L_4 \), and \( K_3 \) and \( K_4 \), respectively, and join \( L_1 \) with \( U_1 \), and \( K_1 \) with \( V_1 \) by edges. Join \( U_1 \) and \( V_1 \) by a path in the \( a \)-part also. Only quadrangles are cut off. Analogously choose a point \( U_2 \) and \( V_2 \) between the vertices \( L_4 \) and \( L_5 \), and \( K_4 \) and \( K_5 \), respectively. Join

**Fig. 16a**
$L_1$ with $U_2$, and $K_1$ with $V_2$ by edges. The vertices $U_2$, $V_2$ are not joined by an edge because they are on the boundary of the opening. But balancing the number of vertices of the other openings must be performed at once. This is repeated until $K_1$ and $L_1$ have degree $(q - 1)$; after identifying the vertices on the borders $K_1$, $L_1$, $K_{x+1}$, $L_{x+1}$, the vertices $K_1$, $L_1$ have degree $q$.

If $z > q$, then $z - q = 4f$ for some positive integer $f$. First the vertices $K_1$ and $L_1$ are made of degree $q - 1$ by the preceding construction. Then the degree of $L_1$ is increased by $4f$ (Fig. 16b). A little computation shows that the $(q - 4)$-th edge which we have drawn from the vertex $L_1$ does not meet the boundary of an opening.

From the preceding construction it follows that the vertices $U_{q-4}$ and $V_{q-4}$ lie between the vertices $L_{q-1}$ and $L_q$, and $K_{q-1}$ and $K_q$, respectively. Join the point $L_1$ with a point $Z_2$ lying between $L_q$ and $L_{q+1}$, with a point $X_1$ lying between $V_{q-4}$ and $K_q$, with a point $X_2$ lying between $K_q$ and $K_{q+1}$, and with a point $X_3$ lying between $L_{q+1}$ and $K_{q+2}$. A path $X_1Z_1Z_2X_3$ is formed where $Z_1$ and $Z_3$ lie between $U_{q-4}$ and $L_q$, and $L_{q+1}$ and $L_{q+2}$, respectively. So the degree of $L_1$ is increased by 4, and only new quadrangles and 4-valent vertices have been introduced. If necessary, we now further increase the degree of $L_1$ by 4. This is done analogously, except that, in a certain sense, the roles of the vertices $Z_i$ and $X_i$ are interchanged. Join $L_1$ with a point $X_4$ lying between $K_{q+2}$ and $K_{q+3}$, and with a point $Z_4$ lying between $L_{q+3}$ and $L_{q+4}$. Form a path $Z_4X_4X_5Z_5$, where $X_4$ and $X_6$ lie between $X_3$ and $K_{q+2}$, and $K_{q+3}$ and $K_{q+4}$, respectively. Proceed in this way until $L_1$ has degree $z$. In the $b$-part of the map $R$ there now occurs a $(2s + 8 - z - q)$-gon. From this polygon all required $k$-gons, $k > 5$, are cut off and the remaining required vertices are constructed as in subcase (a).

(c₃) For some $k \equiv 2 \pmod{4}$, $v_k \neq 0$. 
In this case the starting map $R$ must yet again be changed, as shown in Fig. 17. This requires an explanation. The graph in Fig. 17b is placed in the cross-hatched area of the graph in Fig. 17a. (If $k \equiv 2 \pmod{8}$ the two paths $\mu_1$ and $\mu_2$ do not occur there.) In the map there are a $(k - 1)$-valent vertex $V$, and $\frac{1}{2}(k - 2)$ openings $O_1, \ldots, O_{(k-2)/4}$. All openings are octagonal, except $O_1$ which is a quadrangle. Two vertices of each of the openings $O_2, \ldots, O_{(k-2)/4}$ are joined with $V$ by an edge. The opening $O_1$ is also joined with $V$ by an edge, and further two belts consisting of quadrangles surrounding $O_1$ and all $O_2, \ldots, O_{(k-2)/4}$ are inserted. The vertex $V$ is now of degree $k - 1$. So a map is obtained in which formation of the face-part and the dualized vertex-part (except one vertex of degree $k$) can be performed as already described. But some additional explanation is needed concerning balancing the numbers of vertices on the openings and on the borders.

If in a step of the construction the number of vertices of an opening $O_1$ is increased by two, the same is done as above in case 1 (Fig. 12) with all other openings, with the
possible exception of \( O_1 \). (Pairs of openings to the right and to the left from \( O_1 \) are joined by pairs of paths.) The opening \( O_1 \) cannot be joined by paths with the border \( K_1 L_1 \). In this way all the openings \( O_2, \ldots, O_{2k-2} \) have, after forming the face-part and the dual of the vertex-part, the same number \( r \) of vertices. Let \( y < r \) be the number of vertices of the opening \( O_1 \). Join the openings \( O_1 \) and \( O_2 \) by \((r - y)\) mutually non-intersecting paths; do the same with the opening \( O_{2k-2} \) and the border \( K_{s+1} L_{s+1} \). After this operation the two openings \( O_2 \) and \( O_{2k-2} \) have the same number \((2r - y)\) of vertices and can be joined by a handle. All other openings have \( r \) vertices. It remains to balance the number of vertices of the borders, i.e. to add an even number of vertices to the border \( K_1 L_1 \). This is done as depicted in Fig. 17b by dashed lines. Mutually non-intersecting paths join points of the edge \( V L_1 \) forming quadrangles only.

\((c)_1\) \( p_3 \) or \( v_3 = 1 \). For example, we consider the case \( v_3 = 1 \). From the assumptions made it follows that an odd number \( n \) of odd-valent vertices of degree \( \geq 5 \) is required. If \( n \geq 3 \) at least two of them satisfy condition \((c)_3\). If \( n = 1 \), so that \( v_3 = 1 \) for some odd \( d \), the sequence \( v \) is replaced by the sequence \( v' \) defined as follows: \( v'_0 = 0, v'_{d-1} = v_{d-1} + 1, v_0 = 0, v'_i = v_i \) for all \( i \neq 3, d - 1 \). The sequences \( p, v \) satisfying condition \((c)_3\) are realized, and the vertex \( V \) (Fig. 17) is \((d - 1)\)-valent. The degree of \( V \) is increased by one and a trivalent vertex is created as in case III(c) of the proof of Theorem 1 (Fig. 8b).

\((d)\) \( t = 3 \) or \( 7 \). From this condition it follows that an odd number of \( k \)-gonal faces \((k \geq 5 \) odd\) is required, so that \( p_3 \geq 1 \). Suppose, for example, that \( j \geq 5 \) and \( p_j \) are both odd numbers. Consider the sequence \( p' \) defined as follows: \( p'_0 = p_3 - 1, p'_{j-1} = p_{j-1} + 1, p'_j = p_{j-1} - 1, p'_i = p_i \) for all \( i \neq 3, j - 1, j \). The sequences \( p', v \) satisfy the conditions of subcase \((c)\) and are realized as described above with the following additional demand: the last polygon cut off from the face-part is a \((j - 1)\)-gon \( J \). As we already know, the last cutting arc does not meet an opening. Let \( O_i \) be the last opening contained in the face-part (so it must be adjacent to the polygon \( J \)). Arrange this opening as it was done in subcase \((b)\) (Fig. 2a). Then the operations of balancing the number of vertices of the remaining openings and the borders, dualizing the vertex-part, joining the vertex-part with the face-part, etc., can follow.

This completes the proof of Theorem 2, and so of the Main Theorem also.

5. Remark. With very small modifications the procedure employed in proofs of our Theorem 1 and 2 can be used in the proof of the Main Theorem for \( g = 0 \) and 1(cf. [1]). Thus the proof of the whole Main Theorem could be put in a unified form.

References


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