A construction of magic cubes

MARIÁN TRENKLER

In this paper a *magic cube* of order n is a 3-dimensional matrix $\mathbf{Q}_n = \{\mathbf{q}_n(i,j,k);\, 1 \leqslant i,j,k \leqslant n\}$ containing natural numbers 1,2, ... n^3 such that the sums of the numbers along each row (n-tuple of elements having the same coordinates on two places) and also along each of its four great diagonals are the same. This contrasts with a previous article [1] which considered magic cubes with no requirement for the diagonal sums.

In Figure 1 a magic cube $\mathbf{Q}_3$ is depicted. The sum of the three numbers in every row is 42. The sums of the numbers on the diagonals (the triplets {8, 14, 20}, {19, 14, 9}, {10, 14, 18} and {6, 14, 22}) are 42 too.

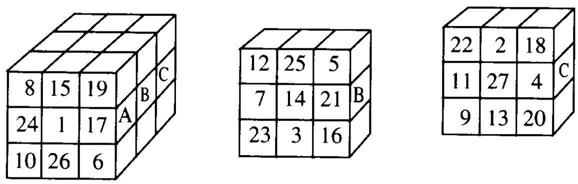

FIGURE 1 – Magic cube $\mathbf{Q}_3$

Using a pair of orthogonal Latin squares, it was proved that such matrices of order n exist for each $n \neq 2$. Probably the first mention of a magic cube (of order 4) appeared in a letter from Fermat on April 1, 1640 (see [2, p. 365].) More information on magic squares and cubes can be found in books [2] and [3]. In this paper we describe, in three steps, a construction of a magic cube $\mathbf{Q}_n$ for every integer $n \neq 2$. (In a similar way we can construct a magic square $\mathbf{M}_n$ for every integer $n \neq 2$.)

Step 1.

If n is an odd integer, then a magic cube $\mathbf{Q}_n$ can be constructed using the following formula

$$\mathbf{q}_n(i,j,k) = \left[(i-j+k-1) - n\left\lfloor \tfrac{i-j+k-1}{n} \right\rfloor\right] n^2 +$$
$$\left[(i-j-k) - n\left\lfloor \tfrac{i-j-k}{n} \right\rfloor\right] n + \left[(i+j+k-2) - n\left\lfloor \tfrac{i+j+k-2}{n} \right\rfloor\right] + 1.$$

The symbol $\lfloor x \rfloor$ denotes the integer part of x.

The formula was derived using two mutually orthogonal Latin squares of odd order n, $\mathbf{R}_n = \left\{\mathbf{r}(i,j) = (i+j+a) - \left\lfloor \tfrac{i+j+a}{n} \right\rfloor\right\}$ and $\mathbf{S}_n = \left\{\mathbf{s}(i,j) = (i-j+b) - \left\lfloor \tfrac{i-j+b}{n} \right\rfloor\right\}$ where a and b are two constants and the formula is taken from [1, p. 57]. This formula can be rewritten as

$$\mathbf{q}_n^*(i,j,k) = \mathbf{s}(i, \mathbf{s}(j,k))n^2 + \mathbf{s}(i, \mathbf{r}(j,k))n + \mathbf{r}(i, \mathbf{r}(j,k)) + 1$$

The constants a and b were chosen so that for $m = (n+1)/2$

$$\mathbf{s}(m, \mathbf{s}(m,m)) = \mathbf{s}(m, \mathbf{r}(m,m)) = \mathbf{r}(m, \mathbf{r}(m,m)) = \tfrac{n-1}{2}.$$

The proof of the correctness of our formula is similar to [1, p. 58]. The sum on every diagonal is the same because for each triple (i, j, k) from the definition of $\mathbf{Q}_n$ it follows ($\bar{x}$ denotes the number $n + 1 - x$)

$$\mathbf{q}_n(i, j, k) + \mathbf{q}_n(\bar{i}, \bar{j}, \bar{k}) = \sum_{k=0}^{2} (n - 1)n^k + 2 = n^3 + 1$$

and

$$\mathbf{q}_n\left(\tfrac{n+1}{2}, \tfrac{n+1}{2}, \tfrac{n+1}{2}\right) = \tfrac{n^3+1}{2}.$$

The sum of numbers on each diagonal is $\frac{(n-1)}{2}\left(n^3 + 1\right) + \frac{(n^3+1)}{2} = \frac{n(n^3+1)}{2}$.

Step 2.

If $n = 4k$, $k = 1, 2, 3, \ldots$, then a magic cube $\mathbf{Q}_n$ can be constructed by the following formulas

$$\mathbf{q}_n(i,j,k) = (k-1)n^2 + (j-1)n + i \quad \text{if } \mathcal{I}(i,j,k) \text{ is odd,}$$
$$\mathbf{q}_n(i,j,k) = (n-k)n^2 + (n-j)n + (n-i) + 1 \quad \text{if } \mathcal{I}(i,j,k) \text{ is even,}$$

where $\mathcal{I}(i,j,k) = \left(i + \left\lfloor \frac{2(i-1)}{n} \right\rfloor + j + \left\lfloor \frac{2(j-1)}{n} \right\rfloor + k + \left\lfloor \frac{2(k-1)}{n} \right\rfloor\right)$.

In Figure 2 a magic cube $\mathbf{Q}_4$ is depicted. The sums of the four numbers in each row are 130. The sums of the numbers on the diagonals (the quadruples {1, 43, 22, 64}, {4, 42, 23, 61}, {13, 39, 26, 52} and {16, 38, 27, 49}) are 130 too.

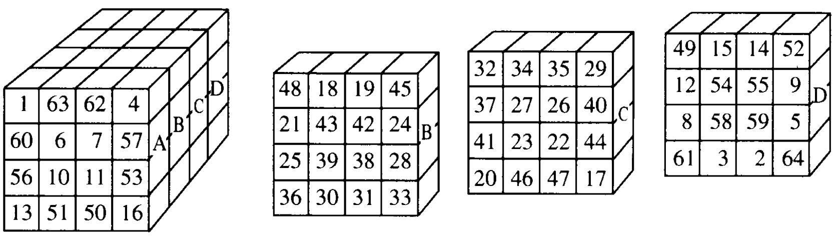

FIGURE 2 – Magic cube $\mathbf{Q}_4$

The proof of the correctness of our formulas follows from the following three facts:

(i) No two elements with different coordinates are the same because $\mathcal{I}(\bar{i}, \bar{j}, \bar{k})$ is odd, if and only, if $\mathcal{I}(i, j, k)$ is odd.

(ii) The sums of numbers in the rows are the same, because, for every odd coordinate i (or j, or k):

$$\mathbf{q}_n(i,j,k) + \mathbf{q}_n(i+1,j,k) = n^3 \quad \text{or} \quad n^3 + 2$$
$$\mathbf{q}_n(i,j,k) + \mathbf{q}_n(i,j+1,k) = n^3 - n + 1 \quad \text{or} \quad n^3 + n + 1$$
$$\mathbf{q}_n(i,j,k) + \mathbf{q}_n(i,j,k+1) = n^3 - n^2 + 1 \quad \text{or} \quad n^3 + n^2 + 1.$$

In every row there are $n/4$ pairs of elements whose sum is n^3 (or $n^3 - n + 1$ or $n^3 - n^2 + 1$) and the same number of pairs whose sum is $n^3 + 2$ (or $n^3 + n + 1$ or $n^3 + n^2 + 1$).

(iii) The sums on the diagonals are the same, because, for every triple (i, j, k):

$$\mathbf{q}_n(i, j, k) + \mathbf{q}_n(\bar{i}, \bar{j}, \bar{k}) = n^3 + 1.$$

Step 3.

If $n = 4k + 2$, $k = 1, 2, 3, \ldots$, then a construction of a magic cube $\mathbf{Q}_n$ starts from a magic cube $\mathbf{Q}_{n/2}$ and an auxiliary cube $\mathbf{V}_n = \{\mathbf{v}_n(i, j, k)\}$ of order n. First we describe the construction of $\mathbf{V}_n$. In Figure 3 six layers of $\mathbf{V}_6$ are shown.

0	5	5	0	6	5
3	6	3	6	3	0
3	5	5	0	3	5
6	0	3	6	6	0
6	0	0	6	3	6
3	5	5	3	0	5

1st Layer

7	2	2	7	1	2
4	1	4	1	4	7
4	2	2	7	4	2
1	7	4	1	1	7
1	7	7	1	4	1
4	2	2	4	7	2

2nd Layer

0	5	5	6	0	5
3	6	0	3	3	6
6	0	3	6	6	0
3	5	5	0	3	5
6	0	3	6	6	0
3	5	5	0	3	5

3rd Layer

7	2	2	1	7	2
4	1	7	4	4	1
1	7	4	1	1	7
4	2	2	7	4	2
1	7	4	1	1	7
4	2	2	7	4	2

4th Layer

5	0	5	0	6	3
3	6	3	6	3	0
0	3	3	6	3	6
5	6	5	0	0	5
5	0	5	3	3	5
3	6	0	6	6	0

5th Layer

2	7	2	7	1	2
4	1	4	1	4	7
7	4	4	1	4	7
2	1	2	7	7	2
2	7	2	4	4	2
4	1	7	1	1	7

6th Layer

FIGURE 3

This cube consists of 27 cubelets of order 2, each containing the numbers 0, 1, … ,7. The arrangements of the numbers in cubelets is such that the sums of numbers in each row and each diagonal are the same, i.e. 21.

If $n = 6(2k + 1)$ for $k = 1, 2, 3, \ldots$, then $\mathbf{V}_n$ is obtained by the composition of $(2k + 1)^3$ copies of $\mathbf{V}_6$ such that

$$\mathbf{v}_n(i, j, k) = \mathbf{v}_6\left(i - 6\left\lfloor \tfrac{i-1}{6} \right\rfloor, j - 6\left\lfloor \tfrac{j-1}{6} \right\rfloor, k - 6\left\lfloor \tfrac{k-1}{6} \right\rfloor\right) \text{for all } 1 \leqslant i, j, k \leqslant n.$$

If $n = 6(2k + 1) + 4$ for $k = 0, 1, 2, \ldots$, then our construction starts from $\mathbf{V}_m$, where $m = n - 4$ and six matrices (in pairs of orders

$m \times m \times m \times 2$ and $m \times 2 \times 2$ and $2 \times 2 \times 2$) called *blocks*. Blocks

$$\begin{aligned} \mathbf{A} &= \{\mathbf{a}(i,j,k)\} \text{ and } \overline{\mathbf{A}} = \{\bar{\mathbf{a}}(i,j,k)\}, \quad 1 \leqslant i,j \leqslant m,\ 1 \leqslant k \leqslant 2, \\ \mathbf{B} &= \{\mathbf{b}(i,j,k)\} \text{ and } \overline{\mathbf{B}} = \{\bar{\mathbf{b}}(i,j,k)\}, \quad 1 \leqslant i \leqslant m,\ 1 \leqslant j,k \leqslant 2, \\ \mathbf{C} &= \{\mathbf{c}(i,j,k)\} \text{ and } \overline{\mathbf{C}} = \{\bar{\mathbf{c}}(i,j,k)\}, \quad 1 \leqslant i,j,k \leqslant 2 \end{aligned}$$

are defined by relations

$$\mathbf{a}(i,j,k) = \mathbf{b}(i,j,k) = \mathbf{c}(i,j,k) = \mathbf{v}_m(i,j,k),$$

$$\bar{\mathbf{a}}(i,j,k) = \bar{\mathbf{b}}(i,j,k) = \bar{\mathbf{c}}(i,j,k) = 7 - \mathbf{v}_m(i,j,k),$$

for all define elements with coordinates (i, j, k).

The auxiliary cube $\mathbf{V}_n$ consists of a cube $\mathbf{V}_m$, three pairs of blocks $\mathbf{A}$ and $\overline{\mathbf{A}}$, six pairs of $\mathbf{B}$ and $\overline{\mathbf{B}}$ and four pairs of $\mathbf{C}$ and $\overline{\mathbf{C}}$. The blocks $\mathbf{C}$ and $\overline{\mathbf{C}}$ are situated by the vertices of $\mathbf{V}_n$, blocks $\mathbf{B}$ and $\overline{\mathbf{B}}$ are situated by the edges of $\mathbf{V}_n$ and blocks $\mathbf{A}$ and $\overline{\mathbf{A}}$ at the opposite faces of $\mathbf{V}_m$. In Figure 4a there are the first two layers of the auxiliary cube $\mathbf{V}_n$, in Figure 4b there are the x-th, for $x = 3, 4, 5, \ldots, n-2$, layers and in Figure 4c there are the last two (the $(n-1)$-th and the n-th) layers.

Every diagonal of $\mathbf{V}_n$ is formed from diagonals of $\mathbf{C}$, $\mathbf{V}_m$ and $\overline{\mathbf{C}}$. It follows from this construction that the sum of numbers in every row and every diagonal of $\mathbf{V}_n$ is greater by 14 than the sum in any row of $\mathbf{V}_m$.

$\mathbf{c}(1,1,x)$	$\mathbf{c}(1,2,x)$	$\mathbf{b}(1,1,x)$	…	$\mathbf{b}(m,1,x)$	$\bar{\mathbf{c}}(1,1,x)$	$\bar{\mathbf{c}}(1,2,x)$
$\mathbf{c}(2,1,x)$	$\mathbf{c}(2,2,x)$	$\mathbf{b}(1,2,x)$	…	$\mathbf{b}(m,2,x)$	$\bar{\mathbf{c}}(2,1,x)$	$\bar{\mathbf{c}}(2,2,x)$
$\mathbf{b}(1,1,x)$	$\mathbf{b}(1,2,x)$	$\mathbf{a}(1,1,x)$	…	$\mathbf{a}(1,m,x)$	$\bar{\mathbf{b}}(1,1,x)$	$\bar{\mathbf{b}}(1,2,x)$
…	…	…	…	…	…	…
$\mathbf{b}(m,1,x)$	$\mathbf{b}(m,2,x)$	$\mathbf{a}(m,1,x)$	…	$\mathbf{a}(m,m,x)$	$\bar{\mathbf{b}}(2m,1x)$	$\bar{\mathbf{b}}(m,2,x)$
$\bar{\mathbf{c}}(1,1,x)$	$\bar{\mathbf{c}}(1,2,x)$	$\bar{\mathbf{b}}(1,1,x)$	…	$\bar{\mathbf{b}}(m,1,x)$	$\mathbf{c}(1,1,x)$	$\mathbf{c}(1,2,x)$
$\bar{\mathbf{c}}(2,1,x)$	$\bar{\mathbf{c}}(2,2,x)$	$\bar{\mathbf{b}}(1,2,x)$	…	$\bar{\mathbf{b}}(m,2,x)$	$\mathbf{c}(2,1,x)$	$\mathbf{c}(2,2,x)$

FIRST AND SECOND LAYER OF $\mathbf{V}_n$, $x = 1, 2$.

$\mathbf{b}(x,1,1)$	$\mathbf{b}(x,1,2)$	$\mathbf{a}(x,1,1)$	…	$\mathbf{a}(x,m,1)$	$\bar{\mathbf{b}}(x,1,1)$	$\bar{\mathbf{b}}(x,1,2)$
$\mathbf{b}(x,2,1)$	$\mathbf{b}(x,2,2)$	$\mathbf{a}(x,1,2)$	…	$\mathbf{a}(x,m,2)$	$\bar{\mathbf{b}}(x,2,1)$	$\bar{\mathbf{b}}(x,2,2)$
$\mathbf{a}(x,1,1)$	$\mathbf{a}(x,1,2)$	$\mathbf{v}_m(1,1,x)$	…	$\mathbf{v}_m(1,m,x)$	$\bar{\mathbf{a}}(x,1,1)$	$\bar{\mathbf{a}}(x,1,2)$
…	…	…	…	…	…	…
$\mathbf{a}(x,m,1)$	$\mathbf{a}(x,m,2)$	$\mathbf{v}_m(m,1,x)$	…	$\mathbf{v}_m(m,m,x)$	$\bar{\mathbf{a}}(x,m,1)$	$\bar{\mathbf{a}}(x,m,2)$
$\bar{\mathbf{b}}(x,1,1)$	$\bar{\mathbf{b}}(x,1,2)$	$\bar{\mathbf{a}}(x,1,1)$	…	$\bar{\mathbf{a}}(x,m,1)$	$\mathbf{b}(x,1,1)$	$\mathbf{b}(x,1,2)$
$\bar{\mathbf{b}}(x,2,1)$	$\bar{\mathbf{b}}(x,2,2)$	$\bar{\mathbf{a}}(x,1,2)$	…	$\bar{\mathbf{a}}(x,m,2)$	$\mathbf{b}(x,2,1)$	$\mathbf{b}(x,2,2)$

$(x+2)$-th LAYER OF $\mathbf{V}_n$, $x = 1, 2, \ldots, m$

$\bar{\mathbf{c}}(1,1,x)$	$\bar{\mathbf{c}}(1,2,x)$	$\bar{\mathbf{b}}(1,1,x)$	...	$\bar{\mathbf{b}}(m,1,x)$	$\mathbf{c}(1,1,x)$	$\mathbf{c}(1,2,x)$
$\bar{\mathbf{c}}(2,1,x)$	$\bar{\mathbf{c}}(2,2,x)$	$\bar{\mathbf{b}}(1,2,x)$	...	$\bar{\mathbf{b}}(m,2,x)$	$\mathbf{c}(2,1,x)$	$\mathbf{c}(2,2,x)$
$\bar{\mathbf{b}}(1,1,x)$	$\bar{\mathbf{b}}(1,2,x)$	$\bar{\mathbf{a}}(1,1,x)$	...	$\bar{\mathbf{a}}(1,m,x)$	$\mathbf{b}(1,1,x)$	$\mathbf{b}(1,2,x)$
...	...	...	...	...	...	...
$\bar{\mathbf{b}}(m,1,x)$	$\bar{\mathbf{b}}(m,2,x)$	$\bar{\mathbf{a}}(m,1,x)$	...	$\bar{\mathbf{a}}(m,m,x)$	$\mathbf{b}(m,1,x)$	$\mathbf{b}(m,2,x)$
$\mathbf{c}(1,1,x)$	$\mathbf{c}(1,2,x)$	$\mathbf{b}(1,1,x)$	...	$\mathbf{b}(m,1,x)$	$\bar{\mathbf{c}}(1,1,x)$	$\bar{\mathbf{c}}(1,2,x)$
$\mathbf{c}(2,1,x)$	$\mathbf{c}(2,2,x)$	$\mathbf{b}(1,2,x)$	...	$\mathbf{b}(m,2,x)$	$\bar{\mathbf{c}}(2,1,x)$	$\bar{\mathbf{c}}(2,2,x)$

$(n - x + 2)$-th LAYER OF $\mathbf{V}_n$, $x = 1, 2$

FIGURE 4

If $n = 6(2k + 1) + 8$, then we repeat the previous construction.

We define a magic cube $\mathbf{Q}_n$ by the following relation

$$\mathbf{q}_n(i,j,k) = \mathbf{v}_n(i,j,k)\tfrac{n^3}{8} + \mathbf{q}_{n/2}\left(\left\lfloor\tfrac{i+1}{2}\right\rfloor, \left\lfloor\tfrac{j+1}{2}\right\rfloor, \left\lfloor\tfrac{k+1}{2}\right\rfloor\right).$$

A construction of magic squares

Similarly we find that a magic square $\mathbf{M}_n = \{\mathbf{m}_n(i,j); 1 \leqslant i,j \leqslant n\}$ of odd order n can be constructed using the following formula

$$\mathbf{m}_n(i,j) = \left[\left(i - j - \tfrac{n-1}{2}\right) - n\left\lfloor\tfrac{i-j+\frac{n-1}{2}}{n}\right\rfloor\right]n + \left[\left(i + j + \tfrac{n-3}{2}\right) - n\left\lfloor\tfrac{i+j+\frac{n-3}{2}}{n}\right\rfloor\right] + 1,$$

and, for $n = 4k$, by the formulas

$$\mathbf{m}_n(i,j) = (i-1)n + j \qquad \text{if } \mathcal{I}(i,j) \text{ is odd,}$$
$$\mathbf{m}_n(i,j) = (n-i)n + (n-j) + 1 \quad \text{if } \mathcal{I}(i,j) \text{ is even,}$$

where $\mathcal{I}(i,j) = \left(i + \left\lfloor\frac{2(i-1)}{n}\right\rfloor + j + \left\lfloor\frac{2(j-1)}{n}\right\rfloor\right)$.

Remark

Analogous formulas can be derived for magic d-dimensional hypercubes of order $n \neq 4k + 2$, for every integer $k \geqslant 1$, and every $d \geqslant 4$. (See [1].) For example, if $d = 4$ and n is odd, then we start from the formula

$$\mathbf{q}_n^*(i,j,k,l) = \mathbf{s}(i, \mathbf{s}(j, \mathbf{s}(k,l)))n^3 + \mathbf{s}(i, \mathbf{s}(j, \mathbf{r}(k,l)))n^2 + \mathbf{s}(i, \mathbf{r}(j, \mathbf{r}(k,l)))n + \mathbf{r}(i, \mathbf{r}(j, \mathbf{r}(k,l))) + 1.$$

Let $n = 4k + 2$ for $k = 1, 2, 3, \ldots$. Analogously to a magic cube $\mathbf{Q}_n$, we can construct a magic square $\mathbf{M}_n$ of order $n = 4k + 2$. It is constructed using $\mathbf{M}_{n/2}$ and an auxiliary square $\mathbf{W}_n = \{\mathbf{w}_n(i,j)\}$. The construction of $\mathbf{W}_n$ starts from $\mathbf{W}_6$ (situated in the middle of Figure 5). For example, in Figure 5, $\mathbf{W}_{10}$ is depicted which was obtained using $\mathbf{W}_6$. If $n = 6(2k + 1) + 4$ or $6(2k + 1)$, then (to preserve the sum of numbers on its diagonals) the last two rows are changed. The elements of $\mathbf{M}_n$ are

$$\mathbf{m}_n(i, j) = \mathbf{w}_n(i, j)\tfrac{n^2}{4} + \mathbf{m}_{n/2}\left(\left\lfloor \tfrac{i+1}{2} \right\rfloor, \left\lfloor \tfrac{j+1}{2} \right\rfloor\right).$$

1	2	1	2	0	3	0	3	2	1
0	3	0	3	2	1	2	1	3	0
1	0	1	2	0	3	0	3	2	3
2	3	0	3	2	1	2	1	1	0
0	2	3	0	3	0	3	0	3	1
3	1	2	1	2	1	2	1	0	2
0	2	1	0	2	3	0	3	3	1
3	1	2	3	0	1	2	1	0	2
3	0	2	0	1	2	1	2	0	3
2	1	3	1	3	0	3	0	1	2

FIGURE 5 – AUXILIARY SQUARE $\mathbf{V}_{10}$

References

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MARIÁN TRENKLER
Šafárik University, Jesenná 5, 041 54 Košice, Slovakia