ON FUZZY RANDOM VARIABLES: EXAMPLES AND GENERALIZATIONS

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ABSTRACT. There are random experiments in which the notion of a classical random variable, as a map sending each elementary event to a real number, does not capture their nature. This leads to fuzzy random variables in the Bugajski-Gudder sense. The idea is to admit variables sending the set $\Omega$ of elementary events not into the real numbers, but into the set $\mathcal{M}_1^+(\mathbb{R})$ of all probability measures on the real Borel sets (each real number $r \in \mathbb{R}$ is considered as the degenerated probability measure $\delta_r$ concentrated at $r$). We start with four examples of random experiments ($\Omega$ is finite); the last one is more complex, it generalizes the previous three, and it leads to a general model.

A fuzzy random variable is a map $\varphi$ of $\mathcal{M}_1^+(\Omega)$ into $\mathcal{M}_1^+(\Xi)$, where $\mathcal{M}_1^+(\Xi)$ is the set of all probability measures on another measurable space $(\Xi, \mathcal{B}(\Xi))$, satisfying certain measurability condition. We show that for discrete spaces the measurability condition holds true. We continue in our effort to develop a suitable theory of $ID$-posets, $ID$-random variables, and $ID$-observables. Fuzzy random variables and Markov kernels become special cases.

INTRODUCTION

Information about quantum structures can be found, e.g., in DVUREČENSKIJ and PULMANNOVÁ [4], FOULIS [5], and the references therein. In this section we present some basic information about $D$-posets and effect algebras related to fuzzy random variables. The purpose is twofold: to make the paper more self-contained and to point out some ideas from GUDDER [13], BUGAJSKI [1, 2], FRIC [11], PAPČO [15], leading to generalized fuzzy random variables discussed in the last section of the present paper.

$D$-posets have been introduced in KÔPKA and CHOVAŁEC [14]. Recall that a $D$-poset is a partially ordered set $A$ possessing the top element 1, the bottom element 0, and carrying two additional algebric structures: a partial operation $\odot$ called difference ($a \odot b$ is defined iff $b \leq a$) and a complementation sending $a \in A$ to $a' = 1 \ominus a$. The following axioms are assumed:

(D1) $a \odot 0 = a$ for all $a \in A$;
(D2) $c \leq b \leq a$ implies $a \odot b \leq a \odot c$ and $(a \odot c) \odot (a \odot b) = b \odot c$.

Canonical examples are the unit interval $I = [0, 1]$, the $D$-posets of fuzzy sets $I^X$ carrying the pointwise order, difference, and complementation ($g \leq f$ whenever $g(x) \leq f(x)$ for all $x \in X$).
such that the composition space morphisms. If \( p : (X, Y) \to A \) is a duality between \( ID \)-measurable spaces and \( E \) into \( B \) can be translated to the corresponding effect algebras of fuzzy functions. The reader is referred to DVUREČENSKIJ and PULMANNOVÁ [4] and FOULIS [5] for more details. As pointed out in FRIČ [10] categorical constructions for \( ID \)-posets can easily be translated to the corresponding effect algebras of fuzzy functions.

Let \( (Ω, A) \) be a (classical) measurable space (i.e. \( A \) is a \( σ \)-algebra of subsets of \( Ω \)). Denote \( Ev(Ω) \) the set of all measurable functions into \( I \). Then \( Ev(Ω) \) carrying the partial addition \( + \) (the sum \( f + g \) is defined iff \( f(x) + g(x) \leq 1 \) for all \( x \in Ω \) and then \( f \oplus g = f + g \) ) is a canonical example of an effect algebra.

Let \( (Ω, A), (Ξ, B) \) be measurable spaces. We identify each measurable set with its characteristic function so that \( A \subset Ev(Ω) \) and \( B \subset Ev(Ξ) \). Then (cf. FRIČ [7]) each classical measurable map \( f : Ω \to Ξ \) induces a sequentially continuous boolean homomorphism of \( B \) into \( A \). Suitable effect algebra homomorphisms of \( B \) into \( Ev(Ω) \) are called fuzzy observables in GUDDER [12] and play a crucial role in the fuzzy probability theory developed by S. Gudder, S. Bugajski and others (cf. GUDDER [13], BUGAJSKI, HELLWIG and STULPE [3], BUGAJSKI [1, 2]).

1. Examples

In this section we present four examples of random experiments in which the notion of a classical random variable does not capture their nature. Recall that in
Consider ten objects of all probability measures on degenerated point measure on point $\omega$ variables – the idea is that for some random experiments it is natural to map a probability distribution $\mathcal{B}(\Omega)$ of the probability that particular segment (when activated) remains “off” or being “on” and all other elements are “on”; other numbers are announced via their usual coding. Each element, after being activated, can remain “off” (independently on the other ones) with a given probability. This results in a distorted outcome. It can be any of $2^7$ combinations; its probability can be calculated as the product of the probabilities that particular segment (when activated) remains “off” or becomes “on”. Denote $\Omega$ the set of all $2^7$ possible distorted outcomes. In the classical model (no distortions) we would have 10 standard outcomes and a classical random variable sending the non distorted digit to the number $i$ corresponding to the object $O_i$. In the fuzzy model (distortions) we need a fuzzy random variable mapping each distorted outcome $\omega \in \Omega$ to some probability distribution $P_\omega$ indicating the probability that $O_i$ has been drawn (input $O_i$) provided $\omega$ is on the display (output $\Omega$).

We can proceed as follows. Knowing the probabilities of (independent) failures of each of the seven segments of the display, for each $\omega \in \Omega$ and each $O_i$, $i \in \{0, \ldots, 9\}$, we can calculate the conditional probability $P(\{\omega\}|\{O_i\})$ of the output $\omega \in \Omega$ given that $O_i$ has been drawn. This enables us to construct an auxiliary probability space $\Omega \times \{O_i; \; i = 0, 1, \ldots, 9\}$ and probability $P$ on it. Indeed, $P(\{\omega\}) = P(\{O_i\})$ is the probability of the “simultaneous occurrence of $\omega$ and $O_i$”. The Bayes rule does the rest. This way we can calculate
The previous example can be generalized in several directions. First, instead of the one-digit display represented by seven elements, each admitting two states (i.e. 2^7 outputs), we can consider a more general set of outputs. Second, we can have a different number of objects, say n, drawn at random (i.e. n inputs); naturally, n is at most the number of outputs. Assume that the display now consists of k elements and each element can be exactly in one of m states i ∈ {1, . . . , m}. Hence we have altogether m^k outputs of the form of k-tuples (a(1), . . . , a(k)), a(j) ∈ {1, . . . , m}. We assign each object o_i, i ∈ {1, . . . , n}, its code c_i represented by a k-tuple of states (c_i(1), . . . , c_i(k)), where c_i(j) ∈ {1, . . . , m}, meaning that the j-th element, j ∈ {1, . . . , k}, is in the state c_i(j). Further, we have a probability distribution {p_i; i ∈ {1, . . . , n}}, where p_i is the probability that the object o_i has been drawn. If o_i has been drawn, then we send to each segment of the display a signal to put it into the corresponding state assigned by the code c_i, i.e. the j-th segment should be in state c_i(j). Due to some random noise, instead of c_i, some distorted outcome can occur. One of the possible ways how the input can be distorted is the following.

For each i ∈ {1, . . . , n} and for each c_i(j), j ∈ {1, . . . , k}, we have a probability distribution p_{ij}; l ∈ {1, . . . , m}, where p_{ijl} is the probability that if o_i has been drawn and the j-th coordinate of the code c_i is c_i(j), then the j-th segment is in the state l (instead of c_i(j)); the distribution depends only on i and j.

Now, our model can be viewed as a random walk. Indeed, at the starting point s we proceed (random draw) according to the probability distribution p_i; i ∈ {1, . . . , n} to some object o_i, from o_i we proceed to (o_i, a(1)), a(1) ∈ {1, . . . , m}, with the probability p_{i1a(1)}, from (o_i, a(1)) we proceed to (o_i, a(1), a(2)), a(2) ∈ {1, . . . , m} with probability p_{i2a(2)}, and so on, until we reach (o_i, a(1), . . . , a(k − 1), a(k)), where a(1), . . . , a(k − 1) are already fixed and a(k) ∈ {1, . . . , m} occurs with probability p_{ika(k)}; since the steps are stochastically independent, the probability of the given path (from s to o_i) and, “step by step through a(j)”, j ∈ {1, . . . , k − 1}, to the outcome (a(1), . . . , a(k)) is p_i · p_{i1a(1)} · . . . · p_{ika(k)}. This yields an auxiliary (discrete) probability space points (the elementary events) of which are all possible paths of our random walk. Since we know the probability of each path, we can calculate the conditional probability that the object o_i has been drawn (input) given that we see the distorted version (a(1), . . . , a(k)) of the code c_i, i ∈ {1, . . . , n} on the display (outcome). So, we have two classical probability spaces: the space of objects and the space of outputs on the display. It is natural to consider a fuzzy random variable, mapping each output not to a particular input (object o_i) but to the conditional probability distribution on objects. Since the space of objects is discrete (events are subsets of \{o_1, . . . , o_n\}), it is enough to know the probabilities of singletons \{o_i\}, i ∈ {1, . . . , n}. As we shall see, such mapping always satisfies the corresponding measurability condition, hence it gives rise to a fuzzy random variable in the Gudder-Bugajski sense (mapping probabilities to probabilities).

Example 1.2. The previous example can be generalized in several directions. First, instead of the one-digit display represented by seven elements, each admitting two states (i.e. 2^7 outputs), we can consider a more general set of outputs. Second, we can have a different number of objects, say n, drawn at random (i.e. n inputs); naturally, n is at most the number of outputs. Assume that the display now consists of k elements and each element can be exactly in one of m states i ∈ {1, . . . , m}. Hence we have altogether m^k outputs of the form of k-tuples (a(1), . . . , a(k)), a(j) ∈ {1, . . . , m}. We assign each object o_i, i ∈ {1, . . . , n}, its code c_i represented by a k-tuple of states (c_i(1), . . . , c_i(k)), where c_i(j) ∈ {1, . . . , m}, meaning that the j-th element, j ∈ {1, . . . , k}, is in the state c_i(j). Further, we have a probability distribution {p_i; i ∈ {1, . . . , n}}, where p_i is the probability that the object o_i has been drawn. If o_i has been drawn, then we send to each segment of the display a signal to put it into the corresponding state assigned by the code c_i, i.e. the j-th segment should be in state c_i(j). Due to some random noise, instead of c_i, some distorted outcome can occur. One of the possible ways how the input can be distorted is the following.

For each i ∈ {1, . . . , n} and for each c_i(j), j ∈ {1, . . . , k}, we have a probability distribution p_{ij}; l ∈ {1, . . . , m}, where p_{ijl} is the probability that if o_i has been drawn and the j-th coordinate of the code c_i is c_i(j), then the j-th segment is in the state l (instead of c_i(j)); the distribution depends only on i and j.

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the output) have been calculated. There are four possible outputs: \((0, 0), (0, 1), (1, 0), (1, 1)\) and if the second element \(s_2\) is “out”, then we know for sure that \(s_2\) has been drawn. This is due to the fact that if \(s_1\) has been drawn, then we try to activate (with positive probability success) only the first element \(s_1\). Hence, if \(s_2\) is “on”, then \(s_1\) is out of question. If we view the situation as a random walk, then there are two types of path. If \(a_1\) has been drawn, then we stop after the next step and there are only two possible paths. If \(a_2\) has been drawn, then we stop after two additional steps (we proceed from \(a_2\) to \(s_1\) and then to \(s_2\)). Our auxiliary probability space of paths now has six elementary events (points).

Our next example describes a random walk with no returns in a finite set \(\{s\} \cup M\) starting in a fixed points \(s\). It can be straightforwardly generalized to a countable set.

**Example 1.4.** Let \(M\) be a finite set (containing at least four points) and let \(s\) be a point, \(s \notin M\). Let \(k\) be a natural number (i.e. a positive integer) and let \(M\) be the union of mutually disjoint nonempty sets \(M_i, \ i \in \{1, \ldots, k + 1\}\). For each \(i \in \{1, \ldots, k + 1\}\), let \(M_i\) be the union of disjoint sets \(T_i\) and \(F_i\). We assume that \(T_i \neq \emptyset\) for all \(i \leq k\) and \(T_{k+1} = \emptyset\). The random walk starts at the point \(s\), proceeds to a point of \(M_1\) (step 1) and then, inductively, from a point of \(T_i\) to a point of \(M_{i+1}\) (step \(i\)). Points of \(T_i\) are transitional, points of \(F_i\), \(i \in \{1, \ldots, k + 1\}\) are final – the walk stops whenever we reach a point of \(F_i\) (after \(i\) steps). We consider a transitional probability distribution \(\{P_i(s, y) ; y \in M_1\}\), where \(P_i(s, y)\) is the probability that our random walk proceeds from \(s\) to \(y\) in \(M_1\). For each point \(x \in T_i\), we consider a transitional probability distribution \(\{P_i(x, y) ; y \in M_{i+1}\}\), where \(P_i(x, y)\) is the probability that our random walk proceeds from \(x\) to \(y\) in \(T_1\) to a point of \(M_{i+1}\) (independently of the path from \(s\) to \(x\)).

We can visualize \(\{s\} \cup M\) as a subset of the Euclidean plane \(E_2\). We identify \(s\) and \([0, 0]\), \(T_i = \{(i, j) \in E_2 ; 0 < j \leq n_i\}, F_i = \{(i, j) \in E_2 ; n_i < j \leq n_i + l_i\}, \ i \in \{1, \ldots, k\}, M_{k+1} = F_{k+1} = \{(k + 1, j) \in E_2 ; 0 < j \leq l_{k+1}\}\), where \(n_i\) is the cardinality of \(T_i\), \(l_i\) is the cardinality of \(F_i\) (possibly \(l_i = 0\) for some \(i \in \{1, \ldots, k\}\)); our walk randomly proceeds from \(s = [0, 0]\) to the right to a point \(a = [1, j] \in T_1 \cup F_1\) with probability \(P_1(s, j)\). If \([1, j] \in F_1\), our random walk stops; otherwise we proceed to \([2, m] \in T_2 \cup F_2\) with probability \(P_2([1, j], [2, m])\). If \([2, m] \in F_2\), our random walk stops; otherwise we proceed in a similar way until we stop in some point of \(F_i, \ i \in \{1, \ldots, k + 1\}\).

Each path (trajectory) starts at the point \(s = [0, 0]\), ends at some point \([i, j_i] \in F_i\) and it will be represented by an ordered \(i\)-tuple \((j_1, j_2, \ldots, j_i)\), \(0 < i \leq k + 1\), where \(j_i\) indicates the position after step \(i\). E.g., if the walk ends in the step 1, then its path is represented by \((j_1)\), \(n_1 < j_1 \leq n_1 + l_1\), if the walk ends in the step \(k + 1\), then its path is represented by \((j_1, \ldots, j_k, j_{k+1})\), where \(0 < j_i \leq n_i\) for \(i \in \{1, \ldots, k\}\) and \(0 < j_{k+1} \leq l_{k+1}\). Denote \(\Omega\) the set of all possible paths. It is easy to see that for \(\omega = (j_1, \ldots, j_i)\) the probability \(P\) of \(\omega\) is equal to the product of the corresponding transitional probabilities: \(P([0, 0], [1, j_1]) \cdot P_2([1, j_1], [2, j_2]) \cdot \ldots \cdot P_{k+1}([k, j_k], [k + 1, j_{k+1}])\), and \((\Omega, exp, \Omega, P)\) is the corresponding discrete probability space.

Denote \(H_n = \{(j_1, \ldots, j_n) \in \Omega; j_1 = n\}, n \in \{1, \ldots, n_1, \ldots, n_1 + l_1\}\), i.e., \(H_n\) is the set of all paths such that in the first step we reach \([1, n]\) \(\in M_1\). Clearly, \(H_n \cap H_m = \emptyset\) for \(n \neq m\) and the union of all \(H_n\) is the set \(\Omega\), hence \(\{H_n; n \in \{1, \ldots, n_1 + l_1\}\}\) is a partition of \(\Omega\). Denote \(F = \bigcup_{i=1}^{k+1} F_i\) the set of all final points. For \(f \in F\), denote \(E_f = \{(j_1, \ldots, j_i) \in \Omega; f = [i, j_i]\}\), i.e., \(E_f\) is the
set of all paths stopping at \( f \). Since \( F_i \cap F_j = \emptyset \) for \( i \neq j \), \( \{ E_f, f \in F \} \) is also a partition of \( \Omega \).

Observe that, in a sense, the previous examples are special cases of the construction of \( (\Omega, \exp \Omega, P) \). Indeed, Example 1.1 can be recovered as follows. Objects \( O_0, \ldots, O_9 \) can be visualized as the points of \( M_1 \) (\( F_1 \) being empty), the set of outcomes at the one digit display (consisting of seven elements, enumerated by digits 0, \ldots, 9), the set \( M_2 \) (\( F_2 \) being empty) consists of two states “on”, “off” of the first segment, the set \( M_3 \) consists of four states (\( 4 = 2^2 \)) of the first and the second segments, \ldots, the set \( M_7 \) consists \( 2^6 \) states of the first up to the sixth segments; our random walk starts with drawing (step 1) one of the objects \( O_i, \ i \in \{0, \ldots, 9\} \), the second step is the activation of the first segment of the display according to \( O_i \) (\( O_i \) sends a signal “yes” or “no” to each of the seven segments of the display), \ldots, the eighth step is the activation of the seventh segment of the display according to \( O_i \). Other examples (Example 1.1 and Example 1.3) can be reconstructed analogously.

In all four examples we have the points of \( M_1 \) as the input and the points of \( F = \bigcup_{k=1}^7 F_k \) as the output. The random walk represents sending signals (“on”, “off”) to the segments of the display. If we start at a given point of \( M_1 \), we can end up at different points of \( F \). For each path, its probability is given by the product of corresponding transitional probabilities, i.e., probabilities of success and failure when activating the corresponding segment according to the input. The fuzzy random variable resulting from our model of the random walk sends each output to the conditional probability that the walk proceeds through the respective points of the input set \( M_1 \) given that we have finished our walk at the given output; it can be calculated via the Bayes rule related to sets \( H_n \) and \( E_f \).

### 2. Discrete fuzzy random variables

Let \( (X, \mathcal{X}) \) be an ID-measurable space (we always assume that \( \mathcal{X} \) is reduced, i.e., if \( x \neq y \), then \( u(x) \neq u(y) \) for some \( u \in \mathcal{X} \)). Recall that \( (X, \mathcal{X}) \) is called sober if each state on \( \mathcal{X} \) is fixed (i.e. for each sequentially continuous \( D \)-homomorphism \( s \) on \( \mathcal{X} \) into \( I \) there exists a unique point \( x \in X \) such that \( s \) is the evaluation \( ev_x \) at \( x \) defined by \( ev_x(u) = u(x) \), \( u \in \mathcal{X} \subseteq I^X \)). Define \( X^* \) to be the set of all states on \( \mathcal{X} \) and consider \( X \) as a subset of \( X^* \). Each \( u \in \mathcal{X} \subseteq I^X \) can be protracted to \( u^* \in I^{X^*} \) by putting \( u^*(s) = s(u) \), \( s \in X^* \). Then (cf. Papčo [15]) \( (X^*, X^*) \) is a unique (up to an isomorphism) sober \( D \)-measurable space such that \( X \subseteq X^* \), \( X^* \upharpoonright X = X \), and \( (X, \mathcal{X}) \) and \( (X^*, X^*) \) have the same states; it is called the sobrification of \( (X, \mathcal{X}) \). Further, if \( f \) is a measurable map of \( (X, \mathcal{X}) \) into an \( ID \)-measurable space \( (Y, \mathcal{Y}) \), then there exists a unique measurable map \( f^* \) of \( (X^*, X^*) \) into the sobrification \( (Y^*, \mathcal{Y}^*) \) of \( (Y, \mathcal{Y}) \) such that \( f^* \) restricted to \( X \) is equal to \( f \).

The \( ID \)-posets \( X \) and \( X^* \) are isomorphic and \( u \mapsto u^* \) yields an isomorphism. For each sequentially continuous \( D \)-homomorphism \( h \) of \( Y^* \) into \( X^* \) there exists a unique measurable map \( f^* : X^* \rightarrow Y^* \) such that \( h(u^*) = u^* \circ f^* \). The correspondence between \( h \) and \( f^* \) yields a categorical duality between \( ID \)-posets and sober \( ID \)-measurable spaces. If \( X \) is sequentially closed in \( I^X \), then \( X \) is said to be closed (or sequentially complete). There exists a unique closed \( ID \)-poset \( \sigma(X) \subseteq I^X \) such that \( X \subseteq \sigma(X) \) and \( X \) and \( \sigma(X) \) have the same states. Passing from \( X \) to \( \sigma(X) \) yields an epireflector, also called the sequential completion of \( ID \)-posets into
the category of closed $ID$-posets. In particular, each sequentially continuous $D$-homomorphism $h : Y \rightarrow X$ can be uniquely extended to a sequentially continuous $D$-homomorphism $h_u : \sigma(Y) \rightarrow \sigma(X)$.

Example 2.1. Let $(\Xi, B(\Xi))$ be a classical measurable space and let $M^+_1(\Xi)$ be the set of all probability measures (nonnegative and normed) on $B(\Xi)$. Consider $\Xi$ as a subset of $M^+_1(\Xi)$ consisting of point measures. To each $B \in B(\Xi)$ there corresponds a unique fuzzy subset $u_B \in I^{M^+_1(\Xi)}$ defined by $u_B(p) = p(B)$, $p \in M^+_1(\Xi)$. It is known (cf. Section 3 in PAPČO [15]) that if $X = M^+_1(\Xi)$ and $\mathcal{X} = \{u_\omega, \ B \in B(\Xi)\}$, then $(X, \mathcal{X})$ is a sober object of $ID$ and each $p \in M^+_1(\Xi)$ is a morphism of $ID$. Let $(\Xi, B(\Xi))$ and $(\Omega, B(\Omega))$ be classical measurable spaces and let $f$ be a measurable map of $\Omega$ into $\Xi$. Denote $(X, \mathcal{X})$, $(Y, \mathcal{Y})$ the corresponding observable (cf. BUGAJSKI [1, 2], GUDDER [12], FRIČ [10, 11]).

An efficient way how to describe a fuzzy random variable is via a Markov kernel. Let $(\Omega, B(\Omega))$ and $(\Xi, B(\Xi))$ be classical measurable spaces, let $E(\Omega)$ and $E(\Xi)$ be the effect algebras of all measurable functions to $I = [0, 1]$. Recall (cf. GUDDER [13]) that a map $K : \Omega \times B(\Xi) \rightarrow I$ is a Markov kernel if $K(\cdot, B)$ is measurable for each $B \in B(\Xi)$ and $K(\omega, \cdot)$ is a probability measure for each $\omega \in \Omega$. Observe that $K(\cdot, B)$ yields a sequentially continuous $D$-homomorphism of $B(\Xi)$ into $E(\Omega)$ sending $B \in B(\Xi)$ to $K(\cdot, B) \in E(\Omega)$ and a map of $M^+_1(\Omega)$ into $M^+_1(\Xi)$ sending $p \in M^+_1(\Omega)$ into $K(p) \in M^+_1(\Xi)$ defined by $(K(p))(B) = \int K(\omega, B)d\mu$, $B \in B(\Xi)$.

This is how a statistical map, or an operational random variable, is defined (cf. BUGAJSKI [1, 2]). Markov kernels are easy to handle in case $(\Omega, B(\Omega))$ and $(\Xi, B(\Xi))$ are discrete spaces (i.e. $\Omega$ and $\Xi$ are at most countable and $B(\Omega)$ and $B(\Xi)$ are the fields of all subsets of $\Omega$ and $\Xi$, respectively).

Let $(\Omega, B(\Omega))$ and $(\Xi, B(\Xi))$ be discrete. Let $K_0$ be a map of $\Omega \times \Xi$ into $[0, 1]$ such that $\sum_{\xi \in \Xi} K_0(\omega, \xi) = 1$ for each $\omega \in \Omega$. Define $K : \Omega \times B(\Xi)$ as follows:

$$K(\omega, B) = \sum_{\xi \in B} K_0(\omega, \xi).$$

Lemma 2.2. $K$ is a Markov kernel.

Proof. Since $(\Omega, B(\Omega))$ is a discrete measurable space, each function of $\Omega$ into $[0, 1]$ is measurable. Hence for each $B \in B(\Xi)$ the function $K(\omega, B)$ is measurable. It follows directly from the definition of $K$ that for each $\omega \in \Omega$ the function $K(\cdot, \omega)$ is a probability measure on $B(\Xi)$.

Let $(\Omega, B(\Omega))$ and $(\Xi, B(\Xi))$ be measurable spaces and let $\varphi$ be a map of $M^+_1(\Omega)$ into $M^+_1(\Xi)$ such that, for each $B \in B(\Xi)$, the assignment $\omega \mapsto (\varphi(\delta_\omega))(B)$,
\(\omega \in \Omega, \delta_\omega \in M_+^1(\Omega)\) is the degenerated probability measure concentrated at \(\omega\), yields a measurable function of \(\Omega\) into \([0, 1]\). If

\[(2.0.1) \quad (\varphi(p))(B) = \int (\varphi(\delta_\omega))(B)dp\]

for all \(p \in M_+^1(\Omega)\) and all \(B \in \mathcal{B}(\Xi)\), then \(\varphi\) is said to be a statistical map (cf. BUGAJSKI [1], GUDDER [13]). Let \(\mathcal{E}(\Omega)\) and \(\mathcal{E}(\Xi)\) be the effect algebras of measurable functions into \(I = [0, 1]\). Put \(Y = M_+^1(\Omega), \mathcal{Y} = ev(\mathcal{E}(\Omega)) \subseteq I^Y, X = M_+^1(\Xi), \mathcal{X} = ev(\mathcal{E}(\Xi)) \subseteq I^X\). Theorem 2.4 in FRIČ [11] states that \(\varphi : M_+^1(\Omega) \rightarrow M_+^1(\Xi)\) is a statistical map iff it is measurable as a map of \((Y, \mathcal{Y})\) into \((X, \mathcal{X})\), i.e., for each \(u \in X\) the composition \(u \circ \varphi\) belongs to \(\mathcal{Y}\). From this point of view, it is natural to call elements of \(\mathcal{X}\) and \(\mathcal{Y}\) fuzzy events and \(\varphi\) a fuzzy random variable.

Now, let \(K : \Omega \times \mathcal{B}(\Xi) \rightarrow I\) be a Markov kernel. Then \(K\) sends \(B \in \mathcal{B}(\Xi)\) to a measurable function \(K(\cdot, B) \in \mathcal{E}(\Omega)\). Clearly, its integral \(\int K(\omega, B)dp\) with respect to \(p \in M_+^1(\Omega)\) sends \(p\) into a measure \(K(p)\) defined by \((K(p))(B) = \int K(\omega, B)dp\). Hence \(K\) yields a statistical map of \(M_+^1(\Omega)\) into \(M_+^1(\Xi)\). Since (according to Lemma 2.2) for discrete measurable spaces \((\Omega, \mathcal{B}(\Omega)), (\Xi, \mathcal{B}(\Xi))\) each map \(K_0 : \Omega \times \Xi \rightarrow [0, 1]\) yields a Markov kernel \(K : \Omega \times \mathcal{B}(\Xi) \rightarrow [0, 1]\), each map of \(\Omega \times \Xi\) into \([0, 1]\) determines (a unique) fuzzy random variable mapping \(M_+^1(\Omega)\) into \(M_+^1(\Xi)\). Consequently, the maps in Example 1.1, Example 1.2, Example 1.3, Example 1.4 determine fuzzy random variables.

3. Generalizations

In this section Markov kernels and fuzzy random variables are generalized to ID-posets.

**Definition 3.1.** Let \(\mathcal{X} \subseteq I^X\) and \(\mathcal{Y} = \sigma(\mathcal{Y}) \subseteq I^Y\) be ID-posets. Let \(K\) be a map of \(Y \times \mathcal{X}\) into \(I\) such that

(i) for each \(u \in \mathcal{X}\), \(K(\cdot, u) \in \mathcal{Y}\);

(ii) for each \(y \in \mathcal{Y}\), \(K(y, \cdot)\) is sequentially continuous D-homomorphism of \(\mathcal{X}\) into \(I\) (i.e. a state).

Then \(K\) is said to be an ID-Markov kernel.

**Theorem 3.2.** Let \(K : Y \times \mathcal{X} \rightarrow I\) be an ID-Markov kernel.

(i) There exists a unique sequentially continuous ID-homomorphism \(h : \sigma(\mathcal{X}) \rightarrow \mathcal{Y}\) such that for each \(u \in \mathcal{X}\) we have \(h(u) = K(u, \cdot)\).

(ii) There exists a unique sequentially continuous ID-homomorphism \(h^* : \sigma(\mathcal{X}^*) \rightarrow \mathcal{Y}^*\) such that for each \(u \in \mathcal{X}\) we have \(h^*(u^*) = K(u, \cdot)^*\).

**Proof.** (i) Define a map \(h_0 : \mathcal{X} \rightarrow \mathcal{Y}\) as follows: for \(u \in \mathcal{X}\) put \((h_0(u))(y) = K(y, u)\). It is easy to see that \(h_0\) is a sequentially continuous D-homomorphism. Since \(\sigma\) is an epireflector sending \(\mathcal{X}\) into its sequential completion \(\sigma(\mathcal{X})\), there is a unique extension of \(h_0\) to a sequentially continuous D-homomorphism \(h\) of \(\sigma(\mathcal{X})\) into \(\mathcal{Y} = \sigma(\mathcal{Y})\).

(ii) The sobrification is an epireflector and hence there is a unique sequentially continuous D-homomorphism \(h^* : \sigma(\mathcal{X}^*) \rightarrow \mathcal{Y}^* = \sigma(\mathcal{Y})^*\) such that \(h^*(u^*) = (h(u))^* = K(u, \cdot)^*\). It follows from Corollary 3.3 in FRIČ [10] that \(\sigma(\mathcal{X}^*) = \sigma(\mathcal{X}^*)\) and the assertion follows. \(\square\)
In Fric [11] it was proposed to extend basic probability notions (events, probabilities, random variables, observables) to the realm of $I$-$D$-posets: if $(X, \mathcal{X})$ is an $I$-$D$-measurable space, then the elements of $\mathcal{X}$ are called generalized events, each $I$-$D$-morphism of $\mathcal{X}$ into $I$ is said to be a state, each $I$-$D$-morphism of $\mathcal{X}$ into $Y \subseteq I^Y$ is a generalized observable, and each measurable map of $(Y, Y)$ into $(X, \mathcal{X})$ is said to be a generalized fuzzy random variable. Our results about $I$-$D$-Markov kernels together with the duality between $I$-$D$-posets and sober $I$-$D$-measurable maps provide technical tools for further study of probability on $I$-$D$-structures.

References


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