ON EFFECT ALGEBRAS OF FUZZY SETS

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Abstract. We introduce the category IE of effect algebras of fuzzy sets and sequentially continuous effect homomorphisms and describe its fundamental properties. We show that IE and the category ID of D-posets of fuzzy sets are isomorphic, hence the constructions and properties of ID related to applications to probability theory are valid for the corresponding effect algebras. We describe basic properties of categorical coproducts in ID and dually of categorical products in the corresponding category MID of measurable spaces. We end with remarks on fuzzy probability notions.

Introduction

R. Frič in [11] has suggested to study the subcategory of D-posets cogenerated by the closed unit interval in connection with the generalized probability. In [19] a theory of ID-posets has been developed. ID-posets are D-posets of fuzzy sets carrying the pointwise convergence of sequences. Sequentially continuous D-homomorphisms as morphisms of the corresponding category ID play a key role (cf. [8], [9], [11]). The category ID is natural and suitable to carry out and to describe various categorical constructions in the foundations of probability, e.g., the duality between observables and random variables, completion of the field of events, extension of probability measures.

D-posets (introduced by F. Kôpka and F. Chovanec in [18]) and the effect algebras (developed by D. J. Foulis and M. K. Bennet in [6]) are known to be isomorphic (cf. [5]). In Section 1 we introduce a new category IE of effect algebras and describe its properties. In view of results concerning D-posets (cf. [19], [15], [10], [13], [20]) the objects and morphisms of IE are defined in such a way that IE and ID are isomorphic categories. Note that the unit interval I carries natural isomorphic effect algebra and D-poset structures. Both IE and ID are cogenerated by I. Hence all properties and constructions valid for the D-posets of fuzzy sets are valid for the corresponding effect algebras. Section 2 deals with categorical products and coproducts. In Section 3 we mention some applications to probability theory.

1991 *Mathematics Subject Classification.* Primary 08A55, 08A72, 54B30; Secondary 54A20, 60D05.

Key words and phrases. D-poset, effect algebra, convergence D-poset, sequentially continuous D-homomorphism, convergence effect algebra, sequentially continuous EA-homomorphism, D-poset of fuzzy sets, pointwise convergence, ID-poset, effect algebra of fuzzy sets, IE-algebra, sober IE-algebra, closed IE-algebra, IE-measurable space, measurable map, natural equivalence, duality, monocoreflective subcategory, epireflective subcategory, product, coproduct, generalized elementary event, generalized measurable space, generalized probability measure, state, fuzzy random variable, observable.

Supported by VEGA Grant 1/2002/05.

For the reader's convenience we recall some basic notions in quantum structures and fuzzy probability theory. Our standard references are [5] and [21]. Relevant categorical definitions and constructions can be found in [1] and [2].

Recall (cf. [18]) that a D-poset is a quintuple $(X, \leq, \ominus, 0_X, 1_X)$ where X is a set, $\leq$ is a partial order, 0_X is the least element, 1_X is the greatest element, $\ominus$ is partial operation on X such that $a \ominus b$ is defined iff $b \leq a$, and the following axioms are assumed:

(D1) $a \ominus 0_X = a$ for each $a \in X$;
(D2) If $c \leq b \leq a$, then $a \ominus b \leq a \ominus c$ and $(a \ominus c) \ominus (a \ominus b) = b \ominus c$.

If no confusion can arise, then the quintuple $(X, \leq, \ominus, 0_X, 1_X)$ is condensed to X. A map h of a D-poset X into a D-poset Y which preserves the D-structure is said to be a D-homomorphism.

1. CATEGORY IE

Recall that an *effect algebra* (introduced by D. J. Foulis and M. K. Bennett in [6]) is a system $(X, \oplus, 0_X, 1_X)$, where X is a set, 0_X and 1_X are special elements of X, called the *zero* and the *unit*, and $\oplus$ is a partially defined binary operation on X such that for $p, q, r \in X$ the following axioms are assumed (cf. [5]):

(E1) If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$ (Commutative law).
(E2) If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, then $p \oplus q$ and $(p \oplus q) \oplus r$ are defined and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$ (Associative law).
(E3) For every $p \in X$ there exists a unique element $q \in X$ such that $p \oplus q$ is defined and $p \oplus q = 1_X$ (Orthosupplement law).
(E4) If $p \oplus 1_X$ is defined, then $p = 0_X$ (Zero-one law).

If no confusion arise, then the system $(X, \oplus, 0_X, 1_X)$ is condensed to X. A map h of an effect algebra X into an effect algebra Y which preservers the structure of effect algebras (thereinafter EA-structure) is said to be an *EA-homomorphism.*

Denote I the closed unit interval carrying the usual EA-structure: $p \oplus q$ is defined whenever $p + q \leq 1$ and then $p \oplus q = p + q$, i.e., the system $(I, +, 0, 1)$ is an effect algebra. Analogously, if X is a set and I^X is the set of all functions on X into I, then we consider I^X as an effect algebra in which the partial operation $\oplus$ is defined pointwise: $p \oplus q$ is defined iff $p(x) + q(x) \leq 1$ for all $x \in X$ and $p \oplus q$ is defined by $(p \oplus q)(x) = p(x) + q(x)$, $x \in X$. A subset $\mathcal{X} \subseteq I^X$ containing the constant functions $0_{\mathcal{X}}$, $1_{\mathcal{X}}$ and closed with respect to the inherited partial operation "$\oplus$" is a typical effect algebra we are interested in; we shall call it an *effect algebra of fuzzy sets.*

There is an extensive list of papers devoted to sequential convergence on various algebraic structures. Interesting results about sequential convergence on D-posets and effect algebras can be found in [13].

By a sequential convergence on a set X we understand a subset $\mathbb{L} \subseteq X^N \times X$ satisfying the following conditions:

(S) If $\langle x_n \rangle$ is a constant sequence and $x_n = x$, $n \in N$, then $(\langle x_n \rangle, x) \in \mathbb{L}$.
(F) If $(\langle x_n \rangle, x) \in \mathbb{L}$ and $\langle x'_n \rangle$ is a subsequence of $\langle x_n \rangle$, then $(\langle x'_n \rangle, x) \in \mathbb{L}$.

Let $(X, \oplus, 0_X, 1_X)$ be an effect algebra. Let $\mathbb{L} \subseteq X^N \times X$ be a sequential convergence on X such that:

(EC) If $(\langle x_n \rangle, x) \in \mathbb{L}$, $(\langle y_n \rangle, y) \in \mathbb{L}$ and $x_n \oplus y_n$ exists for all $n \in N$, then $x \oplus y$ exists and $(\langle x_n \oplus y_n \rangle, x \oplus y) \in \mathbb{L}$.

Then (cf. [13]) the system $(X, \oplus, 0_X, 1_X, \mathbb{L})$ is said to be a *convergence effect algebra*.

Additionally, we shall consider the following two properties of a sequential convergence $\mathbb{L}$ on a set X:

(H) If $(\langle x_n\rangle, x) \in \mathbb{L}$ and $(\langle x_n\rangle, y) \in \mathbb{L}$, then $x = y$ (the uniqueness of limits).

(U) If $\langle x_n\rangle$ is a sequence and $x \in X$ is a point such that for each subsequence $\langle x'_n\rangle$ of $\langle x_n\rangle$ there exists a subsequence $\langle x''_n\rangle$ of $\langle x'_n\rangle$ such that $(\langle x''_n\rangle, x) \in \mathbb{L}$, then $(\langle x_n\rangle, x) \in \mathbb{L}$ (Fréchet-Urysohn condition).

In accordance with [13] denote CONE the category whose objects are convergence effect algebras and whose morphisms are sequentially continuous effect algebra morphisms. Since each effect algebra can be considered as a convergence effect algebra carrying the trivial convergence (only almost constant sequences converge) and each EA-homomorphisms is sequentially continuous with respect to the trivial convergence, effect algebras (as trivial convergence effect algebras) form a full subcategory of CONE.

Let $(X, \oplus, 0_X, 1_X, \mathbb{L})$ be a convergence effect algebra. The dual partial operation $\ominus$ defined by the condition

(DIF) $p \ominus q$ exists and equals r if and only if $q \oplus r$ exists and equals p;

yields a D-poset structure on X and $\mathbb{L}$ satisfies the axiom (DC) in [13]. Analogously, let $(X, \leq, \boxminus, 0_X, 1_X, \mathbb{M})$ be a convergence D-poset (cf. [19]). The dual partial operation $\boxplus$ defined by the condition

(SUM) $p \boxplus q$ exists and equals r if and only if $r \boxminus q$ exists and equals p;

yields an effect structure on X and $\mathbb{M}$ satisfies the axiom (EC).

Denote COND the category of convergence D-posets and sequentially continuous D-homomorphisms. The conditions (SUM) and (DIF) lead to two functors F_1 : CONE $\to$ COND and F_2 : COND $\to$ CONE, the composition of which are the identity functors on CONE and COND. Hence the categories CONE and COND are isomorphic (cf. [13, Observation 1.2]).

Fields of sets and bold algebras are classical examples of objects and sequentially continuous (with respect to the pointwise convergence) maps preserving the Boolean or MV-algebra structure are morphisms.

Even though in probability theory the monotone convergence and the sequential continuity with respect to the monotone convergence are standard (cf. [21], [5]), in accordance with [7], [8], [10], [11], [12], [16], we shall work with the usual convergence in I and the pointwise (the product convergence) in I^X. Observe that, for the usual probability domains like σ-fields sets or Łukasiewicz tribes, if an additive generalized probability measure is sequentially continuous with respect to the monotone convergence, then it is sequentially continuous with respect to the pointwise convergence, as well (cf. [12]).

In what follows, I will denote the closed unit interval of real numbers carrying the usual algebraic (also the effect algebra and D-poset, respectively) structures and the sequential convergence.

Let X be a set and let $\mathcal{X}$ be an effect algebra of fuzzy sets with respect to the pointwise sum operation $\oplus$ inherited from I^X, i.e., $0_{\mathcal{X}}, 1_{\mathcal{X}} \in \mathcal{X}$ and if $p(x)+q(x) \leq 1$ for all $x \in X$, then $p \oplus q \in \mathcal{X}$. The next lemma (cf. [13]) is a folklore; its proof is trivial and it is omitted.

Lemma 1.1. *Let $\mathbb{L}$ be the pointwise convergence on $\mathcal{X}$. Then $(\mathcal{X}, \oplus, 0_{\mathcal{X}}, 1_{\mathcal{X}}, \mathbb{L})$ is a convergence effect algebra and $\mathbb{L}$ satisfies the additional conditions (H), (U).*

Definition 1.2. The system $(\mathcal{X}, \oplus, 0_{\mathcal{X}}, 1_{\mathcal{X}}, \mathbb{L})$ is said to be an *E(I)-algebra*.

Denote E(I) the category the objects of which are E(I)-algebras and the morphisms of which are sequentially continuous EA-homomorphisms. Let $(\mathcal{X}, \oplus, 0_{\mathcal{X}}, 1_{\mathcal{X}}, \mathbb{L})$ be an E(I)-algebra. If no confusion can arise, we shall say that $\mathcal{X} \subseteq I^X$, or simply $\mathcal{X}$, is an E(I)-algebra. Each $x \in X$ can be identified with a sequentially continuous EA-homomorphism, called the evaluation of $\mathcal{X}$ at $x \in X$, ev_x: $\mathcal{X} \to I$, defined by $ev_x(p) = p(x),\ p \in \mathcal{X}$. Note that $\{ev_x;\ x \in X\}$ is order determining (cf. [13]).

Analogously as in [19], if for each morphism $h : \mathcal{X} \to I$ there exists $x \in X$ such that $h = ev_x$, then $\mathcal{X}$ is said to be *sober*. If for each $x, y \in X,\ x \neq y$, there exists $p \in \mathcal{X}$ such that $p(x) \neq p(y)$, i.e. p separates x and y, then $\mathcal{X}$ is said to be *reduced*.

Observe that if $\mathcal{X}$ is sober and reduced, than for each morphism $h : \mathcal{X} \to I$ there exists exactly one $x \in X$ such that $h = ev_x$.

Due to the isomorphism between the category COND and the category CONE, the definition of a D(I)-measurable space and measurable map ([19]) can be redefined in the following way.

Definition 1.3. Let $\mathcal{X} \subseteq I^X$ be an E(I)-algebra. Then $(X, \mathcal{X})$ is said to be an *E(I)-measurable space*, or simply a *measurable space*. If $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ are E(I)-measurable spaces and f is a map of Y into X such that $p \circ f \in \mathcal{Y}$ for all $p \in \mathcal{X}$, i.e., $\mathcal{X} \circ f \subseteq \mathcal{Y}$, then f is said to be $(\mathcal{X}, \mathcal{Y})$*-measurable*, or simply *measurable*. If $\mathcal{X}$ is sober or reduced, then $(X, \mathcal{X})$ is said to be *sober* or *reduced*.

Note that if $\mathcal{X}$ and $\mathcal{Y}$ are fields of sets, then $(\mathcal{X}, \mathcal{Y})$-measurability becomes the usual measurability of f.

Let ME(I) be the category the objects of which are E(I)-measurable spaces and the morphisms of which are measurable maps.

Having in mind applications to probability, as in [19] we will work with reduced E(I)-algebras. Let IE be the full subcategory of E(I) consisting of reduced E(I)-algebras. Accordingly, a reduced E(I)-algebra will be called an *IE-object* or an *IE-algebra*, a reduced E(I)-measurable space will be called an *IE-measurable space* and MIE will denote the full subcategory of ME(I) consisting of reduced E(I)-measurable spaces. Note that, in connection with the extension of measures, separation plays the same role as the Hausdorff separation axiom T_2: limits are unique and the continuous extensions of maps from dense subobjects are uniquely determined (cf. [11]). Recall that D(I) is the category of D-posets of fuzzy sets as objects and sequentially continuous (with respect to the pointwise sequential convergence) D-homomorphisms as morphisms. We concentrate on its full subcategory ID of reduced objects. Further, if $\mathcal{X} \subseteq I^X$ and $\mathcal{Y} \subseteq I^Y$ are reduced objects of ID, then a map f of Y into X such that $\mathcal{X} \circ f \subseteq \mathcal{Y}$ is said to be measurable; MID denotes the corresponding category of ID-measurable spaces and measurable maps.

Let $\mathbb{A}$ be a field of subsets of a set Ω. Then each $\omega \in \Omega$ represents a point probability measure on $\mathbb{A}$. Let $P(\mathbb{A})$ be the set of all probability measures on $\mathbb{A}$. For X, $\Omega \subseteq X \subseteq P(\mathbb{A})$, and $A \in \mathbb{A}$ denote $ev_X(A) = \{p(A); p \in X\}$ the evaluation of A. Then $\{ev_X(A); A \in \mathbb{A}\}$ is an ID-poset and, except trivial cases, it is not a field of sets (cf. [19]).

Now let us turn to the relationship between the category IE and the category ID. In view of the isomorphism between categories CONE and COND, in what follows we assume that the partial operations $\oplus$ and $\ominus$ are dual, the partial order and the constants 0_X, 1_X are compatible with $\oplus$ and $\ominus$.

Let X and Y be sets and $\mathcal{X} \subseteq I^X$ and $\mathcal{Y} \subseteq I^Y$ be IE-algebras. To avoid a complicated notation, we will use the following convention. Using the dual conditions (DIF) and (SUM), in what follows IE-algebras will be considered also as ID-posets and, vice-versa, ID-posets will be considered as IE-algebras. The next assertion is trivial.

Lemma 1.4. *Let h be a map of $\mathcal{X}$ into $\mathcal{Y}$. Then h is a sequentially continuous EA-homomorphism iff it is a sequentially continuous D-homomorphism.*

Lemma 1.4 leads to two functors $F_D :$ IE $\to$ ID and $F_E :$ ID $\to$ IE, the compositions of which are the identity functors on IE and ID. Hence the two categories are isomorphic.

Corollary 1.5. *The categories ID and IE are isomorphic.*

Let SIE be the full subcategory of IE consisting of reduced sober E(I)-algebras and let MSIE be the full subcategory of MIE consisting of reduced sober IE-measurable spaces. Their mutual relationships follow from the relationships between ID, SID and MSID (recall that SID and MSID are dually isomorphic and ID and MSID are dually equivalent cf. [19]) and Corollary 1.5. We shall state relevant assertions but the proofs will be omitted.

Let $\mathcal{X} \subseteq I^X$ be a reduced E(I)-algebra. Denote X^* the set of all morphisms of $\mathcal{X}$ into I and, identifying each $x \in X$ with the morphism $ev_x : \mathcal{X} \to I$, consider X as a subset of X^*. For each $p \in \mathcal{X}$, define a mapping p^* of X^* into I as follows: $p^*(x) = x(p)$. Trivially, $p^* = q^*$ iff $p = q$. Clearly, this yields an E(I)-algebra $\mathcal{X}^* \subseteq I^{X^*}$ such that $\mathcal{X}^* \restriction X = \mathcal{X}$. Finally, define a map e of $\mathcal{X}$ into $\mathcal{X}^*$ by putting $e(p) = p^*,\ p \in \mathcal{X}$. The straightforward proof of the next lemma is omitted.

Lemma 1.6. *(i) The map e is an isomorphism of $\mathcal{X}$ onto $\mathcal{X}^*$.*

(ii) The E(I)-algebra $\mathcal{X}^* \subseteq I^{X^*}$ is reduced and sober.

(iii) Let h be a sequentially continuous EA-homomorphism of $\mathcal{X}$ into a sober E(I)-algebra $\mathcal{Y} \subseteq I^Y$. Define a map h^* of $\mathcal{X}^*$ into $\mathcal{Y}$ as follows: $h^*(p^*) = h(p)$. Then h^* is a sequentially continuous EA-homomorphism of $\mathcal{X}^*$ into $\mathcal{Y}$.

Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be E(I)-measurable spaces and let f be an $(\mathcal{X}, \mathcal{Y})$-measurable map. Then $p \circ f,\ p \in \mathcal{X}$, defines a map of $\mathcal{X}$ into $\mathcal{Y}$; denote it $f^{\triangleleft}$.

Lemma 1.7. *The map $f^{\triangleleft}$ is a sequentially continuous EA-homomorphism of the E(I)-algebra $\mathcal{X}$ into the E(I)-algebra $\mathcal{Y}$.*

Lemma 1.8. *Let $f_1,\ f_2$ be $(\mathcal{X}, \mathcal{Y})$-measurable maps and let $f_1^{\triangleleft},\ f_2^{\triangleleft}$ be the corresponding sequentially continuous EA-homomorphisms of $\mathcal{X}$ into $\mathcal{Y}$. If $\mathcal{X}$ is reduced, then $f_1 \neq f_2$ implies $f_1^{\triangleleft} \neq f_2^{\triangleleft}$.*

Lemma 1.9. *Let $\mathcal{X}$ be reduced and sober and let h be a sequentially continuous EA-homomorphism of $\mathcal{X}$ into $\mathcal{Y}$. Then there exists a unique $(\mathcal{X}, \mathcal{Y})$-measurable map g such that $h = g^{\triangleleft}$.*

Observe that if $\mathcal{X} \subseteq I^X$ fails to be sober, $\mathcal{X}^* \subseteq I^{X^*}$ is the E(I)-algebra from Lemma 1.6, and $e : \mathcal{X} \longrightarrow \mathcal{X}^*$ is the isomorphism sending p into p^*, then there is no $(\mathcal{X}, \mathcal{X}^*)$-measurable map g of X^* into X such that $e = g^{\triangleleft}$. Hence the soberness of $\mathcal{X}$ in Lemma 1.6 cannot be omitted.

In what follows, a blanket assumption is made that *all* E(I)*-algebras are reduced.*

Theorem 1.10. *The categories SIE and MSIE are dually isomorphic.*

Theorem 1.11. *(i) The categories IE and SIE are naturally equivalent.*
(ii) The categories IE and MSIE are dually equivalent.

Next, we construct two types of extensions of an IE-object $\mathcal{X} \subseteq I^X$, one extending the domain set X and the other one adding more functions to $\mathcal{X}$.

Definition 1.12. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be IE-measurable spaces. Let X be a subset of Y and let $\mathcal{X}$ be the restriction $\mathcal{Y} \restriction X$ of $\mathcal{Y}$ to X. Then $(X, \mathcal{X})$ is said to be a subspace of $(Y, \mathcal{Y})$. If $p = q$ whenever $p,\ q \in \mathcal{Y}$ and $p \restriction X = q \restriction X$, then $(X, \mathcal{X})$ is said to be a *dense* subspace.

Let $(X, \mathcal{X})$ be an IE-measurable space and let $(X^*, \mathcal{X}^*)$ be the sober IE-measurable space described in Lemma 1.6. If X is sober, then $X^* = X$ and $\mathcal{X}^* = \mathcal{X}$.

Lemma 1.13. *(i) $(X, \mathcal{X})$ is a dense subspace of $(X^*, \mathcal{X}^*)$.*
(ii) Let f be a measurable map of $(X, \mathcal{X})$ into a sober IE-measurable space $(Y, \mathcal{Y})$. There exists a unique measurable map f^* of $(X^*, \mathcal{X}^*)$ into $(Y, \mathcal{Y})$ such that $f^* \restriction X = f$.

Theorem 1.14. *MSIE is an epireflective subcategory of MIE.*

Definition 1.15. We say that $(X^*, \mathcal{X}^*)$ is the *sobrification* of $(X, \mathcal{X})$.

Corollary 1.16. *SIE is a monocoreflective subcategory of IE.*

Definition 1.17. Let X be a non-empty set and let $\mathcal{X} \subseteq I^X$. Define
- (i) $c(\mathcal{X}) = \{g \in I^X;\ g = \lim g_n,\ g_n \in \mathcal{X}\}$;
- (ii) $o(\mathcal{X}) = \{\mathcal{Y} \subseteq I^X;\ \mathcal{X} \subseteq \mathcal{Y},\ \mathcal{Y} \in \mathrm{IE}\}$;
- (iii) $d(\mathcal{X}) = \bigcap_{\mathcal{Y} \in o(\mathcal{X})} \mathcal{Y}$;
- (iv) for each ordinal number ξ define σ^ξ inductively:
  - (1) $\sigma^0(\mathcal{X}) = \mathcal{X}$,
  - (2) $\sigma^\xi = d(c(\sigma^{\xi-1}(\mathcal{X})))$ if ξ is isolated,
  - (3) $\sigma^\xi = \bigcup_{\eta<\xi} \sigma^\eta(\mathcal{X})$ if ξ is a limit ordinal, and denote
  - (4) $\sigma(\mathcal{X}) = \sigma^{\omega_1}(\mathcal{X})$.

Example 1.18. Let $\mathbb{A}$ be a field of subsets of X. Let $\mathcal{X}$ be the set of all characteristic functions χ_A of sets belonging to $\mathbb{A}$. It is known that $c(\mathcal{X})$ is a field of subsets of X and $\sigma(\mathcal{X})$ is the set of all characteristic functions χ_A of sets belonging to the generated σ-field $\sigma(\mathbb{A})$.

Example 1.19. Let $\mathcal{X} \subseteq I^X$ be a bold algebra. Then $c(\mathcal{X}) \subseteq I^X$ is a bold algebra and $\sigma(\mathcal{X})$ is the generated tribe (cf. [12]).

Lemma 1.20. *Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be IE-measurable spaces and let $h,\ h'$ be sequentially continuous EA-homomorphisms of $\sigma(\mathcal{X})$ into $\sigma(\mathcal{Y})$ such that $h(g) = h'(g)$ for all $g \in \mathcal{X}$. Put $e(\mathcal{X}) = \{g \in \sigma(\mathcal{X});\ h(g) = h'(g)\}$. Then*
(i) $e(\mathcal{X})$ is a sequentially closed subset of I^X;
(ii) $e(\mathcal{X})$ is an IE-algebra;
(iii) $h = h'$.

Lemma 1.21. *(i) $d(\mathcal{X})$ is an IE-algebra; it is the smallest IE-algebra in I^X which contains $\mathcal{X}$.*
(ii) $\sigma^\xi(\mathcal{X})$ is an IE-algebra for each ordinal number ξ.

(iii) $\sigma^{\omega_1}(\mathcal{X})$ *is sequentially closed; it is the smallest sequentially closed IE-algebra in* I^X *which contains* $\mathcal{X}$*.*

(iv) $\sigma^{\omega_1+1}(\mathcal{X}) = \sigma^{\omega_1}(\mathcal{X})$*.*

Let $\mathcal{X} \subseteq I^X$ and $\mathcal{Y} \subseteq I^Y$ be IE-algebras and let $f : Y \longrightarrow X$ be an $(\mathcal{X}, \mathcal{Y})$-measurable map.

Lemma 1.22. *(i)* $\mathcal{Z} = \{g \in I^X;\ g \circ f \in d(c(\mathcal{Y}))\}$ *is an IE-algebra.*

(ii) $d(c(\mathcal{X})) \subseteq \mathcal{Z}$*.*

(iii) $d(c(\mathcal{X})) \circ f \subseteq d(c(\mathcal{Y}))$*.*

Let η be an ordinal number, $\eta \le \omega_1$. For $g \in \sigma^\eta(\mathcal{X})$ put $h_\eta(g) = g \circ f$. This defines a map $h_\eta : \sigma^\eta(\mathcal{X}) \longrightarrow I^Y$.

Lemma 1.23. *(i) If* $g \in \sigma^\eta(\mathcal{X})$*, then* $h_\eta(g) \in \sigma^\eta(\mathcal{Y})$*.*

(ii) $f : Y \longrightarrow X$ *is* $(\sigma^\eta(\mathcal{X}),\ \sigma^\eta(\mathcal{Y}))$*-measurable.*

Corollary 1.24. *Let* $\mathcal{X} \subseteq I^X$ *and* $\mathcal{Y} \subseteq I^Y$ *be IE-algebras and let* $f : Y \longrightarrow X$ *be an* $(\mathcal{X}, \mathcal{Y})$*-measurable map. Then* f *is* $(\sigma(\mathcal{X}),\ \sigma(\mathcal{Y}))$*-measurable.*

Definition 1.25. Let $\mathcal{X} \subseteq I^X$ be an IE-algebra. If $\mathcal{X} = c(\mathcal{X})$, then $\mathcal{X}$ is said to be *closed*. The measurable space $(X, \mathcal{X})$ is said to be *closed* if $\mathcal{X}$ closed.

Denote CIE, resp. CSIE, the full subcategory of IE the objects of which are closed, resp. closed sober, IE-algebras. Denote CMIE the full subcategory of MIE the objects of which are closed measurable spaces.

The construction of $\sigma(\mathcal{X})$ and its properties lead to the following functor σ_M : MID $\longrightarrow$ CMID. If $(X, \mathcal{X})$ is a measurable space, then $\sigma_M(X, \mathcal{X}) = (X, \sigma(\mathcal{X}))$. Further, according to Corollary 1.24, each measurable map f of $(Y, \mathcal{Y})$ into $(X, \mathcal{X})$ is also measurable as the map of $(Y, \sigma(\mathcal{Y}))$ into $(X, \sigma(\mathcal{X}))$; denote it $\sigma_M(f)$. Clearly, the identity maps and compositions are preserved.

Theorem 1.26. *The functor* σ_M *is a monocoreflector.*

Corollary 1.27. *CMIE is a monocoreflective subcategory of MIE.*

Corollary 1.28. *CSIE is an epireflective subcategory of SIE.*

2. Products and coproducts

In this section we construct the product of ID-measurable spaces, resp. the coproduct, in the category ID. Such constructions play important roles in the fuzzy probability theory. Details can be found, e.g., in [2], [17], [3], [4], [15], [20], [14]. The isomorphism between the category ID and the category IE allows us to translate the results into the realm of effect algebras.

Let X be a D-poset and let $x \in X$. If $x = 1 \ominus x$, then x is said to be a *middle* of X. Consider the coproduct (horizontal sum) of two 2-chains, i.e. the diamond $D = \{0, a, a^*, 1\}$ (cf. [5]). Then a and a^* are the middles of D. The proof of the next lemma is trivial and it is omitted.

Lemma 2.1. *Let* x *be a middle of a D-poset* X *and let* h *be a D-homomorphism of* X *into* I*. Then* $h(x) = h(1 \ominus x) = 1/2$*.*

Corollary 2.2. *(i) The diamond* D *does not admit an order determining set of D-homomorphisms into* I*.*

(ii) The coproduct of two D-posets admitting order determining sets of D-homomorphisms into I need not admit any order determining set of morphisms into I.

The dual isomorphism between the category SID of sober ID-objects and the category MSID of sober ID-measurable spaces leads to the construction of the coproduct in the category ID.

Lemma 2.3. *Let V be a set, let $\mathcal{V}_0$ be a subset of I^V and let $\mathcal{V}$ be the minimal sub-D-poset of I^V containing $\mathcal{V}_0$ (i.e. the intersection of all sub-D-posets of I^V containing $\mathcal{V}_0$).*

(i) Let $(Y, \mathcal{Y})$ be an ID-measurable space and let f be a map of Y into V. If $u \circ f \in \mathcal{Y}$ for each $u \in \mathcal{V}_0$, then f is $(\mathcal{V}, \mathcal{Y})$-measurable.

(ii) Let $(Y, \mathcal{Y})$ be an ID-measurable space. Let p, q be sequentially continuous D-homomorphisms of $\mathcal{V}$ into $\mathcal{Y}$. If $p(u) = q(u)$ for all $u \in \mathcal{V}_0$, then $p = q$.

Proof. (i) Clearly, the set $\mathcal{V}_1 = \{u \in \mathcal{V};\ u \circ f \in \mathcal{Y}\}$ contains $\mathcal{V}_0$. Further, it contains the constant functions $1_{\mathcal{V}}$ and $0_{\mathcal{V}}$. Assume $u, v \in \mathcal{V}$, $v \leq u$, $u \circ f \in \mathcal{Y}$, and $v \circ f \in \mathcal{Y}$. Since $(u \ominus v) \circ f = (u \circ f) \ominus (v \circ f) \in \mathcal{Y}$, $\mathcal{V}_1$ is a sub-D-poset of I^V which contains $\mathcal{V}_0$ and hence also $\mathcal{V}$. Thus f is $(\mathcal{V}, \mathcal{Y})$-measurable.

(ii) Similarly, the set $\mathcal{V}_2 = \{u \in \mathcal{V};\ p(u) = q(u)\}$ is a sub-D-poset of I^V containing $\mathcal{V}_0$, hence also $\mathcal{V}$. □

Let $\{(X_s, \mathcal{X}_s);\ s \in S\}$ be a family of ID-measurable spaces. Let $X = \prod_{s \in S} X_s$ be the product of the underlying sets and let $pr_t, t \in S$, be the t-th projection of X into X_t sending $\{x_s;\ s \in S\}$ to x_t. Given $t \in S$, for $u \in \mathcal{X}_t$ define $u_t \in I^X$ as follows: $u_t(\{x_s;\ s \in S\}) = u(pr_t(\{x_s;\ s \in S\}))$. Put $\mathcal{Y}_t = \{u_t \in I^X;\ u \in \mathcal{X}_t\}$. Let $\mathcal{X} \subseteq I^X$ be the minimal sub-D-poset of I^X containing all $\mathcal{Y}_s, s \in S$, carrying the pointwise sequential convergence.

Theorem 2.4. *(i) $(X, \mathcal{X})$ is an ID-measurable space.*

(ii) Each projection $pr_s, s \in S$, is an $(\mathcal{X}_s, \mathcal{X})$-measurable map.

(iii) $(X, \mathcal{X})$, together with the projections $\{pr_s;\ s \in S\}$, is the product of the family $\{(X_s, \mathcal{X}_s);\ s \in S\}$ in the category MID.

Proof. (i) and (ii) follow immediately from the construction of $(X, \mathcal{X})$.

(iii) Let $(Y, \mathcal{Y})$ be an ID-measurable space. Assume that for each $s \in S$ there is an $(\mathcal{X}_s, \mathcal{Y})$-measurable map f_s of Y into X_s. We have to prove that there exists a unique $(\mathcal{X}, \mathcal{Y})$-measurable map f of Y into X such that $pr_s \circ f = f_s$ for each $s \in S$. First, f is uniquely determined by sending $y \in Y$ into $\{f_s(y);\ s \in S\}$. Second, it suffices to verify that f is $(\mathcal{X}, \mathcal{Y})$-measurable. Consider $\{u \in I^X;\ u \circ f \in \mathcal{Y}\}$. Let $\mathcal{X}_0$ be the union of all $\mathcal{Y}_s \subseteq I^X$, $s \in S$. Since $\mathcal{Y}_s \circ f \subset \mathcal{Y}$, it follows from (i) in Lemma 2.3 that f is $(\mathcal{X}, \mathcal{Y})$-measurable. □

Recall (cf. [19]) that for each ID-measurable space $(V, \mathcal{V})$ there exists its sober epireflection $(V^*, \mathcal{V}^*)$, called the sobrification of $(V, \mathcal{V})$, such that $V \subseteq V^*$, each $v \in \mathcal{V}$ can be uniquely extended to $v^* \in \mathcal{V}^*$, and the map e sending v to v^* is a D-isomorphism of $\mathcal{V}$ onto $\mathcal{V}^*$; if $(V, \mathcal{V})$ is sober, then $(V, \mathcal{V})$=$(V^*, \mathcal{V}^*)$.

Observation 2.5. Let $\{a\}$ be a singleton and let $(X, \mathcal{X})$ be a sober ID-measurable space. Then the one-to-one correspondence between measures on $\mathcal{X}$ (i.e. sequentially continuous D-homomorphisms of $\mathcal{X}$ into I) and the evaluation measures on $\mathcal{X}^*$ at points of X^* induces a one-to-one correspondence between measures on $\mathcal{X}$

and the $(\mathcal{X}^*, I^{\{a\}})$-measurable maps $f^{(x)} : \{a\} \longrightarrow X^*$ sending a to $x \in X^*$. Clearly, we can identify I and $I^{\{a\}}$.

Theorem 2.6. *Let $\{(X_s, \mathcal{X}_s);\ s \in S\}$ be a family of ID-measurable spaces and let $(X, \mathcal{X})$, together with the projections $\{pr_s;\ s \in S\}$, be their product in the category MID. Let each $\{(X_s, \mathcal{X}_s);\ s \in S\}$ be sober. Then $(X, \mathcal{X})$ is sober.*

Proof. Let p be a measure on $\mathcal{X}$. For each $t \in S$, let p_t be the measure on $\mathcal{X}_t$ defined as follows: for $u \in \mathcal{X}_t$ put $p_t(u) = p(u_t)$ (remember $u_t \in \mathcal{Y}_t$ is defined by $u_t(\{x_s \in X_s;\ s \in S\}){=}u(x_t)$). Let $\{a\}$ be a singleton. Since all $(X_t, \mathcal{X}_t)$ are sober, there are uniquely determined $(\mathcal{X}_t, I^{\{a\}})$-measurable maps $f_t : \{a\} \longrightarrow X_t$, $t \in S$ such that p_t is the evaluation measure on $\mathcal{X}_t$ at the point $f_t(a) \in X_t$. Consider the evaluation measure on $\mathcal{X}$ at the point $\{f_s(a) \in X_s;\ s \in S\} \in X$. Since it coincides with the measure p on each $\mathcal{Y}_t, t \in S$, it follows from (ii) in Lemma 2.3 that p is the evaluation measure at the point $\{f_s(a) \in X_s;\ s \in S\}$. Thus $(X, \mathcal{X})$ is sober. $\square$

Theorem 2.7. *Let $\{(X_s, \mathcal{X}_s);\ s \in S\}$ be a family of ID-measurable spaces. Let $\{(X^*_s, \mathcal{X}^*_s);\ s \in S\}$ be the family of their sobrifications and let $(\overline{X}, \overline{\mathcal{X}})$, together with the projections $\{pr_s : \overline{X} \longrightarrow X^*_s;\ s \in S\}$, be their product in the category MID. Let $e_s, s \in S$, be the map sending $u \in \mathcal{X}_s$ to $u^* \in \mathcal{X}^*_s$. Put $\kappa_s = pr^{\triangleleft}_s \circ e_s$ (sends $u \in \mathcal{X}_s$ to $(u^*)_s \in \overline{\mathcal{X}}$ via $u^* \circ pr_s = (u^*)_s$), $s \in S$. Then $\overline{\mathcal{X}} \subseteq I^{\overline{X}}$, together with the coprojections $\{\kappa_s;\ s \in S\}$, is the coproduct of $\{\mathcal{X}_s;\ s \in S\}$ in the category ID.*

Proof. The assertion follows from Theorem 2.4, Theorem 2.6, and the duality between the categories ID and MSID (cf. [19]). Recall (Lemma 1.10 in [19]) that if $(U,\mathcal{U})$ and $(V, \mathcal{V})$ are ID-measurable spaces, ψ is a sequentially continuous D-homomorphism of $\mathcal{U}$ into $\mathcal{V}$, and $(U,\mathcal{U})$ is sober, then there exists a unique $(\mathcal{U}, \mathcal{V})$-measurable map f of V into U such that $\psi = f^{\triangleleft}$ (defined by $f^{\triangleleft}(u) = u \circ f$, $u \in \mathcal{U}$).

Since the spaces $(X^*_s, \mathcal{X}^*_s), s \in S$, are sober, their product $(\overline{X}, \overline{\mathcal{X}})$ is sober, too. Let $(Y, \mathcal{Y})$ be an ID-measurable space. For each $s \in S$, let ψ_s be a sequentially continuous D-homomorphism of $\mathcal{X}_s$ into $\mathcal{Y}$. Then $\varphi_s = \psi_s \circ (e_s)^{-1}, s \in S$, is a sequentially continuous D-homomorphism of $\mathcal{X}^*_s$ into $\mathcal{Y}$. Then there are uniquely determined $(\mathcal{X}^*_s, \mathcal{Y})$-measurable maps f_s of Y into X^*_s such that $\varphi_s = f^{\triangleleft}_s$, $s \in S$. Since $(\overline{X}, \overline{\mathcal{X}})$ is the product of $\{(X^*_s, \mathcal{X}^*_s);\ s \in S\}$, there exists a unique $(\overline{\mathcal{X}}, \mathcal{Y})$-measurable map f of Y into $\overline{X}$ such that $pr_s \circ f = f_s$ for all $s \in S$. First, $f^{\triangleleft}$ is a sequentially continuous D-homomorphism of $\mathcal{X}$ into $\mathcal{Y}$ and, for each $u^* \in \mathcal{X}^*_s$, $u^* \circ pr_s \circ f{=}u^* \circ f_s$. Equivalently, $(f^{\triangleleft} \circ pr^{\triangleleft}_s)(u^*){=}f^{\triangleleft}_s(u^*)$ and hence $f^{\triangleleft} \circ pr^{\triangleleft}_s{=}\varphi_s$. Second, let φ be a sequentially continuous D-homomorphism of $\overline{\mathcal{X}}$ into $\mathcal{Y}$ such that $\varphi \circ pt^{\triangleleft}_s{=}\varphi_s$ for each $s \in S$. Given $u^* \in \mathcal{X}^*_s$, $s \in S$, φ sends $(u^*)_s = u^* \circ pr_s$ to $\varphi_s(u^*) = u^* \circ f_s$. Since also $f^{\triangleleft}$ sends $(u^*)_s$ to $u^* \circ f_s$, it follows from (ii) in Lemma 2.3 that $\varphi = f^{\triangleleft}$ and hence φ is uniquely determined. But, since $\varphi_s = \psi_s \circ (e_s)^{-1}, s \in S$, φ is also a uniquely determined sequentially continuous D-homomorphism of $\overline{\mathcal{X}}$ into $\mathcal{Y}$ such that $\varphi \circ \kappa_s{=}\varphi \circ pr^{\triangleleft}_s \circ e_s = \varphi_s \circ e_s = \psi_s$ for each $s \in S$. $\square$

3. APPLICATIONS

We briefly mentioned possible applications of the category IE to probability theory. We claim that IE (equivalently ID) is a natural category in which the fuzzy probability, as developed by S. Bugajski (cf. [3], [4]) and S. Gudder (cf. [17]), can be developed further. Fundamental notions of the classical kolmogorovian probability (events, random variables, probability measures…) become special cases (cf. [13], [15], [19], [20]).

As already observed in Section 1 (see [19]), fields of sets can be considered as ID-objects, probability measures are ID-morphisms into I, and classical measurable maps are ID-morphisms. Further, let $\mathcal{B}(\Omega)$ be a field of subsets of Ω and let X be a subset of the set $M_1^+(\Omega)$ of all probability measures on $\mathcal{B}(\Omega)$ such that $\Omega \subseteq X$. Then the evaluation of $\mathcal{B}(\Omega)$ defines an ID-poset $ev_X(\mathcal{B}(\Omega)) = \{ev_X(A); A \in \mathcal{B}(\Omega)\} \subseteq I^{\mathcal{B}(\Omega)}$. Let $(\Xi, \mathcal{B}(\Xi))$ be another classical measurable space and let f be a measurable map of Ω into Ξ. Then the distribution D_f is a map of $M_1^+(\Omega)$ into $M_1^+(\Xi)$ defined by $D_f(p)(B) = p(f^{\leftarrow}(B))$, $B \in \mathcal{B}(\Xi)$, $p \in M_1^+(\Omega)$. Fuzzy random variable in the sense of Bugajski-Gudder is a map T of $M_1^+(\Omega)$ into $M_1^+(\Xi)$ satisfying certain measurability condition (cf. [15]) and D_f is a special case (sending $\omega \in \Omega$ into $f(\omega) \in \Xi$). In general, $T(\omega) \in M_1^+(\Xi)$ can be a nondegenerated probability measure on $\mathcal{B}(\Xi)$. Denote $\mathcal{M}(\Omega)$ the set of all measurable functions of Ω into I. For $u \in \mathcal{M}(\Omega)$ let $ev(u)$ be a map of $M_1^+(\Omega)$ into I defined by $(ev(u))(p) = \int_\Omega u dp$, $p \in M_1^+(\Omega)$. For $X = M_1^+(\Omega)$, $\mathcal{X} = \{ev(u);\ u \in \mathcal{M}(\Omega)\}$, $Y = M_1^+(\Xi)$, $\mathcal{Y} = \{ev(u); u \in \mathcal{M}(\Xi)\}$, the measurability of $T : X \longrightarrow Y$ means (cf. [15]) exactly $\mathcal{Y} \circ T \subseteq \mathcal{X}$ (i.e. $u \circ T \in \mathcal{X}$ for all $u \in \mathcal{Y}$).
NOTE: since ID and IE are isomorphic, fuzzy random variables are intrinsic notions in the realm of IE.

Previous observations lead to the following generalizations. Let $(X, \mathcal{X})$ be an IE-measurable space. Then $\mathcal{X}$ is said to be a *generalized field of events*, each $x \in X$ is said to be a *generalized elementary event*, each IE-morphism of $\mathcal{X}$ into I is said to be a *generalized probability*, or *state*, on $\mathcal{X}$. Let f be a measurable map of $(X, \mathcal{X})$ into another IE-measurable space $(Y, \mathcal{Y})$. Then f is said to be a *generalized fuzzy random variable*. Each IE-morphism h of $\mathcal{Y}$ into $\mathcal{X}$ is said to be a *generalized observable*. Remember, if $(Y, \mathcal{Y})$ is sober, then there is a unique generalized fuzzy random variable f such that $h = f^{\triangleleft}$.

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