XIIth Czech-Polish-Slovak Mathematical School

Derivative of a function at a point

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Abstract:
We present a concept of differentiable functions and derivatives. The notion of a differentiable function f at a point x is based on the existence of a function φ such that $f(x+u) - f(x) = \varphi(u)u$ for all u from some neighbourhood of 0 and φ is continuous at 0.

Key words and phrases. Derivation of linear and polynomial functions, definition and basic properties of a differentiable function, existence of the derivative of a function at a point

Introduction

The notion of a derivative of a function at a point is often introduced via the limit of a function at a point. This is usually too difficult for understanding by students at secondary grammar schools (see (1), (3), (7)), since the definition of a limit is difficult for them, too. We use the continuity of a function at a point as a prime notion (see (2), (4), (6)).

The difference of the value of a function f at a point x and at another point in a neighbourhood of the point x is given by the formula $f(x+u) - f(x)$, where u is distance from the point to x. To quantify the velocity of the change, we use the function $\varphi(u) = \dfrac{f(x+u) - f(x)}{u}$. This function is defined for all $u \neq 0$ for which is $x + u$ from a set, on which f is defined. Is it possible to define the function $\varphi(u)$ at the point 0, so that it will be continuous?

Function $\varphi(u)$ and its properties

Linear function f(x)

If $f(x)$ is a linear function, then $f(x) = ax+b$, where a,b are real numbers. This function is defined for all real numbers. For $u \neq 0$ we have:

$$\varphi(u) = \frac{f(x+u) - f(x)}{u} = \frac{\left(a(x+u)+b\right) - \left(ax+b\right)}{u} = \frac{au}{u} = a\,.$$

This means, that the function $\varphi(u)$ does not depend on the variable u, that is, $\varphi(u)$ is a constant function. The graph of a linear function $f(x)$ is a line. The value of the function $\varphi(u)$ is the slope of this line.

If the function $f(x)$ is interpreted as the law of motion of a particle on a straight-line, then x and $x+u$ represent instants of time and the values $f(x)$ and $f(x+u)$ the corresponding positions of the particle. The difference $f(x+u) - f(x)$ is the displacement of the particle during the time--interval between the instants x and $x+u$. The particle moves at a constant velocity given by the function $\varphi(u)$. The velocity is the rate of displacement.

Let $f(x)$ be the costs of producing x units of the given commodity, f is the costs function of this commodity and $\varphi(u)$ is the marginal costs.

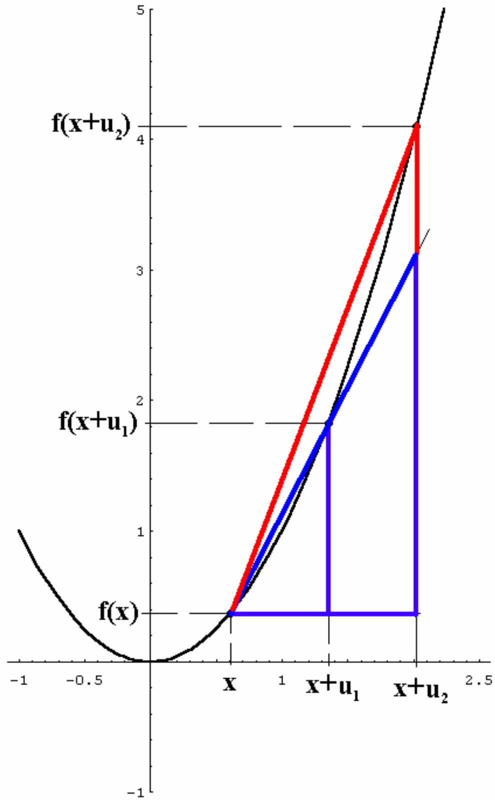
Let $f(x)$ be the amount of heat needed to raise the temperature of a unit mass of the substance from 0 to x (measured in degrees). Then $\varphi(u)$ is the amount of heat needed to raise the temperature of a unit mass of the substance by one degree; $\varphi(u)$ is the specific heat of the substance.

Temperature extensibility can be approximated by linear function $l=l_0(1+\alpha\Delta t)$. The value of the function $\varphi(u)=l_0\alpha$ describes the change of longitude of a solid according to the unit change of temperature.

Non-linear function f(x)

If the function $f(x)$ is non-linear, then $\varphi(u)$ is not constant. For instance, consider the function $y=x^2$.

Fig. 1



If $0 < u_1 < u_2$, then $\frac{f(x+u_1)-f(x)}{u_1} < \frac{f(x+u_2)-f(x)}{u_2}$ and hence $\varphi(u_1) < \varphi(u_2)$.

Theorem (Principle of continuous extension)

Let f and g be functions continuous (left-continuous, right- continuous) at a point a.
If in every neighbourhood (left- neighbourhood, right- neighbourhood, respectively) of a there is a point $x \neq a$ such that f(x) = g(x), then f(a) =g(a).

Theorem implies that the value at *a* of a function continuous at the point *a* is uniquely determined by the values at points $x \neq a$ in a neighbourhood of *a*. If the values of a function at points $x \neq a$ are given and if it is possible to choose the value at a so that the function becomes continuous at a, then it can be done in only one way.

The next example illustrates the theorem.

Example: What is f(3), if you know that the function f is continuous at 3 and

$f(x) = \frac{2x}{x^2-9} - \frac{1}{x-3}$ for every $x \neq$ -3 and $x \neq 3$?

Solution:

Because $\frac{2x}{x^2-9} - \frac{1}{x-3} = \frac{2x-(x+3)}{x^2-9} = \frac{x-3}{x^2-9} = \frac{1}{x+3}$, for every $x \neq$ -3 and $x \neq 3$,

the function defined by $x \mapsto \frac{1}{x+3}, x \neq -3$, is continuous at 3, hence the value $f(3)$ is given

by $f(3) = \frac{1}{3+3} = \frac{1}{6}$.

The function $f(x) = \begin{cases} \frac{2x}{x^2-9} - \frac{1}{x-3} & x \neq 3,-3 \\ \frac{1}{6} & x = 3 \end{cases}$ ***is continuous at the point 3.***

Definition: A function f is said to be differentiable at a point x if there exists a function φ, continuous at 0, such that $f(x+u) - f(x) = \varphi(u)\,.\,u$ for every u in some neighbourhood of 0.

Hence, the problem of differentiability of a function f at a point x reduces to the problem of a suitable choice of $\varphi(0)$. The principle of continuous extension says that if $\varphi(0)$ can be chosen so that φ becomes continuous at 0, then such a choice unique. The number $\varphi(0)$ is then called the derivative of the function f at the point x and is denoted by D$f(x)$.

Example: Let $f(x) = x^3$ for every $x \in (-\infty, \infty)$. Let us show that the function f is differentiable at the point 2 and the derivative of function f at 2 is equal to 12.

Solution: Note that $f(2+u) - f(2) = (2+u)^3 - 2^3 = 12u + 6u^2 + u^3$ for every u. Hence $u.\varphi(u) = 12u + 6u^2 + u^3$. If $\varphi(u) = 12 + 6u + u^2$ for every u then the function is continuous at 0 and $f(2+u) - f(2) = \varphi(u)\,u$ for every u. By definition this means that the function f is differentiable at the point 2. The derivative of function f at 2 is equal to 12 because in this case $\varphi(u) = 12 + 6u + u^2$, for every $u \in (-\infty, \infty)$ and hence $\varphi(0)=12$.

Existence and non-existence of derivation of a function at a point

Our approach to the introduction of derivation of a function at a point was tested for mistakes among students of the fourth class at the secondary grammar school. The function $\varphi(u) = \frac{f(x+u)-f(x)}{u}$ was replaced by the function of the slope of chord given by formula $s_{f,a}(x) = \frac{f(x)-f(a)}{x-a}$. We illustrate our procedure in next example.

Teacher: Calculate the derivation of the function $y = x^2$ at the point 1 from the definition!

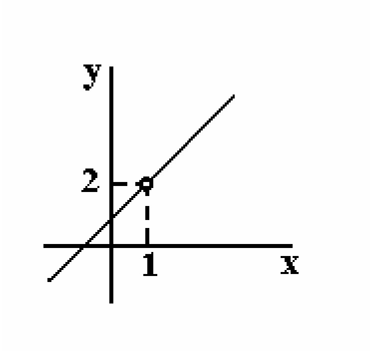
Robert: $y = x^2, a = 1$.

$$s_{f,1}(x)=\begin{cases}\dfrac{x^2-1}{x-1} & x\neq 1,\\ k & x=1.\end{cases}\qquad \frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{x-1}=x+1$$

$$s_{f,1}(x)=\begin{cases}x+1 & x\neq 1,\\ k & x=1.\end{cases}$$

Teacher: Do you know to describe the graph of the function $y=x+1$?
Robert: The line.
Teacher: More precisely.
Robert: The straight line.

Fig. 2



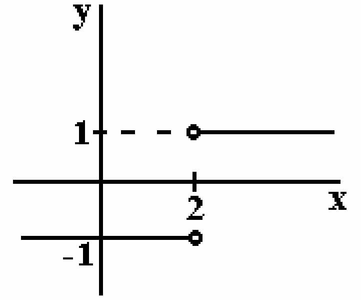
Teacher: What is it possible to add so that the previous function becomes continuous?
Miroslava: We have to fill the circle.
Teacher: How?
Ivan: By number 2.
Teacher: What does it mean for the value of derivation of the function $y=x^2$ at the point 1?
Robert: It is equal to 2.
Teacher: We considered functions with derivation at every point of the domain. Now, we are going to deal with functions having no derivation at least at one point.

$f(x)=|x-2|\quad f'(2)=?$

Paul: $s_{f,2}(x)=\begin{cases}\dfrac{|x-2|}{x-2} & x\neq 2,\\ k & x=2.\end{cases}$

$$x\in(2;\infty):\frac{|x-2|}{x-2}=\frac{x-2}{x-2}=1$$
$$x\in(-\infty;2):\frac{|x-2|}{x-2}=\frac{-(x-2)}{x-2}=-1$$

Fig. 3



Teacher: Is it possible to extend the function (to define its value at 2) so that it becomes continuous?
Luke, Lucy: No, it isn't.

Teacher: What does it mean for the derivation at the point 2?
Paul: It doesn't exist.

Conclusion

At the end we borrow few lines from (5):
"If the reader does not value mathematics and mathematical analysis more than a comfortable feeling that the way calculus is taught at his and other famous universities is essentially all right, then for him the present paper does not have much to say."

We feel that the quality and the amount of intellectual activities needed to transform the mathematics understood (derivation of a function at a point) into the mathematics suitable for teaching should never be undervalued. The effort needed to understand mathematical knowledge matches the effort to invent it. If one wants to write a good calculus book, he has to carry out a mathematical research in the usual sense of the word. In our paper we wanted to follow the idea cited above.

We believe that much more effort is needed to help students to understand and to master the process of derivation. We have a positive experience with teaching derivation based on continuity. This way the process of derivation is visualised. Students practise graphs of functions and it is easier for them to decide whether the derivative of a function exists.

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