Remarks on continuous fractions

Mgr. Ján Gunčaga

1 Introduction

In connection with teaching limits at the secondary school level, many teachers often make a mistake and neglect or omit a suitable motivation and a proper introduction to the topic.

The history of mathematics provides many inspiring examples and approaches to sequential convergence which can help in understanding the limit processes. There are paralells between historical mathematical thinking and the development of mathematical thinking in the mind of students.

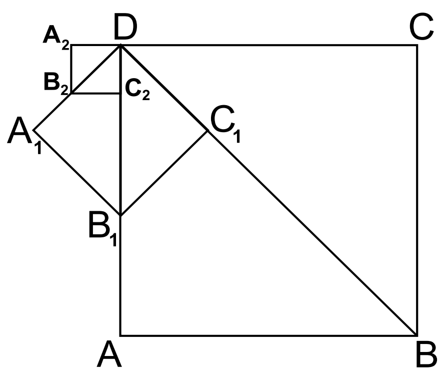
Continuous fractions can serve a teacher at a secondary school as a good material for a motivation and an introduction to sequences and limits.

2 Ancient Greek

About three centuries B. C., in the time when nobody talked on limits and convergence, the ancient Greek mathematicians used continuous fractions to calculate the values of irrational numbers. The fact is illustrated by the following example.

Consider a square $ABCD$(see Picture Nr. 1). Because $|AB| < |CD|$, there exists a point $C_1 \in BD$ with $|AB| = |BC_1|$. The perpendicular at the point C_1 crosses the side AD in the point B_1. Clearly $|\angle ADB| = |\angle AB_1D| = 45°$. The triangle DB_1C_1 is isosceles and rectangular, so $|B_1C_1| = |C_1D|$. Next, consider a square $A_1B_1C_1D$. The length of the side of this square is $|BD| - |AC|$. Again $|A_1B_1| < |C_1D|$ and we can repeat the same construction in the square $A_1B_1C_1D$ and get a square $A_2B_2C_2D$. If repeat the process, the sides of appropriate squares "go" to nil.

Picture Nr.1



Volkert shows in [3] how we can use this construction to aproximate $\sqrt{2}$. Let $|A_iB_i| = a_i$ for $i \in N$ and let $|AB| = a_0 = 1$. So $|BD| = \sqrt{2}$ and $|BD| = |BC_1| + |C_1D|$, hence $\sqrt{2} = 1 + a_1$. Further $|B_1D| = |B_1C_2| + |C_2D|$, hence $|AD| - |AB_1| = |B_1C_2| + |C_2D|$

and we can write $1 - a_1 = a_1 + a_2$, which implies $a_0 = 1 = 2a_1 + a_2$. We can continue: $|B_2D| = |B_2C_3| + |C_3D|$, hence $|A_1D| - |A_1B_2| = |B_2C_3| + |C_3D|$. We get $a_1 - a_2 = a_2 + a_3$, which implies $a_1 = 2a_2 + a_3$. In general, we get for all nonnegative whole numbers $n: \; a_n = 2a_{n+1} + a_{n+2}$. From the preceding equations, we get:

$$\sqrt{2} = 1 + a_1$$
$$1 = 2a_1 + a_2 \Rightarrow \frac{1 - a_2}{a_1} = 2 \Rightarrow \frac{1}{a_1} = 2 + \frac{a_2}{a_1}$$
$$a_1 = 2a_2 + a_3 \Rightarrow \frac{a_1 - a_3}{a_2} = 2 \Rightarrow \frac{a_1}{a_2} = 2 + \frac{a_3}{a_2}$$
$$a_2 = 2a_3 + a_4 \Rightarrow \frac{a_2 - a_4}{a_3} = 2 \Rightarrow \frac{a_2}{a_3} = 2 + \frac{a_4}{a_3}$$
$$\vdots$$
$$a_n = 2a_{n+1} + a_{n+2} \Rightarrow \frac{a_n - a_{n+2}}{a_{n+1}} = 2 \Rightarrow \frac{a_n}{a_{n+1}} = 2 + \frac{a_{n+2}}{a_{n+1}}$$
$$\vdots$$

We substitute:

$$\frac{1}{a_1} = 2 + \frac{a_2}{a_1} = 2 + \frac{1}{\frac{a_1}{a_2}} = 2 + \frac{1}{2 + \frac{a_3}{a_2}} = 2 + \frac{1}{2 + \frac{1}{2 + \frac{a_4}{a_3}}} = 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{a_5}{a_4}}}} = \ldots =$$

$$= 2 + \frac{1}{2 + \frac{1}{2 + \ldots \frac{1}{2 + \frac{a_{n+1}}{a_n}}}} \Rightarrow \frac{1}{a_1} = 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ldots}}} \Rightarrow a_1 = \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ldots}}}}$$

This way we construct a continuous fraction for the number $\sqrt{2}$:

$$\sqrt{2} = 1 + a_1 = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ldots}}}}$$

If we understand this continuous fraction as a sequence of aproximate values of $\sqrt{2}$, we can calculate :

$$1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{2}{5}}} = 1 + \frac{1}{2 + \frac{5}{12}} = 1 + \frac{12}{29} = \frac{41}{29} \doteq 1,413793$$

$$1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}}} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{5}{12}}} = 1 + \frac{1}{2 + \frac{12}{29}} = 1 + \frac{29}{70} = \frac{99}{70} \doteq 1,414286$$

So we get the inequality $\frac{99}{70} > \sqrt{2} > \frac{41}{29}$. From the wiewpoint of school mathematics it is important to observe that it is possible to write this continues fraction with a recurrent sequence:

$$a_1 = \frac{3}{2}, \quad a_{n+1} = 1 + \frac{1}{1+a_n} \text{ for } \forall n \in N \ .$$

3 Leonhard Euler

Leonhard Euler (1707-1783) deals with continuous fractions in his book *Introductio in analysis infinitorum* and the last chapter in the first part of the book is entitled *Continuous fractions.* Let x be a positive real number. If

$$\text{If } x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}}, \text{ than } x = \frac{1}{2+x}$$

We get a quadratic equation $x^2 + 2x = 1$ and its solution $x = \sqrt{2} - 1$.

Euler generalizes this example as follows. If a is a positive real constant, then

$$x = \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \dots}}}}, \text{ hence } x = \frac{1}{a+x} \text{ and } x^2 + ax = 1, \text{ implies } x = \frac{\sqrt{a^2+4} - a}{2}.$$

Also in 18th century the calculation of continuous fractions was a videspread means for getting the values of irrational numbers.

Euler constructed an algorithm how change an infinite series with the changing signs to form continuous fractions. He explains it as follows.

Let $x = \frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \dots$. Consider a continous fraction of the form

$$\cfrac{1}{a + \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \dots}}}}$$

The partial summs of the infinite series are $\frac{1}{A}, \frac{B-A}{AB}, \frac{BC - AC + AB}{ABC}, \dots$ and

the expressions in the continous fraction are $\frac{1}{a}, \frac{b}{ab+\alpha}, \frac{bc+\beta}{abc + a\beta + \alpha c}, \dots$

Comparing the corresponding expressions, we get a system of equations. We explain the procedure on the first three steps:

$$\frac{1}{A} = \frac{1}{a}, \frac{B-A}{AB} = \frac{b}{ab+\alpha}, \frac{BC - AC + AB}{ABC} = \frac{bc+\beta}{abc + a\beta + \alpha c}.$$

Hence
$a = A$
$b = B - A$
$AB = ab + \alpha$
$BC - AC + AB = bc + \beta$
$ABC = abc + a\beta + \alpha c$
The first two equations are simple. If we substitute b into the third equation, we have $A(B-A) + \alpha = AB$ and hence $\alpha = A^2$. We simplify the fifth equation $ABC = a(bc+\beta) + \alpha c$. If $BC - AC + AB = bc + \beta$, then $ABC = a(BC - AC + AB) + \alpha c = A^2(BC - AC + AB) + \alpha c$, which implies $c = C - B$. We substitute now into the fourth equation for b and c. We get $BC - AC + AB = (B-A)(C-B) + \beta$, hence $\beta = B^2$. Euler generalizes these equations:

$$\frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \ldots = \cfrac{1}{A + \cfrac{A^2}{B - A + \cfrac{B^2}{C - B + \cfrac{C^2}{D - C + \ldots}}}}$$

He illustrates this interesting construction using the Leibniz´s series :

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \ldots}}}}$$

4 Summary

We selected the above algorithms and examples from a great number of interesting items from the history of mathematics. In my opinion, in Slovakia unsatisfactory attention is paid to the often ingenious and seminal work of great mathematicians of the past. It would be rather useful to include into mathematical textbooks more such examples representing "arts of thinking" also an integral part of culture and history. This is in particular suitable when introducing some important notions and constructions, or when teaching " how to solve mathematical problems". An interested reader can find more information about teaching mathematics "via history" in [6].

References

[1] Kvasz, L. : *Prednášky z dejín matematickej analýzy*, in: www.matika.sk.

[2] Znám, Š. a kol.: *Pohľady do dejín matematiky*. Bratislava, Alfa 1986.

[3] Volkert, K. : *Geschichte der Analysis*. Zürich, Bibliographisches Institut & F.A. Brockhaus AG 1988.

[4] Cantor, M. : *Vorlesungen über Geschichte der Mathematik. Band 1.* Leipzig, Teubner Verlag 1900.

[5] Edwards, C. H. : *The historical development of the calculus*. New York-Heidelberg-Tokyo, Springer Verlag 1937.

[6] Eisenmann, P. : *Propedeutika diferenciálního a integrálního počtu ve výuce matematiky na střední škole I.* In: *Matematika, Fyzika, Informatika*, 1997, č.7, Praha, Prometheus, 353-359.

[7] Cantor, M. : *Vorlesungen über Geschichte der Mathematik. Band 3.* Leipzig, Teubner Verlag 1901.