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Equations $f(x) = f^{-1}(x)$ as a generator of mathematics teaching problems

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Abstract
Within the secondary school mathematics, the notion of an inverse function and its relationships to the original function does not attract much attention. In this article we deal with equations of the type $f(x) = f^{-1}(x)$ as a source of problems the solution of which leads to a better understanding of the notion of an inverse function. We make use of the PC programs Derive and WinPlot.

1. Introduction

In this article we show, how to use the method of generating problems in teaching calculus at secondary level. The method has been described in [2]. A problem is given to a student an he is assisted when solving the when solving the problem as far as it is necessary. Afterwords, the student is motivated to ask himself further question and to generate problems related to the original one. Exactly, this activity of generating related problems is the care of the method. The related problems can be obtained via analogy, variation, generation, etc.

Let us illustrate the method of generating problems on an example. Consider an equation of the form $f(x) = f^{-1}(x)$, where f is a function and f^{-1} is its inverse function. Solving such equation provides a student of a secondary school a broad variety of topics and activities leading to a much better understanding the notions and properties of functions, inverse functions, equations with parameters, and so on. Various computer programs supporting drawing graphs of functions will effectively help in such activities.

2. Linear function

Assume that f is a linear function $f(x) = ax + b$, $a \neq 0$. Then the inverse function f^{-1}: $x = ay + b$. Hence

$x - b = ay$

$\dfrac{x-b}{a} = y$

The equation $f(x) = f^{-1}(x)$ has a form

$$ax + b = \frac{x-b}{a}. \tag{1}$$

Hence

$$\begin{aligned} x - b &= a^2x + ab \\ x(1-a^2) &= ab + b \end{aligned}$$

If $a = 1$, we get $0 = 2b$. If $b = 0$, the solution are all real numbers. If $b \neq 0$, the equation (1) does not have any solution. If $a = -1$, we get $0 = 0$. For every real number b, the solution of the equation (1) are all real numbers.

If $a \neq 1$ and $a \neq -1$ we get

$$\begin{aligned} x &= \frac{b(a+1)}{1-a^2} \\ x &= \frac{b(a+1)}{(1-a)(1+a)} \\ x &= \frac{b}{1-a} \end{aligned}$$

If $a \neq 1$ and $a \neq -1$ the equation (1) has one solution.

These solutions have also geometrical interpretation. We know that graphs of the function f and the inverse function f^{-1} are axial symmetric with axis $y = x$. If $a = 1$ and $b = 0$ we have a function $y = x$, the graph of which is axis of this symmetry. The inverse of this function is also $y = x$. Therefore the solution of (1) are all real numbers. If $a = 1$ and $b \neq 0$, the graphs of function $f(x) = x + b$ and $f^{-1}(x) = x - b$ are two parallel lines, which are parallel with $y = x$ and do not have common point. Therefore the equation (1) does not have any solution.

If $a = -1$, the graphs of functions f are lines perpendicular to axis $y = x$ and these lines are in this axial symmetry the isometric sets of points. Therefore the solution of (1) are all real numbers. If $a \neq \pm 1$ the graphs of $f(x)$ and $f^{-1}(x)$ are non-parallel lines with common point on the axis $y = x$. Therefore in this case the equation (1) has one solution.

3. The rational functions of type $\dfrac{ux + v}{px + r}$

Every function $f(x) = \frac{ux+v}{px+r}, x \neq -\frac{r}{p}$ $(p, r, u, v$ are real numbers), can be written in the form $f(x) = a + \frac{k}{x-b}, x \neq b$ $(a, k, b$ are real numbers). For this function, the equation $f(x) = f^{-1}(x)$ has for $a \neq 0$ and $b \neq 0$ a form

$$a + \frac{k}{x - b} = b + \frac{k}{x - a} \tag{2}$$

If we solve this equation, we get

$$\begin{aligned}
a + \frac{k}{x - b} &= b + \frac{k}{x - a} \\
a(x - a)(x - b) + k(x - a) &= b(x - a)(x - b) + k(x - b) \\
(a - b)(x - a)(x - b) + k(b - a) &= 0 \\
(a - b)\left((x - a)(x - b) - k\right) &= 0
\end{aligned}$$

If $a = b$, we get $0 = 0$ and the solution are all real numbers. If $a \neq b$, then

$$\begin{aligned}
(x - a)(x - b) - k &= 0 \\
x^2 + (-a - b)x + (ab - k) &= 0 \\
x_{1,2} &= \frac{a + b \pm \sqrt{(a - b)^2 + 4k}}{2}
\end{aligned}$$

Now, we have three possibilities:

1. If $k > -\frac{1}{4}(a-b)^2$, then the equation (2) has two solutions

$$x_1 = \frac{a+b+\sqrt{(a-b)^2+4k}}{2}, x_2 = \frac{a+b-\sqrt{(a-b)^2+4k}}{2}.$$

   Notice that $(a-b)^2 > 0$ and for every positive number k the equation (2) has two solutions.

2. If $k = -\frac{1}{4}(a-b)^2$, then one solution is $x = \frac{1}{2}(a+b)$.

3. If $k < -\frac{1}{4}(a-b)^2$, then (2) does not have any solution.

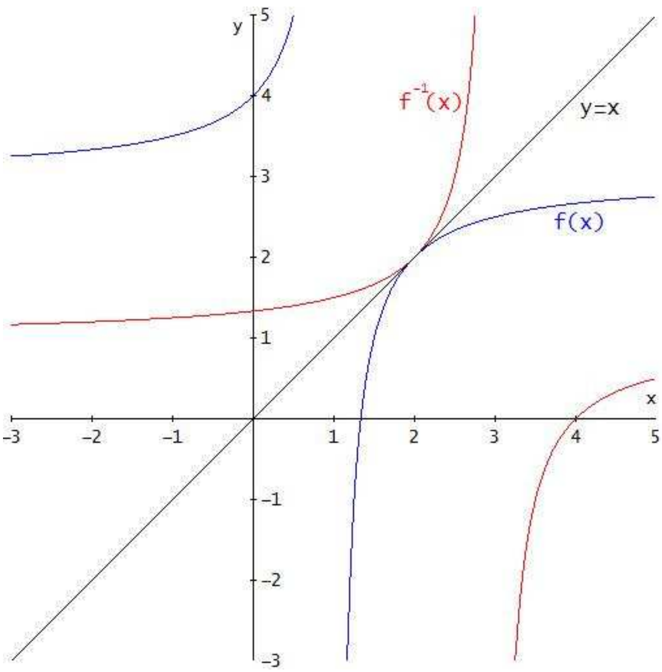

Figure 1

The solutions have a geometrical interpretation. Graphs of the functions f and f^{-1} are hyperbolas. The question is, how many common points these hyperbolas have?

In case $a = b$, the hyperbola - graph of the function f is symmetrical with respect to the axis $y = x$. Therefore the solution of equation (2) are all real numbers.

In case of $a \neq b$ we have three possibilities. First, the hyperbolas have two common points. For every positive number k the hyperbola - graph of the function f has two common points with the axis of axial symmetry $y = x$. These common points are the common points with the hyperbola - graph of the function $f^{-1}(x)$. Second, one arm of the hyperbola f touches the one arm of the hyperbola f^{-1}. They have a common tangent $y = x$ at the common point. This situation we explain by the function $f(x) = 3 - \frac{1}{x-1}$ (see Figure 1).

Third, the hyperbolas do not have any common point.

4. Conclusion

These examples illustrate, how it is possible to use the method of generating problems with generator problem - solving equations $f(x) = f^{-1}(x)$ for different types of functions f. We connect with this teaching mathematical analysis, analytical and synthetical geometry in school mathematics. It is very important that the students' knowledge be not isolated.

Teacher has a possibility to explain the students the notions of inverse function, graphs of different types of functions. The students can see the graphs of the function f and function f^{-1} are symmetrical with respect to the axis $y = x$. The drawing these functions can help us the computer programs. We can use this programs during the teaching process also for finding the numerical solutions the equations $f(x) = f^{-1}(x)$ for some types of functions (trigonometrical, exponential functions, etc.)

The method of generating problems we can use in other parts of school mathematics. That we can see in [1], [3], [4], [5], [6].

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