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Ružomberok 2008

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THE USE OF GEOMETRIC PLACE IN PROBLEM SOLVING

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Abstract. Geometry is a wonderful area of mathematics to teach. It is full of interesting problems and surprising theorems. It is open to many different approaches. The present talk gives the possibility of using the geometric place for the teaching of the conics. We describe some facts from the analytic method of the conics in solving various construction problems.

ZDM Subject Classification: G40, G70.

1. Introduction

During the tenth century, a Persian mathematician, physicist and astronomer Abu Sahl al-Quhi performed intensive studies on tangent circles as a new method for defining conics. In fact, he was able to define every conic section as the locus of centers of tangent circles to two given elements (points, lines or circles). The most interesting part of his work involved solutions for centers of circles tangent to two given circles.

At present, conic sections are generally introduced at secondary school after students gain sufficient skills in algebra and algebraic manipulation. This is because of the approaches and the definitions used to teach the conics. In this article, we look at an approach for the teaching of the conics, and demonstrate a method in which conics are used as tools in problem solving. The driving force behind this approach is Geometric Constructions, which creates the opportunity to introduce and develop the very important concept of Geometric Place. The main advantage of our approach to conic sections is that a dynamic interactive geometry software (such as Cabri Geometry) can be effectively used to conduct investigations to explore the properties of these important curves through plotting, sketching and the use of transformations. By using an informal approach to conics, building geometric constructions related to tangent circles, plotting and sketching of conics by free hand or via the aid of a computer, and solving tangency problems by trial and error are just some of the activities that can be conducted. We also describe some facts from the analytic method of the conics in solving various construction problems.

2. Concept of Geometric Place

We begin with the construction of a circle tangent two given circles, where one circle is contained in the other and where the desired circle is tangent to the larger given circle at a given point C. There are actually two such circles, an internal circle and an external circle. The internal circle contains one of the given circles, and the external circle does not (figure 1).

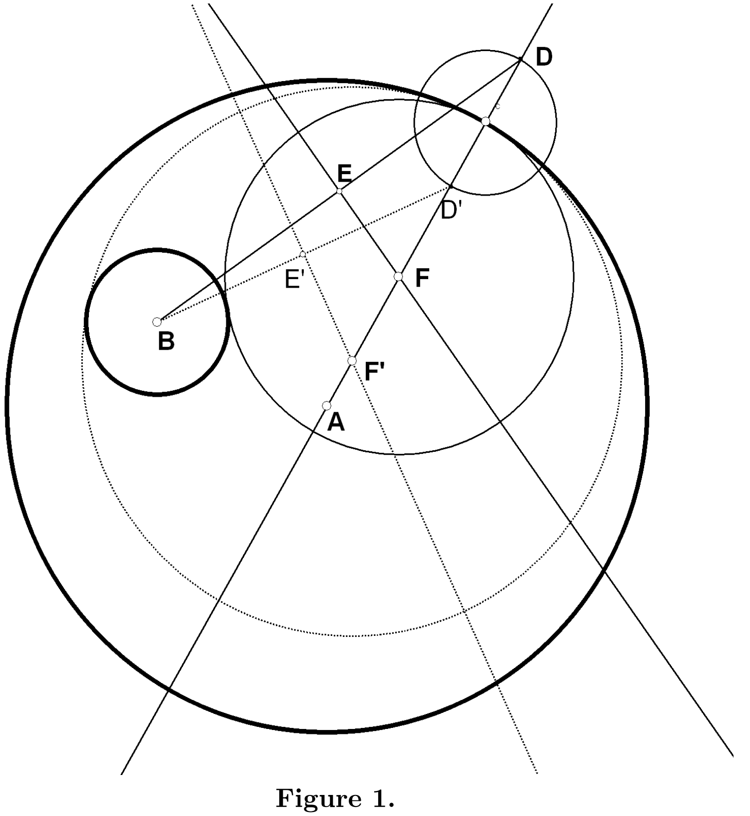

Figure 1.

The following procedure depicts the construction of the external tangent circle. The construction of the internal tangent circle is identical except we use D' instead of D. The center of the desired tangent circle must lie on the line passing through the center A of the large circle and the given point of tangency C. Also, this desired center is equidistant from the smaller circle and the given point C. We do not know which point on the smaller circle is closest to the desired center, but we do know that a radius of the desired circle extends through that point to the center B of the smaller circle. This prescribed segment will have the same length as AD. Now we can construct the base BD of an isosceles triangle BDF, where the point F is unknown. However, if we construct the perpendicular bisector of the base BD, we get the third point F of our isosceles triangle. The construction for the internal tangent circle is similar.

For this case, where one circle is contained in the other, the locus of all points F forms an *ellipse*. The focal points of this ellipse are the centers A and B of the given circles. The Major Axis of the ellipse $2a = R + r$, where R and r are the radii of the given circles. This is true for the locus of internal and external tangent circles. An *ellipse* is usually defined as the set of all points whose sum of distances from two given points (focal points) is fixed. Now we can formulated following problem.

Problem 1. Two internally tangent circles are given. Construct a circle with radius r to be tangent to both of them as shown in Figure 2.

Solution. The centres of the given circles are represented by O_1 and O_2 and their radii r_1 and r_2 respectively. Suppose S is the centre of the circle to be constructed (see Figure 3).

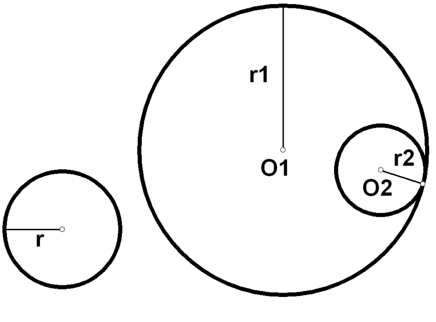

Figure 2.

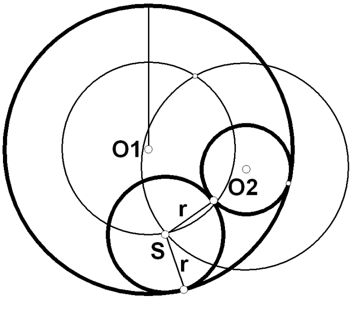

Figure 3.

(1) S will be $r_1 - r$ from O_1, therefore it will be on a circle with radius $r_1 - r$ and centre O_1.

(2) S will also be $r_2 + r$ from O_2, therefore it will be on a circle with radius $r_2 + r$ and centre O_2.

If we construct the circles described in (1) and (2), their intersections S and S' are the centres of two circles forming the two answers to the problem.

Activity 1. Two internally tangent circles $C_1(O_1; r_1 = 5cm)$, $C_2(O_2; r_2 = 2cm)$ are given. A set of tangent circles are to be constructed tangent to both of them (tangent to the larger internally). In Table 1, r shows the radius of the desired circle and $r_1 - r$ $[r_2 + r]$ represents the distance of its centre to the centre of the given circle C_1 $[C_2]$.

(a) For the values given in the table construct the circles and plot their centres.

(b) Sketch the curve that contains the centres of all constructed circles.

r	0	0.5	1.0	1.5	2.0	2.5
$r_1 - r$	5	4.5	4.0	3.5	3.0	2.5
$r_2 + r$	2	2.5	3.0	3.5	4.0	4.5

Table 1.

Figure 4 shows the completed task.

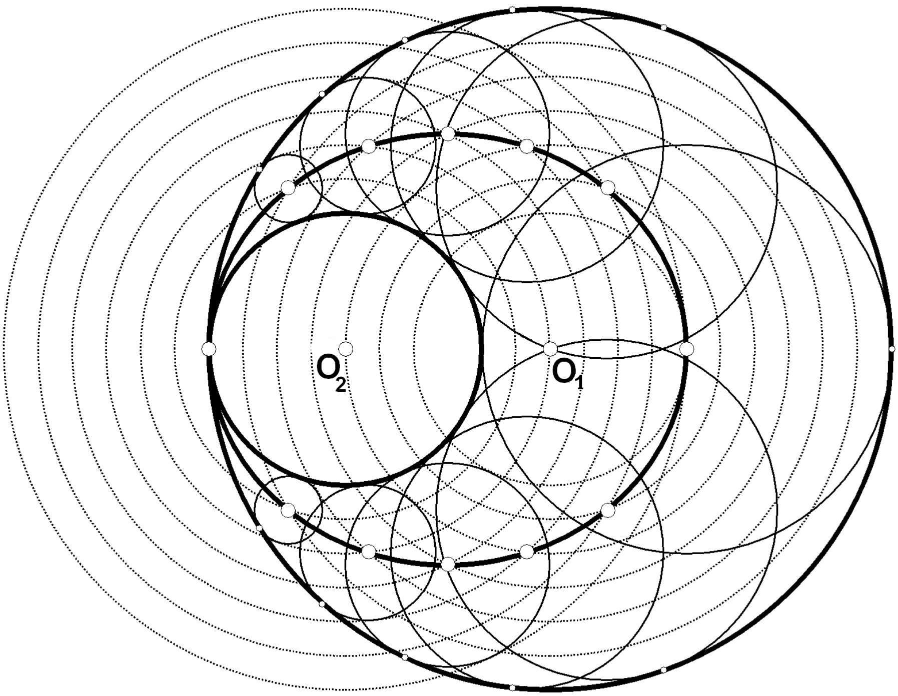

Figure 4.

Equation of the ellipse. Let the centre of the drawn circle be $S(x, y)$ and its radius r. The centres of the given circles are represented by O_1 and O_2 and their radii r_1 and r_2 respectively. The point of contact is assigned to be the Origin and the line joining the centres of the given circles the x–axis (see Figure 5).

$$SO_1 + SO_2 = (r_1 - r) + (r_2 + r) = r_1 + r_2 = 2a \quad \text{(Constant)}$$

therefore the locus of S is an *ellipse*. From right-angled triangles O_2PS and O_1PS:

$$y^2 = (r_2 + r)^2 - (x - r_2)^2 \tag{1}$$

$$y^2 = (r_1 - r)^2 - (r_1 - x)^2 \tag{2}$$

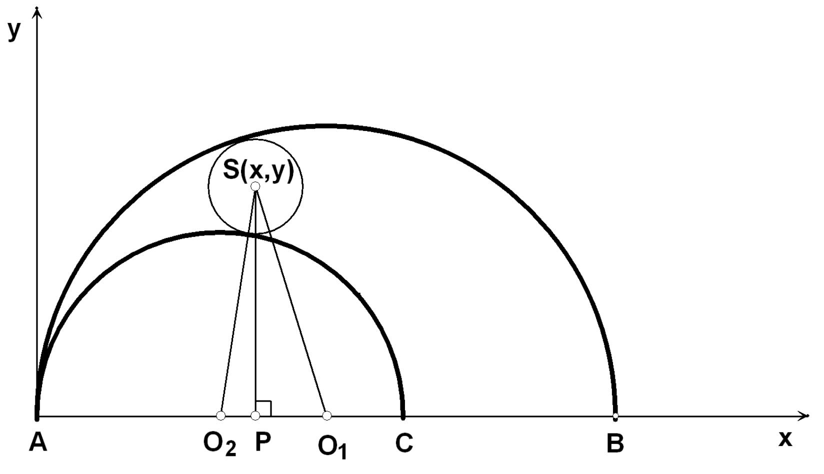

Figure 5.

By eliminating y from (1) and (2) we get:

$$x = \frac{r_1 + r_2}{r_1 - r_2} r. \tag{3}$$

If we substitute x in (3) into either one of (1) and (2) we obtain the equation of the ellipse in the form $y = f(r)$, e.g.,

$$y = \frac{2}{r_1 - r_2} \sqrt{r_1 r_2 (r_1 - r_2 - r) r}. \tag{4}$$

In (3) if r is made the subject; its substitution in (1) or (2) yields

$$y = \frac{2}{r_1 + r_2} \sqrt{r_1 r_2 (r_1 + r_2 - x) x}. \tag{5}$$

We proved the following theorem.

Theorem 1. *Two internally tangent circles are given. A third circle is drawn to be tangent to both of them as shown in Figure 5. The locus of all centre points $S(x, y)$ forms an ellipse.*

3. Conics in problem solving

Problem 2. Two congruent circles are outside each other, but inside a third, larger circle. Each of the three circles is tangent to the other two and their centres are along the same straight line. Given R the radius of the larger circle centered at O and the contact points of the congruent circles and the larger one are P and Q. Construct a circle to be tangent to the given three circles.

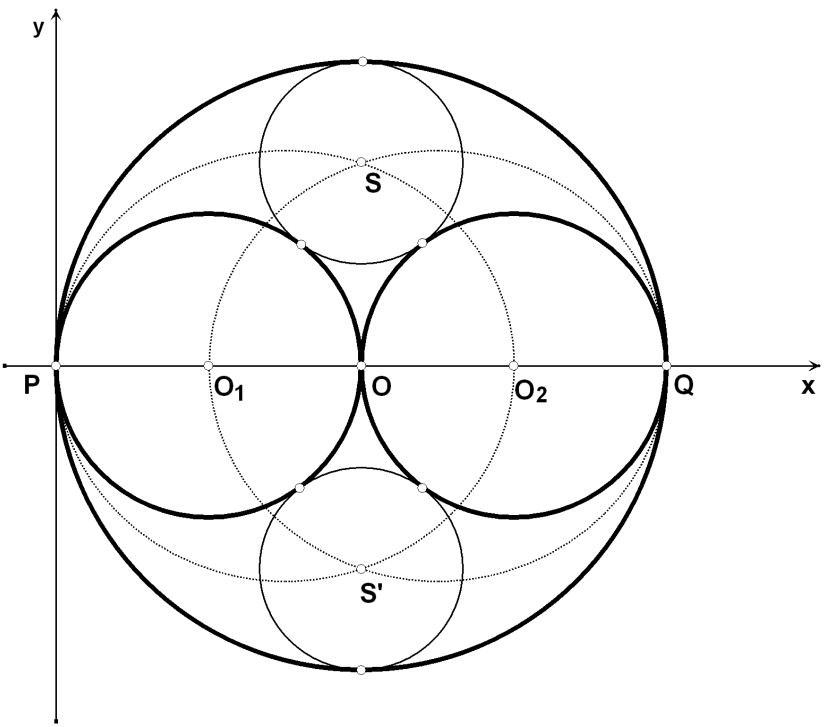

Figure 6.

Solution. Let the line contains the centres O_1, O_2 and O of given circles be the x–axis, and P the Origin (Figure 6). The equation of the ellipse associated with the circles centered at O_1 $[O_2]$ and O is

$$y^2 = \frac{4}{9}x(3R - 2x) \qquad \left[y^2 = \frac{4}{9}(R - 2x)(x - 2R)\right].$$

Solving the two equations simultaneously yields: $x = R$, $y = \pm\frac{2}{3}R$ and $r = \frac{R}{3}$ is the radius of the circle sought.

Extending Problem 2. Two congruent circles, but inside a third, larger circle are given and their centres are along the same straight line. Each of the congruent circle is tangent (internally) to larger circle. Construct a circle to be tangent to all of them.

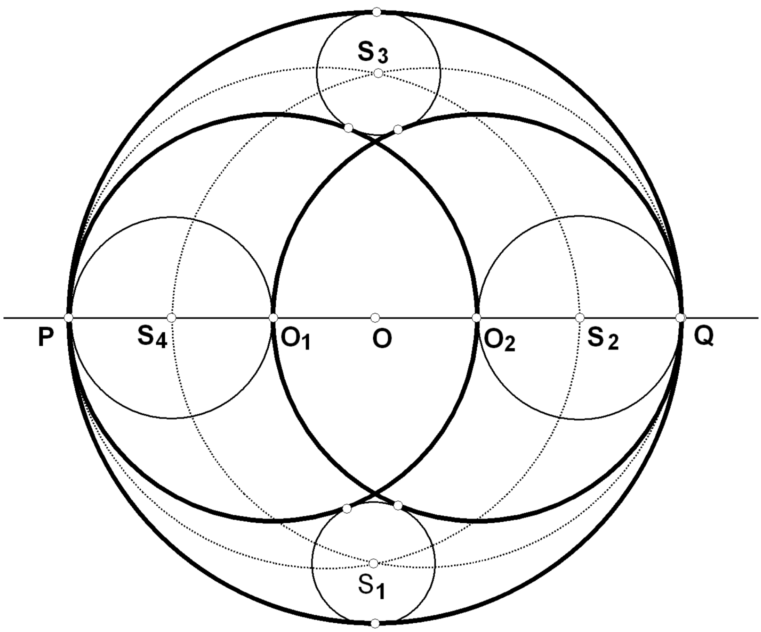

Figure 7.

We leave the solution of this problem (see Figure 7) to the reader.

4. Conclusion

We can demonstrate our approach for the teaching of the conics in two steps. In the first, conics are plotted, sketched and their properties investigated based on geometric constructions. This is an invaluable experience in forming and developing the concept of Geometric Loci. ”Given two or three things, find the locus of the centres of all circles tangent to all of them” can be an excellent scheme of work to start with. By ”a thing” we mean a circle, a line, or a point.

Although the examples given in this paper are related to the ellipse, the method is also applicable to parabolas and hyperbolas. In some cases combinations of two out of the three conics can be used.

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POLYNOMIALS

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Abstract. The article arose as the need of a reaction on problems which occur by the introduction of polynomials at a school. On the base of a teaching with computer, as a game, we demonstrate the method of visualization of polynomials as a geometric interpretation.

ZDM Subject Classification: D40.

1. Introduction

The majority of the population uses computers as a production tool; we use them in every-day civil and private life. For students, they are a natural means, a helper, and also a tool at school where the computers create reliable and attractive environment. We have made a program which creates such environment and which is a suitable complement when presenting the topic of *Expressions and Polynomials* at the lower secondary level.

What is the role of technologies in the teaching and learning process, how is it possible to use them in the teaching and learning of mathematics? According to (Kutzler, 2003), it is possible to use them when simplifying, experimenting, for visualisation and concentration. A computer simplifies the execution of arithmetic and algebraic operations, creation of graphs, it is possible to "discover" new pieces of knowledge in the same vain of the contemporary psychological theories of the teaching and learning which consider the learning to be an inductive process, in which experiments play a key role. The role of computers is also important in classes of geometry. We do not want to say that we get rid of pairs of compasses and rulers. However, the one who learnt geometry using a sketchbook appreciates the possibility of visualisation of such situations which can be hardly demonstrated on the board during a single period. We do not have to limit the visualisation only to geometry. In this paper, we would like to introduce a program which serves as a didactical material for the teaching of polynomials at the lower secondary level. According to (Vaníček, 2005) " it is also important from the educational point of view to show children who use their computer at home only for entertainment the advantages of the possible computer use in applicable problems from every-day life (for a child it is the utilisation in common classes). The use of computers in classes of other subjects has a world-view importance for children: the computer is not a toy separated from the world but a normal working tool making work easier and more effective."

2. Motivation

The mathematical program **Mathhill** was created as a reaction to the situations which are typical of a majority of pupils at the primary and lower secondary level. The example which we want to use to explain the situation is taken from the teaching practice of teacher trainees in a class of mathematics in a lower secondary level school; in class 8.B (pupils aged 14).

A teacher trainee presented the following example showing the advantage of polynomials introduced in the previous lesson.

Classes 8.A and 8.B have decided to go together for a school trip to the Hluboká Castle in South Bohemia. The trip is organised by a teacher who is not a mathematician. How much money is it necessary to collect for the admission, if he does not know exactly how many children and adults will participate in the trip? Prepare a table in Excel for the teacher.

Hluboká nad Vltavou - castle	
Guiding tour in Czech:	
Adults	CZK 130
Children 6 - 15years and students	CZK 70
Elderly over 65 years	CZK 80

The expected solution for the creation of the table was in the form of:

$$130 \cdot u + 70 \cdot s + 80 \cdot d$$

where u is the number of teachers, s is the number of students, and d is the number of elderly people.

The pupils reacted in accordance with their assumption of the teacher's expectation: 13 out of 24 pupils wrote the answer that: *The teacher should collect CZK 7.280 for admission* (as $130 \cdot 70 \cdot 80 = 728\ 000$, so 7.280).

The reaction in accordance with the teacher's expectation is in its essence a reaction in the lines of the didactical contract. G. Brousseau stated in 1980 the definition of the didactical contract: the contract corresponds with "the set of the teacher's behaviours (specific to the taught knowledge) expected by the student and the set of the student's behaviour expected by the teacher. (Sarrazy, Novotná, 2005) say that "this contract is not real contract in that sense of the world, in fact it was never contracted either explicitly or implicitly between the contractors, but also to the intent that its regulations and its criteria of satisfaction can never be really précised either by one party or the other.

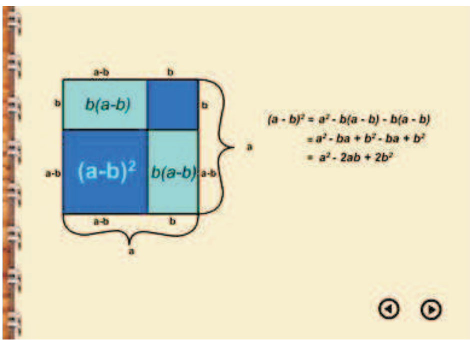
The pupils in our sample seem to be looking for such a solution, which they are used to. If they solve word problems, the solution is a number and their desire to find the correct answer ends with answers in the form of a number and not an expression. We brought up a question if it is possible to improve the understanding of the concept of polynomial and if it could be achieve using a kind of its visualization. The means of the correction could be the program **Mathhill**.

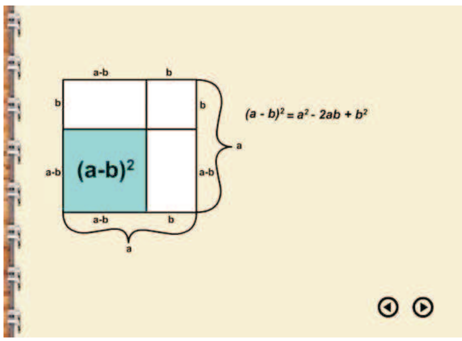
3. About Mathhill Program

Mathhill is created in program Macromedia Flash. Flash is a program designed for creation of interactive multimedia animations. It is attractive for more and more user thanks to its simplicity and various possibilities. The most common products of this program are above all flash banners, on line games, whole web-sites, and other applications, which can work both on-line and off-line. What is interesting on this program is the fact that it connects the graphic and programming and therefore it is suitable for people who create only graphical animations, banners or various effects as well as people who work with ActionScriptem (programming language Flash).

The menu of the application has been created in a winter environment containing four buttons which are links to particular games or animations and whose names are situated under their pictures. The first button of **Formulae Animations** will take us into the environment suitable for utilisation in classes of mathematics. Here are animations and visualisations of particular algebraic expressions taught and used in the secondary level schools. Another link **Revision** can serve to pupils in classes of mathematics or at home for home revision. Here are exercises on operations with number expressions and introduction of a variable.

The third link is **Rational Expressions** which is a game based on calculation with rational expressions. The game goes through all the operations with rational expressions starting with the addition, subtraction to multiplication and division of the expressions. The last link **Calculation with Expressions** contains exercises on all operations with polynomials (sum, difference, factorization by factoring out as well as by using formulae).





The program has been tested in several lower secondary schools and the following passage presents in brief the results of the experiment.

First Experiment:

18 students were working with program Mathhill. A half of them finished the program, the fastest pupils managing to go through it in 85 minutes. Nobody of the pupils succeeded in the game Rational Expressions. The fastest pupils got to the sixth task. All of the pupils managed the game

Revision. The fastest was a boy and a girl, who finished the game in 35 minutes. The others were working about 40 minutes. The pupils were working individually and they were trying to reach the right answer. In our opinion, they are not used to more difficult tasks presented in the game and they are not also used to using computers in classes of mathematics. In spite of the fact that a half of the pupils did not finish the game, they evaluate the game positively.

Second Experiment:

Program Mathhill was used by 20 pupils and almost all of them finished it. The fastest girl managed to finish the game in 60 minutes. The others were working about 80 minutes. In the rest of the time, some of the pupils were browsing *Formulae Animations* or they were playing the game *Revision.* They were so captured by the calculation that they even spent the break between particular lessons playing the game *Rational Expressions.* We were happy about the course of the class. In the class and during the break, there was a working atmosphere. Again, the pupils were working individually and they were trying to reach the right answer. The level of mathematical knowledge of the pupils was excellent even though they had been dealing with this study matter six months before the experiment.

4. Conclusion

The lower secondary school in which we conducted the second experiment is significantly different from the first one. The equipment of the school is above standard, a half of the classes are equipped with interactive tables including the classroom where we conducted our research. The school obtained from the Norwegian Fund 30 new laptops for their students. The pupils for whom there was no school desktop could work on a laptop. Thanks to the laptops which are available to the pupils, each student could work individually and solve given problems in his/her pace. It is therefore obvious that the work with a computer, if it is taken for granted, is of a significant benefit to the teaching mathematics. We think that the interactive multimedia CD - Mathhill can develop important competences of pupils, teach the pupils to be active in classes and enables the teachers to have an individual approach to their pupils. The teachers who saw the program were enthused by it and positively evaluated the animations which they explain in a difficult way by the drawing on the board.

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THE ROLE OF USING MANIPULATIVES IN SPACIAL SENSE DEVELOPMENT AT THE EARLY STAGES

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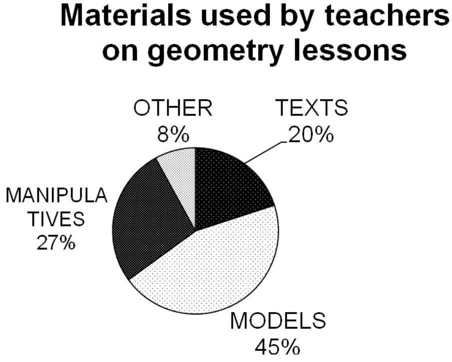
Abstract. The aim of this article is to report the results from our research. We tried to prove hypothesis that *unless pupils at early stages use manipulative materials during their geometry lesson there will be hardly any progress in their spatial sense development.* After ten alternative geometry lessons by using different manipulatives the pupils of the 4^{th} grade at elementary school have shown fine improvement in spatial thinking.

ZDM Subject Classification: C70.

1. Introduction

It is obvious that spacial imagination should be developed at school during geometry lessons. It is stated in elementary school curriculum for mathematics: *devotes to content and methods on teaching mathematics, which contributes to development of creative skills, mental operations,* ***spatial images****, and also social abilities, etc.* [2]. It is not stated how to provide this. So it is up to the teachers what methods they choose for gaining this goal. We think the one of the possibilities is using **manipulative facilitation.**

We were curious how elementary teachers deal with this fact and asked them *what materials besides textbooks they use during their geometry lessons.* We have spread 120 questionnaires and we have received 71 answers. According to character of materials teachers use we have divided answers into three groups. The first group of teachers (20%) use textbooks, pictures, flash cards and drawing tools. Typical for the second group (45%) is using models such as geo solids, video presentations of geometry models, software programs, etc. The group of 27% teachers mentioned manipulatives or handling materials, e.g. clay, string, wire, toothpicks, skewers, sticks, clips, straws, etc. The group of 8% teachers did not answer or had rare answers: *"making own examples", "using fantasy", "using grid paper"*. For viewing the results see the following graph.



To prove the importance of using manipulatives for developing spatial sense we have organized an experiment that 16 fourth graders participated. On 30^{th} January 2007 all 16 pupils were given 10 geometry problems to solve and on 15^{th} June 2007 they were given the same problems. Between the testing half of the pupils (experimental group) were taught geometry using manipulatives. The other half (control group) were not working with manipulatives. We have compared results of both groups at the end of testing.

2. Test Tasks

All 10 test tasks were chosen according to released tasks from the TIMSS 2003 (Trends in International Mathematics and Science Study)[1]. We have tried to cover three main geometry areas: lines and angles; two- and three-dimensional shapes; location and movement. In the area of angles and lines the pupils were asked to identify and draw the parallel and perpendicular lines. In the area of two- and three- dimensional shapes they were supposed to determine number of edges in the cube (not all the edges were seen in the picture); recognize relationships between three–dimensional shapes and their two–dimensional representations. In the area of location and spatial relationships the pupils were asked to locate point in a plane; recognize and draw figures in symmetry; recognize and draw reflections and rotations of figures.

[1] http://timssandpirls.bc.edu/timss2003i/released.html

3. Test Tasks

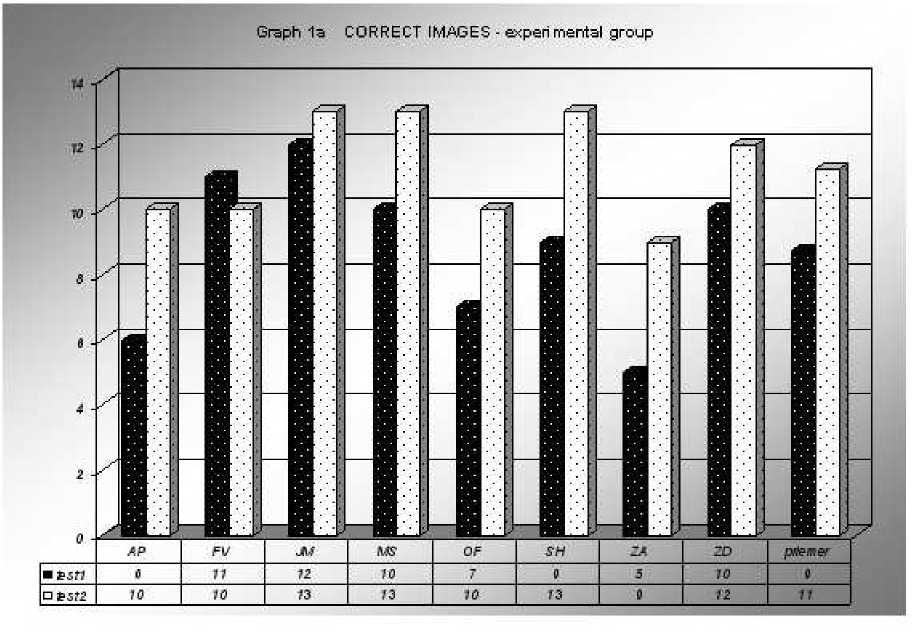
We used grounded theory for analyzing and identifying phenomenon that the pupils achieved when testing, recording their way of reasoning and interpreting their level of spatial ability. We have found a large number of categories during *open coding* which we reduced into only few subcategories during *axial coding*:

- mental manipulation – MM; physical manipulation – PM;
- correct comprehension – CC; false comprehension – FC;
- correct answer – CA; false answer – FA;
- correct explanation – CE; false explanation – FE;
- correct manifestation – CM; false manifestation – FM;
- correct image – CI; false image – FI.

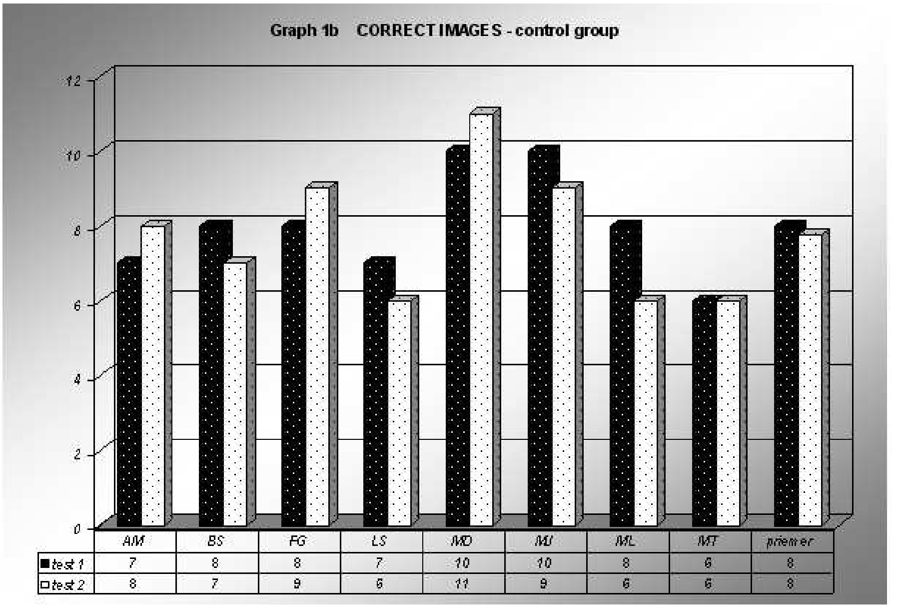
We have marked the codes for individual subcategories in pupils' demonstrations and created two tables for each pupil. One for the results from the pretest and another one for posttest. During *selective coding* we have chosen two main categories that would best represent pupils' progress in spatial abilities: CORRECT ANSWER and CORRECT IMAGE. We have come up to interesting results that are further described.

4. Results

After analyzing the test results in both groups (experimental and control) we have projected them into graphs.

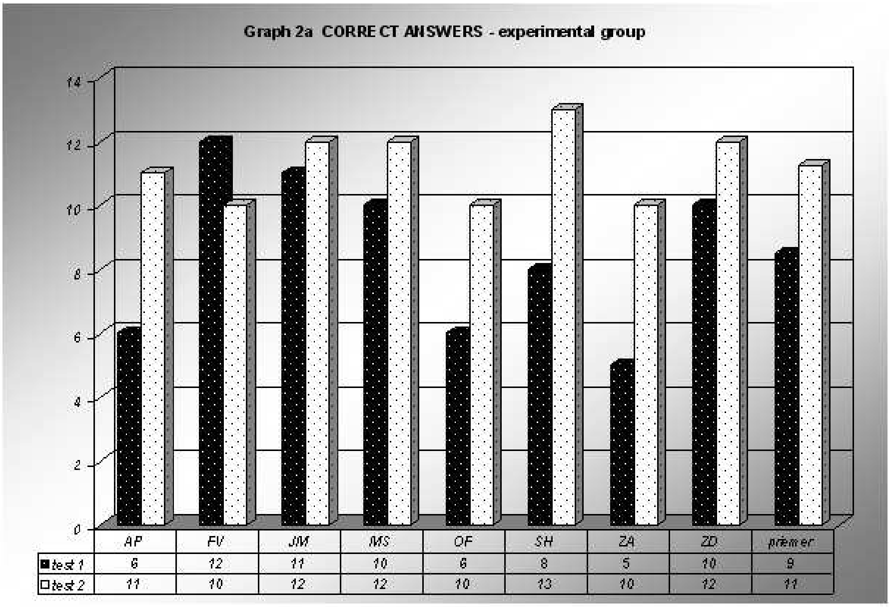


Graph 1a exemplifies the level of spatial imagination of pupils from experimental group before and after using manipulatives during 10 geometry lessons. All pupils from this group has shown some progress but the pupil FV. Pupils AP, MS, OF, SH a ZA have achieved the most progressive results.

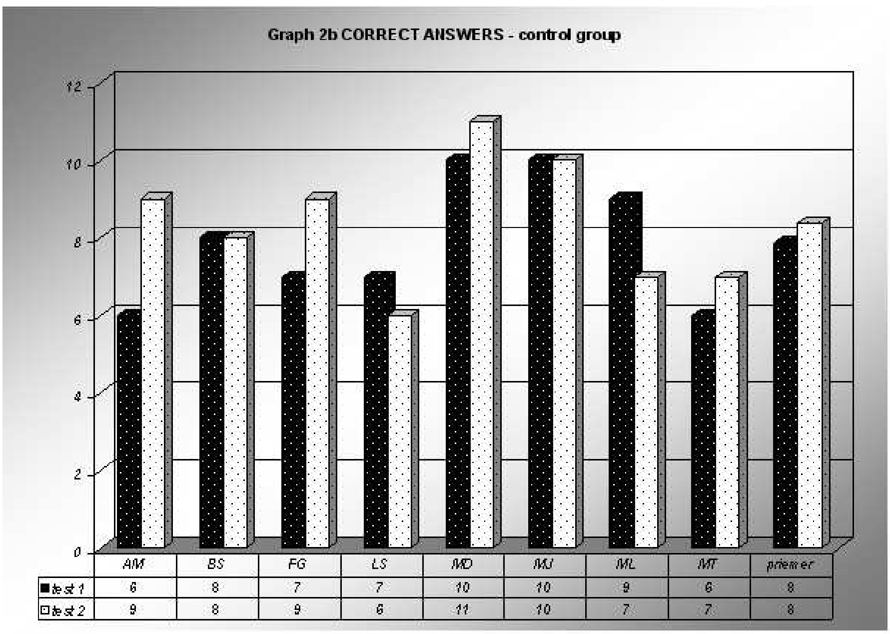


Graph 1b presents achieved level in the control group which did not work with manipulatives. There was not much progress shown. A little progress occurred only by few pupils AM, FG a MD.

The second examined category in both groups was correct answer. In case of correct answer the child was given plus and in case of wrong answer he/she was given minus. Those results were then projected into graphs. Graph 2a represents number of correct answers in experimental group. Again most of the pupils have achieved some progress but a pupil FV. We have tried to find the reasons of FV failure. It is obvious that this pupil has shown the best results within the both groups after the first testing and was the first who completed both tests and one extra task. Our surprise was even bigger as this pupil has shown a great sense of spatial imagination during our experimental lessons. We think one of the possibilities of his failure was an effort he was giving to not loosing his reputation of being the first to completing all the tasks. The second reason could be his two absences. Other reasons could have psychological background typical for boys of his age: being lazy to do the tasks for the second time or short concentration when doing repeated activities. The pupils AP, OF, SH a ZA had the most progressive results.



Graph 2b represents situation after the testing in control group. The number of progresses was less evident than in experimental group. Only few pupils from control group (AM, FG, MD a MT) have shown some progress.



4. Conclusion

Perhaps to create new interesting theory according to the role of using manipulatives at early stages we would need to analyze pupils' works in more details and on less exercises. However, we did prove the hypothesis that ***unless pupils at early stages use manipulative materials during their geometry lesson there will be hardly any progress in their spatial sense development.***

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STRATEGIES OF SOLVING TASKS WITH PERCENTAGES

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Abstract. In this paper we discuss the issue of (non) rules of understanding and using the idea of percentage in various situation contexts. We present the results of empirical investigation carried out among students - the future mathematics teachers.

ZDM Subject Classification: D50.

1. Introduction

Z. Krygowska writes (1977): "a pupil creates such a conception of mathematics as it appears to him by a prism of problems soluted by him". The choice of tasks is as important as the methodics of working with them. The key role in teaching have the text tasks. All these tasks concern some situation described by its contents. The description or the situation itself can be strictly mathematical or can concern a situation outside the mathematics. The methodology of solving tasks with a realistic context is specific (see G. Trelinski 2006). As it is seen in practice, the typical tasks are solved properly by pupils, who are taught of schemes of solving these tasks. Schemes are not sufficient in more difficult problems - here the understanding of ideas which are put in deep structures is necessary (see R. Skemp 1989).

In this paper we will present the results investigations, which aimed at showing and characterization of the strategies[2] used while solving of text tasks dealing with percentages.

The investigation included the group of 41 students of mathematics of full-days studies of teacher specialization[3]. These students were given the set of 12 tasks, taken from different various mathematical contests and from sets of tasks for elementary and lower secondary schools. In the article we show only these tasks, which were the most difficult for students. The list of tasks is shown below.

Task 1. In the marketplace I have bought apples by 20% cheaper than the seller wanted. Selling these apples I want for them the same price as the previous seller demanded. What is the percentage of my profit?

Task 2. More than 94% of participants of the mathematical circle which Joan attends are boys. How many students at least have to participate in the circle?

Task 3. The half of passengers, who got on a tram on the initial tram stop sat down on sitting places. At the next tram stop the number of passengers increased by 8%. It is known, that in the tram there is a room for at most 70 passengers. How many passengers got on the tram on the initial tram stop?

Task 4. Two students, Tom and Luke came out of the same house to school. Tom's step was by 20% shorter than Luke's one, but Tom managed to take in the same time by 20% more steps than Luke. Which of these students will get to school earlier?

Task 5. There were 1000 things produced in the factory. 8% of produced things were defective. The acceptable norm of the amount of defective products is 5%. How many things have to be produced additionally if the acceptable norm must not be exceeded?

[2]By as a strategy we understand every way of a student's behaving, acting, thinking that is to lead him/her to the fulfillment of a role of person who faced the necessity to solve mathematical tasks.

[3]14 people have studied the second-degree studies and therefore they have had the pedagogical qualifications and have been prepared for teaching mathematics in primary and lower secondary schools. However, (both) the strategies used by them and the errors they made do not differ from the results of the remaining people surveyed. For this reason we do not distinguish these groups.

From the point of view of practical applications percentages are very important. They concern the problems which often appear in the real world. Nevertheless, the problem of percentages is difficult in teaching and in learning because of its complex nature.

2. The strategies of solving problems related to percentages

2.1 The strategy of changing the task by imposition of additional conditions

The fairly typical mistake was an incorrect treatment of the situation shown in the task and of the set of information described by the text of the task. For some students "the content of a mathematical problem with a realistic context" was not only what they concluded from announced set of information concerning objects, events, processes and relations between them, but also what was associated by them with the described situation. For different students the same task carried different contents. The construction of the different theoretical models was caused by various understanding of the text and by introducing additional assumptions. Let us look at a few ways of solving the task 5.

Example 1
Solution 1

1000 – the number of things produced
$8\% \cdot 1000 = 80$ – the number of defective things produced
x — the number of things, which have to be produced additionally
5% – the acceptable norm
$\frac{80}{1000+x} \cdot 100\% = 5\%,$
$80 = 0,05x + 50,$
$x = 600.$

Solution 2

$8\% \cdot 1000 = 80$ defective,
$80 = \frac{5}{100}(100 + x),$
$x = 600,$
$x < 600$ not to exceed 5%.

Solution 3

1000 – the number of things produced
$8\% \cdot 1000 = 80$ – the number of defective things produced
x – the number of things, which have to be produced additionally
p – the percentage of deficiencies in the amount of things, which

have to be produced additionally
5% - the acceptable norm
$\frac{80+px}{1000+x} \cdot 100\% = 5\%,$
$80 + px = 0,05x + 50,$
$(0,05 - p)x = 30.$
If $p \in \langle 0; 0,05)$ then $x = \frac{30}{0,05-p}$.
Not enough data, we do not know what will the percentage of defective things be like.

Solution 4

1000 – the number of things produced
$8\% \cdot 1000 = 80$ – the number of defective things produced
x – the number of things, which have to be produced additionally
y – the number of defective things in produced additionally set
5% – the acceptable norm
$\frac{80+y}{1000+x+y} \cdot 100\% \leq 5\%,$
$80 + y \leq 50 + 0,05x + 0,05y,$
$0,05x \geq 30 + 0,95y,$
$x \geq 600 + 19y.$
This task could not be solved because of the lack of data.

Solution 5

1000 - the number of things produced
$8\% \cdot 1000 = 80$ - the number of defective things produced
x - the number of additional things
5% - the acceptable norm of defective things
$1000 - 80 = 920$ - not defective products
$1000 - 80 + x = 1000 - 5\% \cdot 1000,$
$920 + x = 950,$
$x = 30.$

Differences between the solutions result from different understanding of the task's meaning, objects appearing in it and relations between them. The set of information included in the task was differently interpreted by students. They introduced (consciously or not) additional assumptions, accordingly to their own understanding of the text, and, as a result they changed the contents of the task. In the solution number 1, 3 and 5 students understood, that number of things produced additionally should be big enough to equal the number of defective things to the acceptable norm. Although the solution number 2 was presented as an equation, the student was aware, that the answer should be presented as an inequality. We can see it in his last notation. This method is often used in lower and higher secondary school during working out square inequalities, for example. Without understanding of the essence of this proceeding, the habit of solving an equation in a place of inequality led up to the incorrect solution. It was additionally assumed in solutions number 1 and 2, that extra produced things have to be not defective. In the solution number 5 we can see, that the student imposed the condition, in which the number of produced things cannot be changed. It was analysed how many of defective things should be replaced by not defective ones, to reach the norm of 5%. Solutions number 3 and 4 are correct. We suppose that answers given by students do not indicate the lack of solutions, but the lack of unambiguity of answers.

2.2 The strategy of making the general problem a concrete

The situation presented in the task, especially when characterized by some degree of generality was not understood by some students. In these cases students made a specification of the problem and considered the concrete case by substitution of the number chosen by themselves in a place of a letter. Although initial situation was considerably simplified, this way of dealing with a problem allowed them to understand the situation and sometimes led to a solution. Examples of this behaviour we will show below.

Example 2
Solution of the task number 2

$94\% \cdot (x + y) < x$,
$0,94y < 0,06x$,
for $y = 1$ we have $x > 16$, then $x + 1 \geq 17$.

Solution of the task number 4

x – the step of the first boy
$x - 20\%x = 0,8x$ – the step the second boy
for example: for 10 steps we have $10x$,
$0,8x \cdot 12 = 9,6x$.
The first boy came to school earlier.

2.3 The trials and errors strategy

The general situation shown in the task number 2 was too difficult for students, therefore they worked out the concrete problem for creating intuitions concerning described situation. They used the strategy of trials and errors, which was preceded by a conscious choice of numbers. This choice took into account the real situation - the amount of participants in the mathematical circle is not usually bigger than about 30 people, so the least numerous (accordingly to conditions of the task) it would be in the situation, when the only girl in the circle is Joan (therefore the numerator has to be fewer than the denominator by 1). These trials were made until the correct number of participants was chosen.

Example 3

$\frac{30}{31} \approx 0,97$, $\frac{24}{25} \approx 0,96$, $\frac{15}{16} \approx 0,93$, $\frac{16}{17} \approx 0,94$.

2.4 The strategy of the ready scheme

2.4.1 Mechanical usage of designs

Some students tried to make use of methods of proceedings learnt at school, with omission of the problem's essence. The transfer of learnt on maths lessons method of proceedings to the new situations led to a mistake. In many textbooks we can see the scheme of transformation numbers to percentage transformation formed by the rule "If a number is to be transformed into percentage, this number should be multiplied by 100%" (H. Lewicka, E. Roslon, 2000, p.103). In the solution of the task number 1, solutions that appeared were of such kind:

$$\frac{x}{x + 20\%x} 100\% = 0,83 \approx 83\%.$$

In the task number 5 majority of students used the proportion, as a comfortable means of describing the given shown dependencies. We saw some solutions of that kind:

$100\% - 8\%$ defective,
$x - 5\%$ defective,
$x = \frac{500\%}{8\%}$,
$x = 62,5\%$.

It appears that the way of solving the tasks with percentages with the use of proportions had been learnt by students during chemistry lessons; then it was mechanically transferred to the shown example, and as we can see, was contradictory to the common-sense approach to the problem.

2.4.2 The additive approach to percentages

The additive approach to percentages is usual in the real world, for example we can see inscriptions like "reduction by 70%", "minus 30%" in the shop windows. It creates numerous mistakes in the stage of mathematising of information given in a natural language, as it is shown in the examples of solutions of the task number 4.

Example 4
Solution 1

x – the length of the I boy's step
$120\%x$ – the length of the II boy's step
y – the number of II boy's steps
$120\%y$ – the number of I boy's steps
$120\%yx$ – the way traveled by the I boy
$120\%xy$ – the way traveled by the II boy
The boys will come to school at the same time.

Solution 2

x – the length of Tom's step
n – the number of Tom's steps
$x + 20\%x$ – the length of Luke's step
$n - 20\%n$ – the number of Luke's steps
$t_T = xn, t_L = 1,2x0,8n = 0,96nx$.
At the same time Luke will cover a distance equaled to 0,96 Tom's one. Tom will come to school first.

Similarly, in the solution of the task number 1, the additive approach to the percentage led the solution to the statement: I bought by 20% cheaper than the seller wanted, I sell by 20% more expensive, so I have an initial 100% of profit. The behaviour of students was influenced by a school identification of percentage with a fraction. It results in approval of making operations with percentages, without connecting them with the measurement described by percentages,as we can see in the above solution. In quoted examples it is shown that for students the expression: "a is by b% bigger than c" means the same as "c is by b% smaller than a". They do not notice that the argument of b% function has been changed (see W. Wawrzyniak-Kosz 1997).

2.5 The strategy of „backward” reasoning

Some students at first advanced a hypothesis and than in the course of action they to confirm or refute it. Here preferably used was the reasoning of the reductive type. We can see such an approach in the way of working on the tasks number 2 and 4.

Example 5
The student at first introduced symbols:

x — the step
y — the length of a step
$x, y > 0$

Next she wrote:

T: $80\%x, 100\%y$
L: $100\%x, 120\%y$

And next she advanced the hypothesis that Luke came to school earlier, so in the same time he had to cover longer distance than Tom. The student signaled this fact by the question mark:

$$y{\cdot}0{,}8x \overset{?}{<} x{\cdot}1{,}2y,$$
$$0{,}8xy \overset{?}{<} 1{,}2xy,$$
$$0{,}8 \overset{?}{<} 1{,}2.$$

She received the true inequality which confirmed the correctness of the hypothesis.

2.6 The strategy of avoiding difficulties

Students made effort of solving a problem but as difficulties appeared they did not analyse the situation once again, did not check if every information was used by them, did not wonder at the correctness of text interpretation. For example during working out the task number 2 the student writes: "x — the number of pupils, $0,94x$ — the number of boys, $0,06x$ – the number of girls including Joan. It cannot be solved". This student left out the information, that the least possible number of mathematical circle's participants should be appointed. In the solutions of the task number 4 there are only answer: "It is a task with a lack of data – the answer depends on the distance between the house and the school", "It is not known, how long the distance to cover is and how much time boys need to do it".

2.7 The strategy of acting only within a mathematical model

The mathematical model had been correctly constructed by the great majority of students in the solution of the task number 3. Moreover, the students made correctly the admissible equivalent transformations of an inequality. But when they got the solution - the range, they ceased here, for example:

Solution

$x + 8\%x \leq 70,$
$\frac{108}{100}x \leq 70,$
$x \leq \frac{7000}{108}$ – it was the largest amount of passengers, who could get on the tram.

Those sampled, who sought to interpret the obtained mathematical results in situations described by the text of the task, referred only to the part of information the fact that the number of passengers must be a natural number so they gave the answers such as: "64 people or less could get on at the initial tram stop". Only 2 people took into account the assumption that 8% of the searched number must be a natural number as well.

3. Conclusion

In school education percentages appear in two senses - as a notation number (in this sense 300% is the number 3) and as the operator (which is a function of the form) (see W. Wawrzyniak-Kosz 1997, H. Siwek, E. Wachnicki 1998). Understanding the situation in which the percentages are in one of these two meaning is a necessary (but not sufficient) condition for an effective and proper tasks solving. Analysis of the results of research shows that students often did not have the awareness of the limited application of each of these aspects in specific task situations. Those students who treat percentage as a specific fraction, do equations typical for fractions in the situation when the percentage is connected with the variable volume so it days the role of an operator. Moreover, the research show that the students who use the additive approach to percentages (the school approach) make errors as naive as a primary school students, such as: "Tom's step is by 20% shorter than Luke's one, so Luke's step is by 20% longer than Tom's step". It should be noticed that the process of systematic forming of the percentage concept understanding finishes at the secondary school stage. Solving real problems concerning the percentage after finishing the school education can change this understanding; it may enrich it, or it may do just the opposite. Nevertheless, this image of the percentage concept is then not influenced by any didactic activities of mathematics teachers. It is therefore worth taking care of the prior preparation of some solid foundations for the deeper understanding of the percentage idea. In the light of these potentially forming errors behaviours of the mathematics students, teachers-to-be, it should be feared that they are not properly prepared for the realization of such an aim. Shown in the research strategies of behaviour indicate that it is very likely that certain routine actions will be passed to students from their teachers. Consequently, the students will probably share the errors committed by the teachers.

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DIAGNOSTIC – METHODOLOGICAL ANALYSES OF ITEMS CONTAINING THE OFFER OF ANSWERS INCLUDING A BRIEF OPEN-ENDED ANSWER

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Abstract. This paper presents a methodology of diagnostic and methodological procedures of (secondary) analyses of test items for a large number of tested pupils containing the models of usage in mathematics at primary schools. This is a model of analyses of tests including items containing the offers of answers including a brief open-ended question, where we merely know the answers given by the pupils, but not the processes of thoughts, by which the pupils tested arrived at their answers.

ZDM Subject Classification: D30.

1. Introduction

One of the points of departure foe achieving change of content in pedagogical documentation should be represented by empirical pieces of knowledge acquired by regular monitoring or any other type of testing of educational outcomes achieved. The diagnostic-methodological (or secondary) analysis of items should be the final stage of the process of testing. Even though this is a highly significant stage in the process of testing, which provides for a feedback for teachers, methodologists, and curricular specialists, the State Institute of Education has not until now made or published any diagnostic-methodological analysis of ro items on a nation-wide scale fo testing (Monitor 9, external school-leaving tests, the same Institute even having failed to create favourable conditions for doing this. This appears one of the reasons why we are unable to lean against any empirical results (apart from international measurements) in exerting our topical endeavours as changes in content, even though there does indeed exist a sufficient number of national measurements.

Diagnostic-methodological analyses of items of school tests are as a matter of rule performed by teachers, who personally present such an analysis to pupils. The teacher in this instance usually administers the test, assesses it, and is frequently an author of the same. The teacher analyses the success rate of individual test questions and the mistakes made by the pupils, draws diagnostic-methodological conclusions for further teaching. The teacher is an analyst here, the pupils are the target group.

A different situation arises in the case of nation-wide testing. The results of the diagnostic-methodological analysis of testing are intended for teachers, methodologists, and curricular specialists, who may employ the results of the latter in their respective methodological procedures or in the changes made in the pedagogical documentation. The primary analysis of items will show us suitability of values of their properties – successfulness, sensitivity, discrimination index, insolvability of items, incomplete structure of the test (unsolved final test questions), frequency of distractors, etc. and should lead to modification or exclusion of items containing unsuitable value properties of the test. The chief criterion is mainly increasing the reliability of the test prior to main testing, which may be performed by a statistician, not necessarily an expert in the given subject. If such an analysis is performed after the main testing, it should contain consequences for excluding the items containing unsuitable value properties in assessing the main testing. The primary analysis of items merely offers an external view on the item, it does not penetrate within the items (tasks) themselves.

2. Diagnostic – methodological Analysis of Items (Secondary Analysis)

<u>The diagnostic-methodological analysis of items may be understood as a process of:</u>

- analysis of mistakes made by the pupils in individual items and looking for the reasons for these;
- proposing possible modifications in methodological procedures in teaching, in accumulating the relevant pieces of knowledge in several items, up to the proposals for modifications in pedagogical documentation.

<u>In order to be able to draw as many conclusions as possible from the diagnostic-methodological analyses, it is specifically important that:</u>

- dispose with a suitable methodology of he diagnostic-methodological analysis, or, eventually, that we have at our disposal a suitable software product simplifying the analysis in question;

- the process of testing is properly documented and the background documentation is given to the analyst for their disposal.

In case of nation-wide testing (Monitor 9, external school-leaving tests), what usually prevails or is employed on an exclusive basis, are the items containing a brief open-ended question and the items containing an enclosed offer of answers. The tests are assessed in such a way that the answer sheets of the pupils are scanned and processed. This means that the following are at the disposal:

- numerical or verbal answers given by individual pupils in items containing a brief open-ended question;
- answers by individual pupils containing multiple-choice offers A, B, C, or, eventually, E in items containing an offer of answers....

No procedures or calculations employed by the pupils illustrating their way of having arrived at a particular answer are at the disposal. This is one of the reasons why we have to lean against statistical processing of the above, emphasising the significance of documentation of the item creator, including the *target-oriented description of the choice of distractors.* The most convenient way of determining the distractors is in our giving the items as tasks during the pre-piloting stage, the tasks containing the open-ended answers to questions. We transform the most frequent mistakes made by the pupils into a suitable offer of distractors in such a way that the choice of any of these was a reflection of the pupil's mistake in solving the task. The items containing the distractors are then tested in the pilot order. If time or tools are not available for doing this, we may also rely on experienced teachers (Butaš 2005).

3. Methodology of Diagnostic-Methodological Analysis of Items

In the items containing an offer of answers, we analyse probable mistakes made by the pupils in individual items and look for probable reasons for making the mistakes. We probably employ words due to the fact that this type of items is most heavily burdened by guesses. The frequency of any given answer is in all instances given by the sum total of answers chosen by the pupils by guessing and which they arrived at by a correct or incorrect consideration or calculation. However, the proportion in the above sum total remains unknown. Guessing might be limited by awarding negative points for incorrect answers, a suitable number could be 1 (n-1), where n is a number of answers offered in each of the questions (Burjan 1999,

p. 37), however, this negative allotment of points has not yet been carried out on a large-scale testing in the Slovak Republic. Despite that, we hereby recommend to employ the following procedure, within which we create two tables, from which we later depart:

[1] The table of measured percentage frequencies of answers by pupils to items by offered choices (A, B, C, D or, eventually E ...) uniformly distributed into five performance groups according to scores achieved in the tests. The equal number of performance groups with 5 assessment degrees is not inevitable, but still preserves specific advantages. *We divide the pupils into performance groups, the main reason for this being to enable us to see, whether a given mistake is increasing proportionately to the decreasing performance of pupils, or whether one and the same mistake is made almost equally by both pupils of higher and those of lower performance, or, eventually, whether any other observable functional dependence exists here.*

[2] The table of interpretations of value of offers of answers distributed in advance into items (distractors). *The authors of the tests should have a very clear idea about the reason of choice of each individual distractor.*

One should say that this diagnostic interpretation of the quantitative data measured is only of approximate probability, although we do believe that this is one of the highest achievable in the items containing the offer of answers.

One proceeds in a similar way in the diagnostic-methodological analysis of items containing brief open-ended answers; the frequency of distractors is replaced by frequencies of the most frequent mistakes made by the pupils, while simultaneously striving at explaining the reasons, since the procedures employed by the pupils are not at our disposal (we recommend to consider all of those having ca at least 2 per cent frequency).

4. Examples of Diagnostic-Methodological Analysis of Items

By way of example, we present herein the diagnostic-methodological analyses of results of 3 tasks of specific tests in mathematics for the 9th grade of primary schools.

1st task: The original price of petrol was SKK 32.00 per litre. Due to the change of price of crude oil, the price of petrol went up to SKK 40.00 per litre. What is the percentage of the increased price of petrol?

a) 25% b) 20% c) 8% d) $\frac{40}{32}\%$

Author's solution: $x = 40/32.100\% - 100\% = 125\% - 100\% = 25\%$, thus, 100% is deducted from the percentage part calculated by the rule of three sum.

The above task is tp the item in the standard Arithmetics – 5.5. To be able to solve adequate verbal tasks as percentages.

1	Total	One fifth of pupils according to success rate				
		1.	2.	3.	4.	5.
A	45,0%	85,2%	60,6%	44,4%	26,5%	8,5%
B	15,1%	9,6%	18,1%	16,7%	18,8%	12,3%
C	27,3%	2,7%	13,5%	24,4%	35,6%	60,4%
D	10,1%	2,5%	6,7%	11,9%	15,2%	14,4%

Table 1.1
Choice of answers to the 1st task

Choice of distractors
Brief description of the solution mistake
Correct answer
Replacement of basis for percentage part
The difference in koruna expressed, instead of percentage
Multiple of increase expressed (proportion)

Table 1.2
Interpretation of answers offered

Analytical expressions:

- The tasks contains a specific transfer, the average success rate achieved by the pupils žiaci amounts to 45%, which is even lower than the minimum standard (50%).
- The difference among the respective performance groups is high, even equal and amounts to almost 20 per cent.
- The performance of the 4th group is on the level of probability of accidental ticking, the 5th group even consider the correct solution the least probable one.
- The mistake contained in mistaking the percentage basis for percentage part is not dominant here, it is made as non-growing and relatively comparably by the pupils in all the performance groups.
- The dominant mistake is represented by the replacement of the difference of percentage to the difference in price in koruna, which is increasingly growing in performance groups. The weaker the pupils, the more prone they are to adopt simple solutions represented by the difference in numbers emerging in the text. This significant growth could have been increased by less attentive reading by the text of the task with understanding.
- The least frequent mistake (10.1%) was represented by the proportion between the increase of the price, which has mildly gone up proportionately to the decreased performance of the group.

Methodological recommendations for teachers:

- Do not underestimate the percentage tasks, the pupils are still unable to solve them on an appropriate level, include mainly the tasks from real life.
- Devote chief attention to less advancing pupils (4th and 5th performance groups) to that they understand especially the difference between the increase (decrease) of the value of the price of petrol) to the number of per cent and the increase (decrease) of percentage.

2nd task: Solve the equation $\frac{8x+2}{2} - \frac{5x-4}{3} = 0$

a) $x = 1$ b) $x = -\frac{2}{7}$ c) $x = \frac{1}{7}$ **d)** $x = -1$

Author's solution 1:

$$\frac{8x+2}{2} - \frac{5x-4}{3} = 0 \quad /.6$$

$$3.(8x+2) - 2(5x-4) = 0$$

$$24x + 6 - 10x + 8 = 0$$

$$14x = -14$$

$$x = -1$$

Author's solution 2:

$$\frac{8x+2}{2} - \frac{5x-4}{3} = 0$$

$$\frac{8x+2}{2} = \frac{5x-4}{3} \quad /.6$$

$$24x + 6 = 10x - 8$$

$$14x = -14$$

$$x = -1$$

The tasks is to the item in standard Algebra – 2.3. To be able to make equivalent modifications of the equation and by means of this solve linear equations containing one unknown.

E10	Total	One fifth of pupils according to success rate				
		1.	2.	3.	4.	5.
A	11,8%	1,5%	6,9%	10,6%	16,0%	23,8%
B	14,6%	2,1%	12,3%	15,6%	19,4%	23,5%
C	15,9%	4,2%	11,5%	16,9%	21,5%	25,4%
D	55,4%	92,1%	68,5%	56,3%	39,6%	20,6%

Table 2.1
Choice of answers to the 2nd task

Choice of distractors
Brief description of the mistake in solution
Error in sign during „transfer" onto the second part of the equation
0.a=a, error in multiplication of the right-hand side
Error in sign in removing the second fraction
Correct answer

Table 2.2
Interpretation of answers offered

Analytical expressions:

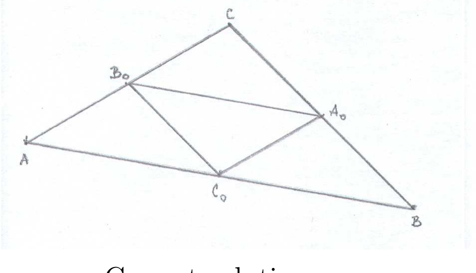
- The pupils acquired the overall minimal standard (55.4%), optimal standard was only achieved by the 1st group pupils (92.1%).
- Only the 1st group solved the task without errors/mistakes, the correct answer was least probable for the 5th group.
- All of the errors/mistakes – those pertaining to sign and in multiplication by zero – were made by the pupils in almost equal frequency (11.8%, 14.6%, 15.9%) and the error/mistake rate increases in almost a linear way to the decreasing performance of pupils (in groups 2 – 5).
- We consider the error/mistake rate in solving simple linear equations high; this is the task for the lowest cognitive level applied – understanding, where we would justifiably anticipate at least the optimal standard (75%).

Methodological recommendation for the teachers:

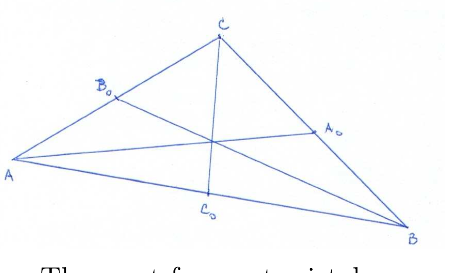
In solving the equations of this type (the difference in fractions equals zero), prefer as the first modification addition of the second fraction (author's solution 2), which in fact eliminates the possibility of errors/mistakes presupposed by the choice of distractors B and C.

3rd task: Into how many isosceles triangles will be divided each of the triangles by its median traverses?

Author's solution: It is enough to draw a general triangle with its 3 median traverses and we can see 4 triangles identical by twos according to the sss proposition.



Correct solution



The most frequent mistake

Task to the item in the standard Geometry – 2.4. Get to know the properties of the median traverse of the triangle, 6.7 Be able to promptly use the propositions on the concordance of 2 triangles.

Solution	Total	One fifth of the pupils according to success rate					Possible interpretation of the answer
		1.	2.	3.	4.	5.	
4	43.6%	84.0%	56.1%	38.7%	25.2%	14.0%	Correct answer
6	35.7%	12.4%	32.1%	44.0%	47.4%	42.6%	Mistake for the median
3	5.9%	0.8%	3.6%	4.5%	8.5%	12.2%	Apart from the central triangle
2	2.5%	0.6%	1.5%	2.6%	3.1%	4.8%	1 median to 2 formations?

Analytical expressions:

- The pupils failed to achieve even the minimal standard (43.6%).
- Optimal standard was only achieved by the pupils of the 1st performance group ad the minimal standard only by the pupils of the 1st and 2nd performance groups.
- Frequency of correct solutions decreases in a convex manner with the decreasing performance of pupils.
- The most frequent solution was the 6th one, which is represented by the mistake of median traverses for medians; this mistake was made equally by almost all the pupils of the 3rd, 4th, and 5th groups, who even preferred this mistake to correct solution. This solution was even not blocked by the fact that our question pertained to isosceles triangles.
- The remaining two mistakes were episodic and increased on general in a linear way proportionately to the decreasing performance of pupils.
- The notion of median traverse of the triangle has probably not been appropriated permanently.

Methodological recommendation for the teachers:

- Devote more time to explaining the notion median traverse and its difference from the triangle median,
- devote specific attention to the less prospering part of the class,
- observe reading the tasks with comprehension by the pupils, concentrate on forming the awareness of all the conditions which are to be contained within the formation considered.

4. Conclusion

Methodology has been proposed for the diagnostic-methodological analysis of tests containing the offer of answers including a brief open-ended answers.

The examples of the item analysis have been presented for the assessment of tests scored by 1 point for each correct answer. We anticipate that in awarding negative points for incorrect answers (1/(n-1), where n is a number of answers offered) we would contribute to a more precise analysis.

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COGNITIVE CONFLICT AS A TOOL OF OVERCOMING OBSTACLES IN UNDERSTANDING INFINITY

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Abstract. The article is devoted to obstacles in the understanding of the concept of infinity. Both the attributes of infinity and fundamental types of epistemology obstacles are formulated. The cognitive conflict and its role in the overcoming of obstacles are analyzed. The illustration of the problems by means of real experimental interviews is given.

ZDM Subject Classification: D70.

1. Introduction

The concept of **infinity** is a key concept in mathematics and its teaching. In the history of mathematics the development of knowledge about infinity creates crucial milestones important for its further development. If we accept the hypothesis of the so called genetic parallel, which means that the ontogenetic development is not independent of the phylogenetic development, it is possible to assume that the obstacles we find in the phylogeny of the concept of infinity, can be identified also in the ontogeny and that the overcoming of these obstacles is a necessary component of the cognitive process of an individual. Understanding of many concepts which pupils meet during their learning process is dependent on the gradual concept of the building of infinity (point, line, line segment, decimal notation of numbers etc.).

At present, within the framework of the three-year GAČR project, our attention is drawn to the study of both the phylogenetic concept development of infinity and also to the study of ontogenetic development of the understanding of infinity in our contemporary population. The research should be crowned with a formulation of suggestions of how to overcome the identified obstacles.

In the paper Cihlář & Eisenmann & Krátká & Vopěnka 2007, the researchers specified basic attributes of infinity and the current summary of obstacles in the understanding of infinity, both epistemological as well as didactical, based on the experience from the previous research (Eisenmann, 2002, Krátká, 2005). We understand an obstacle as a set of knowledge grounded in the knowledge structure of an individual which can be successfully used in one context or situation but in another, they fail or give wrong results (Brousseau, 1997).

In this paper, we elaborate above all the theory of the so called cognitive conflict as a tool for overcoming obstacles in understanding infinity.

2. Cognitive conflict

The pieces of knowledge which are obstacles that fail in a certain context produce wrong answers. This failure cannot be considered by an individual. It means that he/she does not know that the obtained result or answer is wrong. There must be someone who knows that the answer is wrong. We will talk about an objective observer. It can be an experimenter, teacher or mathematician.

At first the experimenter identifies the **contradiction** between the answer of an individual and some objective truth (valid in a given mathematical theory), in our case it is the truth valid in school mathematics. It is important to remark that different individuals can face the same contradiction caused by various obstacles.

If the individual is aware of the contradiction, we talk about the inducing of the **cognitive conflict**[4] (CC). The CC is not a disagreement between a pupil's wrong answer and an objective truth (it is a contradiction), but a conflict between the knowledge producing this wrong answer and the pupil's different knowledge. Therefore the CC is a disagreement in the knowledge structure of an individual. The inducing of the CC means that an individual is aware of the fact that his/her utterances (based on the inconsistence of

[4]The conflict is according to Psychological Dictionary "contradiction, disagreement, simultaneous clash of the opposite tendencies, an offer of two or more attractive aims". Cognitive conflict related to infinity is first mentioned by Tall (see Tall, 1976, 1977). So called cognitive conflict teaching approach in connection with the comparison of infinite sets is discussed by Swan (see Swan, 1983).

his/her knowledge structure) are not in accordance. We should note that not every wrong answer can be brought to the CC. For example, a four to six-year old child thinks that there is a decrease in the amount of the liquid poured from a narrow glass to a wider one. The child evaluates only the final state when the column of the liquid is lower. We cannot talk about the CC, as a child of this age has not created such basic knowledge yet. The argument that it is the same water, or that nothing has been added or taken (identity), or that it is possible to pour the liquid back to see that nothing has changed (inversion), or that the column is higher but the glass is narrower (reciprocity) can be used successful when talking with children at the age of seven or more, when these preservation operations start to develop gradually (Piaget & Inhelderová, 1997).

Tall and Schwarzenberger speak about the so called conscious and subconscious conflicts induced by two "close" (mathematical) mutually irreconcilable concepts. They say that conflicts are induced by the transfer of mathematical ideas into the teaching and learning process when these are inevitably deformed. In a greater detail: in some cases "... the cause of the conflict can be seen to arise from purely linguistic infelicity ... ", in another case "... the cause of the conflict arises from a genuine mathematical distinction.", and finally "... the conflict arises from particular events in the past experience of an individual pupil." (Tall & Schwarzenberger, 1978, p. 49). If we compare this with the theory of obstacles, these obstacles are of didactical origin. We assume that the cause of these conflicts can also lie in other types of obstacles, more particularly in epistemological ones. We take into account both source types of the CC. The authors say that if the conflict is conscious we can expect the following reaction: "... the existence of two ‚nearby' concepts can cause mental stress arising from the emergence of unstable thoughts" (Tall & Schwarzenberger, 1978, p. 44). An individual makes an effort to remove this tension, which can be effectively used in the process of overcoming obstacles.

If an individual faces a cognitive conflict, we distinguish the following possibilities[5]:

- **A vain attempt** at removing the conflict: the individual finds out after several trials that he/she has not got sufficient means. The obstacle is so internalised that the individual does not admit that the given piece of knowledge could be the cause of the CC. He/she refuses to remove the CC and the disagreement in his/her knowledge

[5]We should remark that if an individual manages an induced CC is influenced also by the instant psychosomatic state and by other external conditions, especially by motivation or tiredness.

structure prevails. However, this possibility of facing the CC is not valueless. It has a propedeutic meaning for other conflicts induced with the aim of overcoming the given obstacle.

- **A wrong attempt** at removing the conflict: the individual tries to change his/her cognitive structure in a way that he/she adjusts it to the wrong thesis of the conflict. The individual corrects in a wrong way his/her right knowledge. The obstacle does not change or it changes in an undesirable way, and prevails further on. It confirms the presence of the obstacle, for it shows a resistance to conflicts and it changes always as little as possible.

- **A successful attempt** at removing the conflict: the individual ruins his/her idea about the rightness of the obstacle and changes it in the way that the obstacle produces no more wrong answers in the given context.

By removing the **cognitive conflict**, the individual gradually overcomes the given **obstacle** that causes the noticed contradiction and the inducing of the CC. A single obstacle can induce different CC in various contexts. By removing one CC, the given obstacle is not overcome totally, it is, on the contrary, usually necessary to induce more CC (in different contexts) and to remove them in order to overcome the obstacle.

The cognitive conflict has two functions:

- with the help of the conflict we can identify a particular obstacle (the **diagnostic function**),

- after the conflict is induced and removed, we can overcome the obstacle and thus we can improve pupils' knowledge structure (the **educational function**).

A teacher or an experimenter can induce such situations when he/she assumes that an individual gets into the CC. His/her aim is to prove the existence of the assumed obstacles (the diagnostic function) or overcome the assumed obstacle (the educational function).

A teacher or an experimenter sometimes gets into a situation when a contradiction or the CC appears unexpectedly, or unintentionally. Then the teacher or experimenter should identify if it is caused by an obstacle. If so, the obstacle should be well diagnosed and he should induce other cognitive conflicts in suitable contexts, make pupils remove them and in this way reach the gradual overcoming of the given obstacle.

Our aim is to find and identify obstacles so that teachers noticing a contradiction have a tool for inducing the CC and in this way they create a situation suitable for the overcoming of this obstacle, as required by Tall a Schwarzenberger: "...the role of the teacher in finding a suitable resolution will be crucial, and more decisive than such factors as choice of syllabus, textbook...". Unlike them we do not think that the teaching process should be conducted in a way that enables to avoid cognitive conflicts. This is connected with the fact that the authors focus only on the conflicts of didactic obstacles. Like Tsamir or Swan we appreciate their function when creating knowledge (Tsamir, 2001, Swan, 1983) and above all when overcoming obstacles.

The attributes of infinity are mentioned by Tall and Schwarzenberger when they talk about the following (conscious and subconscious) conflict concepts: between decimals and limits, between decimals and fractions, between numbers and limits, and between progressions and series (Tall & Schwarzenberger, 1978). As we understand the concept of the CC in a wider sense, we do not restrict ourselves only to the conflict between two concepts related to the attributes of infinity, but we also observe conflicts between any concept or knowledge, where at least one of them is connected with a determined attribute or observed object.

3. Research methodology

The research is conducted at two levels. Firstly, **we probe, using a questionnaire, the initial reactions of respondents** to identify the key phenomena connected with the creation of ideas of objects introduced in school mathematics related to the concept of infinity (line, point, line segment, plane and its parts, numbers sets). Thus, also with the attributes of infinity. This will lead to more exact specification of the formulation and completion of possible obstacles when building the concept of infinity. The mentioned questionnaire is designed for basic school children, years 3, 5, 7, and 9, for secondary school students of levels 2 and 4, and for university students. For that reason we have five different versions (the questionnaire for both secondary school and for university students is the same). The respondents answer in writing and anonymously. The formulation of particular items has been proved on a sample of 30 respondents in each of the above mentioned categories. Besides, we applied the guided interview recorded on a camcorder with one representative of each age category after filling in the questionnaire. In the course of the year 2008 the researchers plan to carry out a wider questionnaire investigation of about 150 respondents in each age category. This sample will enable the researches to accept or refuse the stated hypotheses with adequate infallibility.

Secondly, we conduct, using the above mentioned questionnaires, **guided experimental interviews** with respondents of each age category, also recorded on a camcorder. These recordings are transcribed into protocols after watching. Using the protocols, we set analyses which serve us to elaborate more precise work terminology and to correct the developing theory, which means mainly the formulation of obstacles in understanding infinity and the way of overcoming them. Besides that, we look here for suitable examples of particular illustrations of how to face the cognitive conflict. When setting approximate scenarios of experimental interviews, we always used the method of the so called constructed reactions of pupils (Hejný & Stehlíková, 1999), when we tried in our research team to prepare for all possible versions of the dialogue with pupils and students concerning the given topic. The experience acquired in the process of the creation and evaluation of the questionnaires helped us significantly to improve the assumptions about the reactions of pupils and students.

4. Illustration examples

In the following three examples from real experimental interviews conducted in the year 2007, we illustrate inducing of the CC and make comments on the way pupils faced them. The number in brackets indicates a pause in seconds.

Example 1 Jakub, school year 5

Attribute of boundedness (distinction of bounded, very large and unbounded sets)

In the picture, there is line p and points A and B lying in opposite half-planes given by the straight line p.

E42: *So, if you stood here* (pointing to point A) *at this point and you would like to get there,* (2) *would it be possible to get to point B anywhere on this paper?*

J43: *Yes, it would be possible.*

E44: *Show me that.* (Jakub gestures around the line.)

E46: *Hm. You can draw it, if you want.* (Jakub draws an arch around the picture of the straight line.)

E48: *Hm.* (2) *So that the straight line p does not intersect the line AB?*

J49: (2) *Hm, well, it does not.*

The experimenter notices a contradiction that it is possible to link two points in the opposite half-planes given by the straight line p with a line so that the line does not intersect the straight line. He thinks that the wrong answer is caused by the fact that Jakub understands a straight line to be its image on the paper, which correlates with the obstacle of the exchange of a geometric object and its model (in this case it is the picture).

E52: *Hm. And so... So I will ask you once again* (drawing two intersecting straight lines without their point of intersection). *Here is one straight line and here is another one* (drawing), *I will name them, for example,* (2) *m and n. OK.* (4) *Will they intersect?* (3) *Would you find their intersection?*

The experimenter tries to induce a cognitive conflict. To do that, he uses an easier task for Jakub without leaving the context. He assumes that Jakub is able to solve the problem correctly.

J53: (2) *I would find it, if I made it longer.*

J56: *Can I draw it?*

E58: *Of course, you can.*

J59: (drawing) *So and so.*

E60: *And have you found their intersection?*

J61: *Here.* (pointing at the paper with a pencil)

Jakub succeeded in solving the problem in the given context. However, he is not able to confront his answer with the solution of the previous problem. For this reason, the experimenter makes an explicit reference to the original answer.

E62: *OK. Hm. And would it be possible to find an intersection for your line* (pointing to the paper) *AB and for the straight line as well?*

J65: *I would also make this line longer.* (He considers the straight line *p*.)

E67: *And can you make it longer?*

J68: (4) *Well, it is a straight line, so yes, I can.*

E69: *OK.*

After the repetition of the question (E62), Jakub is aware of the fact that the same algorithm can also be used here and so he changes his original answer. In a short time period, there is an inducing of the CC as well as its removal.

J72: *Can I, this way.* (drawing)

E73: *Of course, you can.*

J74: *And here the intersection has been created.*

E75: *Hm, it is the point. So now it intersects line AB. Would it be possible to draw another line, which would not intersect the straight line?*

J76: Not intersect (3). Uf.

E77: *You know how it was before.* (pointing to the paper) *Before, you drew line AB so that there was no intersection. Now, you have made the straight line longer and so the intersection is again there. Would it be possible to draw a different one, not this one* (pointing to the paper)?

Even though in the previous part of the interview Jakub corrected his wrong answer in the right way, the experimenter wants to be sure that the conflict in this context has been really removed and so he asks in a similar way again (E77 and further on E81).

J78: *No.*

E79: *Wouldn't it be possible?*

J80: *It wouldn't.*

E81: *Couldn't it be for example this one?* (drawing an arch on the opposite side)

J82: *Well, not this one, as the straight line could again be made longer, up to this place* (showing on the paper).

Jakub confirms that he can use the procedure of making the straight line longer securely (J82). The cognitive conflict was induced and then successfully removed. However, we cannot think that the obstacle itself has been overcome. It is possible (and also probable) that it will appear again, this time in a more complex context.

Example 2 Michal, school year 9

Attribute of cardinality (distinction of finite, very numerous and infinite sets)

E21: And what about people, all people in the whole world, is their number infinite?

M22: *No, it is not. There was a census of people, flats and households. And these are finite.*

E23: *And what about fish, all fish worldwide?*

M24: *Well, this is infinite, it cannot be counted.*

E25: *And what about (3) ants, ants on the Earth?*

M26: *The ants on the Earth are infinite.*

At this moment, the experimenter noticed a contradiction. Michal uses for the infiniteness of a set the criterion *They cannot be counted* (understand the elements of the given set), *therefore they are infinite.* The right or the wrong use of this criterion is lying in the meaning of the statement *They can (cannot) be counted.* For Michal it means the real realization of the process of counting. This can be proved by statements in positions M22 and M24. Even though the set of all people in the Earth is relatively numerous and he is not able to count them himself, he has the experience that the number of people has already been determined. For "less numerous" sets, the principal and real possibility of counting overlap. Even more, they overlap also for infinite sets (by applying both real and principal counting of the elements of the set the pupil gets the same, i.e. the negative answer). In these contexts, Michal solidifies himself that his knowledge is right. And this knowledge - the understanding the meaning of "it is possible to count it" as a real possibility of counting - is an obstacle for Michal. The experimenter tries to induce a cognitive conflict.

E27: *And ants in the Czech Republic?*

M28: *It's also infinity.*

E29: *And in one forest?*

M30: *It's also infinity, still infinity.*

E31: *And in one ant-hill?*

M32: *It's also infinity.*

Michal perceives the idea of counting the ants in an ant-hill as unreal.

E33: *And if I took some ants into a glass, how many of them would be in the glass?*

M34: *Well,* (4) *I do not know.*

It is possible that the CC has been induced. Michal was aware of the inconsistency of his answers, but it is possible that it has not happened yet.

E35: *Is the number of the ants finite or infinite?*

M36: *Well, finite.*

E37: *And in one ant-hill?*

M38: *Well, perhaps also finite.*

Michal starts to reconsider his answers.

E39: *And how is it with the ants in the entire world?*

M40: *No they must be also finite.*

Michal has changed his previous answer, which leads him to the CC and has corrected it in the right way. It means that in this context he understands a finite set as follows: "Its elements can be principally counted". Michal understands that it is not necessary to know the number of set elements but he can imagine the way to determine it. The obstacle has not been totally overcome yet, as it can be seen in the following utterance.

E51: *And how many atoms are on the whole Earth?*

M52: *There is an infinite number of atoms there.* (contradiction)

E53: *And how many atoms are for example in this nail?*

The experimenter tries to induce again the CC. He assumes that Michal would use the analogy from the previous situation.

M54: *There is an infinite number as we cannot count them because we cannot even see them.*

Michal cannot imagine how to count the atoms, they are unimaginably small, impalpable that they cannot be counted; they are beyond the horizon of discriminability (Vopěnka, 2000 a, 2004). The idea of counting still prevails in his mind, moved further thanks to the previous experience with the removal of the CC but not the wholly overcoming of it. The ability of children to assess the cardinality of real sets does not relate, in our opinion, directly to age. For example, two years younger Marek has a clear opinion about the question of the cardinality of real sets – about the set of all ants in the world, grains of sand in a desert, or molecules in a room. He says:

It will be a big number, a limited one, but it won't be infinity.

The counting must end somewhere.

The grains of sand in a desert, they are not infinite, it would be possible to

count them, it would be really hard, the numbers would be really high, but it would end somewhere
... Only the number of numbers is infinite.

Marek is sure about this question, he talks about a limited number. He holds out and thinks that the only thing that could be infinite is the number of stars. But he adds *Well, maybe.*

Marek says: *The counting would always end somewhere.* In these considerations, he is at such a level that he abstracts away from unreality the process of real counting of set element to the advantage of the principal possibility of counting.

We dare to raise the hypothesis that the ability of thinking in this way is far less dependent on the age of children than the ability of the right consideration of other attributes of infinity, for example the order density. Here, the creation of right ideas is more connected with school education than with the consideration of set cardinality. During our research, we met A-level grammar school students who unrelentingly defended their opinions that the number of grains in a sand-pit or bugs in the garden is infinite while nine year old children had a clear idea about set cardinality. It can be illustrated by the following sample of an interview between two pupils of school year 3, Jeník and Mařenka:

E1: *Let's go to another page. What is infinitely many?*

M2: *Numbers.*

J3: *Well, yes. There are infinitely many numbers.*

E4: *Well.* (5) *And do you think that you would find anything else that is infinite in number?* (3) *Apart from numbers?*

J5: *Twaddle.* (laugh)

E6: *Well, let's try something that we can imagine better. OK?* (4) *What about* (2) *people?*

J7: *No.*

E8: *All people in the whole world.*

M9: *No.*

J10: *There are about five billion of them, so their number cannot be infinite.*

E11: *Clear.* (3) *And what about people's hair.*

M12: *It is not infinite.*

E13: *No, why?*

J14: *It must be finite, the number of hairs must end, as the number of people is finite.*

E16: *Well.* (3) *And what about something tiny, little, minute. Grains of sand in all deserts and all sandpits.*

M17: *No, they are not infinite.*

J18: *No. They must end.*

E19: *How does it come?*

J20: *Well, as the desert ends, the grains must end too.*

M21: *No, if you counted them, you would not count infinity. It would have to end one day, the counting.*

Example 3 Klára, grammar school year 3

Attributes of order (order density), infinite process, convergence

The question "Is it true that $0.\overline{9} < 1$ or $0.\overline{9} = 1$? Explain why.", Klára answers:

K172: *When rounded, the number would be equal one, but without rounding it is not the same.*

There is a contradiction. We will try to diagnose the obstacle, which is the cause of this contradiction, and we will try to induce a cognitive conflict.

E173: *Hm. And why is it not the same?*

K174: *Zero point nine periodic is not the same as one as it is a bit smaller than one, it is a smaller number.*

E175: *Hm.*

K176: *And it is rounded, we must round it up, we round it to one, but there will be a bit missing.*

E177: *Well. OK. I will write zero point nine and one.* (The experimenter writes on the sheet of paper 0.9 and next to it 1). *When you see these next to each other, which of the numbers is smaller?*

K178: *The* 0.9.

E179: *Well. And what is the difference between* 1 *and* 0.9*?*

K180: *A tenth.*

E181: *A tenth. It is a positive number bigger than zero. We can say that one is bigger than* 0.9 *as the difference is the positive number, it that so?*

K182: *Hm, yes.*

E183: *Yes, and if you say that* $0.\overline{9}$ *is smaller than one, tell me what the difference is between the one and the* $0.\overline{9}$.

K184: (2) *So there is zero point something, I do not know how much, a millionth, I do not know.*

E185: *A millionth?*

K186: *Well, it is not the one tenth; it is smaller, much smaller.*

E187: *And what exactly?*

K188: (7) *Well, there are more* 9*'s, it is* 0.99*, I do not know how far it can go.*

The CC has not been induced yet. We expected that it could be induced on position 183, but we can assume that Klára struggled not only with the usual pupils' and students' obstacle *If the number starts with* 0*, it cannot be* 1, but also with her idea of what the bar above 9, indicating the period, means.

E189: *Well. How many 9's are in the number expansion of* $0.\overline{9}$*?*

K190: *Infinitely many.* (2) *No, I do not know, there are many of them, plenty of them.*

E191: *And many, or infinitely many?*

K192: (3) *I do not know if we can determine their number exactly, or we cannot.*

E193: *Look, Klára, the period means that there is an infinite number of* 9*'s and the decimal expansion never ends, you know, the digit* 9 *recurs for ever.*

The experimenter in the following part of the interview, which is not presented here, reminded Klára what the period means and its use in the example with the division of 2 by number 9. The obstacle of the contradiction presented at the beginning of the interview is also the fact that Klára understands the infinite amount of 9's in the decimal expansion of $0.\overline{9}$ as naturally infinite (see Vopěnka, 1996, 2000 b), thus it is "extremely big, enormous" (see position K 190).

E302: *Well, let's go back to the question whether* $0.\overline{9}$ *is less than* 1 *or whether it is equal.*

K303: *Less, because* $0.\overline{9}$ *will never be 1, as there is still a bit missing.*

E304: *What bit? The bit between* $0.\overline{9}$ *and* 1*?*

K305: *Yes, clear, and we cannot say exactly how big it is.*

The presented words prove that Klára does not understand, even after the above mentioned explanation, the decimal extension of $0.\overline{9}$ as infinite. The cognitive conflict was not induced in the intended way (using the difference between 1 and $0.\overline{9}$). The experimenter tried to induce the CC in another way.

E306: *Write down, Klára, one divided by nine, next to each other and divide these two numbers. How much is it?*

K307: (Dividing on the paper.) *Zero point* (3), *one* (4), *zero point one periodic.*

E308: *OK. And two ninths?*

K309: (3) *Zero point two* (2) *periodic.*

E310: *Yes. And three ninths?*

K311: *Zero point three periodic.*

E312: *Well, and so on, yes, and eight ninths?*

K313: *Zero point eight periodic.*

E314: *Hm. And nine ninths?*

K315: *Zero point nine periodic.*

E316: *Hm. And but is it possible to express nine ninths in a different way, to express it with different words?*

K317: *One.*

E318: *I see. But now you say that one is the same as* $0.\overline{9}$.

K319: (2) *Well, you are right.*

The experimenter succeeded in inducing the CC.

E320: *Then, is it the same or not?*

K321: (2) *Well, so, it is like that.*

E322: *Like what? What do you mean? Either it is equal to one or it is not, isn't it?*

K323: *Well, I think that it should not be equal to one as it starts with* 0.9 (3), *but using the fraction, it equals one.*

Klára's obstacle If it starts with zero, it cannot be one, has not been attacked by the above induced CC. We explain this by the fact that the way of inducing the CC does not attack explicitly the obstacles which are its cause. It is in our terminology, the so called vain attempt at removing the conflict: The individual refuses to remove the CC and the disagreement in his/her knowledge structure endures. Even this version of facing with the CC has its value for him/her, as it is important, from the propedeutic point of view, for other conflicts induced in order to overcome the given obstacle gradually.

5. Conclusion

The aim of our research is to specify the process of forming the concept of infinity. This concept is grasped by using other concepts – objects (point, straight line, number etc.) as we learn about its attributes – set cardinality, set boundedness, set measure, ordering, infinite process and convergence. These attributes are in the focus of the structured interview questions.

For the explanation of the process of forming the concept of infinity, we used the theory of epistemological obstacles (Brousseau, 1997) that are fundamental for the given concept. Thus we should not avoid obstacles but we should intentionally look for them. It is typical of an obstacle that it resists disagreement with which it is confronted. The key moment for overcoming the obstacle is the inducing and subsequent removal of the cognitive conflict. Only after the removal of many cognitive conflicts in different contexts can we overcome the given obstacle and create "better" pieces of knowledge.

In this article we present three illustrative examples of interviews conducted in the year 2007 with pupils of primary and secondary schools and with one grammar school student. The interviews focus on the attributes of infinity – set boundedness and cardinality, ordering (order density), the infinite process and convergence.

The first sample (Jakub, school year 5) demonstrates the inducing of the cognitive conflict, which followed the indicated contradiction caused by an obstacle, i.e. by a replacement of a straight line and its image as a model. The respondent was able to successfully remove the CC.

In the second sample (Michal, school year 9), the interview is focused on the attribute of set cardinality. Also here, the inducing of the CC was successful after noticing the contradiction by the experimenter. The obstacle is the understanding of the criterion "it is possible to count" as a real possibility of counting. The respondent was successful in removing the induced CC, however only in the given context. In a different context, the respondent fails as the obstacle has not been overcome yet.

The last sample (Klára, grammar school year 3) illustrates the case when first the inducing of the CC failed as there was a lack of knowledge of the infinite decimal expansion in Klára's knowledge structure. After that the CC was induced in another context, unfortunately with the causing obstacles being attacked. For that reason, it was a vain attempt at CC removal.

For a successful process of forming the concept of infinity it is necessary to overcome obstacles. A teacher needs to know possible obstacles in this process. To overcome them gradually, the teacher has to create such didactical situations when cognitive conflicts in different contexts are repeatedly induced and removed.

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ICT AS A SUPPORT FOR CREATING WORKSHEETS FOR THE MATHEMATIC ACTIVITIES CENTRE

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Abstract. The contribution is focused on the possibilities of ICT using for creating worksheets for the Mathematic Activities Centre within the programme "Let's Begin Together."

ZDM Subject Classification: U20.

1. Introduction

This paper deals with creating worksheets for the Mathematic Activities Centre. The Activities Centres can be found within the programme "Let's Begin Together." The organization unit in this programme is a day, not a lesson, as it is common in regular classes.

The day usually begins with a morning session which generally takes twenty to thirty minutes. A teacher meets pupils, they greet each other and share their experience, feelings or new ideas and pupils are familiarized with the schedule for the day.

This meeting also provides an opportunity for frontal work when deducing new phenomena in the Czech language or Mathematics.

The morning session is usually followed by group work – Czech language and Mathematics blocks lasting forty to fifty minutes. For organization reasons there is a common forty minute break. Lesson and relaxation time is the same as in any other ordinary class.

The brake is followed by pupils' work in so-called activities centres, which takes approximately sixty minutes. In these centres, there are prepared tasks related to the topic in hand. There are certain rules within the centres and each pupil should go through all the centres. Pupils are drawn to groups according to the centres in advance. They can sign in a table prepared beforehand, which constitutes a schedule that maps work of all the pupils in the classroom throughout the week. Each pupil is responsible for the work they have done.

Pupils cooperate on given tasks. Various tasks have been prepared to lead pupils to think and do practical activities at the same time. Activities where they can learn how to cooperate, communicate mutually, solve problems, etc.

Pupils usually work in one centre for a day and the next day they proceed to another centre. At the end of the day pupils are gathered again for 35-40 min and present the results of their work and evaluate how well they have done or what problems they have encountered.

2. The Centre of Mathematics

Children need to gain some concrete practical experience in "mathematics from the real world" they are surrounded by in order to learn how to comprehend mathematical patterns and perceive their relations.

In the Centre of Mathematics the following aids are used: math workbooks and textbooks, sheets of papers, graph paper, rulers, a tape measure, a thermometer, scales, a string, matchsticks, natural products used for counting for children with SpLDs (beans, grits, etc.) coins, dice, varicoloured geometric shapes and objects and their coverings, brain-teasers and riddles, cross-words (word-searches, labyrinths, sudoku, etc.), building blocks and games.

The class environment arranged in the centres of activities feels much more comfortable, provides some privacy and feeling of safety. Everyone is able to find their own little nook.

Pupils are enabled to learn in an active way. They are involved in problem solving and learn upon their own experience. Thanks to this, pupils comprehend the topic in depth and their pieces of knowledge cease to be superficial. There is pupils' direct experience behind their acquired knowledge. They are able to pass what they have learned on to the others.

Working in these centres of activities lead pupils to greater independence. They can choose the way of working, distribute their roles and tasks among themselves in a team, and plan a strategy they would use. They do not hesitate to use their initiative or to help each other. Pupils are taught not to avoid stressful situations and consider them a challenge.

Inspiring class environment is one of the most important conditions for a successful realization of ideas and requirements defined in RVP ZV (Framework programme for Education at Elementary Schools).

Project lessons have already been in use for several years. Pupils work on a project as individuals or in a group. This project comes from a real world and needs pupils to use their knowledge they have gained in different life situations. They need to plan their work, they analyse and process the information gained. At the end, they evaluate and present their results.

However, there each work and methods bring about both good sides and difficulties. In my opinion the main difficulty about working in the centres is the fact that some rules, strategies and principles are not followed, are omitted or difficult to achieve. It can be very difficult to teach children to accept the differences of other children and to work with someone they do not get on with. It is also very difficult to teach them to overcome a problem and not to give in and, last but not least, absent-mindedness and impetuosity when solving a problem.

3. Worksheet creating with the help of ICT

Fulfilling a task in the centres of activities within the programme "Let's Begin Together." can be done with the help of worksheets as well. These worksheets can be put together using some topics from workbooks, making copies, and also with the help of a multimedia computer.

Adobe Photoshop CS3

It is possible to use a program Adobe Photoshop CS3 to create a worksheet as this program is widespread and enables to process a picture in various graphic formats and modes. It is possible to create photomontages using layers or graphic and text effects.

Photoshop CS3 offers a wide range of drawing and effects tools for creating, retouching and colour modifications of a coloured or black and white picture. It is better to use a coloured picture when creating a worksheet as I assume this way of graphic format is more suitable and effective for elementary school children.

4. Sequence of operations when creating the worksheets

The first step is to open a new page in Adobe Photoshop CS3 which has the following parameters: width – 29.7 cm, height – 21 cm, resolution – 150 dpi. The layout and texts of a worksheet should be prepared beforehand. These texts come from the curriculum of a certain subject and grade. Some of the worksheets are parts of a project and are made for a particular centre of activities. Other worksheets are directly applicable to revision or e.g. a five-minute activity in the Mathematics class.

This was followed by the worksheet creation, done with the help of a multimedia computer. Firstly, you need to paste a text into a document, this forms the first layer of a page. The layers of a pictorial part make up the second, third and other layers of the worksheet. Transparent layers are pasted among the pictorial layers as they enable you to move or colour a picture, etc. There are several ways of creating a picture. Some pictures can be uploaded, you can use a black and white picture and colour it in this program. Other pictures can be drawn and uploaded. Another method is to draw a picture directly in Adobe Photoshop CS3. You can use clip-arts as well.

5. The benefits of worksheets for pupils

The advantage of the worksheets is the fact that you can use them not only during the lesson, but also before and after the lesson. The texts perform different functions according to their types; e.g. for a warm up, it is advisable to use factual texts containing some information from magazines and books, or texts like that. If you want your pupils to express their opinions and debate certain points, use some texts "for thought". The worksheets are useful for pupils when there is too much information to be put in various charts and graphs or when there is a problem solving task or a laboratory practice.

Worksheets used during the lesson serve as a reinforcement of the lesson's most essential content or as a means to support independent work.

Worksheets are an indicator of teacher's inventiveness when a teacher is not entirely dependent on pencil and paper. Computers offer them to make the lesson more comprehensible, well-arranged and more aesthetic; text, graphic and spreadsheet programs are thus very important here. However, you need to maintain all the aspects important for creating worksheets, i.e. comprehensibility of a text – using short, concisely formulated sentences, stimulating additions, and transparent interpretation of a content. Factual content must be cognitively structured.

6. Conclusion

Creating worksheets via ICT is truly an ideal solution as some worksheets readily available may not be suitable for pupils due to their educational methods, graphic layout or motivational tools.

A teacher can adjust all their materials directly for their pupils. It might be useful to teach the older pupils how to create their own concepts of worksheets as well. Time spent at a computer can be very helpful but it might be rather damaging when this type of work is overused. Parents and teachers should choose the activities carefully for their pupils as well as motivate them and encourage them to use a computer for useful purposes.

I would like to conclude with thanks to Mgr. Jana Všetečková, a diploma thesis writer who enabled me to look into the educational project "Let's begin together" and provide you with an interesting insight into the possibilities of using ICT in Mathematics lessons.

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TECHNOLOGICAL SUPPORT FOR THE REALIZATION OF INTERACTIVE MATHEMATICAL EDUCATIONAL ACTIVITIES

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Abstract. New information and communication technologies provide means for support of the learning process based on active work with information and on realization of new forms of students' dialog with educational environment. Electronic learning materials can be a source of educational content, but also the means for implementation of modelling activities, interactivity and communication with the educational environment, which offers together with feedback possibilities for guiding the learning process. Two interactive learning activities focused on absorbing and consolidating geometrical knowledge are described in the paper. A combination of different technological tools containing authoring tool ToolBook, web technologies and Cabri Geometry is used for implementation of corresponding degree of interactivity and factors of feedback into the learning materials.

ZDM Subject Classification: D04, D54, U74.

1. Characteristic of feedback in e-content

The preparing of various electronic learning materials (e-content) is an important premise for wider application and effective use of e-learning in education. E-content is considered as key pillar of e-learning, because it enables realization of educational activities and processes based on the use of electronic media and web technologies. Therefore, in present day the main attention is devoted to didactic aspects of e-learning connected with goals, content and methods of education.

The searching and providing learning assistance are important factors for planning and creating individual learning paths in interactive learning materials. If student need assistance, he is guided step by step to finding answers to questions and to solving problems, but he should perform essential part of solution independently. The searching and providing assistance have by Mareš [1] these specific features:

- student gets running feedback information about his own learning process,
- system provides to student certain freedom in decision making about next action,
- student does not think about formulation of request for help, but system is not sufficiently sensitive for providing assistance,
- requisition for providing assistance can decrease student's motivation for deeper reflection about searching assistance.

The design and implementation of simple versatile components based on schemes, in which learning processes and ways of reactions of system in dependence on learning efforts are implemented, have important position by the creation of electronic learning materials.

2. Absorption of knowledge with using ICT

We have used the theory of contingent tutoring [1] as theoretical base for proposal of interactive activities focused on absorption of knowledge. If student needs assistance, system acts as mentor, which provides advices and guides him to solution of a problem. The assistance should be structured into levels so that the system might stimulate active learning and to enable student to discover maximum of the solution independently. The four levels of the assistance are recommended. The first three levels should have increasing rate of strictness gradually explaining basis of the solution. Field of knowledge used for solving is indicated at the beginning. Then student is gradually guided to explanation of the main step of solving problem. The final level of assistance contains detailed explanation of process of solving. If student does not understand process of solving and defaults also occur in next work, it is necessary to provide for him unasked assistance from teacher.

We applied this model to educational activity useful for absorption of knowledge. Educational content is divided into smaller units. Transition between them is often conditioned on correct solving of the task. We also tried to include into the system of assistance exploration of dynamic constructions which enables to visualize examined objects.

Described activity is focused on the use of the relation between sizes of central and circumferential angle in a circle for derivation of the property of angles in cyclic quadrilaterals. The problem situation raised at the beginning of activity serves to arouse interest of students. Students are required to circumscribe a circle about a parallelogram. Student can use the dynamic construction, with which can search circles circumscribed about parallelograms. After realization of the fact, that it is not possible to circumscribe a circle about any parallelogram, justification based on the use of the property of pair of opposite angles in cyclic quadrilateral follows. The derivation of this property is the main goal of this activity. Fig. 1 shows page containing the problem, incomplete relation, which student has to complete and buttons for system reaction.

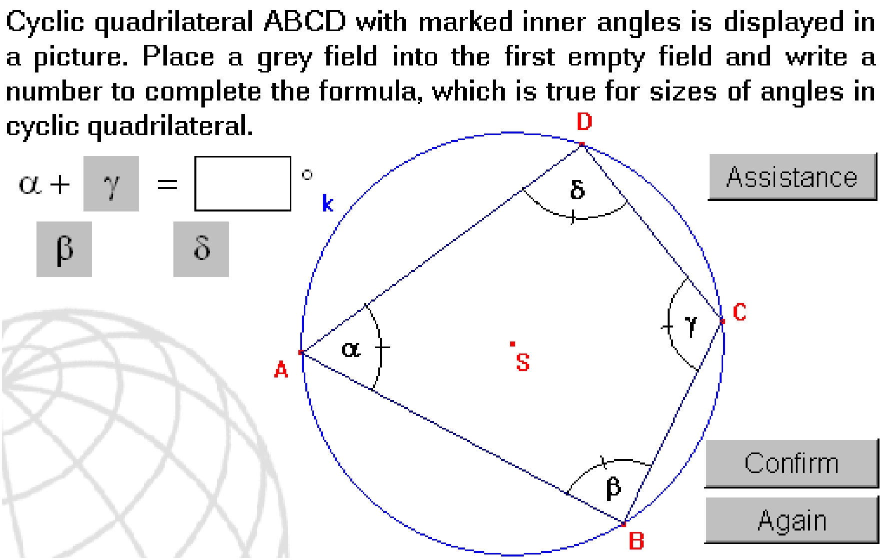

Figure 1: Part of the educational activity

Student can require the first level of assistance by one's own reflection. This level also includes an exploration of dynamic construction. Another three levels are provided automatically after incorrect answer. If student does not know to use advice that he should exploit for pair of opposite angles relation between sizes of central and circumferential angle in a circle, then the main step of problem solving follows. Basis of this level of assistance is the picture of a circle with marked central angle subtending arc BAD, near which the size 2γ is stated. Student obtains the advice that he should mark central angle corresponding to circumferential angle α and to determine a sum of sizes of angles α, γ. If the answer is incorrect again, detailed explanation of solving follows. Student has to show understanding of derived relation by solving alternate task, which involves, apart from derived relation, Thales' theorem, because one side of examined cyclic quadrilateral is diameter of a circle. If student solves also alternate task incorrectly, he gets a recommendation that he should ask for teacher assistance.

3. Educational activities for exercising and fixing of knowledge

Apart from the theory of contingent tutoring, there are other approaches to providing guidance information. McKendree [2] has suggested the following classification of feedback:

[1] minimal feedback, which only warns the student that his answer was wrong,

[2] signalizing feedback, which also warns the student that he did not respect principles,

[3] initiating feedback, which also informs the student about requirements, which are necessary for elimination of the mistake and indicates next working,

[4] feedback, which represents the combination of the second and third type of feedback.

Interactive tests (www.univie.ac.at/future.media/moe/tests.html) on the web portal Maths-online are an example of the minimal level of feedback. Learning activity Cube Nets, which is free accessible on web portal of NCTM (illuminations.nctm.org/ActivityDetail.aspx?ID=84), is a suitable example of a simplified signalizing feedback. Student's task is to mark pictures, in which a cube net is shown. After choosing a right picture, the corresponding cube net is filled with colour, otherwise a message box with feedback information reasoning, why the selected picture does not represent a cube net is displayed.

We have used the fourth level of feedback from the presented classification in learning activity focused on exploration of a set of all points of given property. We have created a scheme, which contains statement of the task and four possible answers, of which exactly one is correct. After choosing incorrect answers, contextual guidance information warning of wrong consideration and hints suggesting process to the correct answer are provided. The process, how the student got to the correct answer is also regarded by evaluation through points.

The described activity consists of two tasks whereby the second one is an extension of the first task and therefore its statement will not be displayed until the first task is solved. The statement of the initial task is following: *Line p containing point A and different point S, B, which lie in the same half-plane stated by the line p are given. Explore the set of all vertices C of parallelogram ABCD with the centre S for arbitrary points A.* Students can use a corresponding CabriJava applet, in which they can move point A along line p and explore particular cases of parallelograms. Fig. 2 shows two offered incorrect answers.

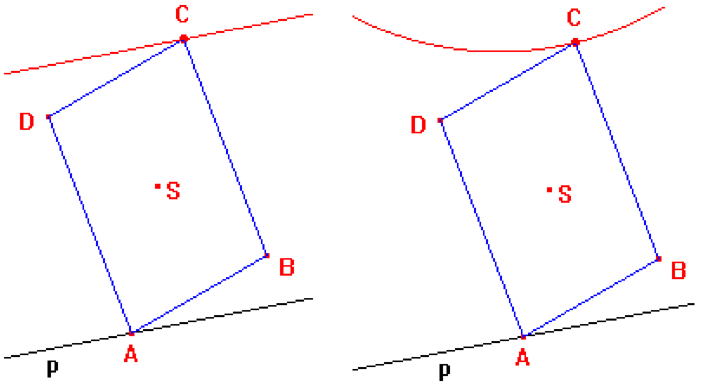

Figure 2: Two of four offered answers

Additional two answers display the explored set as a line concurrent with line p and as a line without one point parallel to line p, which is the correct solution of the task. Clicking on each picture causes displaying of feedback information dependent on the chosen answer. If student marks the curve as the set of vertices C, he will get this assistance information: *Because parallelogram ABCD is symmetric with respect to point S (intersection of diagonals) vertices C lie on a line, which is symmetric to the line p with respect to point S.* If he chooses the picture with the concurrent line, then he will get this help: *Vertices C lie on a line indeed, but this line is not concurrent with line p. In symmetry determined by point S vertex C is image of vertex A, which can acquire any position on the line p. Consider, how the line p will be projected in this symmetry.* And finally, if student selects the picture with the whole parallel line, then this guiding information will be shown to him: *There is a concrete position of point A, when points A, B, C, D do not form vertices of parallelogram. Set of vertices C therefore does not consist of all points of this parallel line.* In the second task the students should find a position of point B, when the solution is the entire line parallel to line p. This happens, when distance of points B, S to line p is equal.

4. Conclusion

We have used schemes, on which both described learning activities are based by creating of learning materials, which are components of e-learning course aimed for future mathematics teachers. The main goal of this course is to develop knowledge and skills of students with using ICT for support of mathematics teaching.

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TOPOLOGY: WHY OPEN SETS?

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Abstract. A high school teacher of mathematics should be familiar with all principal areas of mathematics. He is definitely not educated as a specialist in some particular field of research, but he rather should understand what a particular field is about and what are general trends in mathematics, often correlated to the history of mathematics. We sketch some trends in the development of topological structures.

ZDM Subject Classification: I99.

1. Why open sets?

Topology belongs to "continuous mathematics" and it is related to calculus. Calculus of functions of n variables leads to metric spaces and advanced calculus leads to function spaces. The process of abstraction - a fundamental feature of mathematics - leads to topology and topological structures. Practicing abstraction leads to a better student and a better teacher. We start with a formal definition.

DEFINITION 1. Let X be a set and let $\mathcal{O}$ be a system of its subsets such that

(i) $\emptyset, X \in \mathcal{O}$;

(ii) If T is a set and $U_t \in \mathcal{O}$ for each $t \in T$, then $\bigcup_{t\in T} U_t \in \mathcal{O}$;

(iii) If $\{U_i; i = 1, 2, ..., n\}$ is a finite family of subsets of X and $U_i \in \mathcal{O}$ for each $i = 1, 2, ..., n$, then $\bigcap_{i=1}^{n} U_i \in \mathcal{O}$;

then $\mathcal{O}$ is said to be a **topology** on X, the pair $(X, \mathcal{O})$ is said to be a **topological space**, each $U \in \mathcal{O}$ is said to be an **open set** in $(X, \mathcal{O})$ and each complement $U^c = X \setminus U$ of an open set U is said to be a **closed set** in $(X, \mathcal{O})$; if no confusion can arise, then $(X, \mathcal{O})$ will be condensed to X. Let $(X_1, \mathcal{O}_1)$ and $(X_2, \mathcal{O}_2)$ be topological spaces and let f be a map of X_1 into

X_2 such that for each $U \in \mathcal{O}_2$ we have $f^{\leftarrow}(U) = \{x \in X_1; f(x) \in U\} \in \mathcal{O}_1$. Then f is said to be a **continuous map** of $(X_1, \mathcal{O}_1)$ into $(X_2, \mathcal{O}_2)$.

First, to fathom the distance from calculus to topology, we analyze the process of abstraction and describe steps leading to such abstract definition. Second, we describe several equivalent ways how to define topology and, finally, we survey some generalizations of topology such as proximity, uniformity and filter structures. It sounds rather abstract, but we hope to put the development into a perspective and to stress the motivation.

Indeed, for a student who mastered calculus it is hard to appreciate the definition of topology and to understand in this context what does it mean that a set is open and that a map is continuous. Let us list some steps leading from calculus to topology.

STEP 1. Assume that we are interested in an abstract characterization of continuity of a real function defined on an open interval $I = (a, b)$. Traditionally, in calculus the continuity of f is defined either: a) via convergence of sequences, or b) via "ϵ - δ". First as the continuity at a point of I and then as the continuity at each point of I. Clearly, in both cases it is fundamental what is represented by symbol $|x-y|$, $x, y \in I$. Of course, this leads to the distance and the reformulation of both definitions of continuity via distance:

a) a sequence $\{x_n\}$ of real numbers converges to a real number x whenever the distance $|x_n - x|$ "decreases to zero with the increasing index n" (i.e., given $\epsilon > 0$, for sufficiently large n_0 all distances $|x_n - x|$, $n \geq n_0$, are less than ϵ) and f is continuous if $x_n \longrightarrow x$ implies $f(x_n) \longrightarrow f(x)$ (f preserves the convergence of sequences);

b) $\forall(x_0 \in I)$ and $\forall(\epsilon > 0, \epsilon \in R)$ $\exists(\delta > 0, \delta \in R)$ such that if the distance from x to x_0 is less than δ, then the distance from $f(x)$ to $f(x_0)$ is less than ϵ (the distance from $f(x)$ to $f(x_0)$ can be controlled by the distance from x to x_0).

STEP 2. Introduction of a metric space (X, d) via the usual axioms of distance and the definition of the metric convergence of a sequence of points to its limit via open balls $B(a, \epsilon) = \{x \in X; d(a, x) < \epsilon\}$, $\epsilon > 0, \epsilon \in R$ ($a_n \longrightarrow a$ whenever for each ball $B(a, \epsilon)$ all but finitely many elements of the sequence belong to the ball), checking that the continuous real function f on I is continuous with respect to the distance d. This step shows that if we have a distance in the domain and a distance in the range of a map, then we can check whether the map is continuous (whether it preserves the convergence of sequences or whether it maps small balls into small balls) and, indeed, the continuity is as it should be.

STEP 3. Le f be a continuous map of a metric space (X, d_X) into a metric space (Y, d_Y). It is natural to ask the following question. Will a "small" change of d_X and d_Y have an impact on the continuity of f? Of course, if by "small" we understand that the change does not have any impact on the convergence of sequences, then the answer is **NO!** This observation leads to the understanding that the distance is not the ultimate way leading to the abstract notion of continuity. For, it means that by changing metrics (e.g. by zooming) we do not necessarily change continuity and hence metrics (distance) is not a suitable notion to completely characterize continuity.

Next, we shall try to find out a suitable notion invariant with respect to "small" changes of distance and to define abstract continuity of a map from one abstract space into another one in terms of such notion.

STEP 4. Let (Z, d_Z) be a metric space. The following question is crucial. Which subsets of Z are invariant with respect to "small" changes of distance? Since we do not change the limits of convergent sequences, the answer is **CLOSED SETS**! Recall that a subset $S \subseteq Z$ is **closed** whenever for each convergent sequence of points of S also the limit point is a point of S. Observe that also the complements of closed sets - called **open sets** - remain the same if the convergence of sequences is not changed. Further, observe that a change of the system of closed sets would change convergent sequences and hence the identity map would not be continuous anymore.

Now, what are the characteristic properties of closed (open) sets? In our next steps we have to characterize closed sets (equivalently open sets), to develop a mathematical structure in terms of such sets, to show that continuity of a map can be defined via such sets, and that passing from one system of such sets to a differet system will destroy the continuity of at least one continuous map, that is the identity map of Z.

STEP 5. First, it is easy to see that in (Z, d_Z) both $\emptyset$ and Z are always slosed. Second, The intersection of each system of closed sets is always closed (if a sequence is in the intersection, then also its limit is there). Third, the union of a finite system of closed sets is closed (if a sequence $\{x_n\}$ is in the union, then for infinitely many n the point x_n belongs to one of the sets - by the pidgeon hole principle - hence the limit point, to which the corresponding subsequence converges, belongs to the set as well). Since $\bigcup_{n=1}^{\infty}[1/n, 1-1/n] = (0,1)$, we cannot require that the union of a infinite system of closed sets is closed. Finally, this means that open sets (as the complements of closed sets) satisfy the conditions (i), (ii) and (iii) of Definition 1.

STEP 6. The logic of implication helps us to understand the role of a preimage in the definition of a continuous map (Definition 1). We have to guarentee that $x_n \longrightarrow x$ implies $f(x_n) \longrightarrow f(x)$, but not the converse implication. Hence only the preimage of each closed set has to be closed (and not vice versa, the image of a closed set need not be closed). Indeed, assuming contrariwise that the preimage of some closed set is not closed, then some sequence in the preimage has a limit outside the preimage - a contradiction with the assumption that f preserves the limits of convergent sequences. Dually, we have to guarantee that the preimage of each open set is open and this is exactly as stated in Definition 1.

CONCLUSION 1. Continuity can be captured completely by fixing "open" sets. Everything in topology is coded in open sets. On the one hand, the resulting mathematical language of open sets is very efficient but, on the other hand, it is not so natural and intuitive. This is one of the reasons why topology is performed in several "isomorhic topological languages" simultaneously, in terms of: open sets, neighborhoods, closure, ... ; the choice of a particular (otherwise equivalent) language depends usually on a particular topological construction and, for example, how the construction matches our intuition. Open sets of a topological space form a bounded lattice having some additional properties. Such lattices provide a tool to study topological spaces; this leads to the so called pointless mathematics (cf. [4]).

2. Topological structures

First, without going into details, we hint that there are other ways to define a continuous structure so that it is in a one-to-one correspondence to topology and continuous maps are "the same" as the structure preserving maps.

1. **Closure spaces**: we axiomatize the notion "a point is close to a subset"; canonical maps preserve the "closedness" and all points close to a set form its closure. Closure spaces are attributed to K. Kuratowski (cf. [2]). General closure spaces have been introduced by E. Čech (cf. [1]).

DEFINITION 2. Let X be a (nonempty) set and let $\mathcal{P}(X)$ be the set of all subsets of X. Let u be a map of $\mathcal{P}(X)$ into $\mathcal{P}(X)$ such that
(i) $u(\emptyset) = \emptyset$;
(ii) $S \subseteq u(S)$ for each $S \subseteq X$;
(iii) $u(S \cup T) = u(S) \cup u(T)$ for all $S, T \subseteq X$;
(iv) $u(u(S)) = u(S)$ for each $S \subseteq X$.
Then u is said to be a **topological closure operator** for X and $u(S)$ is said to be the closure of S.

THEOREM 1. (i) Let $(X, \mathcal{O})$ be a topological space. For each subset S of X let $u(S)$ be the intersection of all closed sets containing S and let u be the resulting map of $\mathcal{P}(X)$ into $\mathcal{P}(X)$. Then u is a topological closure operator. The set $\{S \in \mathcal{P}(X);\, u(S) = S\}$ is exactly the set of all closed sets in $(X, \mathcal{O})$.

(ii) Let u be a topological closure operator for a set X. Then there exists a unique topology $\mathcal{O}$ for X such that $\{S \in \mathcal{P}(X);\, u(S) = S\}$ is exactly the set of all closed sets in $(X, \mathcal{O})$.

THEOREM 2. Let $(X_1, \mathcal{O}_1)$ and $(X_2, \mathcal{O}_2)$ be topological spaces, let u_1 and u_2 be the corresponding topological closure operators, and let f be a map of X_1 into X_2. Then the following are equivalent.

(i) f is a continuous map of $(X_1, \mathcal{O}_1)$ into $(X_2, \mathcal{O}_2)$.

(ii) If $x \in u_1(S)$, then $f(x) \in u_2(f(S))$, $S \subseteq X, x \in X$.

(iii) $(u_1(S)) \subseteq u_2(f(S))$ for each $S \in \mathcal{P}(X)$.

OBSERVATION 1. Roughly, topology deals with topological invariants, i.e., notions defined via open sets. It follows from Theorem 1 and Theorem 2 that each topological invariant can be alternatively defined via topological closure operators and that the "topological languages" of open sets and topological closure operators are equivalent. Note that the same is true with respect to other equivalent topological structures.

2. **Neighborhood spaces**: the idea is to axiomatize neighborhoods via "open balls without metrics", see the famous book [3] by F. Hausdorff (father of general topology). Recall that by a filter of sets we understand a nonempty system of subsets which does not contain $\emptyset$, it is closed with respect to larger sets and finite intersections.

DEFINITION 3. A **neighbourhood system** on a set X is a system $\mathcal{N}_X = \{c, x \in X\}$ of filters of subsets of X such that

(i) $(\forall U \in \mathcal{N}_x)[x \in U]$;

(ii) $(\forall U \in \mathcal{N}_x)(\exists V \in \mathcal{N}_x)\ (\forall y \in V)\ [U \in \mathcal{N}_y]$.

The pair $(X, \mathcal{N}_X)$ is called a **neighborhood space** and the sets belonging to $\mathcal{N}_x$ are called **neighborhoods** of x. If A is a subset of X such that $(\forall x \in A)(\exists V \in \mathcal{N}_x)[V \subseteq A]$, then A is said to be **open** in $(X, \mathcal{N}_X)$. Let $(X, \mathcal{N}_X)$ and $(Y, \mathcal{N}_Y)$ be neighborhood spaces and let f be a map of X into Y. If $(\forall f(x) \in Y)\ (\forall V \in \mathcal{N}_f(x))(\exists U \in \mathcal{N}_x)\ [f(U) \subseteq V]$, then f is said to be **continuous**.

Note that in a neighborhood space, a filter of neighborhoods $\mathcal{N}_x$ of a point x can have a very complicated order structure (unlikely as in a metric space, where $\bigcap_{n=1}^{\infty} B(x, 1/n) = \{x\}$). Due to this fact, in general the closure of a set cannot be defined via convergent sequences and a sequentially continuous map need not be continuous.

OBSERVATION 2. Topological spaces and neighborhood spaces are "equivalent structures".

Second, to put the development of general topology into a perspective, we sketch some other topological structures and their relationships. More information can be found in the references, in particular in [6].

3. **Proximity spaces**: a proximity space is an axiomatization of notions of "nearness" that hold set-to-set, i.e. for each pair of subsets we declare whether they are "near" or "not near", as opposed to the better known point-to-set notions, i.e. for each point and each subset we declare whether they are "close" or "not close", that characterize topological spaces; canonical are maps preserving the nearness.

The concept was introduced by F. Riesz in 1908 and ignored for a long time. It was rediscovered and axiomatized by V. A. Efremovich in 1934, but not published until 1951. In the interim, in 1940, A. N. Wallace discovered a version of the same concept.

OBSERVATION 3. To each proximity space there corresponds a topological space having "good" properties.

4. **Uniform spaces**: a neighborhood system is assigned not for each point x of a set X, but for the diagonal $\Delta_X = \{(x, x); x \in X\}$ of the product $X \times X$; canonical maps are defined via preimages of neighborhoods.

Before A. Weil gave the first explicit definition of a uniform structure in 1937, uniform concepts, like completeness, were discussed using metric spaces. Famous Nicolas Bourbaki group provided the definition of a uniform structure in terms of entourages in the book Topologie Genrale and J. Tukey gave the uniform cover definition. Weil also characterized uniformities as structures determined via a suitable family of pseudometrics.

OBSERVATION 4. To each uniform space there corresponds a topological space having "good" properties. Each metric space is in fact a uniform space determined by a single metrics.

...

(n+1). **Approach spaces**: the missing link in the topology-uniformity-metric triad developed by R. Lowen (cf. [5]). It is a really conceptual but rather technical unification of continuous structures ; it came perhaps too late ... .

CONCLUSION 2. After a "big boom" in the last century, general topology nowadays does not attract young researchers and some people say that it is not "in". Remember, even though E. Hewitt (fundamental contributions in general topology and abstract analysis, cf. [2]) has said 50 years ago that "general topology is a dead discipline", since then many fundamental results have been achieved. Is general topology really "out"? Well, it depends ... . I do believe that good mathematics is and always will be an integral part of our CULTURE.

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SOME REMARKS ON ABSOLUTE VALUE

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Abstract. This paper treats about problems connected with proper understanding of the notion "absolute value" and with possible definitions of this notion.

ZDM Subject Classification: D50.

The authors of the present paper wish to pay attention to the diversity of possible definitions of absolute value and to pay attention to dangers connected with the brace notation of these definitions. The brace notation is used in the greater part of mathematical textbooks. Here we give some examples:

(1) $$|x| = \begin{cases} x & \text{for} \quad x \geq 0 \\ -x & \text{for} \quad x < 0 \end{cases}, \text{(see [1], [3])}$$

(2) $$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}, \text{(see [2])}$$

(3) $$|x| = \begin{cases} x & \text{where} \quad x \geq 0 \\ -x & \text{where} \quad x < 0 \end{cases}, \text{(see [1], [7])}$$

The above-mentioned definitions are not formal, they are rather intentional. We can guess that the brace performs a role of a logical connective "and" or maybe "or". Such implicite notations may lead to unnecessary misunderstanding.

Before we discuss some problems caused by using the brace in above-mentioned definitions we present possible formal definitions.

The absolute value "$|\ |$" is a binary relation defined on the set $\Re$ satisfying one of the following conditions:

(4) $|x| = \mathrm{odl}(x, 0),$

(5) $|x| = \max(x, -x),$

(6) $|x| = \sqrt{x^2},$

(7) $|x| = (\mathrm{sgn} x) \cdot x$

The union of two sets is a graphic representation of this relation:

$$P_+ \cup P_- = \{(x,y) \in \Re^2 : x \geq 0 \wedge y = x\} \cup \{(x,y) \in \Re^2 : x < 0 \wedge y = -x\}$$

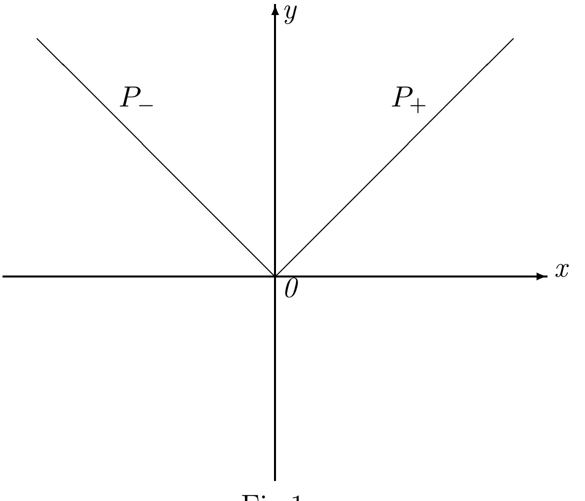

Fig.1

Basing on the above graph we may give a formal definition of the symbol "| |":

(8) $y = |x| \Leftrightarrow [(y = x \wedge x \geq 0) \vee (y = -x \wedge x < 0)].$

The definition (8) written in a brace representation takes the following form:

(9) $|x| = \left\{ \begin{array}{ccc} x & \wedge & x \geq 0 \\ -x & \wedge & x < 0 \end{array} \right. .$

In this case the brace has a role of a logical disjunction "*or*".

The relation "| |" is a function, too. We have to remember that the definition of the function symbol is correct if the definition satisfies the condition of uniqueness and existence (see [4]):

(10) $\forall x \exists_1 y[(x \geq 0 \wedge y = x) \vee (x < 0 \wedge y = -x)].$

First we prove the condition of existence. It is obvious that

$$\forall x \in \Re[(x \geq 0 \vee x < 0) \wedge x = x].$$

From the above we obtain

$$[(x \geq 0 \wedge x = x) \vee (x < 0 \wedge x = x)] \Rightarrow$$
$$\Rightarrow [(x \geq 0 \wedge x = x) \vee (x < 0 \wedge -x = -x)] \Rightarrow$$
$$\Rightarrow [\exists y(x \geq 0 \wedge y = x) \vee \exists y(x < 0 \wedge y = -x)] \Rightarrow$$
$$\Rightarrow \exists y[(x \geq 0 \wedge y = x) \vee [(x < 0 \wedge y = -x)].$$

So, we proved that

$$\forall x \exists y[(x \geq \wedge y = x) \vee (x < 0 \wedge y = -x)].$$

Now we prove the condition of uniqueness:

$$[(x \geq 0 \wedge y_1 = x) \vee (x < 0 \wedge y_1 = -x)] \wedge$$
$$\wedge [(x \geq 0 \wedge y_2 = x) \vee (x < 0 \wedge y_2 = -x)] \Rightarrow$$
$$\Rightarrow [(x \geq 0 \wedge y_1 = x \wedge y_2 = x)] \vee [(x < 0 \wedge y_1 = -x \wedge y_2 = -x)] \Rightarrow$$
$$\Rightarrow [(x \geq 0 \wedge y_1 = y_2)] \vee [(x < 0 \wedge y_1 = y_2)] \Rightarrow$$
$$[(x \geq 0 \vee x < 0) \wedge y_1 = y_2] \Rightarrow y_1 = y_2.$$

From the above considerations it appears that condition (10) is satisfied.

Another frequently used definition of the symbol "$|\ |$" is the following definition:

(11) $y = |x| \overset{\text{def}}{\Leftrightarrow} [(x \geq 0 \Rightarrow y = x) \wedge (x < 0 \Rightarrow y = -x)].$

This definition is represented by the brace notations (2) and (3) (we assume that the word *where* suggests an implication). In the notation (2) and (3) the brace performs a role of a conjunction *and.* Consequently, the notation (1) may be interpreted incorrectly in the following way:

(12) $y = |x| \overset{\text{def}}{\Leftrightarrow} [(y = x \wedge x \geq 0) \wedge (y = -x \wedge x < 0)],$

because the word *for* suggests the logical conjunction *and*. We should remember that using some notations of colloquial language in mathematical definitions may lead to many misunderstandings (see [5]).

Basing on the law of propositional calculus we present one more definition of the absolute value equivalent to the definition (11):

(13) $$y = |x| \overset{\text{def}}{\Leftrightarrow} [(y = x \vee x < 0) \wedge (y = -x \vee x \geq 0)].$$

In the definition (13) we deal with the intersection of two sets of the form:

$$L = \{(x, y) \in \Re^2 : y = x \vee x < 0\} \text{ and } P = \{(x, y) \in \Re^2 : y = -x \vee x \geq 0\}.$$

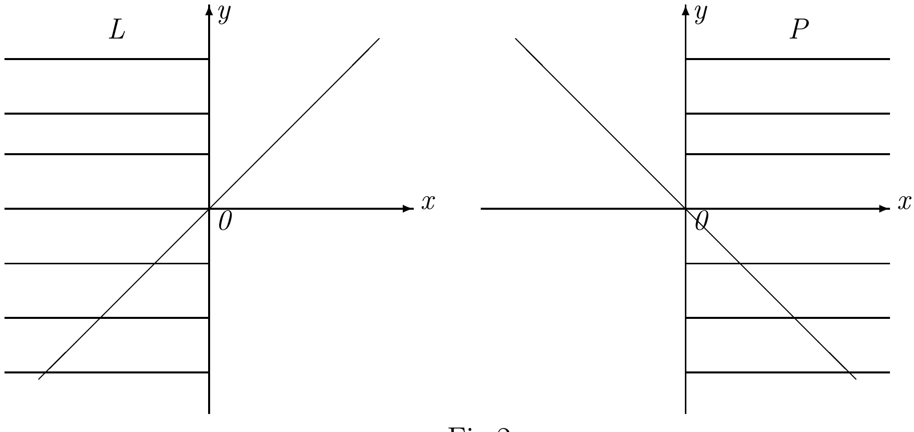

Fig.2

The above mentioned representations of the absolute value symbol seem to be confusing as far as definitions are concerned. Moreover, the confusion is inevitable and cannot be eliminated by simply turning students' attention to a double character that the brace has.

The notion of absolute value as a distance has been introduced by the basic principles of the educational program concerning the basic level of the school-leaving examination administered at higher secondary school (see [9]).

The definition of the absolute value as a function is requirement at the secondary level for the first we should avoid ambiguities and inaccurate definitions.

Students encounter the notion of the brace at a lower secondary level when they use it to solve the system of linear equations, and, that is why it is understood as the conjunction *and*. For this reason it is worth choosing a definition with no brace representation (see [6], [8]). If the definition of the absolute value appears after the lesson of logic, so that students are able to know and understand the logical connectives *and* and *or*, we will be able to work out the definition together with whole class.

We thus understand that this problem is difficult and demands considerations and materials, such as lesson scenarios presenting the absolute value.

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MODERNISATION AND INNOVATION OF THE CALCULUS TEACHING

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Abstract. In this article we show different innovative approaches to the calculus teaching. It is important for us the understanding of notions such limits, derivative and integral. We will show some possibilities of using ICT in calculus teaching.

ZDM Subject Classification: I20, I40, I50.

1. Introduction

In connection with calculus teaching at the secondary school level, many teachers often make a mistake and neglect or omit a suitable motivation and a proper introduction to the topic.

The history of mathematics provides many inspiring examples and approaches which can help in understanding the calculus notions. There are parallels between historical mathematical thinking and the development of mathematical thinking in the mind of students.

ICT brings new possibilities in teaching mathematics. It is important to understand and respect the advantages and limitations of ICT in teaching. In this article we will present some examples and remarks how to motivate calculus. We present also some results of the pedagogical research dealing with the notion of derivative.

2. Obstacles preventing students from understanding the concept of infinite series

In maths lessons at secondary schools, a frequently discussed issue is whether it is true that

$$0.999\ldots < 1 \text{ or } 0.999\ldots = 1?$$

>From my experience as a college teacher, I know that the vast majority of freshmen chooses the first alternative without any hesitation. Their justification is almost always the same: *If a decimal number begins with a zero, it cannot equal one but it is smaller than one.* Similarly, Mundy and Graham mention students' frequent statements that *The number* 0.999... *equals approximately 1, gets closer and closer to 1, but it is not exactly 1* (Mundy & Graham, 1994). The students think that *The difference between* 0.999... *and* 1 *is infinitesimally small, but there is one* or *The number* 0.999... *is the last number before* 1 (see studies Cornu, 1991 and Tall & Schwarzenberger, 1978).

In the subsequent discussion with students (we will come back to it), it might be appropriate to say that 0.999... can be understood as the infinite sum

$$0,\overline{9} = 0,9 + 0,09 + 0,009 + 0,0009 + \ldots \tag{1}$$

This way we are coming to infinite series. The key issue now is the question: Are students capable of accepting the thesis that the sum of the infinite number of positive real numbers is a real number? At this stage, the most frequent answer is: No.

In my view, three basic obstacles (see the theory of the epistemological obstacles in (Brousseau, 1997)) can be specified concerning students' understanding of the concept of infinite series.

First, it is the attitude of students that the infinite series cannot be summed up. This view follows students' experience with finite sums:

It cannot be determined if it goes up to the infinity. It has no end, after all. (A male grammar school student Ivan, 16 years).

It cannot be done to sum the numbers up to the infinity as I always add something more to that, after all. (A female grammar school student Marta, 17 years).

Second, it is the common idea of the majority of students that the sequence of partial sums of the infinite series of positive terms grows above all limits:

But when I add one more number, it further grows and it keeps on growing until the infinity. (A male grammar school student Petr, 16 years).

>From the point of the relation of phylogenesis and ontogenesis, this idea corresponds to Zenon's belief (about 490 – 430 BC) that the sum of an infinite number of line segments must be the infinity.

It is possible to overcome this obstacle of students by specific geometrical procedures they are common knowledge (see e.g. Hischer & Scheid, 1995). For example, we can mention Oresme's imaginative method (Nicole Oresme, 1323? – 1382) of cutting up a unit square for equality proof:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1 \tag{2}$$

After mastering the second obstacle, though, the third obstacle occurs (which is closely connected to the first one). For example regarding the equality (2), students often argue:

All the same, it never equals one, it only gets closer and closer to one, but it never gets there. (A female grammar school student Marta, 17 years).

There is always a little bit missing to one – even if we keep on adding, there will always be a little bit missing. (A male grammar school student Ondra, 18 years)

The above mentioned statements of the students clearly show their potential understanding of the infinite process in the task (see also Richman 1977 or Marx 2006). Students have not internalized the sum of a series as a limit of the sequence of its partial sums. However, even the majority of those who did this part in their lessons deals with non-standard tasks similarly. Hence, there is still a long way to go to get a deeper understanding of the limit and thus the infinite sum (see e.g. Tall, 1976).

How to deal with the third obstacle in lessons? Let's go back to the initial issue:

Is it true that $0.999\ldots < 1$ or $0.999\ldots = 1$?

An adequate method would be to explain students the limit of sequence and the sum of series, and to give them the formula to calculate the sum of infinite geometrical series, by the means of which to calculate the sum of series (1):

$$a_1 \cdot \frac{1}{1-q} = 0.9 \cdot \frac{1}{1-0.1} = 1. \tag{3}$$

However, it is advisable to present the above mentioned problem already before dealing with the corresponding parts. Besides, a great number of secondary schools never take the last step, namely to move from the geometrical sequence to the infinite series. Even more significant drawback of this method is, however, that although students superficially master to give the sum of the geometrical series while using the formula (3), they rarely understand the heart of what they are doing and they cannot see the link to the statement 0.999... = 1.

Hence, teachers at secondary schools often adopt the ‘equation’ technique to justify the equality of $0.999\ldots = 1$:

$$\begin{aligned} x &= 0,\overline{9} \qquad \cdot 10 \\ \underline{10x} &\underline{= 9,\overline{9}} \\ 9x &= 9 \\ x &= 1 \end{aligned}$$

Students are usually impressed by this smart solution. However, it involves a little bit of cheating as, in order to remove the infinite part of the decimal number, we multiply and subtract infinite decimal progressions term by term, without asking if we are justified in doing so.

At this point, it would be appropriate to use a simple task to show to the students that some cases of infinite sums cannot be dealt with by this mechanical method:

$$0 = (1-1) + (1-1) + (1-1) + \ldots = 1 - 1 + 1 - 1 + 1 - 1 + \ldots =$$

$$= 1 + (-1+1) + (-1+1) + (-1+1) + \ldots = 1$$

Given the students’ actual perception of the infinite limit process, what I see as something yielding good results is the discussion stimulated by asking the supporters of the statement 0.999... = 1 the following question:

If $0.999\ldots < 1$*, then the difference* $1 - 0.999\ldots$ *must be a positive real number, the same way as it holds that for example*

$$0.9 < 1 \quad \textit{and} \quad 1 - 0.9 = 0.1 > 0.$$

What does then $1 - 0.999\ldots$ *equal to?*

In a very lively discussion students often reach the conclusion that the solution cannot be any number in the form of 0.0000000001, no matter how many 0 it might include (to be precise, arbitrarily, but finitely many). The reason is clear: the sum of such a number with the number $0.999\ldots$ is obviously a number greater than 1. Thus, the suggestions that follow are $0.000\ldots 1$ (explained by the statement *Infinitely many zeros and at the end* 1) or *Ten to the power of the minus infinity* (which proves to be the same thing after the discussion). This discussion still concerns potential and actual perception of the infinite limit process and it is extremely valuable in terms of forming perceptions of the limit process.

In conclusion, I would like to express my firm belief that a suitable procedure for teaching the discussed parts is the following sequence of steps:

Motivation by presenting the problem of the sum of the infinite series (e.g. $0.999... = 1$) $\rightarrow$ the limit of a sequence $\rightarrow$ the sum of the series.

3. Derivative of the function at a point - qualitative research

This research was realized at the St Andrew secondary school in the school year 2003/2004. The goal of the research was to analyze the students' mistakes and to find their roots. The problems we have solved with students are usually not contained in typical mathematical textbooks. Similar research has been described in (Hejný, 2001) and (Sierpinska, 1987).

Hischer-Scheid (1995) use for the function f a function of slope-secant.

Definition 1. *Let f be a function defined at a point a and at least at an one point different from a. Let D_f be a definition range of the function f. The function $s_{f,a}(x) = \frac{f(x)-f(a)}{x-a}$ for every $x \in D_f - \{a\}$ is called the function of slope-secant of the function f at a point a.*

For example, if $f(x) = x^2$, then $s_{f,a}(x) = x + a$ for every $x \in R - \{a\}$. This definition we use in the next definition of derivative of function at a point.

Definition 2. *Let f be a function defined at an interval I, $a \in I$ and k is a real number. If the function of slope-secant of the function f defined at interval I*

$$s_{f,a}(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a} & \textit{pre } x \neq a, \\ k & \textit{pre } x = a \end{cases}$$

is continuous at the point a, then the function f is differentiable at a point a. The number k is called the derivative of a function f at a point a.

The idea to define the notion of derivative via continuity was successfully used in the calculus concept by Igor Kluvánek (see Kluvánek, 1991). This definition we use in the experimental teaching with the pupils. We solved with using graphs next examples:

Teacher: Calculate the derivation of the function $y = x^2$ at the point 1 from the definition!
Robo: $f(x) = x^2$, $a = 1$

$$s_{f,1}(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1} & \text{pre } x \neq 1, \\ k & \text{pre } x = 1, \end{cases}$$

$$\frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x - 1} = x + 1,$$

$$s_{f,1}(x) = \begin{cases} x+1 & \text{pre } x \neq 1, \\ k & \text{pre } x = 1. \end{cases}$$

Teacher: Do you know to describe the graph of the function $y = x + 1$?
Robo: The line.
Teacher: More precisely.
Robo: The straight line.

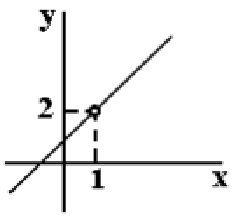


Fig. 1

Teacher: It is not exactly a straight line, because one point belongs not to this straight line. What is it possible to add so that the previous function becomes continuous?
Miroslava: We have to fill the circle.
Teacher: How?
Ivan: By number 2.
Teacher: What does it mean for the value of derivation of the function x^2 at the point 1?
Robo: It is equal to 2.
For pupils was not problem to find the derivative of the function x^2 for each point a.
Teacher: We considered functions with derivation at every point of the domain. Now, we are going to deal with functions having no derivation at least at one point: $f(x) = |x-2|$ $f'(2) = ?$

Pavol: $s_{f,2}(x) = \begin{cases} \dfrac{|x-2|}{x-2} & \text{for } x \neq 2, \\ k & \text{for } x = 2 \end{cases}$

for $x \in (2;\infty)$ $\qquad \dfrac{|x-2|}{x-2} = \dfrac{x-2}{x-2} = 1$

for $x \in (-\infty;2)$ $\qquad \dfrac{|x-2|}{x-2} = \dfrac{-(x-2)}{x-2} = -1$

Teacher: Is it possible to extend the function (to define its value at 2) so that it becomes continuous?

Lukáš, Lucia: No, it isn't.
Teacher: What does it mean for the derivation at the point 2?
Pavol: It doesn't exist.

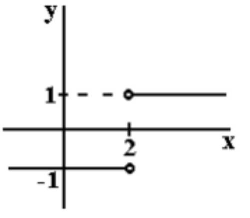


Fig. 2

4. Counterexamples in mathematical analysis and their visualization

When teaching the elements of mathematical analysis in secondary schools and universities, it is not the *generalization,* i.e. the transition from particular entities to abstract concepts, that usually causes the greatest difficulties, but it is the *concretization.* It is thus very useful to take into consideration that majority of mathematical analysis concepts are normally defined and introduced in sufficiently *general and abstract form.* Consequently, one of the key roles of the mathematical analysis teacher rests mainly in *concretization,* i.e. in the improvement of the craft to *see* in the concepts of mathematical analysis and their definitions the *actual patterns,* which enable to visualize the exact notion with sufficiently wide *content* of elementary features of this concept. *The transition from abstract concept to its concretization* can be distinctly made easier with the use of selected *specific* examples, in which *the irrelevant features and attributes of the concept are being varied.*

Let us note, that – generally – the *combined procedure* of transition *from exact to abstract and vice versa – from abstract to exact* – is applied within the process of introduction and consequent clarification of elementary concepts of mathematical analysis. In this procedure, the *concept* is *defined* on the basis of analysis of rather small amount of *situations (or more precisely, the features of exact objects)* serving as *motivational examples* for introduction of the selected notion. Through solving specific problems, in which both the *irrelevant features* of this concept are being varied and the comparison of the given concept with *proximate concepts* is done, the *selected concept is elaborated.* This process often ends in finding the so-called *counterexamples.*

What do we understand under the term *counterexample?* In mathematics, the statement in form *"Each element of the set (class) A belongs also to the set (class) B"* is very frequent. To prove the truth of this statement means to prove the validity of the inclusion $A \subset B$. To prove that the proposition cannot hold means to *find the counterexample*; i.e. to find such an element $x_0 \in A$, that $x_0 \notin B$. To illustrate this, let us present well-known statement usually proved in mathematical analysis courses: *"If the function is differentiable at the point (i.e. if the derivative of the function at this point exists), then the function is at this point continuous."* Quite naturally appears the question, whether also the converse is not valid, that is, whether the following statement is not true, too: *"If the function is continuous at the given point a, then it is differentiable at this point."* It is easy to verify that the function $f : f(x) = |x|$ is the counterexample for the mentioned statement as the function f is continuous at the point $a = 0$, but it does not have the derivative at a.

On the part of verification, the following statement is much more complicated: *"Any continuous function in the interval is differentiable at one point at least."* To find the counterexample for this statement means to prove the statement: *"There exists the function continuous in the interval* (a, b) *that does not have the derivative at any point of this interval."*

In university courses of mathematical analysis, the student usually learns that such functions really exist. The function constructed by German mathematician Karl Weierstrass (1815 – 1897) in 1872, or the one constructed already in year 1834 by Czech mathematician Bernard Bolzano (1781 – 1848) serve usually as the example functions of mentioned type. From the didactic point of view, interesting examples of functions were devised in 1903 by Japan mathematician Teiji Takagi (1875 – 1960) and in 1930 by *B. L. van der Waerden* (1903 – 1996).

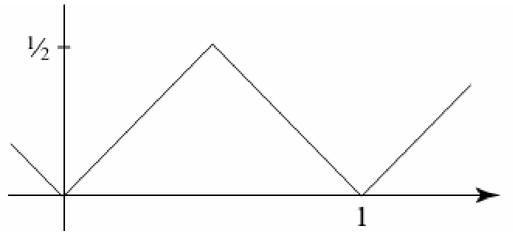


Fig. 3 Graph of function $y = f_1(x)$

The starting point for the construction of *Takagi function* (in English literature this function is usually referred to as *Blancmange function*) and *Waerden function* is the function mentioned above – the function $f_1(x) = |x|$ which is continuous at point $a = 0$, but is not differentiable at this point. Afterwards, the *partial function* to this function defined in the interval $\langle -\frac{1}{2}, \frac{1}{2} \rangle$

is taken and is periodically prolonged (with the period 1) throughout the whole numerical axis, i.e. we define $f_1(x+k) = f_1(x)$, for $k \in Z$ (see Fig. 3).

For $n > 1$ we lay $f_n(x) = 2^{-n+1} f_1(2^{n-1}x)$ in case of *Takagi function*, or $f_n(x) = 4^{-n+1} f_1(4^{n-1}x)$ in case of *Waerden function.* Finally, the demanded function f is defined as the *sum of the corresponding functional series:*

$$T\colon\ T(x) = \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{f_1(2^{n-1}x)}{2^{n-1}},$$
$$\text{or } W : W(x) = \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{f_1(4^{n-1}x)}{4^{n-1}}.$$

It is natural, that students are interested in how do the graphs of functions f_n and $S_n = f_1 + f_2 + \dots + f_n$ look like in the interval $\langle 0, 1\rangle$ at least for small values of n (all the stated functions are periodic with the period 1). To draw by hand the graphs of the functions f_n, S_n, f is certainly quite problematic, if even not impossible. However, it is not any problem at all with the use of suitable graph plotter (e.g. software *WinPlot*) or *Computer Algebra System* (e.g. *Maple* or *Mathematica*). Let us focus on *Takagi function.* With the use of *Maple*, programme of computer algebra, we draw the graphs $f_2, f_3, S_2, S_3, S_{20}$, and finally the *Takagi function* T itself.

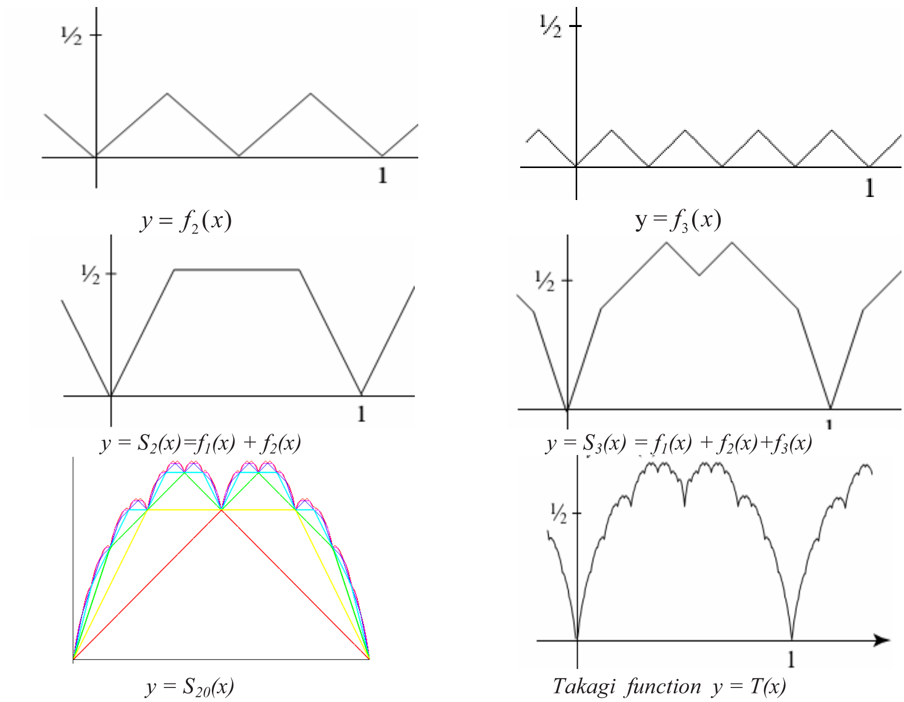


Fig. 4

Seeing the graphs of these functions persuades students that with growing value of n the number of "spires" in graphs of continuous functions f_n rises, and thus also in the graph of function T. These are the points in which the given functions do not have the derivative. After this inner assurance about the veracity of the stated hypothesis (i.e. that the function T is continuous and nowhere differentiable), we can realize its actual proof.

It can be proven easily that *Takagi function* is continuous. For all $n \in N$ are functions f_n *continuous* and for all $x \in (-\infty, \infty)$ holds $|f_n(x)| \leq \left(\frac{1}{2}\right)^n$, which according to the Weierstrass criterion means that the stated functional series is *uniform converges* in the interval $(-\infty, \infty)$, and thus the *Takagi function* is continuous.

We shall even prove, that *Takagi function* is not differentiable in any point $a \in \langle 0, 1\rangle$. This arbitrary number a can be apparently expressed in binary numeration system in the form

$a = (0,\ a_1a_2a_3...)$, for a_n equal either 1 or 0, where by $a = \sum\limits_{n=1}^{\infty} \frac{a_n}{2^n}$.

Define the sequence$(x_n)_{n=1}^{\infty}$, so that

$$x_n = \begin{cases} a - 1/2^n, & \text{if } a_n = 1, \\ a + 1/2^n, & \text{if } a_n = 0. \end{cases}$$

It is obvious, that $\lim\limits_{n\to\infty} x_n = a$, but $\lim\limits_{n\to\infty} \frac{T(x_n)-T(a)}{x_n - a}$ does not exist, consequently $f'(a)$does not exist, too.

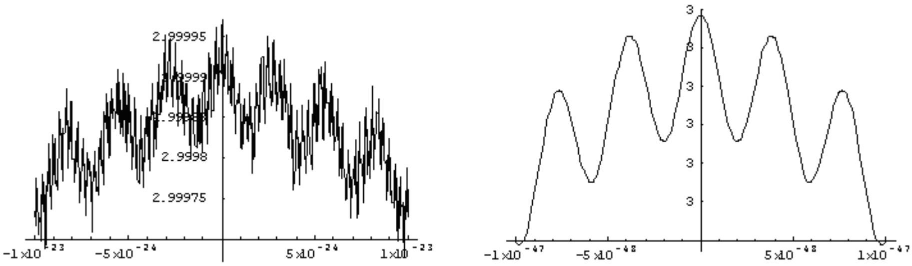
Now we return back to the *Weierstrass function.* This function (W_E) was defined by *Karl Weierstrass through the infinite trigonometric series:*

$$W_E : W_E(x) = \sum_{n=0}^{\infty} A^n \cos(B^n \pi x), \text{ for } A \in (0,1), \text{ odd } B \text{ and } AB > 1+3/2\pi.$$

Special case of this function is the function

$$W_{E1} : W_{E1}(x) = \sum_{n=0}^{\infty} (2/3)^n \cos(9^n \pi x) \qquad (*)$$

that we are going to study more closely. With the help of program system *Mathematica* we plot the graphs of partial sums S_{50} of the series (*). They approximate the *Weierstrass function* in the neighborhood of the point $x_0 = 0$ with appropriately high accuracy (it can be examined that $W_{E1}(0) = 3$). After several experiments with the length of these neighborhoods, the following graphic representations of the stated partial sums are obtained.



$y = S_{50}(x), x \in \langle -10^{-23}, 10^{-23} \rangle$ $\quad$ $y = S_{50}(x), x \in \langle -10^{-47}, 10^{-47} \rangle$

Fig. 5

The properties of the above mentioned graphs quite strongly support the hypothesis (but they do not prove it!) that the *Weierstrass function* does not have the derivative in any point of the setR. In incredibly small interval of length only 2.10^{-47}, the function S_{50} changes its monotony 11times, and thus also the number sign of the first derivative changes, whereby their amplitudes for the given k are constant. We note that it is different situation from the one induced by function

$$g : g(x) = \begin{cases} x^2 \sin(1/x), & \text{for } x \neq 0, \\ 0, \text{ for } x = 0, \end{cases}$$

the graph of which is plotted in the following figure.

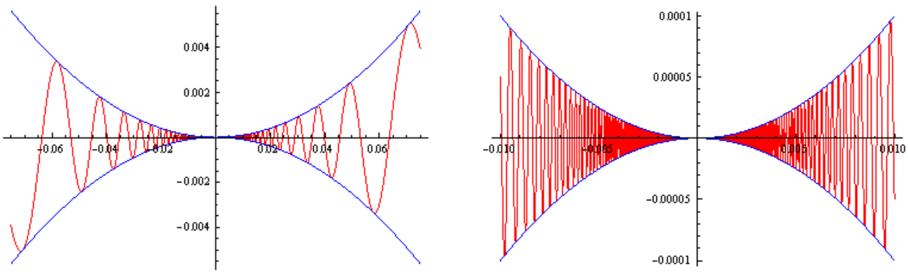


Fig. 6 *Graphs of function g*

The graph of this function in any neighborhood of the point $x_0 = 0$ has an infinite number of oscillations, and thus the *monotony and number sign of the first derivative changes infinitely times.* In spite of that, this function has the derivative at the point $x_0 = 0$,what can be proven analytically, too.

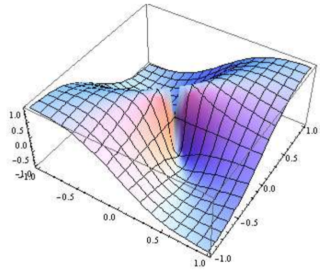
The geometric justification is obvious from the graph of this function. As can be realized from the graph of the function, its oscillations are realized only in between the graphs of functions $y = -\, x^2$, $y = x^2$, which have one tangent in common at the point (0,0) – line $y = 0$ (the gradient equals 0). Therefore the derivative g'(0) exists, whereby apparently g'(0) =0.

Another important field of mathematical analysis (especially in university education of intending mathematics teachers) in which it is possible to apply information technologies successfully is the *theory of function of more variables*, mainly of *two variables.* It is true, that the *graphs of functions of two variables* are not of that high importance as the *graphs of functions of one variable* in mathematics education. That relates to the historical fact, that except for few exceptions it has been problematic to plot the graphs of these functions by hand, therefore, this tool for simplification of ideas did not spread in textbooks that much. Nowadays, however, with help of suitable software, that is no problem at all. Moreover, the graph produced by the computer has another great advantage. Majority of programmes allow to experiment with the visualization of the graph of the function of one and two variables (it is possible to trace the graph details with the help of *zoom* for example, or in case of functions of two variables it is even possible to select the points for the perspective of looking at the graph, that is, to choose the standpoint of the examiner). If we consider, that the theory of *function of two variables* is, naturally, more complicated than the theory of *function of one variable* and if we realize, that within lectures students come across many *counterexamples* concerning various features of functions of two variables, whereby these facts are being proved only analytically, usually without geometrical interpretation and without graphs of these functions, it seems that we do not make use of all the possibilities available to make studies easier and for deeper understanding of stated problems. We suppose that the visual insight into the stated problem could make the knowledge more accessible and thus also more persisting. This can be done in accordance with the economy of education and its effectivity, without the increase of time and material demands on education with the use of computer with suitable software.

Let us present few well-known *counterexamples* from the theory of function of two variables together with their geometric representation, visualization with the use the software *Mathematica.*

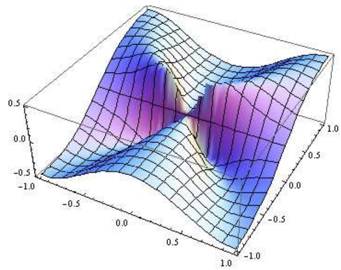
A non-differentiable function (a function without a limit at the point(0,0)) with partial derivatives everywhere ($f_x(0,0) = f_y(0,0) = 0$)

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$



A function without a limit (at the point (0,0)), although limits exist along all lines

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$



A function whose mixed partials are unequal ($f_{xy}(0,0) = -1, f_{yx}(0,0) = 1$)

$$f(x,y) = \begin{cases} \frac{xy^3-x^3y}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

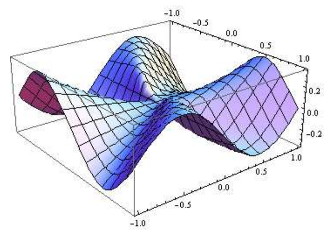


Fig. 7

6. Conclusions

The experimental teaching about derivative shows that much more effort is needed to help students to understand and to master the process of derivation. We have a positive experience with teaching derivation based on continuity. This way the process of derivation is visualised. Students practise graphs of functions and it is easier for them to decide whether the derivative of a function exists.

An objection might be raised against the above mentioned traditional procedure (with the exception of the initial motivation step) as to the fact that this procedure does not respect the principle of congruence between the phylogenesis and ontogenesis. The sequence limit is viewed here as a simpler basic concept, the mastering of which is necessary for the understanding the sum of the infinite series. All the same, for example Archimedes summed up the infinite series without having (and needing) the limit. To be more precise, he did not have the definition of the limit at his disposal – the limit process itself existed in similar lines of thoughts already before Archimedes (e.g. in Antifonos and his calculation of the area of a circle by means of gradual filling the circle with polygons (5^{th} century BC), or in Eudoxus and his exhaustion method (4^{th} century BC) – for details see e.g. Hischer & Scheid, 1995). In their lessons, however, students do not encounter a sufficient number of models of the limit process before they go into the infinite series, and therefore they lack something to follow. Moreover, the principle of genetic parallel is not a universal principle in maths teaching (see e.g. Marx 2006, p. 18), which is also a fact documented by e.g. Knoche and Wippermann (1986, p. 73) on the traditional procedure from differential calculus to integral calculus.

It is essential to realize, that the geometric approach to the new knowledge acquisition with the help of graphic objects and visual geometric conceptions could be a suitable complement of teaching, but not the only basis of *verbal-logical style,* which the teachers of mathematics (usually former successful students of mathematics at school) would instinctively prefer. However, this style need not be the most effective one. It must not be forgotten that the *visual assurance* of the truth of mathematical theorem leads students to the higher degree of trust not only to the achieved conviction about the accuracy of the given statement, but also to the higher degree of self-confidence of weaker students, who often lack the experience with *separated models of corresponding mathematical objects.* It is thus necessary to apply not only the *abstract*, but also *the visual (geometric) approach* within the process of revealing mathematical theorems in mathematical lessons. The excessive emphasis and continuous application of the *abstract approach* can make the pedagogical process uninteresting, ineffective and inoperative. On the other hand, exaggerated leaning upon the geometric visualization, upon the geometric interpretation of the stated fact, can *inhibit the need for the mathematical proof of the statement of students:* as from the figure it is *"obvious and clear".* With the acceptation of this approach the students could accustom themselves to superficial thinking of a type *"I see, thus I believe"* which is strange and sometimes even harmful for mathematics.

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THE APPLICATION OF THE THEORY OF PROBABILITY AND MATHEMATICAL STATISTICS IN PHYSICS

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Abstract. In physics and in natural sciences the term statistics is understood as an analysis of multiple phenomena which makes use of the theory of probability. It is used wherever the formation of objective conclusions based on experimental data is concerned. Mathematical statistics and probability have their noticeable use. The part of physics that uses mathematical statistics and probability for evaluation of physical experiments is called the experimental physics. A field of theoretical physics which cannot do without mathematical statistics and probability is statistical physics and quantum mechanics. It is a part of physics that studies the phenomena of macroscopic solids as complexes of a large number of particles.

ZDM Subject Classification: M50.

1. Introduction

Physics has always been and still is interested in the phenomena and regularities taking place in the nature regardless of the fact that these are the phenomena of the sky or the Earth. The phenomena and processes occurring regularly, and those that occur under the same manner in the same conditions, have always been and still are subject to particular attention.

The theoretical formulations of physical findings and theses are required to correspond with the experimentally assessed data. The physical laws accepted in such a manner are termed as *deterministic*. Based on the deterministic character of the physical laws it is possible, on the ground of the present status, to determine explicitly the future course and status of the phenomena and the processes, and to perform physical experiments based on that.

For research of these phenomena, the theory of probability in natural sciences, and thus in physics, too, utilizes mathematical methods, with the help of which it analyses and describes multiple phenomena. *Mathematical statistics* creates probability models for statistical data, it verifies correspondence of the data and models, estimates free parameters in the models, tests the hypothesis regarding them, analyses variability of the data, evaluates the correlation of the data, determines the errors, etc. It is used wherever objective conclusions from bulk experimental data are made. Theory of probability and mathematical statistics are an important part of the mathematical apparatus, without which the today's modern theoretical, experimental, but also applied physics cannot do (Hanisko, 2007).

2. Random Phenomena

Carrying out scientific studies of physical phenomena and processes, a special kind of phenomena and processes occurs, and those are named random. *Random phenomenon* is such a phenomenon, the result of which, in case of multiple repeating of the experiment, may gain a certain value, while this value can not be determined in advance (Ventceľová, 1973, s. 23). Such random phenomenon is then understood as a phenomenon, which at multiple repeating of the one and the same experiment, occurs in a different way, and leads to various possible results (Jurečková, Molnárová, 2005, s. 27).

In physics, randomness is viewed from two different aspects. On one hand, there is an *ontology* aspect, which attributes randomness to real phenomena and processes, and on the other hand *gnoseological* aspect, which attributes randomness only to the results of scientific learning, hence incompleteness of scientific findings (Staríček, 1988, s. 242). At the same time, it is assumed, that reality is influenced by deterministic laws. If randomness is attributed to the phenomena themselves, then the deterministic character of the physical laws must be regarded as statistical demonstration of a great number of random phenomena.

Physical theories of random phenomena are based on a great number of experiments, and they are analysed through methods of mathematical statistics. When there is a high number of experiments, it is possible to find certain regularities in their results, while among the individual experiments, certain regularities are observed. However, these regularities do not apply to one particular experiment; they apply to a whole series of experiments, which can be considered as one *statistical experiment*. The higher the number of experiments performed, the greater statistical regularity there is. Then we refer to it as *multiple phenomena* (Jurečková, Molnárová, 2005, s. 9) and regularities valid for great statistical sets.

To enable assessing multiple phenomena from the physical point of view, it is necessary to express the statistical relations mentioned above through numerical data, which is possible only through a suitable mathematical model, which is provided by mathematical statistics.

The whole science of physics is based on deterministic character of physical laws. The laws of classical physics, however, do not apply to microcosm. In such case, the categories of phenomena of macro-physical character, to which the laws of classical physics apply, are considered a special case of the broader category of the phenomena, studied by micro-physical methods. *Macroscopic phenomena* are in such case understood as demonstrations of sets of *micro-physical phenomena*. These sets are not, however, only simple multiples of their micro-physical elements, but rather their complex functions, which are not immediately measurable, nor they are quantifiable. In such cases statistical methods are of help, as they understand the contribution from one micro-physical phenomenon into a macro-physical set as a *statistical phenomenon* (Staríček, 1988, s. 242 - 243). Based on such statistical formulations of the macroscopic sets' characteristics and regularities, it is possible to derive the relations formally identical with the relations of the classical physics, which have the same physical content and meaning.

A great number of physical phenomena and processes indicate, that there might exist a few alternatives of some change or process. In such case it is possible, for the particular alternatives, to determine, in a suitable way, the probabilities.

It is possible to divide probability, on one hand, according to its determination based on some theoretical model – in such case we refer to it as probability *a priori*, and on the other hand, if the probability is determined experimentally – then we refer to it as probability *a posteriori*. If for a statistical model of a certain phenomenon, the probability a priori corresponds with the measured probability a posteriori, then in such case it is possible to regard the statistical model under consideration as a part of *statistical physical theory* (Staríček, 1988, s. 245).

3. Mathematical formulation of physical theories of random phenomena

Random experiment is mathematically characterised by a set of its possible results a_1, a_2, $\dots$, a_n. During an elementary experiment there always occurs one of the mentioned possible results, but during a multiple repeating of the same experiment, every result may occur few times. At the number of all the experiments n the i result occurs n_i–times. Then the fraction $\frac{n_i}{n}$, determines its *relative frequency*, which, at the sufficient number of experiments, is possible to be regarded as equal to its probability. The estimation of how great of a part out of the high number of the performed experiments (observations) gives an expected result, is understood as the probability of a certain result of an experiment (observation). We can talk about the probability of a certain event only if it is a result of some repeatable experiment (Feynman, Leighton, Sands, 2000, s. 78).

We can talk about the probability of particular possible results of the series of random experiments only if during repeating the whole series of n experiments, the relative frequencies remain stable, and if with the rising number of experiments they differ less and less. In case of unlimited growing number of experiments it is possible to assume, that relative frequencies of the particular results in all the series will be the same. In real situations, however, it is possible to perform only a limited number of experiments, but in case of sufficient number of experiments it is possible to identify the possible relative frequencies determined from one series of experiments with the corresponding probabilities.

The calculated probabilities are then assigned suitable symbols, for which there are defined adequate mathematical operations, so that through them it would be possible to calculate also the overall probability given by the sum of particular probabilities. It is possible to apply the mathematical structure established in such a way in different areas of physics, when particular symbols are assigned a physical meaning.

It is possible to characterise the statistical set of measured values as a whole, through suitable parameters, of which those that are most important for physics are *arithmetic mean*:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

where n is a range of the measured (observed) values of the set and x_i is a measured value of the i–th measuring, *variance*:

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

and *standard deviation*:

$$\sigma = \sqrt{\sigma^2},$$

which states, how much, in average in the given set of measured values, the particular measured values deviate from average value (Jurečková, Molnárová, 2005, s. 17 – 20).

In many cases, especially in case of broad set of experimental data, it is possible to perform the real measuring only using few elements forming a selection from a statistical set, and for this selection to determine a posteriori, i.e. based on the performed measuring, the selected arithmetic mean and the standard deviation of the selection. Through methods of mathematical statistics it is possible, from selection values, to determine for one and only selection the most probable values for the whole statistical set, and those are, in mathematical statistics, called *estimations*.

It is possible to use the estimations also when calculating the most probable division of the measured values in the whole statistical set, if the division is known for a particular selection. These most probable divisions usually have a form of rather simple mathematical functions. In physics there is mostly used a so called *normal* or *Gauss distribution of density of probability* (Feynman, Leighton, Sands, 2000, s. 87).

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{\frac{-x^2}{2\sigma^2}},$$

where *p(x)* is the density of probability and s is standard deviation. In physics, also other functions of distribution of probability density of the certain parameter values are also used. However, those must be theoretically justified, as well as experimentally confirmed. An example of this can be distribution of molecule velocity in ideal gas.

4. Statistical physics and quantum mechanics

Physical disciplines, which can not do without theory of probability and mathematical statistics, are *statistical physics* and *quantum mechanics*. The subject of statistical physics is searching for and explanation of the regularities and characteristics of macroscopic solids, as sets of a great number of particles (molecules, atoms, electrons, etc.) which comprise these solids; these occur in different forms (gases, liquids, solids, biological organisms of different forms and structures, etc.). The basic task of the statistical physics is the calculation of macroscopic parameters (pressure, temperature, magnetization, electric conductivity, etc.), which characterise the system based on the laws of the motion of particles, which comprise this system.

A great amount of particles comes into every macro-process. But what is important, is that the given macro-process does not depend on which particular particles are involved in the given process, but only on the mean number of these particles. Statistical physics deals with calculation of mean values of macroscopic parameters. This fact enables circumventing a complex problem of solution of motion equations of a great number of particles, and instead, using the calculation of probability and mathematical statistics. The higher the number of these particles of a given system is, the less sensitive the macro-status (macro-process) is to the changes in the behavior of a small number of particles. Therefore, it is natural that general laws of the macro-system have a character of statistical laws (Kvasnica, 1998, s. 11).

Character of the laws of macroscopic solids depends, to a great extent, on a possibility to describe the motion of particles by the laws of the classical or quantum mechanics. Statistical physics based on motion of particles, which are described by laws of classical mechanics, is called *classical statistical physics*. Statistical physics based on quantum mechanics description of the particles movement is called *quantum statistical physics* (Čulík, Noga, 1993, s. 11).

Although statistical physics takes into consideration the molecule structure of the studied objects, it enables to explain and to understand their microscopic origin, determine the borders of laws validity, and most of all it enables to determine the theoretical procedures and those macroscopic characteristics and relations (thermal, electrical, magnetic, state equations, etc.), which are derived based on experiments. It is possible to successfully use the statistical methods in studying equilibrium systems (reversible as well as irreversible). By statistical physics it is usually understood the part of physics, which deals with *reversible processes*. The statistical theory of irreversible processes is called *physical kinetics* (Kvasnica, 1998, s. 9).

The ideas of theory of probability are useful in studying and description of behavior of the order digit 10^{22} of gas molecules. With such number of particles, the experiment of determination of each molecule location or velocity would be inconceivable. Therefore, the theory of probability is necessary for description of phenomena and processes at the level of atoms. According to quantum mechanics, there always exists a certain inaccuracy when it comes to the determination of the particles' location or velocity. At the best it is possible to say, that there exists a certain probability, that the particle will be located near a certain point x (Feynman, Leighton, Sands, 2000, s. 89).

It is possible to determine the probability density $p_1(x)$, where $p_1(x)\Delta x$ represents the probability of a particle occurrence between x and $x + \Delta x$. In the same way, it is possible to determine also the velocity of a particle through probability density $p_2(v)$, while $p_2(v)\Delta v$ is the probability of the fact, that the velocity of the particle is from the interval v and $v + \Delta v$. (Feynman, Leighton, Sands, 2000, s. 89).

One of the basic results of *quantum mechanics* lies in the fact, that it is not possible to select the functions $p_1(x)$ and $p_2(v)$ independently. If Δx

and Δv express uncertainty of the location and velocity of a particle, then conjunction of these uncertainties should be greater than or equal to the number $\frac{h}{m}$, where m is the weight of the particle and h is *Planck constant.* It is possible to write down this relation in the form (Feynman, Leighton, Sands, 2000, s. 89):

$$\Delta x \cdot \Delta v \geq \frac{h}{m},$$

which is a mathematical expression of *Heisenberg Principle of Uncertainty.* The principle of uncertainty in general expresses the natural vagueness, which characterizes every attempt to describe natural phenomena and processes. Based on that, it is possible to state, that the most exact description of natural phenomena and processes is through probability.

5. Conclusion

Today's stage of natural sciences and technology development is characterized by wide usage and application of statistical methods in all areas of knowledge. It constantly receives new stimuli from the science and technology, it develops rapidly, and it is a mathematical discipline, which deals not only with solution of mathematical problems of theory of probability and mathematical statistics, but mainly their application in solution of practical problems of not only physical and technical character, but also in other, very often socio-scientific disciplines. This fact is quite natural, since when studying and exploring the phenomena and processes, it necessarily comes into a period, when it is needed not only to explain the basic laws, but also analyse their possible deviations. Today there is no natural science which would not use the methods of the theory of probability and mathematical statistics. Modern physics (especially statistical physics and quantum physics) is based on theory of probability. More and more often the probability and statistical methods are used in astronomy (stellar statistics), geophysics, meteorology, electronics, etc.

In its effort to as exhaustive knowledge of nature as possible, the modern physics came to the cognition, that certain things can not be known with certainty. Many of the findings will remain uncertain; at most it is possible to know their probability.

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MATHEMATICAL PREPARATION OF INCOMING UNIVERSITY STUDENT

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Abstract. Our research was performed in the form of an inquiry. Here opinions of teachers of mathematics of several universities, on preparation of students of mathematics were found out and investigated.

ZDM Subject Classification: B25.

Mathematics is a subject which can be studied separately; this study is oriented either to an extension of a mathematical horizon or to applications of mathematics. But mathematics is also a supporting and helpful tool by study of different disciplines, mainly at faculties of science and at faculties of a technical character. Students are contacted with mathematics during elementary and secondary schools in a various quantity and quality in dependence on different schools. My research wants to contribute to a creation of a bridge in teaching of mathematics between a secondary school and a university. In accordance with this basic goal I performed a pre-research in the form of an inquiry. In the inquiry, opinions and judgements of teachers of mathematics of several universities, on preparation of students of mathematics were found out and investigated; concretely, the inquiry consisted of five questions. Primarily, there were verified views and experiences of teachers, e.g., which parts of mathematics are to be emphasized at secondary schools and, specially, which types of exercises are to be solved by students in the first rank and which types of exercises are not well handled by students. Further, I wanted to determine, whether university teachers are familiar with secondary schools curricula and how, by their opinion, have the curricula modified with time. 28 respondents from Košice and Bratislava became involved in the pre-research, namely from

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First question: *Which part of mathematics of secondary schools is important for you, and to which depth? What would you suggest to add or to omit from these fields? What concrete types of exercises do you need to be handled and solved by students?*

Aim: The main aim of the question was to determine the areas of mathematics, which are necessary as a base for university mathematics. We expected that university teachers will consider as the most necessary mainly the areas concerning algebraic expressions, equations, inequalities and elementary functions.

Mostly mentioned areas: ◇ Simplifying of algebraic expression ◇ Operations with fractions (though it is to be known from elementary schools) ◇ Solving of equations and inequations; systems of linear equations ◇ Elementary functions, graphs (at least a linear and quadratic function) ◇ Verbal exercises ◇ Division of polynomials ◇ Number theory; divisibility ◇ Work with parameters ◇ Intersection of a system of intervals *Less occurring areas:* ◇ Complex numbers ◇ Analytic geometry in a plane ◇ Work with percents ◇ Logics, prepositional calculus ◇ Conic curves *Recommended to be added:* ◇ To emphasize a logical thinking in any part of mathematics ◇ More lessons of mathematics ◇ Verbal exercises connected with real situations of life *Recommended to be omitted:* ◇ Limits ◇ Derivation and integration ◇ Similarity and congruence transformations ◇ Solid geometry ◇ Combinatorics - either omit or teach differently

Conclusion: 1) From the mentioned fields which are for the university mathematics necessary, it follows that for many of university teachers, the knowledge from elementary schools would be quite satisfactory, but the knowledge handled on a solid level. 2) The assumption about the most necessary areas in mathematics which would be stated by university teachers, was confirmed. 3) More teachers emphasized an importance of a logical thinking; notice their expressions in connection with combinatorics. They do not consider combinatorics as the area absolutely necessary; but they say that here a logical thinking can by very well applied, though by a wrong teaching, there can be also a lot damaged. From experiences of teachers it follows that in combinatorics they most often see, that students do not think at all, they just look for an appropriate formula. Thus combinatorics is recommended either for a total omission or for some other way of teaching.

Second question: *What is your opinion to curricula of mathematics at secondary schools? Do you think that they contain everything, what a student needs entering a university?*

Aim: The main aim of the question was to find out an opinion of university teachers to curricula of mathematics at secondary schools. Also the goal was watched, to what degree university teachers know curricula at secondary schools, hence, if they know what is taught at secondary schools and to what extent. We treated as probable, that there exist also university teachers which are not familiar with curricula at secondary schools at all. In spite of this, the inquiry could lead them to get acquainted with the curricula. In this connection we detected the opinion of university teachers to the extent of mathematics at secondary schools, whether the curricula contain the parts of mathematics, which are necessary for an extension to the university level, and whether the number of lessons of mathematics at secondary school is adequate for having the theme sufficiently worked with and fixed.

Opinions: ◇ Sufficient content, deficiency mainly in the extent, subject too little worked with, too little absorbed; evidently a deficient number of lessons, therefore better to go through less things but properly ◇ Curricula contain everything, but a problem is that the subject can not be taught because of missing lessons (examinations, various actions) and also, students remember only a little ◇ Curricula too large, they implicitly force a student to learn things by memory; I am sad of missing of a usage of "horse-sense" and of a logical thinking ◇ Curricula are all right, though some themes could be changed to facultative; necessary to change the form of teaching (functions, geometry, combinatorics, statistics) ◇ Complex numbers, differential and integral calculus are taught at facultative seminars and students wanting to study mathematics or related subjects will probably choose the seminar

Conclusion: 1) Three respondents did not answer reasoning, that they do not know the curricula. 2) Two respondents used internet to find out the curricula. 3) Some teachers wrote that they are not familiar with the curricula in detail (did not know an extent of taught things), but they were acquainted with the scope; they presented their opinion to the scope. 4) Almost all opinions of respondents coincide in the fact that the content of curricula is in principle good, though there is a lack of lessons and this is the reason that some themes are deficiently worked with and fixed. It is possible to improve the situation by the help of facultative lesson of mathematics. 5) If the taught areas are to wide, then students do not think and they learn just by memory. The respondents agreed on the opinion that it would be convenient better to learn less of material, eventually change the method of teaching (this concerns mainly geometry and combinatorics).

Third question: *Which types of exercises should students know to solve from secondary school mathematics? (You can write concretely - e.g., goniometrical equations, or set a concrete exercise.)*

Aim: By giving the question we wanted to determine more concretely not only the areas, but also the types of exercises, eventually also the concrete exercises, which should be known by secondary school students.

Most often presented types of exercises: ⋄ Simplifying of algebraic expression; application of formulas ⋄ Work with functions and graphs (orientation in a graph, translation in a direction of axis); inverse function ⋄ Solving of linear, quadratic, goniometric, exponential and logarithmic equations and inequations; systems of linear equations ⋄ From an equation determine the type of a curve (line, circle, parabola, . . .); slope of a line ⋄ Students should know to solve the exercises by which it is necessary to think and not only to use an algorithm ⋄ Estimate at least an approximate result ⋄ Solve basic exercises from logics and make simple proofs ⋄ It would be satisfactory to know exercises which are listed in educational standards (on the page of National Institute for Education) ⋄ They should know that 1 divided by 2 is $1/2$ and that it equals 0,5 shown by a calculator, and that $\sqrt{-2}$ can not be counted by a calculator

Conclusion: 1) The answers imply that the areas mentioned by the first question are again emphasized. 2) The first time in the inquiry there appeared a remark of a calculator: on one hand, in connection with concrete computations, but on the other hand, that the students should know to estimate a result. 3) The most of respondents presented their opinions about algebraic expressions and functions, but nobody about exercises from geometry. It is worth to consider, whether geometry is such useless, or whether students are in such a way familiar with geometry that the respondents forgot to mention it. Maybe there should be taken account of an extent of necessity of geometry at secondary schools. 4) More teachers noticed not concrete exercises, but also such, by which it is necessary to think and not only to use some algorithm.

Fourth question: *With what have students in mathematics most problems and you have believed that from secondary schools they should it know to count, deduce, prove, draw,. . . ; thus, what students do not know from mathematics and in what way their unknowing is expressed? Conversely, which typical exercises are managed by students quite well and to which level?*

Aim: The aim was to determine more concretely, which exercises, operations, considerations cause difficulties for students and in which manner their unknowing is expressed. We wanted to verify our hypothesis, that the least problematical will be the exercises, by which an explicit algorithm of solving is given, in contrast to such, where it is necessary to analyze and then apply some logical judgement. Next we expected, that some parts of elementary school mathematics can also be problematical.

Appeared opinions: Problematical: ⋄ Operations with fractions ⋄ Simplifying of algebraic expressions - wrong used known formulas ⋄ Problems with graphs of functions ⋄ Omitting logical thinking, ability of making conclusions, ability to join information from different fields ⋄ Solving of verbal exercises; exercises dealing with percents; exercises which can not be explicitly classified ⋄ Students do not understand connections and usability of mathematics in practice ⋄ Algorithmic thinking - even when their algorithm works just in particular cases ⋄ For arbitrary a, the number a, is always positive and the number $(-a)$ is negative ⋄ They overvalue the role of calculators ⋄ Transcription - they are even unable to dictate from a book ⋄ Problems with a concentration, yielding mistakes *Students know:* ⋄ They handle well standard exercises, where it is enough to use a "recipe from a cookery-book" ⋄ Formulas, but not always correctly applied (e.g., combinatorics) ⋄ Formulas for areas of plane figures; for surfaces and volumes of solid bodies ⋄ Derivations; but for them to differentiate means just "write something to which an apostrophe is added"

Conclusion: 1) Teachers mentioned more the problems. 2) Students perfectly know formulas, but they do not apply them always correctly. 3) Again it was confirmed, that the taught things are not adequately exercised and fixed, because students have often problems with elementary school mathematics. 4) Students often try to apply some algorithm, even when it cannot be applied. 5) They do not understand connections and usability of mathematics in practice. 6) Without a calculator they are lost, they have no idea about an approximate estimate of a result.

Fifth question: *Compare, what was taught of mathematics at secondary schools at the time of your study and now.*

Aim: Since in the inquiry teachers of various age were spoken to, it was evident, that they were taught by different secondary schools curricula. Thus the aim of the question was to determine, whether and how secondary schools curricula was changing with time, e.g., whether the content and the extent of taught mathematics have been changed.

Appeared opinions: ⋄ The curricula were not changed notably ⋄ I met statistics just at the university, so I think that to teach statistics at secondary schools is needless ⋄ The content differs not much, the extent have been "thanks" to the humanization of our educational system substantially reduced ⋄ The difference between the taught things before and the taught things now is not significant. It is notable in the things which were really got up then and now ⋄ The difference of the content is not essential, but it isin what we had to know by memory or by a mental computation. ⋄ We had also derivations, integrals, complex numbers, not taught now; probably it is not necessary ⋄ We had a facultative descriptive geometry which in the present is going to be forgotten, though it well develops space imagination. ⋄ We had no theory of sets, no statistics, but we had more of analytic geometry

Conclusion: 1) Respondents characterized the parts taught before and not taught now: complex numbers, derivations and integrals, logarithms and counting by a calculating ruler; new is statistic and set theory. 3) Facultative subjects can help to become acquainted with some new parts of mathematics (in former times descriptive geometry, in the present statistics). But, mainly from time reasons, a lot of knowledge remains non-fixed. 4) In general, university teachers refuse contemporary trend to count everything by calculators, though mostly they consider calculators as a useful tool.

At the end, let me express the trust that the inquiry would help both to students and to university teachers.

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E-LEARNING IN BASICS OF STATISTICS TEACHING

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Abstract. The paper deals with using e-learning courses in Basics of Statistics teaching. The authors characterize the e-learning course named 'Opisná štatistika' which they prepared. The authors also summarize their experience from using this course in educational process.

ZDM Subject Classification: U75.

1. Introduction

Modern information and communication technologies, which caused significant changes in all fields of human activities, influence also educational process. These technologies brought unimaginable possibilities into education, because they enabled unlimited approach to information from any place at any time. It was Internet that enabled universities to offer modern information and knowledge from different subjects for 24 hours a day. A few years ago the universities distributed information on CDs or placed them on Internet,usually in a form of pdf or doc files. Nowadays, the universities start to use learning management systems, which enable not only the distribution of information, but also creation of virtual classes for students which are hundreds miles away from each other, synchronous and asynchronous communication via chat, discussion groups or internal mail. Moreover, LMS give to teachers or tutors priceless information about students' activities and their progress.

In Slovakia, faculties of education prepare students who will become teachers in few years time. It is very important for these students that the faculties use LMS and modern information technologies in education because the students can see benefits of them on themselves and so they will want to use them also in their future profession. (see for example [2], [3], [4], [5]).

2. Computer supported learning of Basics of Statistics

Similarly to other faculties, our faculty tries to use ICT in mathematics teaching. The authors of this paper have prepared and have been successfully using four e-learning courses in teaching process: Chosen Parts of the Graph Theory, Logic, Sets, and Binary Relations. Students can use these courses in two forms. The on-line versions are integrated in LMS and the off-line versions can be studied from CDs. Although preparation of e-learning course is much more difficult than the preparation of printed textbook, our experience with this method of teaching are positive, so we decided to prepare more e-learning courses. We try to utilize the possibilities of ICT that cannot be used in printed books, for example animations, videos, or interactive tasks.

We decided to prepare e-learning courses for the subject 'Basics of Statistics'. There were several reasons for it:

1. We used material for computer supported learning of this subject for three years and our experience were positive.
2. There are more than 100 students each year who have to master this subject either as compulsory or voluntary.
3. A great amount of our students need statistics for their bachelors works or dissertation thesis.
4. It is effective to use software for operating information. In our course we use Microsoft Excel.
5. During explanation it is effective to use videos of methods how to operate information in Excel.

The content of the subject can be divided into two groups: basics of statistics and hypothesis tests. The course 'Opisná štatistika' deals with the first group, while the course 'Testovanie štatistických hypotéz', which will have been finished by the end of 2008, deals with the second group. The content of the course 'Opisná štatistika' is:

1. presentation of information in form of table (frequency, frequency tabulation, relative frequency, cumulative frequency, interval frequency tabulation),

2. presentation of information in form of graph (frequency histogram, polygon, bar graph, pie chart),
3. mean values and data variability (arithmetic mean, geometric mean, harmonic mean, modus, median, quartile, percentile, average deviation, standard deviation, variance, coefficient of variation),
4. bivariate distribution (correlation, regression, correlation coefficient, regression line),
5. test.

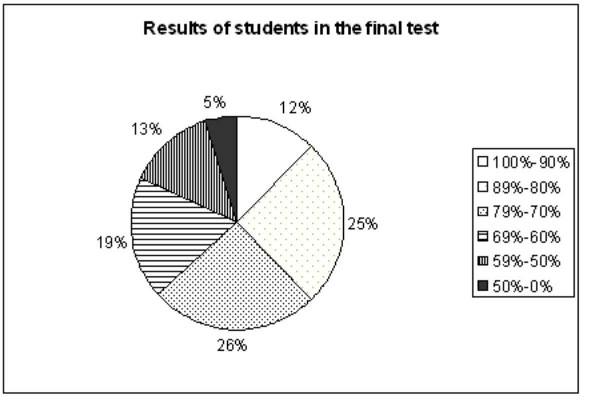
As we have already mentioned, we made an experiment focused on efficiency of computer supported learning of this subject. It was planned that students have 36 contact hours with a teacher, 24 of them in a form of lecture and 12 in a form of practice. Using computer supported learning, we reduced the number of contact hours with teachers to 12 for the students of present form of study and to 8 for students of distance form of study. The experiment lasted for 2 years and was carried out on 254 students from 4 groups:

1. 19 students of present form of Computer Science,
2. 84 students of distance form of Social Pedagogy (these students do not operate with extraordinary skills in working with computers),
3. 67 students of distance form of Computer Science,
4. 84 students of distance form of Social Pedagogy.

The results of the experiment were highly positive, as we can see from Table 1 and Graph 1. From 254 students only 13 did not pass the final test on the first attempt, but all of them were successful on the second attempt. The results proved that the students are able to fulfill the educational goals, although the number of contact hours was reduced to one third. The computer supported learning of 'Basics of statistics' is suitable mainly for students of distance form of study, who study and work at the same time so they are not able to attend a huge amount of contact hours.

Number of points	Group 1	Group 2	Group 3	Group 4	All groups
100-90	1	10	10	10	31
89-80	5	30	12	18	65
79-70	12	24	17	12	65
69-60	1	7	12	27	47
59-50	0	9	16	8	33
49-0	0	4	0	9	13

Table 1: The results of final test



Graph 1: The results of final test

3. E-learning course 'Opisná štatistika'

The results of the experiment persuaded us to prepare e-learning course 'Opisná štatistika' and to integrate it in our LMS. The integration of the course in LMS can offer much more benefits than the computer supported learning. For example, we can communicate with students via internal mail or discussion groups, monitor their progress by tasks and tests, monitor their activities with LMS and our course, or test students directly in LMS.

To lighten the study of the course, we tried to respect the knowledge about the way how people learn from electronic materials. We followed the rules published in [1], mainly:

1. We included both words and graphics.
2. We placed printed words next to corresponding graphics.
3. We avoided interesting but extraneous graphics.
4. We minimized lessons with extraneous words.
5. We used conversational style.
6. We tried to focus attention of students on important information.
7. We tried to use limited capacity of 'working memory' in a proper way.
8. We tried to integrate information from visual channel to existing structure of knowledge in a long-term memory.
9. We included support for students with less developed meta-cognitive skills.
10. We integrated problems into real situations and placed them to the whole course, not only at the end of the course.
11. We enabled students to replay videos as many times as necessary.

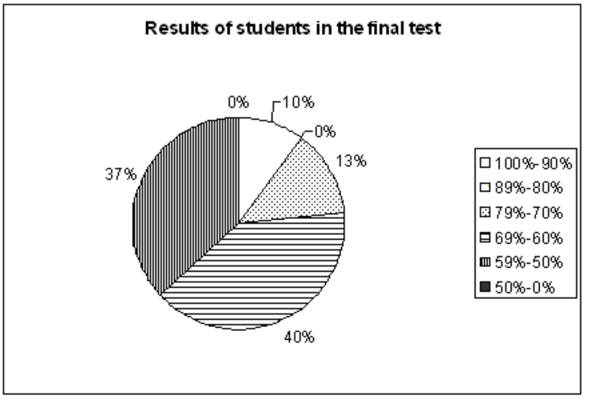
To get feedback about efficiency of the course in educational process, we made an experiment on a sample of 18 students of Computer Science and 12 students of Mathematics (bachelors study). The results of the experiment were positive, as we can see from Table 2 and Graph 2. From 30 students no one failed the final test on the first attempt. The results proved that the course can be considered as an useful material to study basics of statistics.

Number of points	Students of Computer Science	Students of Mathematics	Both groups
100-90	2	1	3
89-80	0	0	0
79-70	2	2	4
69-60	6	6	12
59-50	8	3	11

Table 2: The results of final test

4. Conclusion

We think that ICT can make educational process more effective and interesting to students. We believe that the faculties which prepare future mathematics teachers will cooperate in preparation of electronic courses and materials and that all mathematics curriculum will be covered by these materials.



Graph 2: The results of final test

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MATHEMATICAL COMPETITIONS AT PRIMARY SCHOOLS

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Abstract. The main aim of mathematical competitions is to develop pupils´ critical and logical thinking. It is supposed to have combinatory abilities and imagination to solve mathematical tasks successfully. The primary school pupils in the Czech Republic take part in different mathematical competitions. The international mathematical competition called "The mathematical kangaroo without frontiers" is aimed at winning the pupils´ interest in mathematics. At the same time the competition develops friendly relationships among children from all over the world. The mathematical tasks are amusing, smart and pleasant. "Mathematical Olympic Games" and "Pythagoras Games" are competitions for the pupils talented in mathematics. The pupils succeed in case they have deep theoretical knowledge of mathematics, logical thinking and combinatory abilities. The competitions are aimed at pupils´ creative work and the development of their personalities.

ZDM Subject Classification: D30.

Mathematics is a science having a large practical utilization in the daily life. Each profession in the modern society assumes not only sufficient skills in an activity with numbers but many professional aims require knowledge in logic and a require ability of abstraction at the problem solving. These types of abilities should be developed at students, pupils during the long period of their education in the elementary, basic, secondary schools and universities. The basic abilities should be done as soon as possible in the sufficient range so that the students could use their basic knowledge in their study of other scientific branches. The mathematical knowledge are used not only in physics, astronomy, chemistry and technical branches but more and more they are applied in other natural and social sciences as sociology, linguistics e.g.

That's the reason why to gain the interest of pupils and students in mathematics. It is a tradition to hold a mathematical competition every year and pupils and students of different age take part. The important mathematical competitions are: Pythagorean competition (Pythagoriáda), Mathematical Olympic Games (Matematická olympiáda), Mathematical Corresponding workshops (Matematické korespondenční semináře) and an international mathematical competition Mathematical kangaroo without frontiers (Matematický klokan bez hranic).

The Pythagorean competition and Mathematical Olympic Games (Pythagoriáda and Matematická olympidáda) are mathematical competitions and pupils and students with mathematical talent take part. Preparing for competition students and pupils learn new skills and gain new mathematical knowledge which are higher than knowledge prescribed by school curriculum. The competition trainees usually target the technical branches, physics, mathematics or natural sciences. At present the interest in technical branches or natural branches study is lower then our society needs.

To gain more students and pupils for mathematic and consequently technical and natural branches study we brought in on the preparation and organization of the international mathematical competition Mathematical kangaroo without frontiers. The competition has a funny feature and the main aim is not to learn the pupil something new but entertain him with interesting and not traditional tasks. The task has unusual mathematical content, theme and way of presentation. The tasks don't require extraordinary mathematical knowledge and skills very often but "only careful" reading of text, real judgement, observation, insight into task and experiment. The competition begin is connected with the Australian mathematician Peter O'Hallor. His intention was to gain for mathematics "all" pupils. He wanted to show to children, that mathematics is not boring, dull and dread school subject, that mathematic can give pleasure from the competition of not traditional tasks solving and there is lack of this task in traditional textbooks. His aim it to compare own knowledge with friends of the same age in class, in a school but in a region and in a country. The competition was held in the Czech Republic in the year 1995 for the first time.

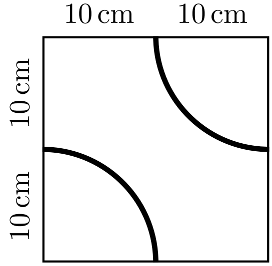
The competition has six categories: Cicada for the 2^{nd} and 3^{rd} class of the elementary schools, Kangaroo for the pupils of the 4^{th} and 5^{th} class, Benjamin for the 6^{th} and 7^{th} class of the elementary school, Cadet for the pupils of the 8^{th} and 9^{th} class. The category Junior is reserved for the 1^{st} and 2^{nd} class of the secondary school and Student for the 3^{rd} and 4^{th} class. The number of participants increases every year. The chart below shows the number of participant in the Czech Republic in the last years.

Mathematical kangaroo development in the Czech Republic from 1995 to 2007.

	Cicada	Kangaroo	Benjamin	Cadet	Junior	Student	Total
1995		6 205	7 834	7 280	2 195	1 297	24 811
1996		18 522	30 819	27 262	6 148	3 938	86 689
1997		61 161	59 314	51 769	8 631	7 349	188 224
1998		62 963	67 417	57 653	11 580	8 484	208 097
1999		87 885	79 717	73 578	16 847	6 606	264 633
2000		95 426	87 304	81 893	20 384	10 319	295 326
2001		93 434	86 458	78 408	20 173	11 228	289 701
2002		99 204	86 785	81 440	20 479	10 428	298 336
2003		83 584	74 112	65 839	19 615	9 879	253 029
2004		78 275	75 609	68 324	17 345	9 729	249 282
2005	11 076	70 886	72 090	69 425	18 333	10 690	252 500
2006	46 832	66 799	69 739	69 104	18 003	9 947	280 424
2007	60 744	70 705	66 840	71 491	17 804	10 274	297 858

At the end of the article we present a task from the category Cadet.

Problem 1.

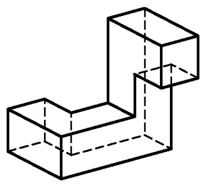


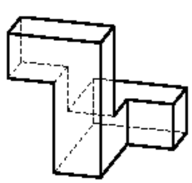
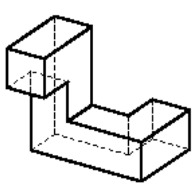
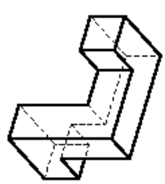
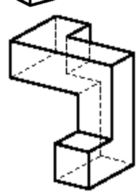
There is a tile on the picture, which dimension is 20 cm $\times$ 20 cm. We want to cover a surface, dimensions 80 cm $\times$ 80 cm with these tiles. On the surface the curve lines (a forth of the circular line) are connecting. At most, how long can be the connected curve line in cm?

A) 75π B) 100π C) 105π D) 110π E) 525π

Problem 2.

Which of the following objects can be created by rotating the given object in space?



W  *X*  *Y*  *Z* 

A) W and Y B) X and Z C) only Y D) none of these E) W,X and Y

These types of the tasks should be submitted not only into the mathematical competitions but in a common mathematical training. Searching for the correct task solution the students and pupil learn to solve the scheduled problem in the creative way and practice their stereo metric imagination. Both skills are the backgrounds for the study of the technical branches. The acquired skills could be used in no technical branches, too.

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ROLE OF LANGUAGE – REASONNING OR JUDGEMENT AMONG 4 TO 7 YEARS OLD CHILDREN

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Abstract. Children - pupils from 5 to 11 years old can solve differents problems. Which role can play the used language in the solving process?

ZDM Subject Classification: C50.

Introduction

Between the age of 4 and 7 years the child's mode of perception changes as well as his way of thinking. The most sizable development of vocabulary is already over, the quality of the language used is progressing, the understanding of the meaning of words is deepening, speech is used for linking different intellectual operations with the surrounding world. Although mixed communication is still predominating, the importance of vocabulary in communication starts to play a growing role. Let us observe the changes occurring in child's judgement, namely to what extent before entering school and during the first year of school attendance the child's vocabulary reflects changes in the process and the quality of his train of thought. In our experiment these changes were observed during two different activities (Sudoku and Zebra type problems).

1. Theoretical starting points

During pre-school age and at the beginning of primary school attendance, the child's way of thinking undergoes a change.

1.1 Train of thought

"Understood as a specific activity of the brain. It concerns the treatment and perception of information stored in memory." Kaslová (1)

”**The train of thought** considered as cognitive process between man and his environment...
a) <u>conscious</u>, man-controlled expressed by images, pictures, signs, i.e. words,
b) <u>unconscious</u>, uncontrollable by consciousness, taking place on the same neuro-physiologic level but through different cerebral mechanisms without using of words, it is difficult to generalise it as a basic intellectual operation.”

Hartl, Hartlová

”**3–6 years** ... here ends the phase of symbolic and pre-conceptual thinking, thinking is still pre-logical i.e. preoperational (it is still linked up with a concrete activity and child’s activity), ... it is egocentric, it lacks differentiation between the objective world and his own world of images and thoughts.” Šulová

Train of thought and its selected sorts (according to Hartl, Arkinson and Kern) applied in intellectual operations among 6 or 7 years old children:

ANALOGIC, ASSOCIATIVE, DIFFUSE (dispersed, chaotic, disconnected, intuitive, replacing logical judgment by sensitively occurring and penetrating contents), DISCURSIVE logico-deductive, gradually knowledge and conclusions are acquired by judgement, IMAGINATIVE conception, mainly visual are part of judgement, INDUCTIVE generalising, deducing by judgement a general opinion, LOGICAL judging according to the laws of logic, CONVERGENT, PRE-NOTIONAL - substituting symbolically reality, CATATIMIC (train of thought distorted under emotional influence, CONCRETE, WISHFUL - not treating information given by reality but by wish, projection, GRAPHICALLY DESCRIBED on the basis of sensual perception of the images perceived (not conceptions) - uniqueness merges with generality, MANIPULTTATIVE sort of concrete thought, MOTORICAL, SENSO-MOTORICAL. To a certain degree the choice of train of thought more frequently used is an individual matter and depends on many factors. In spite of this fact we can observe changes in the given age group.

1.2 Intellectual operations – reasoning and judgement

”Intellectual operation – psychological process unascertainable by outside look, its essence being **working with information**. Reasoning is one of the personality’s intelligence manifestations.” Lafont

”**Reasoning** is a type of thinking which accepts the existence of possibilities, compares and evaluates them according to personal and objective criteria.” Kaslová

”**Judgement** is a mode of using intellectual operations.” Hartl

”Operations which suppose a certain number of (truthful) information (premises) and deduce there from a new information (conclusions) on the basis of structures linked with language and using constructive memory.” Atkinson, Weil-Barais

Among **logical activities** in the framework of intellectual operations we have: deduction, induction, negation, conjunction, ... passing judgement, verification, exclusion or selection according to chosen criteria, use of probability in decision making.

1.3 Activities stimulating reasoning and judgement

SPHERE A Creation of images, memory consolidation, recalling
SPHERE B Development of language, auditory analysis of a longer whole
SPHERE C Problem s of wholes and their parts, hierarchy of part
SPHERE D Accepting the existence of possibilities

1.4 Activities testing reasoning and judgement

It is unnecessary to demarcate Sudoku. **Reduced Sudoku** 4×4 squares was chosen for the experiment. Squares were differentiated one from the other by:

a) 4 colours, b) 4 signs. Solution: a) by manipulation – putting on or pasting (objects, slips of paper with pictures), b) graphically – by colouring, drawing.

Zebras are problems where, with the help of n-groups the child has to build n-groups where every element must become precisely one n-group. One has to take as starting point the given relations and casually or by reasoning he obtains a composition of n-time – in our case threes.

Example: Do you know where he/she sleeps and what he/she eats?

The cat (K) likes cheese. The dog (P) sleeps on a red pillow. The mouse (M) does not sleep on the yellow pillow. The dog does not eat the chocolate cake.

Solution: a) by manipulation, b) by kinetics (dramatisation bring one possibility)

2. Experiment

2.1 Questions

a) Do children use strategies (when solving Sudoku or Zebra problems)?
b) Is the pre-school child able to find the solution? How?
c) In what conditions?

d) Do children use reasoning, judgement?
e) Is there a difference in solution process between pre-school and school children?
f) What is the role of language? Namely that of the level of verbal communication?

2.2 Conditions of the experiment

- 40 children from 2 kindergartens and 2 primary schools
- Individual work, unlimited time
- 8 Sudoku (colour/ objects, drawings/symbols)
- 4 ZEBRA (traditional verbal problems with similar logical structure)

2.3 Used strategies

Using individual strategies is repetition of procedures giving the child a better chance of success or even clear success, it takes as basis the evaluation of experience acquired up to now knowing that there exist several possibilities of solution; a mature strategy is linked with a knowing evaluation of all possibilities and verification of the advantage of the choice made and of the sequence of the different steps.

The strategies have their own evolution depending on the development of the individual's aptitudes, experience with the given type of problem and the appropriate climate for solving in relation with the child's personal specificities.

2.3.1. SUDOKU

H – linear – horizontal
V – linear – vertical
R – rectangular
C – colour
S – symbol
P – words

- LV – linear – vertical
- LH – linear – horizontal
- CO – square / rectangular
- B – one colour / one symbol
- K – combination of both preceding items
- SD – direction upwards
- SH – direction downwards
- SPL/SLP – direction right-left / left right

2.3.2 ZEBRA

BPP – presentation of the couple which is defined by "a verb without negation" (for ex. The dog sleeps on a striped cushion)
H – guesses, does not cogitate, the choice of a couple (trio) according to what he likes best
A – orientation towards animals, persons no matter if the couple is clearly specified
L – orientated towards a certain place in space (dog's kennel, cushion, etc.)
E – exclusion method (in the set the verb is in the negative form)
T – orientation towards creating a couple linked with attributes of nouns (linked with emotions – sweet, red...)
O – creation of a couple follows the sequence of given information although this may not lead to a definite choice (order)
K – combination

2.4 Verbal accompaniment to SUDOKU

- *Then here* (guesses the child deciding quickly and wrongly, the choice fulfils only 2 conditions)
- *Blue may be... Probably here* (the child hesitates, decides between 2 possibilities, the choice neglects the third condition)
- *Either blue or yellow The blue most probably here or ... I am not sure* (The child makes up his mind slowly but correctly, indicating with his hand the different possibilities and waves his hand above the surface, chooses the place hesitatingly)
- *The yellow must be here it can't be anywhere else. The red should be here because ...* Decision-making takes a relatively long time. The child's eyes wonder about the surface, he then talks, a silence followed by the correct choice. At first the gesture is hesitating but then becomes energetic – sure.
- *They must be all ... one is missing here and here also ...* (a hint reminder of the rules, evaluation of the possibilities)
- *If there is a cloud here and the sun there, then we must put ...* He observes the surface as a whole, his eyes moving about the surface, from time to time, he stops, makes comments about the free squares and the occupied ones, he consider choice as a necessity without alternative
- *This symbol (this colour) could be here ...*
- *This symbol (this colour) could be here probably ... or there ... I don't know exactly*
- *This symbol (this colour) must be here ... because ...*
- *If this symbol is here and that one there ... then ... it is necessary ...*

3. Conclusion

The experiment showed that children of that age group in gaining the aptitude of auditory analysis, of auditory memory and of respecting the conditions corresponding to a given activity are able to come to a solution **they are convinced is right**.

Starting with a relatively short period of guessing they pass on to the phase of experiment – error where using auto-correction they become more successful if they remember the rules. Once the child registers all the possible solutions and begins to compare them with the rules/conditions of solution, he then starts to speculate, to evaluate the possibilities but even that is no guarantee that his decision was correct. As the child is more able to describe verbally the given disputation, the context crystallises permitting to pass wilfully from know information to new information. For 89% of the children this means the beginning of use of conditional sentences or they at least get a hint of them. This sudden change is apparent first of all when the conditions of solution are made more difficult. The simplest mode of solution for the child is manipulation where the whole situation holds in one single field of vision. When the space for solution becomes larger, language comes to play an important role in way of decision-making

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HOW TO DIFFERENTIATE?

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Abstract. The different possibilities how to introduce the concept of derivative are dealt in the contribution. The infinitesimal approach of both founders of calculus, Newton and Leibniz, is compared with the present way of introducing the concept. Some attempts to follow or revive the ideas of Newton and Leibniz in Czech and Slovak textbooks are mentioned.

ZDM Subject Classification: I44.

1. Introduction

The concept of derivative is one of the most important concepts of mathematics and it is also a part of any essential course of calculus at high school or university. If we study the approach of both founders of calculus, Leibniz and Newton to this concept, we can find the present way of introducing a derivative at high school or university is usually different from their ideas. A teacher starts with the conception (which is accompanied by a traditional figure) of a tangent line to the graph of a function as a limit location of a secant (by Leibniz) or with the conception of an instantaneous velocity as a special case of average speed for the time difference $t - t_0$ in the formula $v = \frac{s-s_0}{t-t_0}$ tending to zero (by Newton). In spite of this first infinitesimal motivation according to ideas of both founders of calculus the further step consists usually in the definition of derivative in the point x_0 by means of the formula

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}, \tag{1}$$

when the concept of limit was introduced by the $\varepsilon - \delta$ definition before. While Newton and Leibniz drew up their calculus on the basis of infinitesimal calculations, the present school conception leaves very quickly the infinitesimal approach. Let us cite Petr Vopěnka from [9]:

"*The rejecting of Newton and Leibniz conception of infinitesimal calculus by the mathematicians in the 19th and 20th centuries was one of the biggest mistakes not only of mathematics but also of the European science at all. The rejection was caused or by their unwillingness or by the incapability to speculate about and work with the essential concepts that were the building stones of the original conception of calculus.*"

It is also our belief that school mathematics should reflect more the phylogeny of the concept - this approach is for students acceptable and more understandable. Our research in the field of functional thinking of Czech students acknowledges that the ontogeny of the concept evokes the phylogeny and verifies the idea of the onto-phylo genetic parallelism of concept formation.

2. Historical conception of derivative

Let us briefly remind the principles of Newton and Leibniz conception of derivative. Newton (1643-1727) investigated the mechanical motion and he worked out the methods of so called *fluxions* and *fluents* on the base of a motion. The motion of the mass point in a plane splits by him into two straight-line motions called *fluents*. A fluent is such a variable quantity corresponding with the motion-line which is continuously increasing in the dependence on time. The velocities of both fluents x, y are *fluxions* $\dot{x}(t), \dot{y}(t)$. The motion of the point is then described by some relationship between both fluents x, y: $f(x, y) = 0$. If we take this motion-line as a graph of some function, we can find the function f (however Newton did not use the concept of function) is given implicitly. The infinitesimal increments Newton calls *moments*; a time moment denoted o is here especially important. Newton solved then two essential problems of his theory; 1) from the relationship between fluents

$$f(x, y) = 0 \tag{2}$$

he looked for the relationship between fluxions

$$\frac{\dot{y}}{\dot{x}} = F(x, y) \tag{3}$$

(a task of differential calculus seen by present eyes) and reversely 2) he looked for a relation between fluents if the relationship between fluxions is known (problem of integral calculus). The context of his ideas is one thing but the main strength of his considerations, we would like to remind here, was his intuition and certainty with that he neglected infinitesimal quantities in an appropriate moment. The ideas were worked up in the work

Methods of series and fluxions from 1671-1672 (the work was published after his death in 1736 the first time).

Let us remind also Leibniz approach to the concept of derivative. Leibniz (1646-1716) used the method of neglecting infinitesimal quantities, too. He used the context of looking for a tangent line to the curve; his considerations are connected with so called *characteristic triangle*. Leibniz generalized the idea of the characteristic triangle by Pascal (it is a little different from the conception of characteristic triangle today). His ideas were worked up in the work *Nova methodus pro maximis et minimis* from 1684. Leibniz took the curve as a broken line created from infinity number of small segments with amplitudes

$$ds = \sqrt{(dx)^2 + (dy)^2} \tag{4}$$

and the tangent line then as an elongated side of an infinity polygon. For more examples of the considerations and infinitesimal calculations by Newton and Leibniz see e.g. [1], [2].

3. Derivative in Czech school of the 19th century

The concept of derivative and other concepts of calculus begun to penetrate into school mathematics in the Central Europe after the meeting of German scientists in Merano in 1905. Felix Klein (1849-1925) promoted there so called *functional thinking* as an axis of the school mathematics. The elements of calculus became the part of the Austrian school mathematics after the Marchet reform of school curriculum in 1909. Apart from this there was an earlier attempt to present calculus in Czech school mathematics. This is connected with the remarkable personality of Václav Šimerka (1819-1887), chaplain, teacher of mathematics and physics and individual philosopher. Šimerka wrote in 1863 an appendix to a textbook of algebra (*Algebra čili počtářství obecné* - Algebra or general calculations) which was published in 1864 as a separate work ([8]). The main concept of differential calculus is by Šimerka a concept of differential based on intuitive infinitesimal calculations. Differential (notated by δ) is presented here as a very (infinite) small increment of a continuous variable ("*quantity between zero and the smallest fractions we can imagine in practical calculus*"). Šimerka supposed the differential df is an approximation of a real increment of a function $f(x_0 + h) - f(x_0)$ given by the formula (in original notation)

$$\delta y = f(x + \delta x) - fx \tag{5}$$

Many other formulas (e.g. differential of a sum, product, ratio etc.) are derived intuitive way under the presumption that the product of two infinite small quantities is possible to neglect in the comparison with each of them. The concept of derivative is then derived as a derived operation (literally "*odvozený úkon*") by means of the ratio $\frac{\delta y}{\delta x}$.

4. Calculus infinitesimalis of P. Vopěnka

There were some attempts how to revive the infinitesimal calculus in the intentions of Newton and Leibniz in modern mathematics. P. Vopěnka (1935 -) tried to follow the infinitesimal approach in his book *Calculus infinitesimalis* in 1996. Vopěnka operates in so called *geometrical world* which he splits up into *natural* ($\mathcal{N}$) and *classical* ($\mathcal{C}$) one. The difference between both worlds consists in it that the classical world includes also the *classical infinity* which does not exist in the natural world. The important step in this theory is defining of a binary relation *infinity closeness*; two real numbers x, y are mutually infinity close ($x \doteq y$) if the number $x - y$ is infinity small. The concept of derivative of a function f in a point c is then defined as such number $f'(c)$ for which for any infinity small α

$$f'(c) \doteq \frac{f(c+\alpha) - f(c)}{\alpha} \tag{6}$$

is holding. The traditional theorems (included the theorem about derivative of a composed function) are proved in the textbook by means of the definition (6); nowhere the concept of limit is used in this context. The limit and also the elementary functions are explained further - in correspondence with a history of mathematics. Vopěnka shows advantages of the infinitesimal calculus in the comparison with the "modern" calculus (beginning by the personalities of D'Alambert (1717-1783) and especially by Weierstrass (1815-1897) and his $\varepsilon - \delta$ definition of limit).

5. Kluvánek and his derivative

We can find other very interesting conception of derivative corresponding with the historical development of the concept at Igor Kluvánek (1931-1993). The infinitesimal ideas are described in his book *Diferenciálny počet funkcie jednej reálnej premennej* which was edited by the colleagues at Faculty of Education in Ružomberok in 2007. Kluvánek introduced first differentiability of an affine function f for which the following is holding

$$f(x+u) - f(x) = \lambda \cdot u \tag{7}$$

for some coefficient λ representing the rate of a variation of the function f. A general function (not only an affine one) is then substituted by an appropriate affine function in a neighborhood of a point x. The function f is called differentiable at the point x if exists such function φ continuous in the point 0 for which

$$f(x+u) - f(x) = \varphi(u) \cdot u \tag{8}$$

holds. The sense of differentiating consists in the possibility to express the increment of function $f(x+u) - f(x)$ as a product $\varphi(u) \cdot u$ for any u from the neighborhood of 0. The function

$$\varphi : u \to \frac{f(x+u) - f(x)}{u} \tag{9}$$

is not naturally defined for $u = 0$ but under some circumstances it is possible to complete it by such value $\varphi(0)$ to be continuous in this point. If such completing exists we say function f has the derivative in a point x. The concept of derivative is this way introduced naturally without concept of a limit (a derivative is here "derived" concept similar to Šimerka - the main is a differentiability of a function). The relationship

$$f'(x) = \lim_{u \to 0} \frac{f(x+u) - f(x)}{u} \tag{10}$$

presented today in textbooks usually as a definition of a derivative is for Kluvánek a natural consequence of the previous theory.

Let us show e.g. the derivative of a function $f : x \to x^2$ in the point x following the above ideas. The task here is to find $\varphi(0)$ for such continuous function φ satisfying (8). For this function i.e.:

$$(x+u)^2 - x^2 = 2xu + u^2 = (2x+u) \cdot u. \tag{11}$$

It means $\varphi(u) = 2x + u$ and then f is differentiable at the point x; value of derivative in this point is (as we have expected) $\varphi(0) = 2x$.

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PROBLEM-SOLVING STRATEGIES USING INVARIANTS OF TRANSFORMATIONS

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Abstract. There are many strategies which mathematicians use when solving problems. Most of these strategies can be used not only for attacking mathematical problems but also when we solve problems outside mathematics – in physics, chemistry, biology, geography, and many other disciplines. However, here we will discuss a particular strategy which at present seems suitable only for mathematical problems. It is the strategy of *Using Invariants of Transformations.*

ZDM Subject Classification: D50, C30.

If we wish our students to experience what mathematics is, then we should solve problems with them. The most appropriate problems are those that students find interesting, challenging, and acceptably difficult. The solutions should involve mathematics with which they are familiar. Nonetheless, we must always encourage students if they attempt unusual and creative solutions.

Let's see what role problems play in the science of mathematics? We know, for example, that Euler became famous "overnight", when he solved the Basilay problem: what is the sum of the infinite series $1/1^2 + 1/2^2 + 1/3^2 + ...$? This story incident links closely with our question, which we slightly reformulate. What does mathematics consist of? Are these axioms?, Theorems?, Proofs?, Clever ideas?, Notions?, Definitions?, Theories?, Formulas?, Methods? Of course, without these components mathematics cannot exists. It is just a sack full of tricks and rules. But *the main reason* *for existence of mathematics is to investigate problems and their solutions* (American mathematician Paul Halmos). Many famous problems remained unsolved for centuries and there are still many more for which a solution is still unknown. However, the efforts made by mathematicians to solve problems enriches mathematics even when no solution is found. We need to recognise that the same characteristics of mathematics in searching for solutions to problems need to be a fundamental part of school mathematics.

Mathematics is a creative art as well as a rigorous science. We must do mathematics with our students at school in such a way that the creative aspect of mathematics is always to the fore, both in our minds and in that of our students.

We define a problem as a situation in which neither the solution nor the method of solution is readily apparent. One‘s level of mathematical experience will influence whether or not a particular situation is a problem or merely an exercise.

In a problem-solving situation, a student has a goal to achieve but may not have the means to achieve it immediately. Solving the problem consists of constructing or discovering the means. This process sometimes involves unanswered questions, false starts and dead ends. Indeed, any of these can provide ideas for solving the given problem and perhaps other problems. One of the joys of problem-solving is that we often discover new problems to solve as we proceed. Strategies are tools for discovering or constructing means to the goal. Often, a problem can be solved in more than one way.

There are many strategies which mathematicians use when solving problems. For example:

Investigative strategies:	Trial and error
	Guess-check-revise
	Systematic experimentation
Other strategies:	Reasoning by analogy
	Reducing to a simpler case
	Generalizing
	Concretizing
	Working backward
	Examining a related problem
	Drawing a sketch or diagram (geometric approach)
	Writing an equation (algebraic approach)
	Identifying a subgoal
	Reformulating the problem

Most of these strategies can be used not only for attacking mathematical problems but also when we solve problems outside mathematics – in physics, chemistry, biology, geography, and many other disciplines. However, here we will discuss a particular strategy which at present seems suitable only for mathematical problems. It is the strategy of *Using Invariants of Transformations.* We begin with some general statements about this strategy and then we will give three examples of its use.

Given a problem, if we want to use this strategy, we must:

a) Find out some mathematical transformation related to the problem.
b) Isolate the properties of the objects involved in the problem.
c) Find out which property or properties of the objects remain invariant under this transformation.
d) Apply these invariant properties.

A simple early example at the elementary school level requires students to investigate the problem of finding all additions of two whole numbers which have a sum of n. Students choose particular values of n, say 8, listing all those they can find and order them so that the properties of the additions becomes much clearer.

$$\begin{gathered}7+1=8\\6+2=8\\5+3=8\\4+4=8\\3+5=8\\2+6=8\\1+7=8\end{gathered}$$

The properties are the first and second numbers in each addition, the variants, and the third property the sum of each addition, the invariant. Altogether there are 7 additions when the sum is 8. Similar investigations with different values of the sum leads students to the conclusion that the number of additions with a sum of n is $n-1$.

Now we jump to three problems appropriate for secondary students. Our first problem is not a difficult one, but its solution will provide a good demonstration of our strategy[6].

Problem 1: A ladder has length 6 metres and stands on a horizontal floor, leaning against a vertical wall. The base of the ladder slides away from the wall. Describe the path of the ladder's midpoint.

[6]This problem is taken from [1], but it is a little changed.

Solution: If we have enough geometric intuition (which we can gain only with experience), we can determine the path at once. If not, we can draw several diagrams, showing the position of the ladder at different times. A typical situation is shown in Fig. 1. In this moment we use the strategies drawing a sketech and experimentation. The position of midpoint M depends on the amount of time t that has elapsed since the ladder began to slide. For this reason, we denote the position of the midpoint at time t by M_t . To every time t we assign the point M_t; this is our transformation, changing from time to position. One important property of this transformation is the distance of the point M_t from the point $O = [0, 0]$ (see Fig. 2). So we can investigate this distance. After some experimentation or directly, we can make the following conjecture.

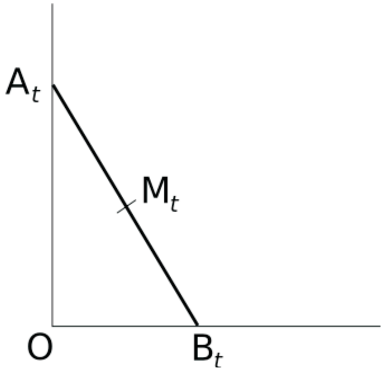

Fig. 1

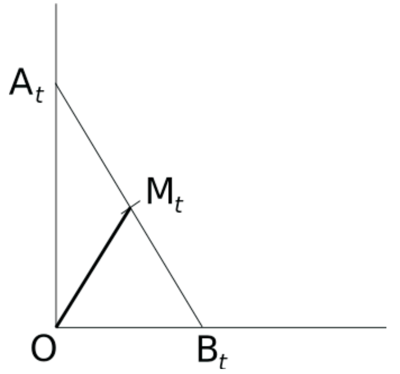

Fig. 2

Conjecture: M describes an arc of the circle with centre O and radius 3 metres.

We claim that the distance from M_t to O is half the length of the ladder, namely 3 metres Why?

If $A_t = [0, y_t]$, $B_t = [x_t, 0]$ then midpoint $M_t = [x_t/2, y_t/2]$. The distance from M_t to O is

$$|OM_t| = \sqrt{(x_t/2)^2 + (y_t/2)^2} = \frac{1}{2}\sqrt{x_t^2 + y_t^2} = \frac{1}{2} \cdot 6 = 3.$$

Since this distance does not change during transformation, we can say, that this distance is *an invariant of the transformation* $T : t \mapsto M_t$, that is, $|OM_t| = 3$ for all t.

Thus, we have shown that our conjecture is true and so we can state it as a (simple) mathematical theorem.

Solution of Problem 1: M describes an arc of the circle with centre O and radius 3 metres.

Our next problem can be presented as one in "zoology" and was published in Kvant [3][7].

Problem 2: There are 48 chameleons on an island; 20 of them are yellow, 15 are grey and 13 are blue. The chameleons wander around, meeting occasionally. The chameleons only meet in pairs. If two chameleons of the same colour meet, their colours remain unchanged. If two chameleons of different colours meet, both change their colours into the third colour (e.g. if a blue chameleon meets a yellow one, they both change their colour to grey).

Could it ever happen that at some time all chameleons on the island have the same colour?

Solution: We can start with experimentation. Every "colour situation of chameleons" can be described as an ordered triple of integers, where in the first position we write the number of yellow chameleons, in the second position the number of grey, and in third position the number of blue chameleons.

Thus, we can describe our starting situation as (20, 15, 13). If, for example, on the first meeting, a yellow chameleon meets a grey one, the situation becomes (20-1, 15-1, 13+2) = (19, 14, 15). Now we see that if it is possible to have all chamelons the same colour, there must be a sequence of meetings where the final situation is either (48, 0, 0) or (0, 48, 0) or (0, 0, 48). If we experiment, either with a plan in mind, or just at random, we will not find a sequence of meetings that produces any of these three triples. Were we not clever enough to find a solution, or is it that none exists? Even though the problem is not yet solved, we have accomplished something very important – we have found a mathematical representation, namely the ordered triple, that describes the situation. Note, that the problem can be also solved with the help of smaller numbers in the starting triples – strategy of a simpler case. The following suggests another approach:

Now we can easily generalize our problem. Recall that generalization is a useful strategy for problem-solving. Let the starting situation be (y_0, g_0, b_0), where y_0, g_0, and b_0 represent the number of yellow, gray, and blue chameleons initially present. Suppose that after k meetings of pairs of chameleons having different colours, the situation is described by the triple (y_k, g_k, b_k). At the $(k+1)^{st}$ meeting, the triple (y_k, g_k, b_k) will change into one of the following triples: $(y_k - 1, g_k - 1, b_k + 2)$, $(y_k - 1, g_k + 2, b_k - 1)$ or $(y_k + 2, g_k - 1, b_k - 1)$. (Explain why.)

[7] The number of chameleons has been changed.

Graphically:

$$T_k = (y_k, g_k, b_k) \longrightarrow T_{k+1} = \begin{cases} (y_k - 1, g_k - 1, b_k + 2) \\ (y_k - 1, g_k + 2, b_k - 1) \\ (y_k + 2, g_k - 1, b_k - 1) \end{cases}$$

We can see that the difference $g_{k+1} - b_{k+1}$ can be represented as

$$g_{k+1} - b_{k+1} = \begin{cases} g_k - b_k - 3 \\ g_k - b_k + 3 \\ g_k - b_k + 0 \end{cases} \tag{1}$$

Note that there are similar patterns for the differences $y_{k+1} - g_{k+1}$ and $y_{k+1} - b_{k+1}$. From Formula (1) we see that the remainder of $g_k - b_k$ upon division by 3 remains *invariant under the transformation*

$$T : k \in \mathbb{N} \mapsto (y_k, g_k, b_k).$$

That is,

$$g_k - b_k \equiv g_{k+1} - b_{k+1} \ (modulo\ 3) \ \text{ for all natural numbers } \ k$$

and so

$$g_0 - b_0 \equiv g_k - b_k \ (modulo\ 3) \ \text{ for all natural numbers } \ k. \tag{2}$$

Formula (2) gives a condition of solvability. We now return to the original situation described in Problem 2. In doing this, we are concretizing (another important strategy for problem-solving), the opposite of generalizing.

The starting situation is (20, 15, 13) and the final situation we had hoped to achieve was one of

$$(48, 0, 0), (0, 48, 0), (0, 0, 48).$$

We use statement (2). At the start, $g_0 - b_0 = 15 - 13 = 2$. Therefore it must be true that $g_k - b_k \equiv 2$ (*modulo* 3) for all natural numbers k. But in the final situation $g_k - b_k$ is one of the numbers 0, 48 or -48. Since none of these numbers gives the remainder 2 upon division by 3, it follows that at no time can all chameleons on the island be of the same colour. We have solved Problem 2.

Now we can create a cluster of problems "generated" by Problem 2. If, for example, we change the numbers of yellow, gray and blue chameleons initially present, we get new problems. Again Statement (2) gives us the means of determining if we can find a time when all the chameleons are the same colour. If the starting situations are any of

$$(7,12,20) \quad \text{or} \quad (17,13,24) \quad \text{or} \quad (31,11,27)$$

then there is no solution. However, if we start with

$$(6,12,10) \quad \text{or} \quad (9,18,15) \quad \text{or} \quad (19,21,27),$$

we can show that the problem has a solution. To demonstrate that this is so, let us take the starting situation $(6,12,10)$. Then $12-10=2$ and if this problem has a solution, than the final triple must be $(28,0,0)$ or $(0,28,0)$ or $(0,0,28)$. We examine the differences $0-0=0$, $28-0=28$, $0-28=-28$. Only the number -28 gives remainder 2 after division by 3, because $-28=3\cdot(-10)+2$. So if after several meetings, all the chameleons are the same colour, that colour must be blue. Here is one possible way that can happen:

$$(6,12,10) \to (8,11,9) \to (10,10,8) \to (9,9,10) \to$$

$$\to (8,8,12) \to \cdots \to (0,0,28)$$

You can see that the chameleons corresponding to the two largest numbers in the initial situation (gray and blue) meet twice to get two numbers the same. When we have equal numbers of two kinds of chameleons, the problem becomes nearly trivial; we arrange meetings of only these chameleons to get the solution.

In general: Begining with the starting triple, we constuct a triple with two numbers the same. This is the strategy of identifying a subgoal, another useful problem-solving strategy.

If the starting situation is (9, 18, 15), then 18 − 15 = 3 and 3 $\equiv$0 (mod 3) . The final triple must be (42,0,0) or (0,42,0) or (0,0,42). Our differences $g_k - b_k$ in these triples are: 0, 42 and -42. All these numbers have a remainder of 0 after division by 3. So, given any one of the three colours, it is possible to find a sequence of meetings after which all the chameleons will be that color. We encourage the reader to find sequences of meetings to obtain each of these three possible outcomes. Here is one solution:

$$(9,18,15) \to (11,17,14) \to (13,16,13) \to (15,15,12) \to$$

$$\to (14,14,14) \to \begin{cases} \cdots \to & (42,0,0) \\ \cdots \to & (0,42,0) \\ \cdots \to & (0,0,42) \end{cases}$$

We leave to the reader the problem of finding conditions on the numbers x, y, z that will lead to a solution of the problem if we begin with the triple (x, y, z). As a second problem, find conditions on x, y, z that will guarantee that the problem has no solution. Here is a hint: if all numbers x, y, z have a remainder of 0 after division by 3, then we can find a solution for each color. That is, we can find a solution where all the chameleons are blue, one where all are gray, and a third where all are yellow. The same is true if all three numbers have a remainder of 1 or if all three numbers have a remainder of 2. If two of the numbers have a remainder of 1 and the third a remainder of 0, the problem has one solution. If one number has a remainder of 0, the second a remainder of 1, and third a remainder of 2, the problem has no solution. Why is this so? Now, find a complete solution.

Our last problem was solved by Copernicus in the Sixteenth Century. You can find it in [1].

Problem 3: A circle c' rolls without slipping along the inside of a stationary circle c. The diameter of c' is half of the diameter of c. A point M is marked on the circumference of c'. Describe the path of M.

Solution: We can start with experimentation. We might draw the circle c' and the point M in several different positions and attempt a conjecture. However, this can be very difficult. Experimentation without the use of a computer will probably not provide much help. The following suggests a better approach:

Figure 3 shows a possible starting situation (again, the strategy drawing a sketech).

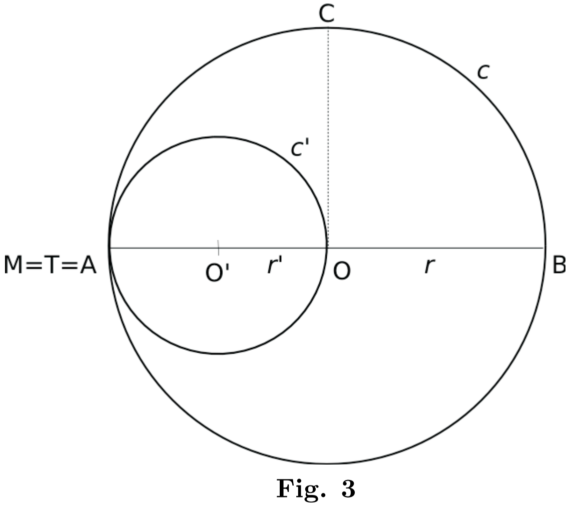

Fig. 3

The centre of circle c is O and the centre of circle c' is O'. Segment AB is a diameter of c. The contact point between c' and c is denoted by T and $|\sphericalangle AOC| = 90°$. In this initial situation, we see that $A = T = M$.

Fig. 4 shows one position of T on its way from A to C. While T is moving along arc AC, the transformation which moves M into new positions also changes the angles $\sphericalangle AOT$ and $\sphericalangle MO'T$. For these angles it is true that

$$|\sphericalangle AOT| = \frac{|\text{arc}AT|}{r} \quad \text{and} \quad |\sphericalangle MO'T| = \frac{|\text{arc}MT|}{r'}$$

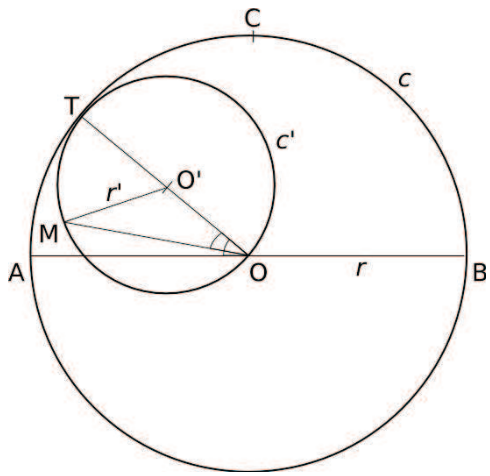

Fig. 4

Because $r = 2r'$ and $|\text{arc}AT| = |\text{arc}MT|$, we can write

$$2|\sphericalangle AOT| = |\sphericalangle MO'T|$$

this means that the ratio $|\sphericalangle MO'T| : |\sphericalangle AOT| = 2$ is *invariant under the transformation*. But it's also true that $2|\sphericalangle MOT| = |\sphericalangle MO'T|$ so we can write

$$|\sphericalangle AOT| = |\sphericalangle MOT|.$$

This last formula says that M lies on the radius AO of c. Thus, Fig. 3 is not drawn correctly. The correct figure is shown in Figure 5.

Conclusion: When T moves from A to C along arc AC, the point M traces the segment AO. Thus, when T moves from C to B along the arc CB , the point M traces the segment OB. When T moves along the remaining half of c, point M moves back along segment BA to its original position. It is very interesting that the point M on c' moves only along the diameter AB of circle c ; M performs a linear motion.

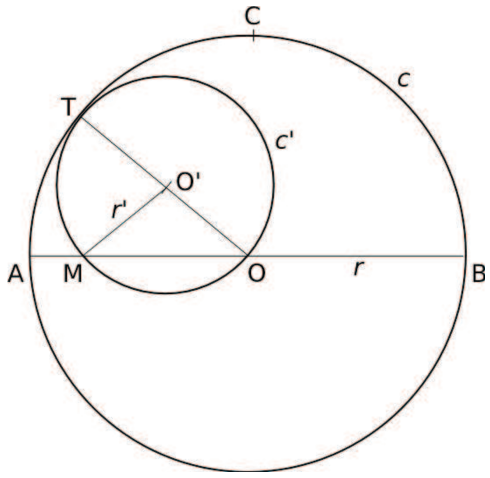

Fig. 5

Now we can create other problems. Describe the path of M, when $r' < \frac{r}{2}$ or when $\frac{r}{2} < r' < r$. We can also consider the situation when $r' > r$. The reader will quickly discover that in these situations, the motion of M is not linear. With the help of these last questions we have created a cluster of problems. However, the method of solution of these new problems is different from that of Problem 3.

Remark: As we can see Problem 1 lends itself to investigating possible solutions through experimentation. However, Problem 3 is an example where an experimental approach using only paper (not a computer) is not possible.

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APPLICATION OF VAGUE PROBLEMS TO IMPROVING EXACT PHRASING

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Abstract. In this paper we have conducted research into improving exact phrasing. We used the vague problems to reveal a necessity for an accurate formulation of words in Maths. In our study we used the term the vague problem to define a mathematical problem expressed in words whose assignment can be interpreted in different ways.

ZDM Subject Classification: D40.

1. Introduction

Mathematics teachers want pupils to learn to be exact in all their work not only in expressing themselves. We can use vague problems as a type of non–standard problem to practise exact phrasing. This is the most suitable example to show importance of exact formulation. With help of vague problems we tried to explain to pupils the need for exact phrasing in assignment and accurate choice of words used in solving problems. We carried out an experiment in the class. At first, we had showed to pupils the answers of an assignment and then asked them to create formulation of assignment by themselves. Their answers were often not clear and definite which as a result led to new vague problems. Some of these newly created assignments we used in discussion with pupils. Together we tried to find out the correct form of assignment that would be more appropriate for the given answers. In the article we show one original assignment, pupils' formulations of the assignment, the most common mistakes and pupils' attitude towards the mistakes they made.

2. Pupils' formulations of the assignment and the most common mistakes made

The experiment was carried out in the secondary school with 49 pupils aged between 10 and 15 years old. We handed them out a solution of solved assignment (Figure 1). They were asked to create assignment for this solution. The answers are underlined numbers showed in the text below. The original assignment is taken from collection of exercises for 9–10 year old pupils preparing for the exams at the last year of primary school [1].
The original assignment:

There are 35 pupils in the class. Each of them can ski or skate. Number of pupils who can ski is 18 and who can skate are 23. How many pupils can both ski and skate as well? How many pupils can ski only?

The solution given to pupils at the beginning:

SOLUTION

Together	....	35 pupils
Skiing	....	18 pupils
Skating	....	23 pupils

a) $(18 + 23) - 35 = 41 - 35 = \underline{\underline{6}}$

b) $18 - 6 = \underline{\underline{12}}$

ASSIGNMENT

Figure 1.

Pupils made a lot of mistakes while formulating the assignment. Sometimes more mistakes at the same time. The most mistakes were connected to the exact exercise. For example, the question created most often was: *"How many pupils do not attempt ski class either skate class?"*
The answer for this question is not our given solution. Furthermore, we are not able to answer this question from information given in assignment.

In the following text we are showing examples of mistakes (incorrect assignments), which were not connected with the exact problem, pupils had made them in different assignments too. For your information there is showed a school year of the pupil.

1 – VI There are 35 pupils, 18 of them ski and 23 of them skate. How many skiers are missing to be equal to the amount of skaters?

2 – IX There are 35 pupils together. On the school trip 18 of them ski and 23 of them skate. How many skiers and skaters are more than pupils together?

Some pupils create *formulation of assignment which does not make any sense* (in the assigment 1). They have some ideas of the assignment but it makes sense only to themselves and nobody else. We assumed that this kind of mistake would make mostly younger pupils but this mistake made pupils of 9th class too.
In the second type of mistake is *disagreement between the assignment and the question* (in the assigment 2). Reason for this type of mistake is misunderstanding the situation. This mistake is more common for younger pupils.

3 – V There are 41 pupils in the class. 18 pupils can ski and 23 pupils can skate. How many pupils can not ski from 18 pupils who can ski?

4 – VIII There are 35 pupils in the class, 20 boys and 15 girls. 18 pupils are going for skiing and 23 pupils are going for skating. How many pupils are going to do any sport?

5 – V 18 pupils can ski and 23 pupils can skate. How many pupils can not do any sport?

The third pupil's assignment is combination of two most common mistakes, the question without the sense and *changing the original information* from assignment (there are only 35 pupils in the class not 41). Pupils do not understand the situation from the assignment therefore they change the information to make sense for them. These mistakes mostly make pupils in 5th class.
Some pupils *add new information* to the assignment which are not essential for solving the problem. This is misleading for a person solving the assignment but it is possible to solve the problem (in the assignment 4).
The exact opposite type of mistake is *missing information* in the assignment (in the assignment 5). If the information is missing the assignment is not possible to solve. There have appeared two types of missing information, missing number or missing word explanation.

6 – V There are 35 pupils in the class, 18 pupils like skiing and 23 pupils like skating. 6 of the pupils can not ski or skate. 12 pupils can not ski.

7 – VI There are 35 pupils in the class. 18 of them can ski and 23 can skate. How many pupils can both ski and skate? 18 pupils can ski, from which 6 pupils can do both.

8 – V There are 35 pupils in the class. 18 pupils can ski and 23 pupils can skate. How many pupils can ski?

Mistakes mentioned above have connection with formulating the question. Often happened that pupils made right assignment but *instead of question they wrote the answer* (in the assigment 6). Other time they wrote *question and the answer as well* (in the assigment 7). In this case it can be explained that the answer is the way of self checking that the question is correct. The next mistake was that pupils made *question for the fact stated in the assignment* (in the assigment 8).

9 – VIII There are 35 pupils in the class. 18 pupils are going skiing and 23 pupils are going skating. How many pupils are going both skiing and skating? How many pupils are going skiing only?

This is an example of the assignment which was the closest to the original assignment. Almost all pupils started to solve the problem in the similar way. The only mistake in this type was that there was information missing that there is no pupil which is not at least in one group. No one from pupils has added this information to the assignment. Only six pupils from the experiment made the assignment like this. Others made major mistakes or they created only one question.
Assigment 9 is an example of *vague problem*. The problem where the solution is not completely clear. Correct answer for the question from this assignment can be: minimum of 6 and maximum of 18 pupils can do both sports. From 0 up to 12 pupils can go for skiing only.

10 – IX There are starting two activity groups in the school. There are accepting 35 pupils all together. For the skiing group applied 18 pupils and for the skating group applied 23 pupils. How many more pupils applied for both groups than it was expected? How many pupils will be accepted for skiing group after deducting pupils which are over the limit?

This was only one *correct assignment without any mistake*.

The most important part of the experiment was a discussion with pupils. From the assignments created by pupils we have chosen a few which were correct and wrong too representing each type of mistake. Pupils were asked to decide if the assignment is correct or wrong. Under correct assignment we understand the assignment which is mathematically correct for the solution given to pupils at the beginning.

Pupils from 5th class did not find out the mistakes in the assignment. They assumed it was correct. We were expecting the similar reaction from young pupils because they did not come across this type of problem. The type of problems they used to solve were always correct and with the information essential for them. Older pupils approached the problem more carefully but they did not find specific mistakes too. It was necessary to point towards the critical places or to ask them appropriate questions to lead them towards the correct assignment. At the end pupils were more relaxed and they were able to find more mistakes.

3. Other way of developing the ability of exact phrasing of assignment

The situation which enabled us to improve the pupils' reading and exact phrasing of assignment was spontaneously created during the process of teaching part Combinatory in the 6ht class in primary school. We were not able to write down this situation nevertheless we are suggesting it as a good idea for the maths teachers.

This thematic part is very interesting for pupils. Even weaker pupils are able to join in and be active in the class.

The process starts with the writing down all possible options, later pupils have to find their own system of writing down the options and the last part is finding out number of all options. Everybody can imagine that problems in this part require more attention from the pupil because it is necessary to think about a lot of important details. For example, if it is possible to repeat elements or not, how many elements need to be assorted and so on. We were able to solve all exercises from the schoolbook in a few class hours despite there are allocated 10 hours in class for this subject. Therefore we gave to pupils their homework. They were supposed to create their own assignment with the knowledge of combinatorics they learned. With the assignments they prepared the answers for it as well. Not all assignments were correct so we had discussion and we had induced pupils what they need to change in their assignment or how to say what they want to say simply and clearly. In our opinion these discussions were more productive than doing more exercises.

4. Conclusions

A person who can ask correct question in clear phrasing can easier orientate in problematic situation. For that reason, pupils should learn to ask correct questions and defend their opinion. Teachers can prepare pupils for similar situation at maths class having discussions about problematic tasks and assignments. We assume if pupils get more in contact with incorrect assignments they will develop critical reading, so important in real life.

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WAITING TIME FOR SERIES OF SUCCESSES AND FAILURES AND FAIRNESS OF RANDOM GAMES

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Abstract. In the article, a concrete random game is analyzed. Using the mean of the waiting time of a particular random variable, it is possible to determine, if the given random game is fair.

ZDM Subject Classification: K50.

Definition 1.
Let $u \in (0,1)$ is an arbitrary real number. We call the random trial modelled with its sample space $(\Omega_{0-1}, p^u_{0-1})$, where

$$\Omega_{0-1} = \{0,1\}, \qquad p^u_{0-1}(1) = u \quad \text{and} \quad p^u_{0-1}(0) = 1-u,$$

the Bernoulli trial and we denote it with δ^u_{0-1}. The results of the Bernoulli trial are denoted with 0 resp. 1 and we call them *failure* resp. *success.*

Definition 2.
we denote each result of m-multiple repetition of the trial δ^u_{0-1} with α which is called *a series of successes and failures with the length of* m. We say that the series of α_1 is a *subseries* of series of α_2, and we write $\alpha_1 \subset \alpha_2$, if α_1 as a string of 0 and 1 is a subseries of series α_2. If α_1 is not a subseries of α_2, we write $\alpha_1 \not\subset \alpha_2$. If $\alpha_1 \not\subset \alpha_2$ and $\alpha_2 \not\subset \alpha_1$, we say that *series* α_1 *and* α_2 *are differential.*

Definition 3.
Let α is the chosen series of successes and failures of length m. We call the

repetition of trial δ^u_{0-1} till the results m of the last trial make series α *the waiting for series* α and we denote it with δ^u_α. The number of repetitions of trial δ^u_α is a random variable T^u_α on set $\Omega_{T^u_\alpha} = \{m, m+1, m+2, \ldots\}$. Number $E(T^u_\alpha)$ is the mean waiting time for series α.

Definition 4.
Let α_1 and α_2 are series of successes and failures. If $E(T^u_{\alpha_1}) = E(T^u_{\alpha_2})$ then we call series α_1 and α_2 *of the same speed at point* u. If $E(T^u_{\alpha_1}) < E(T^u_{\alpha_2})$, than we call series α_1 *faster than series* α_2 *point* u.

Definition 5.
Given two differential series α_1 and α_2 of successes and failures, $|\alpha_j| = m_j$ for $j = 1, 2$. We repeat trial δ^u_{0-1} as long as:

- results m_1 of the last trials make series α_1,
- or results m_2 of the last trials make series α_2,

is called *the waiting for one of two series of successes and failures*, and we denote it with $\delta^u_{\alpha_1-\alpha_2}$ and its probability model with $(\Omega_{\alpha_1-\alpha_2}, p^u_{\alpha_1-\alpha_2})$. If

$$P^u_{\alpha_1-\alpha_2}(\ldots\alpha_1) = P^u_{\alpha_1-\alpha_2}(\ldots\alpha_2),$$

we call series α_1 and α_2 *alike at point* u. If

$$P^u_{\alpha_1-\alpha_2}(\ldots\alpha_1) > P^u_{\alpha_1-\alpha_2}(\ldots\alpha_2),$$

then we call series α_1 *better than series* α_2 *at point* u.

□ Given the following random game. Player G_A tosses a coin till the results of the last two tosses make series lr. Player G_B tosses a coin till the results of the last two tosses make series ll. The winner is the player who tosses first his/her series. Which of the player has a higher chance to win?

Let us denote the tails with 0 and the heads with 1. It is possible to simulate the game course with the rambling of a stone in a stochastic graph. (see [1] and [6]) in Fig. 1.

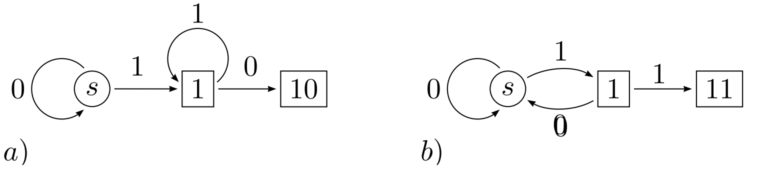

Fig. 1

Using the algorithm for the calculation of the mean of the rambling time in the stochastic graph (see [6], p. 399), we get

$$E(T_{10}) = 4 \qquad \textit{and} \qquad E(T_{11}) = 6,$$

which means that series 10 is quicker than series 11, and so there is a higher chance to win for player G_A.

□ Given another game. Players G_A and G_B toss a coin till the results of the last two tosses make series lr (G_A wins) or series ll (G_B wins). This game represents the waiting δ^u_{10-11}, with its graph in Fig. 2.

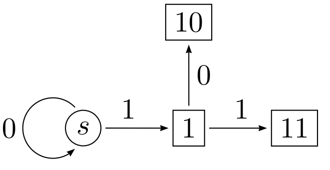

Fig. 2

It seems that if series 10 is quicker than series 11, it must be also better. Paradoxically series 10 and 11 are alike, which is obvious from the symmetry of stochastic graph in Fig. 2.

This begs the question: What random variable connected with the waiting time for series 10 and 11 do we have to think about to be able to make a decision which of the series is better based on its mean?

In Fig. 3, there is a modified graph δ^u_{10-11}.

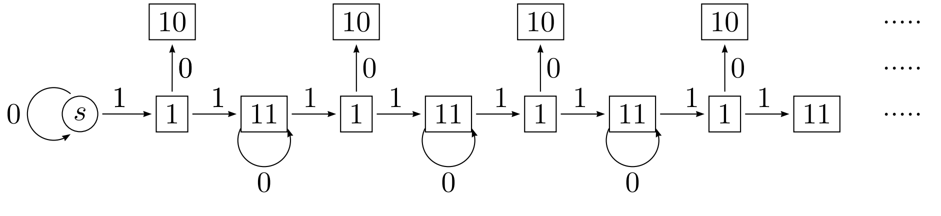

Fig. 3

If we calculate the waiting time for series 10 and we get combination 11, then we start again from node $\boxed{11}$, which becomes the starting node. We obtain a stochastic graph with infinite number of nodes. The time of rambling in such a graph is a random variable T_{10}^{10-11}, which represents also the waiting time for series 10. Fig. 4 shows several consecutive periods of this modifications.

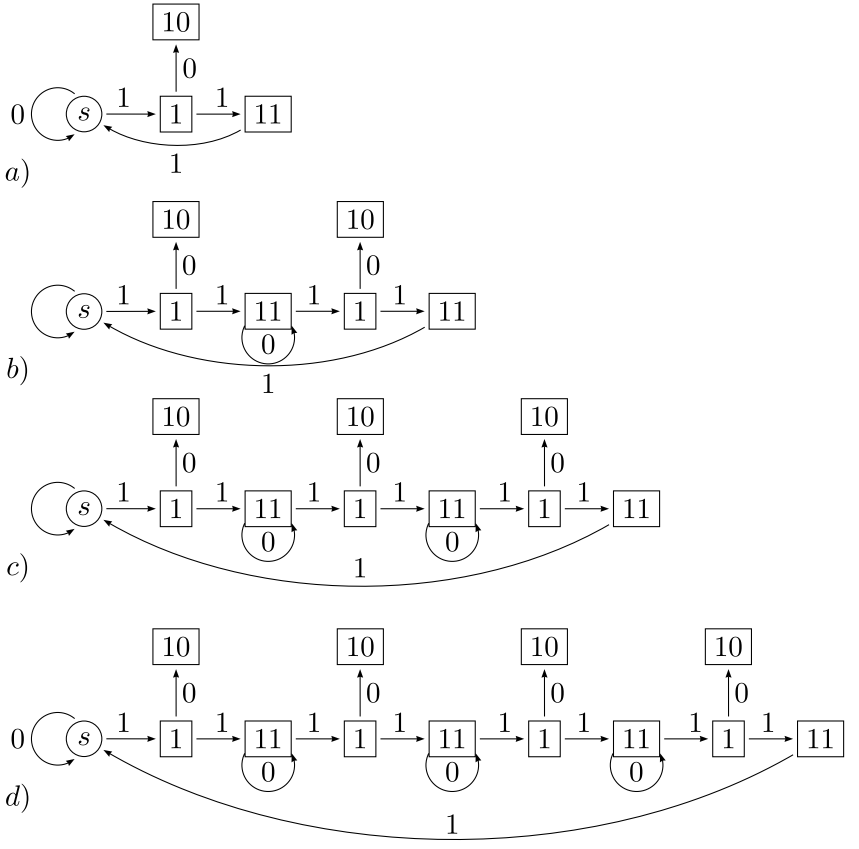

Fig. 4

Mean $E(T_{10}^{10-11})$ of the waiting time in the graph in Fig. 4 can be determined as a limit of a sequence. In the same way, we determine $E(T_{11}^{10-11})$. As $E(T_{10}^{10-11}) = E(T_{11}^{10-11}) = 6$, the both series are alike.

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SURPRISE AS A CATEGORY IN MATHEMATICAL REASONING OF STUDENTS

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Abstract. The paper deals with a problem of *surprise* in learning mathematics at school. What status of a student is called *astonishment*, and what is its cause? Why do students accept any content of school mathematics without any reflection or *surprise*? What are the reasons of the disappearance of the phenomenon of student's *astonishment*? Does anyone need *surprise* in mathematics? The author tries to provide answers to these and related questions. Consequently, thanks to selected examples, certain possibilities in inspiration and enhancement of students to spontaneous *astonishment* are presented, especially during mathematical activeness. Mistakes referred to as *"blessed ones"* in a sense given by Krygowska (errando discimus) can constitute good occasions for it.

ZDM Subject Classification: C70.

Introduction

In didactics of education and development *surprise* did not appear as an important educational category. It seems, that the main reason of the absence of *surprise* in didactics of mathematics is the treatment of this phenomenon as something exclusive, reserved for mathematically talented students. It is generally agreed, that majority of students do not feel any need to learn mathematics through surprise which initializes the process of approaching understanding of concepts, their deep ideas. It has been confirmed by various observations, that students' need to be *surprised* is believed not to appear at school at all, because at different levels of mathematical education nearly everything they learn is taken for granted, true and sure - although often difficult. The same refers to both the content, mathematical reasoning and its interpretation. At the same time, teachers agree, that in the teaching/learning process, spontaneous activity of students plays a very important role; especially valuable and welcome in mathematics, because this is a condition of, too often underestimated, intuition and imagination in their creative activity. Nevertheless, *surprise*, meditation and reflection of children over internal stimuli and external information are sources of spontaneous creativeness. The teacher should understand it and take advantage of this activeness, creating diverse situations and possibilities for students to take up various undertakings. If we admit that students' surprise is a welcome attitude in the process of mathematical reasoning, so a natural seems a question: how to provoke them to develop this type of spontaneous and reflexive attitude?

1. Surprise and the process of mathematicization in teaching mathematics

Surprise, is a state of a man who is astonished with something. To be surprised is to be impressed... astonished, amazed, astounded [9, p. 510]. In philosophy, to be surprised... *is a state of mind specified as a cognitive approach, which is reflection of the lack of confidence or the lack of feeling of obviousness and confusion in a given field* [2, p. 508]. Whereas in pedagogy, *surprise is a spontaneous attitude of a child, but can also be a point of an individual's awareness of his ignorance* [2, p. 509]. *Surprise*, as an emotional category, refers to reaction to novelty, which basically differs from earlier experiences of an individual. Due to intensity of experience of the *surprise*, we can talk about amazement, astonishment, shock, bewilderment, stupefaction. It is worth mentioning, that uncommon, unusual or unexpected situation of *surprise* does not trigger fear, but admiration, and can also lead to curiosity. Woronowicz [13, p.168] finds that... *only small children can be surprised these days (usually at the pre-school age); the older ones, in most instances, have gotten rid of that capacity of world cognition, so have majority of adults.* Since all that children learn at school is clear, unquestionable and at the same time - recognizing teachers' authority – true, so it is obvious for them that there is no surprise. It is an amazing fact, since surprise as a cognitive approach was already appreciated by Plato and Aristotle. The former said: *To be surprised, is a status characteristic for a philosopher. There is no any other outset of philosophy as just that*, whereas the latter, by confirming his master's words added: *It was because of surprise that the contemporary people as well as the first thinkers started to philosophize; they were initially surprised by unusual phenomena found on every day basis, later they slowly approached more difficult problems as, for example, phenomena connected with the moon, the sun and stars and beginning of the universe. And the helpless one being surprised learns about his ignorance* [1, p. 8].

The phenomenon of schools "killing" their conceptual understanding of mathematics by students seems to be the most important reason of decline of students' surprise. Turnau [11, p.161] finds, that teaching mathematics, instead of reinforcing and developing a specific kind of logical and disciplined reasoning, too frequently kills their ordinary, practical thinking based on common sense. The result of such approach at schools is:

1. imposition on students a one sided way of reasoning, practicing and applying school mathematics at the level of surface structures (allowing for excessive use of mathematical symbols, digits, letters, graphs etc. to generate certain notations and calling them correct answers to a given problem). Whereas, the role of thinking and acting at the level of conceptual structures which comprise sensible application of mathematical concepts and their relationship in finding answers to the questions related to those concepts or real objects, decreases. Hence, unsubstantiated reliance of teachers and students on symbolic procedures, often carelessly and irresponsibly. applied without any contemplation;
2. students' belief that mastery of mathematics in the context of surface structures requires less effort and takes less time; and in addition, is sufficient to obtain acceptable or even satisfactory results of tests, class tasks and examinations.

Another important reason for decline of the phenomenon of students' *surprise* can be found in different interpretation of the concepts of teaching of school mathematics by teachers - so called *mathematics for everyone*. Wheeler [12, p. 103] differentiates among three practically preferable approaches to teaching/learning process of *mathematics for everyone*:

1. Mastery of a minimum level of mathematical competence available to any student. However, certain degradation of mathematics takes place then, because it is devoid of such content which cannot be easily learnt. In consequence, trouble free version of mathematics for everyone is left.
2. Teaching each student in the same way, as if any willing student could become a mathematician. In such a case, teachers usually do not care about students' motivation and abilities and ignore the truth, that they know mathematics only as much, as to teach basic mathematics.
3. Teaching students skills of mathematizing - is the best way of to make mathematics accessible to every student. Make mathematic accessible to anyone does not mean that all students are supposed to acquire the same mathematics, because superimposition of learnable answers to

> mathematics questions restrains possible ideas of solutions which can result from their own skills of mathematizing. Only mathematicization which provides students with occasions to display their mathematical skills and talents in the classroom situation is a basic orientation in learning the subject.

The third interpretation of *mathematics for everyone* seems to be one of the most effective ways of evoking *surprise* with students which is necessary to initialize mathematical reasoning without fear. One should remember that a student's fear leads to disturbance of his action and reasoning at the level of intuition and reflection, and negative impact of fear of mathematics leads to apathy and passiveness destroying his motivation, interest, creativeness and ability to be *surprised*. As Siwek alarms [8, p. 13], the situation... *when students do not like mathematics because they are afraid of it, which in consequence leads to cheating, no learning and avoidance*, is disastrous. The nature of mathematical reasoning as a combination of empirical reasoning, intuitive and formal argumentation was presented in literature. Krygowska [4, p. 48] claimed that:... *any instance of mathematical reasoning consists of major and minor cycles; in each one we can (...) distinguish the following stages: observation, mathematicization, deduction, application*. Although, creative act of mathematical thought still fascinates philosophers, psychologists and mathematicians, it is, however, according to Krygowska, mathematical activeness of students which was analyzable, and the analyses enabled exploitation of its components. Generally known constituents of such activeness, such as: seeing analogy, schematization, generalization, deduction and reduction, encoding and algorithmization etc. function within human complex activeness, such as, for example, mathematicization. Mathematizing is always to lead students to mastery of a mathematical method which is more widely applied also outside mathematics. Since, in mathematics and its teaching function two words: mathematizing and mathematicization, I followed Semadeni [7, p. 111] in approach that mathematizing only comprises mental activity, human activeness, while the concept of mathematicization means mathematizing or the product of these activities. The process of mathematizing, according to Krygowska, requires specific cognition by all who aim at effective teaching of mathematics. This knowledge should be result of:

- Its research into and analysis of functioning in creative mathematical work;
- Search of means of its provoking and development at different levels of teaching;
- Research into conditions favorable or unfavorable for its development.

In order to create mathematicization reasonably, one has to know a lot about it. A well prepared teacher who understands that each child can be motivated, attracted, surprised and stimulated to mathematical mental activity will successfully reach them and come up with an adequately prepared educational offer.

What does a student's mathematizing of, lets say, a problem comprise?

Following Wheeler [12, p. 110] one can assume, that **mathematizing is a kind of activity comprising creation of equivalence for an initial problem by means of its transformation**, for example: the data included in it, the whole representation or change of he frame of reference. Constituents of such understood mathematicization by a student in order to solve a problem comprise:

1. ability of perception of interrelations;
2. ability of their idealization into mental constructs;
3. ability of their mental operation in order to create new interrelations.

Thus, students' activeness comprises *passage from a concrete non-mathematical problem originating from a widely understood reality to its clearly specified and adequate mathematical model or description (or attempts to describe) and analysis of relations between identified elements of reality with application of school mathematics but without application of any specific model* [7, p. 112]. An unfortunate alternative for competent mathematizing is a frequent school practice comprising superimposition by the teacher or a course book of ready methods of drawing of e.g.: models of figures, projection of solid figures, analysis from ready graphs of function properties, methods of solving equations their systems or text problems. Such situations, by premature disclosure of finished mathematicizations discourage students from independent attempts of finding their own interpretations, their own mathematical methods or models. Is there a chance then to *surprise* students at all? After all, everything is ready, evident, correct, and perfect, so why to debate, why be surprised?

2. Surprise and the student's intuitive reasoning

Based on the facts quoted above, related to teaching *mathematics for everyone*, two questions arise:

1. Is initializing of such a process of mathematizing possible and within the scope of ability of each student?
2. Having some knowledge on how mathematizing works, can one change teaching in such a way as to make school mathematics more accessible for all students?

Positive answers are possible only when learning mathematics is understood as the act of specific creation, as learners' capacity of specific cognition of phenomena and astonishment, asking questions and solving conceptual and cognitive conflicts. Wheeler [12, p. 111] finds that… *constituents of this process are known, and they work thanks to skills, because every man (…) meets initial conditions necessary to understand and participate in mathematical activity (…) to enter the world of mathematics.* Krygowska, on the other hand, [5, p. 11] says that:… *rational creation of questions directed to oneself and giving oneself rational orders is a vital mechanism of mathematical activeness at each level.* And when do the questions arise? When surprise builds up. *Surprise*, according to Jose Ortega y Gasset [3, p. 9] *is the beginning of understanding … The whole world surrounding us is strange and wonderful if we look at it* […] *eyes wide open with surprise.*

We are surprised when we do not understand something or when we cannot believe something, or when we are not quite sure about something. Astonishment does not allow us to ignore something, it forces us to consider, to think, even for a while. A question is a frequent reaction, and a question is perfect for a basis of any mthematizing. Hence, questions introduce intellectual unrest and enhance need to think; answer, on the other hand, according to general impression, introduce understanding, appeasement, satisfaction, the feeling of success and victory.

Surprise, in teaching mathematics can constitute a natural stimulus to solve problems which do not evoke fear or anxiety in connection with them, but astonishment of students. Whereas, astonishment, can initialize intuitive reasoning which is an act of holistic catching of the sense, meaning or structure of the problem without clear cut application of learned mathematical apparatus. Then, such a situation is a source of invention of fast hypotheses, ideas which are borne in the student's mind, strangest conceptual arrangements possible for further verification. However, as Tock [10, p. 101] suggests – logical thinking (practical - based on a common sense) always preceeds intuitive reasoning since it appears at subconscious level and then all the time shapes the direction of intuition in cognition. When is the student is eager to undertake reasoning inspired by *astonishment and intuition*? Here, the student is eager to undertake it is he is not afraid of bearing irreversible consequences comprising a punishment [10, p. 101]. If the student is not afraid of consequences of his mistakes appearing during creative mathematical reasoning, surprise may become an indispensable element of displayed emotions and may become inspiration on any ideas indispensable in the process of mathematizing of various problems.

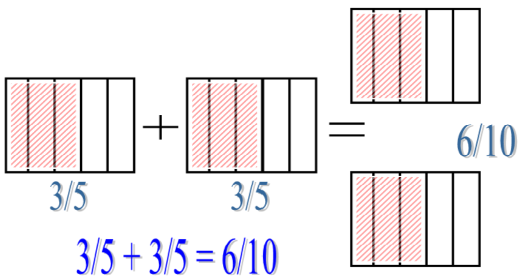
Mistakes made by students and provoked by the teacher are a good occasion for students to become surprised, which Krygowska [6, p. 115] called blessed. *Mistake as a positive instructive constituent of development of knowledge, learning, creativity* (*errando discimus*). Hence, if spontaneous surprise appears when its inspiring stimulus is in contradiction to the world of concepts built up in students, it means, that we have excellent situation in which their correct understanding comes up.

3. Selected examples and problems

Due to the limits of this paper (6 pages) only nine problems were selected.

Problem 1.

What do you think about such explanation of the operation?



Problem 2.

Is the highest double digit number a single digit number?

Is the following statement true: If the sum of a number is divisible by 3, this number is also divisible by 3?

Are the following equations true: $0,99999999999\ldots\ldots = 1$; $a : b \cdot c = a/b \cdot c$; $a : b \cdot c = a/(b \cdot c)$.

Which figure has more points: a segment or a straight line?

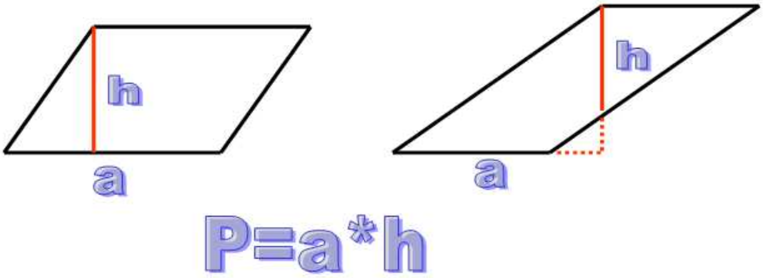
What is "84" equal to, if $8 \cdot 8 =$ "54"?

What is "100" equal to if $5 \cdot 6 =$ "35".

This is only apparent absurdity.

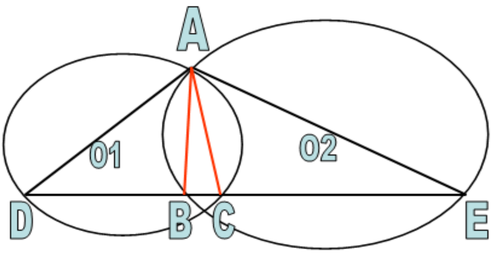
Problem 3.

How can you derive a formula for area of parallelogram of a given base a and the height h. Can this formula be derived for each parallelogram. Look at the figure. What can you see?



Problem 4.

A student has to draw two circles of different diameter, which intersect each other in two points; then specified their centers O_1, O_2; from one (upper) point of intersection A drew straight lines going through the centers of the circles; the straight lines intersected the circle in points D, E; connected points D and E by a section which intersected the circles in two points B and C. Thus, connecting points A, B, C a triangle ABC appeared. On the board, appeared a construction prepared by a student:



Please find, that ABE angle is an angle supporting AE diameter of the right circle, whereas ACD angle is an angle supporting AD diameter of the left circle. What are the sizes of these angles? Are they internal angles of the ABC triangle? Was a triangle constructed which has two right angles? So, is the size of the sum of the internal angles of ABC triangle more than $180°$? Explain whether we missed something?

Problem 5.

A car covered a distance between Warszawa and Kraków at a speed of 60km/h, and next time (coming back) with a speed of 40km/h. What is an average speed of the vehicle?

An apparent simplicity of the problem misleads many students. Great surprise that 50km/h is an incorrect answer. The answer would be true if the time of the journey both ways was the same. From the formula

$S = v \cdot t$, $t = S/v$. The whole journey lasted $t_1 + t_2$ of the time, so we obtain an equation $2S/x = 3/60 + S/40$. Since different than 0, we get: $2x = 1/60 + 1/40$, so $x = 2(1/60 + 1/40)$, $x = 48\text{km/h}$.

Problem 6.

A caterpillar creeps up a tree aiming at a branch situated at the height of 1,5 meters above the ground. During the first minute, it covered a distance of 50 cm, during the second minute a distance of 25 cm, during the third minute, a distance of 12,5 cm and so on. How many minutes will it take for the caterpillar to get to the branch? Draw a graph of the relation of the distance to time.

What was the students' surprise at the moment, when the realized that the caterpillar will never get to the chosen branch. They suspected a trick and they argumented that it would be impossible; the caterpillar must get there.

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Problem 7.

Draw a section, straight line and a circle defined metrically on a surface of coordinated axes XOY depending on metric b which we use to measure the distance of two random points A, B. Let $A(x_1, y_1)$, $B(x_2, y_2)$. We have the following metrics:

- Euclidean metric: $m(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$;
- Municipal metric (Taxi): $m(A,B) = |x_1 - x_2| + |y_1 - y_2|$;
- River metric - the river is an axis OY:

$$m(AB) = \begin{cases} |x_1 - x_2|, & y_1 = y_2; \\ |x_1| + |y_2 - y_1| + |x_2|, & y_1 \neq y_2; \end{cases}$$

- And a railroad metric with a distinguished point W on a plane

$$m(A,B) = \begin{cases} |AB|, & \text{if } A, B, W \text{ are collinear;} \\ |AW| + |WB|, & \text{if points } A, B, W \text{ are not collinear.} \end{cases}$$

The consequences of metrics introduced in this way astonished students,. They had never dealt with anything like that. Correct seems a presentation to students of such a distance, in which a segment, straight line and circle and other metrically defined figures look completely different than the ones which have been rooted in the imagination since childhood.

Problem 8. (Witold Bednarek)

Lotteries! Lotteries! We have two lotteries. The first one is of 10 tickets: 1 winning a computer, 2 entitles to another draw and 7 losers. In the second lottery, there are 8 tickets, 1 – to win a computer and 7 losers, since there are no tickets entitled to further draw. We have money to buy one ticket and we are interested which lottery gives us more chance to win a computer? Are tickets entitled to another draw increase our chance of winning the main prize?

How surprised the students were and could not believe when after solutions had appeared upon application of an ordinary probability tree (answer 1/8), and then generalization of the problem to w - winning tickets and u - entitled tickets and p - loosing tickets (w, u, p belong to the natural numbers) pursuant to mathematical induction they came to conclusion that a chance to win does not depend on the number of tickets entitled to further draw. One can repeat after Cantor who admiring a proof of tracing a segment into a square said: *I can see that, but I cannot believe it!*

Problem 9. (Witold Bolt)

A train covered a distance of 320 km during a period of 4 hours. Prove that the train covered 80 km per hour. An incorrect answer: $320 : 4 = 80$. General astonishment?

Correct solution

I. Observation.

The problem aims to prove that within a four hours period of time, say 8:00 to 12:00 we can find at least one moment, for example 9:54, that in the period form 9:54 to 10:54 the train covered exactly 80 km. The following information surprised the students: we do not know what train we mean, how it looks, but we know that it travels; we do not know if the train had any break from 9:00 to 11:30; we do not know if it went with a constant or changeable speed; we do not know if it changed its direction or stopped a the stations or not; we know only what was given in the content of the problem.

Confusion and surprise was evoked by information that no matter what we do not know, we can beyond any doubt prove that one can find such a moment. Only, we are not able to say precisely, when within that period of four hours the moment occurs and whether there will be only one such hour or more. The following extreme situations could happen theoretically: the train could travel at a constant speed exactly 80km/h and than at any hour covers 80 km. It could however, stand still for 3 hours and during the fourth hour could go at a speed of 320 km/h. In both extreme situations one can provide such a moment, from which the train will have covered exactly 80 km.

II. Conditions.

The problem which seemed to be quite simple and prosaic for fourth year students of mathematics surprised them, since it lead them to consider quite obvious and typical things for them - they dealt with definition of two functions with a uniform set, continuous function and the Darboux quality.

Students generally do not want to ask themselves a question "why?", "what for?", because they are accustomed to treat and consider many things, theorems, phenomena as something normal, unchangeable, which they in fact have no impact on and cannot modify. Astonishment can change it, because it refers to internalization of discrepancy between what a student can see and what he expects. If he is surprised, he will finally be aware of what he deals with is not completely common and evident. One can be a creative student, one has to grow to it.

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CREATIVE MATHEMATICAL ACTIVITIES: A RESEARCH PROJECT FOR MATHEMATICS TEACHERS

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Abstract. This paper presents a research project conducted among mathematics teachers. The aim of the project was to improve the teachers' ability to develop creative mathematical activities. For that purpose, diagnostic activities, workshops and lessons' observations were organized. The results show a considerable improvement in the teachers' ability and their attitude towards mathematics activities.

ZDM Subject Classification: C70, D40.

1. Introduction

Independent and creative thinking of students is an integral part of mathematics teaching. Contemporary mathematics education follows in the direction of mathematical activities. In the newest Polish curriculum (2007) we find following goals of mathematics teaching:

1. Assurance of an education which promotes independent, critical and creative thinking; limitation of schematic and reconstructive actions to minimum.
2. Assurance that every student has the necessary conditions to develop mathematical abilities according to his/her cognitive potential.

The above goals are related to the second level of goals of mathematics teaching according to Krygowska (1986), which refer to the development of creative mathematical activities of students. Krygowska (1986) distinguished three levels of mathematics education goals:

On level I the goals of mathematics teaching concerning basic mathematical knowledge, abilities and skills are formulated. They are usually described in school curricula;

On level II the goals of mathematics teaching concerning elements of mathematical methodology and behaviours unique to mathematical activity are formulated;

On level III the goals concerning the development of approaches and general intellectual behaviours which can be constructed in mathematics and then transferred to everyday life situations are formulated.

However, the research connected with the observation of school reality stresses a worrying aspect that the creative mathematical activities approach is neglecting. Mathematics is perceived not only by the students as a ready-made knowledge (a set of definitions, theorems, procedures), but - which is alarming – also by the mathematics teachers, who treat the teaching of mathematics as the teaching of that ready-made knowledge. Krygowska (1986) underlines that there are *false beliefs that the aims of the second and the third level are realised by themselves almost automatically during knowledge acquisition and the exercise of the skills specified by the national curriculum.*

It is also confirmed by the results of the research carried among the mathematics teachers (Maj, 2006).

2. Theoretical framework

Mathematical activity of a student is *a work of mind oriented to formation of concepts and to mathematical reasoning, stimulated by the situations which lead to formulating and solving theoretical and practical problems* (Nowak, 1989).

It is worth to underline two things, namely that the mathematical activity is firstly a work of mind and secondly that it should be stimulated. Therefore, it is not a work of a student which appears in a natural way.

Krygowska (1986) distinguishes particular elements of mathematical activity, which should play a special role in Mathematics for everybody, namely:

- identifying and using an analogy,
- schematizing,
- defining, interpreting and using rationally a definition,
- deducting and reducting,
- coding, constructing and using rationally symbolic language,

– algorithmizing and using rationally an algorithm.

A more detailed description of mathematical activity was done by Klakla (2002). He distinguishes particular kinds of creative mathematical activities, which are present in an essential way in activities of mathematicians.They are:

(a) hypotheses formulation and verification;
(b) transfer of a method (of reasoning or solutions of the problem onto similar, analogous, general, received through elevation of dimension, special or border case issue);
(c) creative receiving, processing and using mathematical information;
(d) discipline and criticism of thinking;
(e) problems' generation in the process of the method transfer;
(f) problems' prolonging;
(g) placing the problems in open situations.

Mathematics teaching should acquaint the students with all the aspects of mathematical activities so far that it is possible. Particularly, the students should have the opportunity of creative work according to their abilities (Polya, 1975).

The essential condition for the development of the skills of undertaking of different kinds of creative mathematical activities among students is the deep understanding of those issues by the mathematics teachers. To that it is necessary:

– to raise their awareness of the necessity of formation of such activities among their students,
– to develop their skills of organizing the situations which favour the undertaking of different kinds of such activities.

Only then the teachers would form and develop effectively these activities in their work with the students.

3. Methodology

In this paper we present a short description of the research carried among a group of mathematics teachers and also a description of the lesson conducted by one of the teachers who took part in the research, with the analysis of that lesson from the point of view of creative mathematical activities.

A group of seven teachers of mathematics (of gymnasium and high schools) has taken part in a series of workshops from March to September 2006. The workshops were organized as part of the Professional Development of Teachers Researchers (PDTR) project (226685-CD-1-1-2005-PL-Comenius-C2.1), during the mathematics course.

The main aims of the workshops were:

– developing the skills of undertaking creative mathematical activities among teachers of mathematics,
– raising the mathematics teachers' awareness of the need to develop creative mathematical activities and developing the skills of provoking these activities among students,
– showing a model of work of the teachers with the students.

Our purpose was to influence the development of the teachers' skills in organizing situations that - under certain circumstances - can lead to creative mathematical activities which are favourable to be undertaken by the students.

After the end of the workshops the teachers had the task to prepare and conduct a mathematics lesson which main aim was to develop some creative mathematical activities among students. In their previous experience concerning the preparation of the lessons and the determination of the lessons' aims, the teachers were used to focus on the mathematical content enclosed in the curriculum. Now they had to concentrate on mathematical activities which they will form and develop around a theme of a lesson.

The observations of these planned lessons had the aim to show us if the teacher can plan and organize a work of his/her students in that way so that they have the opportunity to undertake different kinds of creative mathematical activities. However, it was less important what class that lesson is conducted in and what mathematical content it is related to.

We will present a short description and analysis of the lesson of the teacher who participated in the workshops. That analysis was conducted in the direction of the answers to the following questions:

– Does the teacher develop any mathematical activities among his/her students and if yes, what kind of activities does s/he develop?
– Does s/he stimulate his/her students to undertake creative mathematical activities?
– Do the ways the teacher acts favour independent and creative thinking of the students?

The data collected comprised of audio recording of the lesson (45 min.) and notes. After this a full transcription of the audio recording and analysis of this transcription was made.

4. Analysis of the lesson

The lesson was conducted on 20th December 2007 (17 students 13-14 years old).

The teacher hanged a plate with four drawings (Figure 1) and asked the students to formulate questions and problems for the introduced situation. This task was based on the assumption that the mathematical experience of a student is incomplete if s/he has never solved a task made by himself/herself (Polya, 1993).

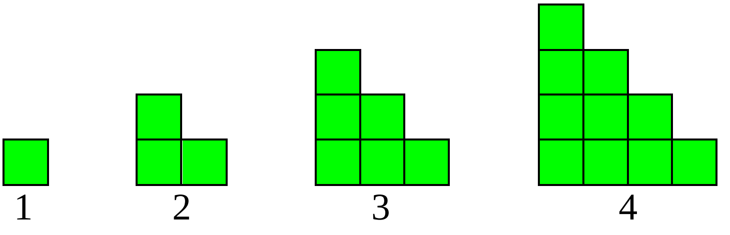

Figure 1. The plate with four drawings

The students undertook the challenge willingly and posed the following questions:

1. How many squares does each figure contain?
2. How many more squares does each next figure contain?
3. What is the perimeter of each figure?

The teacher chose the question in such a way to stimulate the students in the direction of generalizing, at the same time imposed nothing:

Teacher: *And what is the last question you would like to ask?*
Ela: *How many squares are in the tenth or n–th figure?*
Ola: *How many squares...*
Teacher: /nods with approval/
Ela: *We will do a table and in the top we will write ...*
Teacher: *So do not ask too hard questions... What other question we will ask?*
Marcin: *How many squares will the tenth figure have?*

Students: *The fifth !!!*
Janek: *For now fifth. Or tenth or fifteenth...*
Teacher: *For now fifth ...*
Students: *Fifth, sixth, and then it can be tenth.*
Teacher: *The fifth, sixth... and then?*
Kamil: *And n–th!!!*
Teacher: *wow, Kamil.... So let's draw the table. And now... here we will write ... What will be here? The numbers of the figures ...* /he is drawing/
Students: /dictate/: *the first, second, third, fourth, fifth, sixth...*
Teacher: *It can be sixth... and now we jump to?*
Kamil: *the tenth, the fifteenth and n–th!*
Teacher: *So? Here we put dots ... and seventh* /he writes 7 in the table/
Students: *No!!!*
Teacher: *The tenth, right? And what is before n–th?*
Students: *The fifteenth!*
Teacher: *The fifteenth....*
Students: *And n-th!*

The teacher can be characterised by the open attitude; his questions made the students formulate the task by themselves and also prolonged this task. It was a new open situation for the students, in which they did not feel safe. The facts that they decided by themselves what they wanted to calculate, and about the tempo of their work affected their view of the task: it started to be closer to them and they were more motivated, thus they were more active. The teacher was flexible: he tried to reconcile all the propositions of the students, he did not reject any of them - one part of the students were ready after the fourth figure to move to the tenth but the other part still felt the need to prolong for the next two figures and only then they agreed to "jump" to the tenth, the fifteenth and the n–th. It seems that the teacher was aware of that. The ideas of putting the results on the table and the way of their ordering also came from the students, therefore the students also determined a plan of solving the task.

The teacher was interested not only in particular results but also in the way of thinking of the students. His "how", "why" questions forced his students to justify their answers:

Teacher: *Gosia have you thought up of something? Gosia, can you do the fifth? Come and tell us why. Because we are interested in how do you think up that.*
Gosia: /writes/: *15*
Teacher: *Gosia tell us, how did you think?*

Gosia: /shows the drawing number 4 and shows adding the next "stairs" from the top/: *Because here we add...*

Teacher: *So did you draw it in your exercise book or in your mind?*

Gosia: *in my mind*

Teacher: *You drew in your mind ... Michał what result do you have?*

Michał:/very fast, expressive/: *15, but it will be that ... because first we have 1 and then we add 2, and then 3 ... so to every one added before we add 1 more.*

Teacher: *So how did you think up that 15?*

Michał: *First from 6 to 10 we added 4. And here to 4 we added 1 that is 5 ...*

Other students: *And that we added to 10.*

Teacher: *So what was the second question?*

Students: *How many more squares does each next figure contain?*

Teacher: *So when I'm going from the first to the second figure then?*

Michał: *at 2*

Teacher: *Please Michał, it is your idea...* /asks Michał to come to the blackboard/

Michał: /writes/ *here at 2, at 3, at 4, at 5, here at 6* /and writes the number 21 for the sixth figure (the first question)/ (Table 1).

<table>
<tr><th>question \ number</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>...</th><th>10</th><th>15</th><th>n</th></tr>
<tr><td>1. How many squares does each figure contain?</td><td>1</td><td>3</td><td>6</td><td>10</td><td>15</td><td>21</td><td></td><td></td><td></td><td></td></tr>
<tr><td>2. How many more squares does each next figure contain?</td><td colspan="10">2 3 4 5 6</td></tr>
<tr><td>3. Perimeter</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

Figure 2. Table 1

The students expressed two attitudes: Gosia was not able to give a rule, only the way of creating the next drawings, related to geometrical relation; however, Michał noticed the dependence between the numbers. The teacher did not negate any of that ways, gave the students the chance to present their ideas. Michał answered on the teacher's question about the result, but he thought up the general way of counting the figures' areas and wanted to pride oneself on it. His language was not precise, so the teacher wanted him to explain exactly in what way he got the number 15 - such conversations develop the skills of communication about mathematics.

The students noticed a recurrent relation between the numbers by calculating the sums of the next natural numbers. The reaction of the teacher helped them to understand that their way of calculating figures' areas does not work, it is insufficient for the all cases:

Teacher: *Come to the blackboard and show us how you calculated this.*
Kasia: /writes/ *21+7=28 28+8=36 36+9=45 45+10=55* /puts to the table 55/
Teacher: *And fifteenth? Does anybody have the fifteenth?* /the students are calculating - they are adding the next numbers/
Teacher: *Karol?*
Karol: /goes to the blackboard and writes /: *55+11+12+13+14+15=*

66 78 91 105 120

/he erases the numbers in the second line and writes: =120 and puts 120 to the table/
Teacher: *What if we had the three hundredth figure? How long we would wait for the result? How long we would calculate?*
Students: *half an hour....*
Teacher: *So we have a problem how to do it faster, for example for the three hundredth figure. If we take.... maybe before we will take the letter n, then maybe we will take 100* /he writes 100 into a table/ *You have limited time....*
Students: *5 minutes!*

Asking about the three hundredth figure the teacher "forced" the students to search for a variable relation, he provoked them to the trials of generalization. The students stated that the first way does not make sense, it is time-consuming, they imposed themselves the restriction of 5 minutes. The teacher provoked on purpose that situation, in which the students came from arithmetic to algebraic thinking, to start combining. He expected that on the question about n–th figure the students would answer $1+2+\cdots+n$, but they would not be able to write it in the form of a general formula.

This aware action of the teacher resulted in the discovery of two ways of calculating the squares: geometrical and algebraical. The students did generalisation by themselves and wrote the general formula $n(n+1):2$. The effect of that conducted lesson surprised the teacher. In the conversation that followed he said that the reaction of his students was unexpected: all of them were very active even the less gifted students. He admitted that if he had planned the lesson (the questions of the task, the plan of solving) he would not ”go” so far like his students because he would think that it would be too ambitious for them. He said that before his participation in the workshops he never asked the students to formulate questions, or to solve problems in an open situation, because he thought that they would not think up of something sensible. Finally he admitted that he never thought that his students can have so fantastic ideas, ideas which even himself did not have.

5. Conclusions

The choice of the open situation during the lesson let the students pose the questions by themselves. They could show their ability in creative invention. It stimulated them to undertake mathematical activity: to formulate hypotheses, whereas the frequent ”how” and ”why” questions of the teachers resulted in the necessity of verification, thus enhancing the skills of argumentation and critical thinking. The students prolonged the problem by themselves; they were asking questions about the areas of the next figures (5, 6, 10, 15, n) and they could decide by themselves on what they will calculate and on the tempo of their work. All these resulted in a lesson that was around the problem which they created and solved by themselves. It was possible to observe the amazingly big engagement of the students in that process. The question of the teacher about the three hundredth figure motivated them to generalize and directed their thinking to the more abstract level, at the same time without suggesting anything (the students noticed that the recurrent relation did not play any role here because it was too time-consuming). During that lesson the students had the opportunity to discover and construct algebraic formulas, describe some general rules, therefore, they had the opportunity to get the sense of these formulas by discovering recurrent and variable relations.

Our aim was to observe the teacher during his work with the students. During the lesson described the students had the opportunity to undertake a few kinds of creative mathematical activities. The actions of the teacher were limited to the organization of the process of teaching, in order to let the students put the questions, formulate problems, and then discover some mathematical relations. The teacher stimulated the work of the students in the direction of creative mathematical activities; he did not impose his own way of thinking, he was flexible and reacted positively to the ideas of his students. And all these favoured students’ independent and creative thinking.

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REMARKS ON DEFINITIONS AND THE PROCESS OF DEFINING MATHEMATICAL CONCEPTS

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Abstract. This paper deals with some studies of understanding the definitions. This article is divided into two parts. The first one contains the conclusions drawn from the analysis of the questions. The second part of this article contains the analysis of the solutions of the mathematical problems with the absolute value of real number which were solved by the students.

ZDM Subject Classification: B40, B50, D70, E40.

Problems related to introducing and covering definitions of mathematical concepts are widely discussed in literature on teaching mathematics (cf. Konior, 1975; Krygowska, 1977). These issues are important at each stage of teaching mathematics. The problems connected with covering definitions, evaluating the correctness of a definition of a concept or formulating different definitions of one object should be paid special attention to during the education of future teachers of mathematics.

Earlier research shows that some students significantly lack school knowledge, for example the skill of formulating definitions, in particular the skill of reconstructing definitions which they know, the skill of writing the definition of a familiar mathematical concept in a different way, i.e. by providing a definition equivalent to the source one on the basis of, for example, another concept, and the skill of constructing a definition in an appropriate form, for example with respect to the range of applicability of particular formulae (cf. Major, Major, 2004, Major, 2006). It can be stated that some students have shortcomings in using the language of mathematics, in particular they have considerable difficulties in using the set theory symbols.

The research into students' understanding of mathematical concepts also investigated comprehension and the skill of using mathematical definitions. This paper presents how students understand the concept of a definition itself and its place in mathematical theories. It also reveals how students evaluate the correctness of different definitions of one mathematical concept taught in schools of primary/secondary education.

Second year students of mathematics, with their major in teaching, at the Pedagogical University of Cracow were asked the following questions:

1. What is a mathematical definition?
2. What role do definitions play in mathematics?
3. What conditions should a correct definition fulfil?
4. Give examples of definition types which you know.

To answer the first question the students said that a definition is:

- an explanation of a regularity which is accepted in order to solve a problem;
- an explanation of a concept;
- a formal explanation of a concept;
- adopting a certain terminology;
- something that describes a concept;
- a verbal explanation of a concept;
- a formula, a sentence which describes a concept;
- a tool which a teacher uses to explain things;
- a set of information about a mathematical concept;
- something that attempts to describe something.

The research subjects noticed that a definition is a sentence which explains the meaning of a new concept. They also pointed out that when constructing definitions new terminology is introduced. Please note that much as the features of a definition given by the subjects outline what a definition is, they are not exhaustive at all. None of the study subjects said that a definition is a sentence or a description of a concept within a given mathematical theory, i.e. that a definition can function only in a broader structure, within broader relationships.

In the opinion of the students a correct definition should be logical (3 persons), accurate (3 persons), succinct (1 person), comprehensible (6 persons), written in colloquial language (1 person), contain elements typical of a concept (2 persons), clear (2 persons), precise (1 person), transparent (3 persons), useful (2 persons), concise (5 persons), written in an accessible language (3 persons), formulated in such a way that everybody can understand it (1 person).

The study subjects paid their attention mostly to aesthetics of a definition structure. They also hinted at the language in which a definition should be written and its comprehensibility. Hence, they paid attention to features which from the viewpoint of the correctness of a definition are not important. The students mentioned also that a correct definition should be intelligible to everybody, which for obvious reasons is not feasible. Please note that none of the subjects paid attention to the object defined or its uniqueness.

When giving examples of familiar mathematical definitions types most students differentiated between verbal and symbolic definitions. One study subject mentioned definitions through formulas $(a-b)^2 = a^2+2ab+b^2$. One student differentiated between written verbal and oral verbal definitions. One person pointed out that a definition can use primary concepts or it can be constructed by using other definitions. Three subjects gave examples of concepts which are defined by school and university mathematics (the definition of a metric space, a triangle, a natural number, etc.).

Please note that asked to provide examples of familiar definition types, most students differentiated them by the language used to construct a definition (symbolic language, educated language). One study subject singled out definitions in which the meaning of a new concept of a given theory is explained with primitive concepts or concepts defined before. None of the students differentiated definitions by the structure of the sentence which explains the meaning of a new concept (e.g. equivalence, implication).

Summing up this portion of results obtained, a hypothesis can be made that the knowledge of some study subjects about what definitions are and what role they play in mathematics is fragmentary. This is worrying due to the fact that students work with pupils, i.e. they prepare and run classes in schools during which they introduce and cover new mathematical concepts.

Our research has also shown students' great difficulties in evaluating the correctness of a definition correctly. First and second year students of mathematics at the Pedagogical University of Krakow (155 persons in total) worked on the following problem.

Problem:

The question what an absolute value generated the following answers. Evaluate their correctness. Justify your answer.

(a) $|x| = \max(x, -x)$;

(b) An absolute value of a number is the bigger number from among the following numbers: the given number and the opposite number;

(c) $|x| = \begin{cases} x & \text{dla} \quad x > 0, \\ -x & \text{dla} \quad x \leq 0; \end{cases}$

(d) $|x| = \begin{cases} x, & \text{gdy } x > 0, \\ -x, & \text{gdy } x < 0. \end{cases}$ Additionally, we assume that $|0| = 0$;

(e) $|x| = x \cdot \operatorname{sgn}(x)$;

(f) An absolute value of a positive number is the same number; an absolute value of a negative number is the number opposite to it;

(g) An absolute value is an identity function for non-negative numbers and a function which changes a number to its opposite for negative numbers;

(h) An absolute value of a number is the distance between the point which corresponds to the number on the numerical axis and the point which corresponds to number 0 as measured in unit segments;

(i) $f(x) = |x| \iff [(x \geq 0 \wedge f(x) = x) \vee (x < 0 \wedge f(x) = -x)]$;

(j) An absolute value of a given number is the number without its sign.

The data show that definitions a) and b) were found correct by 8% subjects, definition c) by 69% subjects, definition d) by 78% subjects, definition e) by 10% subjects, definition f) by 2% subjects, definition g) by 61% subjects, definition h) by 83% subjects and definition i) by 29% subjects.

definition	answers		
	found correct	found incorrect	without answer
a	13	79	63
b	13	74	68
c	109	41	5
d	123	21	11
e	15	42	98
f	3	148	4
g	96	57	2
h	132	13	10
i	46	2	107

Table 1.

Identical results obtained by definitions a) and b) can suggest that the subjects noticed the similarity between the two conditions, one of which is written verbally and the other symbolically.

Worthwhile pointing out is the fact that a lot of subjects did not evaluate the correctness of definitions a), b), e) and i). In our opinion the reason for it could be that definitions a) and b) use the concept of the maximum of two real numbers whereas definition e) uses the concept of the sign of a number. It can be assumed that these concepts were not known to many (in particular younger) study subjects. In our opinion not attempting to solve problem 14i) can be related to students' difficulty in understanding the definition and so with the lack of the ability to analyze a condition written with symbols. This can be indicative of difficulties in reading and understanding mathematical texts.

It should be mentioned here that virtually all students did not justify their answers. Problem 14f) was the only exception. For this part of the problem 94% subjects stated that condition f) cannot be accepted as a definition of an absolute value. The student justified their answer by saying that the definition given does not make it possible to determine the absolute value of number 0. In our opinion not justifying answers can be related to students' difficulties in demonstrating equivalence of different definitions of one concept.

Significant shortcomings in the education of students of mathematics, with their major in teaching, related to the skill of evaluating the correctness of given definitions of a concept are worrying due to the fact that: knowing structures of different definition types and conditions of their formal correctness is highly required for a teacher. In their work teachers have to analyze, correct and evaluate definitions formulated by their students, react to possible mistakes made by them when formulating definitions (Podstawowe zagadnienia dydaktyki matematyki, 1982).

We think that defining and using definitions which introduce a mathematical method to students as well as formulating definitions and evaluating the correctness of definitions which fulfil certain conditions regarding their structure are important elements in teaching mathematics to future teachers. Solving this type of problems facilitates better understanding of the process of defining concepts itself and provides a possibility to revise and apply knowledge and skills of mathematical logic.

Teaching proposals related to defining concepts of mathematics on primary and secondary level of education which use concepts of higher mathematics are presented in papers (Major, Powązka, 2007, Chronowski, Major, Powązka, 2008). They include examples of defining an absolute value of a real number with equations and functional inequalities.

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DIFFICULTIES OF STUDENTS BY MATHEMATIZATION OF CHOSEN WORD PROBLEMS

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Abstract. Some results of a research, related to solution of 3 concrete word problems, are presented and finally analyzed in this contribution. This research was performed at 4 high schools (of different types) in 2007 and 363 students of second classes participated in it. The main aim was to find out, how our students can use a variable by a solution of a word problem. Concretely we tried to find out, whether our students can establish a variable and subsequently form one linear equation with one variable, eventually system of two linear equations with two variables.

ZDM Subject Classification: H33.

1. Introduction

In general the main aim of education is better preparation for real life. Therefore the teaching should include problems similar to problems, that people are normally solving in their lives. In mathematics that are exactly the word problems. They force students to model a real life situations by mathematical means.

The fact, that solving of word problems (compared with solution of other types of mathematical problems) is for students much more difficult, is widely accepted by teachers and also by students. Consider shortly, what a student must do to solve such a problem. Firstly, he must carefully read the assignment of the problem and he has to understand it. Than he has to transpose it into the mathematical language (he has to design a mathematical model). Finally he must solve this model and find an interpretation of an obtained result. As we just realized, solving of a word problem is prolegomenous and requires many different activities from student.

In this contribution we will present partial results of an extensive research, which was oriented to problems of students in each stage of solving of a word problem.

2. Description of the whole research

The main aim of the research was: on the basis of obtained solutions of word problems, to gain a realistic view on the problems of students by solving a word problem and to decide, what is the main problem of students by solving a word problem. Only two alternatives come to consideration. Either it is a problem with understanding of an assignment (text) of a word problem, or it is an inability of students to design a mathematical model. By problems with understanding of an assignment we mean f.e. following: student do not understand the situation described in a text, student is not able to register from text all necessary connections – objects and relations among them, or student do not understand the challenge created by a word problem – he simply do not understand the question.

This research was performed at 4 high schools (of different types) in 2007 and 363 students of second classes participated in it.

By studying and analysing of 30 various mathematical schoolbooks and collections of mathematical tasks we have selected (in accordance with the main aim of the research) 10 word problems. This 10 problems were split in two series of 5 problems. For solving of each series students became 45 minutes.

In this contribution we direct our attention only on word problems number 2, 4, and 8, those main aim was to find out, how our students can use a variable by a solution of a word problem. Concretely we tried to find out, whether our students can establish a variable and subsequently form one linear equation with one variable, eventually system of two linear equations with two variables.

3. Commentary to 3 mentioned word problems

2th word problem

Peter knew how to guess right a number, which had his classmate in mind. His technique was following: "Choose an arbitrary number, add 5 to it, multiply this sum twice and subtract 10. Tell me now your result." In all cases Peter guess right the number, which had his classmate in mind. Explain how. [3]

The purpose of enlistment of this word problem was to find out, if student can establish a variable in the simplest possible case and if he is able to transcribe the corresponding sentences (without antisignals) into mathematical language.

4th word problem

In the store are 2,5 – times more sheets of paper format A0 as sheets of paper format A2. Storekeeper emited 1450 sheets of paper format A0 and 220 sheets of paper format A2. Then the same amount of sheets from both formats remained in the store. How many sheets of paper from each format were initially in the store? [1]

8th word problem

The family had sons and daughters. Each son had the same count of brothers and sisters and each sister had two times more brothers than sisters. How many sons and daughters had this family? [2]

It seems that the last two word problems have similar mathematical models – models are represented by a system of two linear equations with two variables. But there are also some very significant differences between this two word problems. The assignment of the 8th problem is much more complicated. It contains objects: son, daughter, son's brothers, son's sisters, daughter's brothers, daughter's sisters, resp. their counts, what finally makes 4 objects more as in the 4th problem. Additionally the relations, that the count of sons is 1 bigger than the count of brothers of each son (analogue for daughters), are not explicitly mentioned in the assignment. On the other hand, in the assignment of the 4th word problem are all relations, that are necessary for construction of the system of two linear equations, explicitly mentioned. Result of the 4th word problem are relatively big numbers. The reason is simple, students can not guess this numbers and avoid so the construction and solution of the system. On the other side 8th word problem can also be solved experimentally - by guessing numbers.

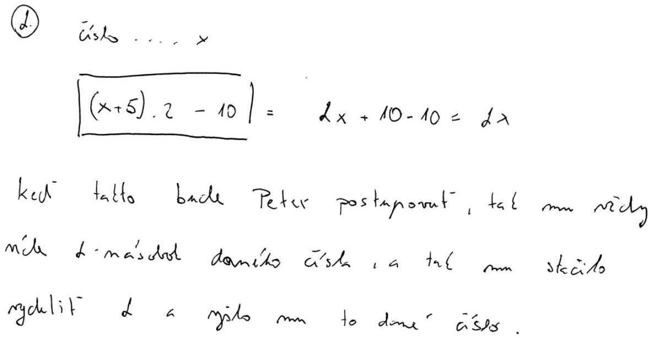
Both word problems contain antisignal. In the assignment of the 4th problem is written, that in the store are 2,5 – times more sheets of paper format A0 as sheets of paper format A2, but the correspondent first equation is $AO = 2{,}5A2$ and not $2{,}5A0 = A2$, as the assignment "tries" us to mislead. Analogical in the assignment of the 8th word problem is written, that each sister had two times more brothers than sisters, but in the model it is necessary (by correspondent equation and variable) to divide by two.

The purpose of enlistment of this two word problems was to find out, whether our students can form a system of two linear equations with two variables, in the simplest possible case and subsequently in the sadden conditions.

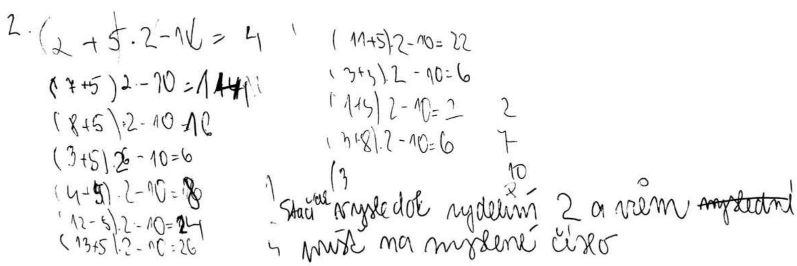
4. Word problems evaluation with samples of typical student's solutions

Solutions of the 2th and 4th word problem we have obtained from 321 students (who solved first series). We have obtained 346 solutions of the 8th word problem (from students, who solved second series).

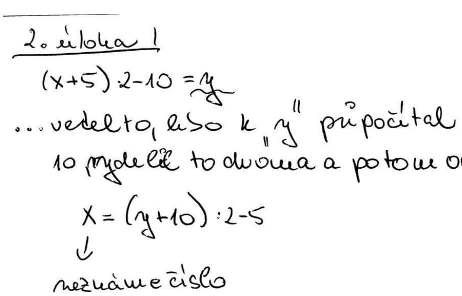
The **second word problem** was correctly solved by 196 from 321 (61%) students. 109 of them (55,6% of successful solutionist; 34% of all solutionist) solved this problem by using a variable.



87 (44,4%; 27,1%) students solved it without using a variable – by testing and guessing.



30 students (9,3%) solved this word problem by reverse progress (with or without using a variable).



We have not expected this kind of solution.

Next sample shows one of the typical student's mistakes by construction of an appropriate mathematical model:

$$
\begin{aligned}
2\ \big((x+5)\cdot 2\big)-10 &= x\\
2x+10-10 &= x\\
2x-x &= 0\\
x &= 0
\end{aligned}
$$

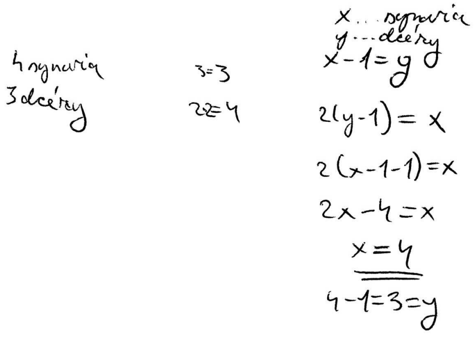
Student used the same variable in two different meanings. Some students actually wrote 0 or 1 on the right side of this equation. Then is the error even more fatal. In this case student do not understand, that the number, which classmate in the end says to Peter, can change on each occasion.

The **fourth word problem** was correctly solved by 138 from 321 (43%) students.

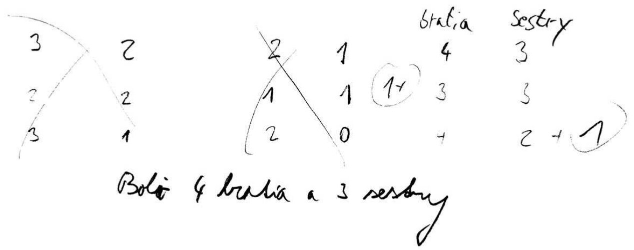
$$
\begin{aligned}
A_0 &= 2{,}5\cdot A_2\\
A_0-1450 &= A_2-220\\
2{,}5\cdot A_2-1450 &= A_2-220\\
2{,}5\cdot A_2 &= A_2+1230\\
1{,}5A_2 &= 1230\\
A_2 &= 820\\
A_0 &= 2{,}5\cdot 820\\
A_0 &= 2050
\end{aligned}
$$

Only 10 from 148 (6,8%) students, who correctly have formed the system of equations, failed by solving it. That means, that solving of a system of two linear equations with two variables does not make any problems to our students. 13 (4%) students failed by transcribing the antisignal into an equation.

The **eighth word problem** was correctly solved by 158 from 346 $(45,7\%)$ students. Only 12 of them (7,6%; 3,5%) solved this problem by using a variable.



146 (92,4%; 42,2%) students solved it without using a variable – by guessing.



18 (5,2%) students failed by transcribing the antisignal into an equation. By comparing it with the corresponding number from the fourth word problem, we came to conclusion, that approximately 1 from 20 students makes such a mistake. 178 from 346 (51,4%) students recognized the implicit relations, but only 18 (10,1%) formed it correctly into the equations.

Complex view at the solution of 3 word problems

66 from 294 (22,4%) students, who have solved both series, did not solve any of mentioned word problems. 60 from 294 (20,4%) students solved all 3 word problems. 122 from 294 (41,5%) students, did not solve any of mentioned word problems by using a variable. Only 5 from 294 (1,7%) students solved all 3 word problems by using a variable. Globally students solved 49,8% of all word problems. By using a variable they solved only 26,2% of them.

5. Conclusion

It is remarkable, that from among 158 students, who understood the assignment of the eight word problem (and also correctly solved this problem), 92,7% decided to find the solution only by guessing numbers. Also by the second word problem 43,4% of all successful solutionists found the solution only by trying concrete numbers, i.e. without using a variable. So only two possibilities are left. Either our students can not use a variable by solving word problems (practical problems), or they feel very unsafety by using it and if they have another possibility to solve the problem, they rather choose this another possibility. Considering the effectivity of our students by solving the fourth word problem (43%), which was not possible to solve otherwise as by using a variable, we come to conclusion, that more than one half of our students simply can not use a variable. The research outcomes pointed out, that students perceive exercises related to simplifying of algebraic terms and solving of equations, resp. systems of equations separately from other tematic units and can not use achieved knowledge by solving practical problems.

From research outcomes result following recommendations for the praxis:

1. To create some word problems related to the simplifying and use of algebraic terms (f.e. similar to the second word problem) and integrate them into the mathematical schoolbooks.
2. In schools it is necessary to devote more time to word problems related to linear equations, systems of linear equations, simplifying of algebraic terms instead of redundant solving of typical exercises to these tematic units.
3. It is necessary to change the structure (spectrum) of word problems, appearing in our mathematical schoolbooks. Quantity of similar "standard" word problems should rather be substituted by unstandard word problems, which force student to create an unstandard mathematical model.
4. It is inept to classify (by teaching) word problems into: "work w.p.", "mixture w.p.", "age w.p.", etc. After such classifications students do not think how to create an appropriate model, but which model is used by this type of word problems. That necessary leads to formalism.

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PROJECTION OF SPATIAL FORMATIONS AND SPACE IMAGINATION AT PRIMARY SCHOOL

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Abstract. The contribution deals with projection of space objects to plane by primary school pupils. It explores in detail the mathematical approach and attempts to compare it with art approach, both with respect of space imagination development. It experimentally explores the way in which space is projected by pupils without projection methods knowledge.

ZDM Subject Classification: D30.

1. Introduction

One of the most complicated situations requiring spatial imagination is the projection of 3-D object into plane. It is performed by various projection methods, which are taught at second stage of basic school. As also the primary school pupils live in 3-D space, we were interested, how they project 3-D objects and which of the projection methods are close to their point of view.

2. Experimental part

A pre-experiment is described in this contribution. It explored how are the 3-D objects projected by pupils 9 – 11 years old, who were not introduced to projection techniques yet. We expected that they will attempt for their own projection of space or that they will use skills from art education, where pupils deal with projection of space from the lowest grades. Because of that, experiments were carried out in mathematics and with a time delay also in art lessons.

In the introductory experiment pupils were required to select two out of sets of tasks and to attempt to project the space objects. We were interested, to which projection methods their work will be similar to. We expected various types of free parallel projection, linear perspective, one of the projection two-plane projection, top or front view, or possibly other own means of projection. We explored the differences between boys and girls and also between older and younger pupils.

After this work these pupils should display set of solids based on models in both Art and Mathematics lessons. A survey on methods and success of projection was made to see the improvement from pre-experiment.

3. Results

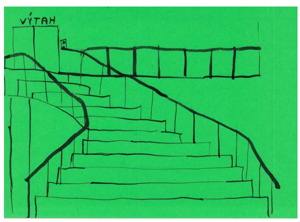
In the pre-experiment we explored how the space aspects were expressed. All samples named as "without space expression" were drawn as front view. Some suggestions of space were most frequently present in solids with circle base. Cubes and parallelepipeds were drawn with side wall only. For expression of space all pupils used (more or less successfully) rules of free parallel projection.

Task: Draw by ink blocks of flaks, stairs, furniture.

Sample solvings 1:

2. gr.

5. gr.



2. gr.

The differences between 2^{nd} and 5^{th} grades are not distinct, 2^{nd} grade girls succeeded even better than 5^{th} grade girls. Boys performed better than girls.

After targeting projection of 3-D objects in projects "Cube" and "Cuboid", pupils attempted to express space more successfully, but with big differences between mathematics and art lessons. We found interesting, that in mathematics all pupils but one used top right view, no-one used shading, lines were thin and most pupils asked for permission to use ruler. In art lessons the projection was more real, surprisingly also top left view and traces of center axonometric projection, usually depending on the position of pupils to the model. Drawings were more distinct, mostly colored and shaded.

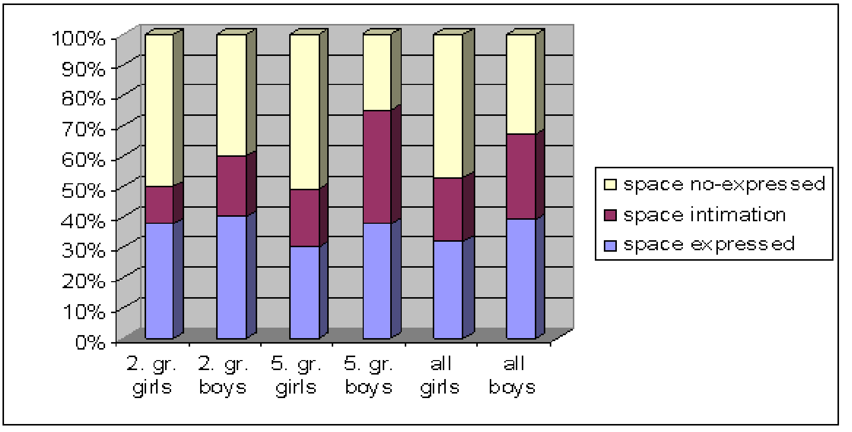
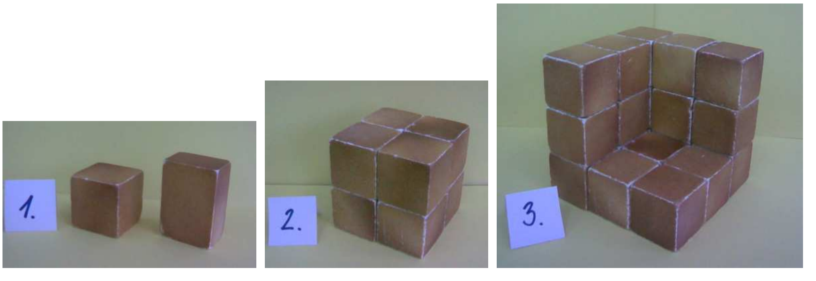
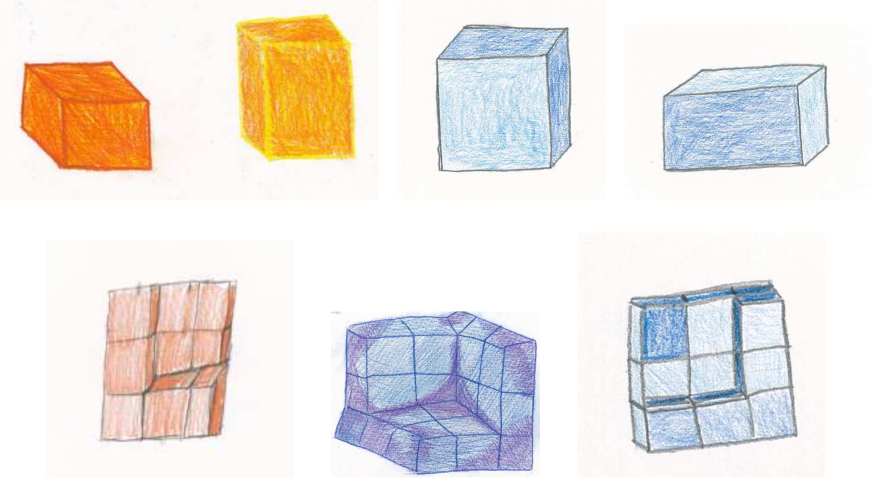


Fig. 1: Expression of space by school grades and sex (%)

Task: Display models of solids in art lessons.



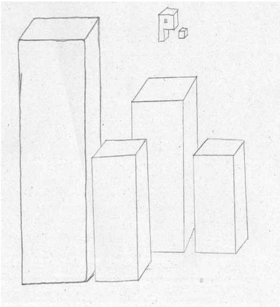
Sample solvings 2:

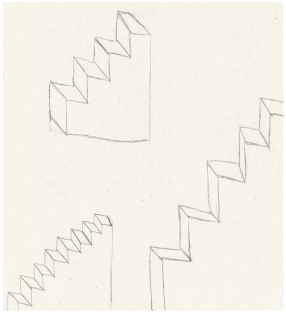


Task: In Mathematics draw blocks of flaks, stairs, furniture.

Sample solvings 2:

2. gr. 5. gr. 2. gr.





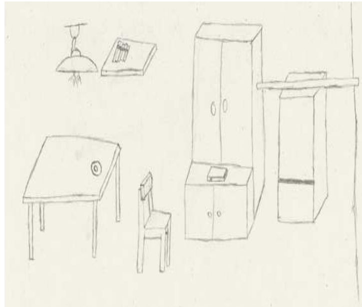


Figure 2 shows the general improvement in space expression compared to pre-test in art lessons for boys and especially for girls.

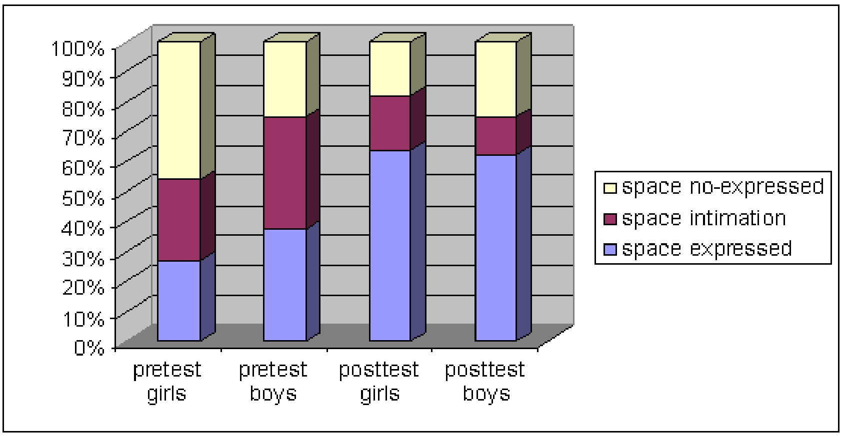


Fig. 2: Comparison of space expression in introductory and final experiment (%)

Term "the same" includes all pair of drawings where the same projection method was applied. It means that pupils used the same (front view, submission of space or expression of space) in mathematics and art lessons. According to figure 3, over half of pupils used the different approach in mathematics and art lessons.

We consider interesting also the difference of approach between boys and girls. While girls preferred projection based by real models, draft drawings were sufficient for boys.

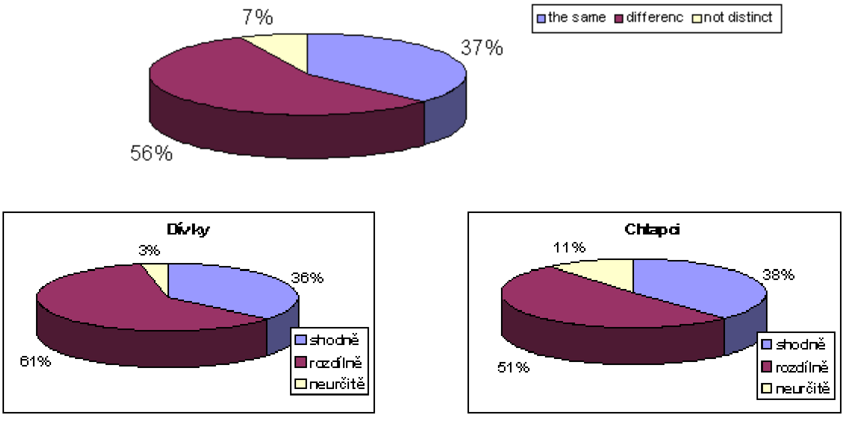


Fig. 3: Comparison of space expression in mathematics and art lessons (%)

4. Conclusion

The contribution suggests that projection of 3-D objects to plane is not an easy task for first stage pupils. They were more successful in projection of single solids. In order to display a set of solids, significant part of pupils retreated to front view. None of the pupils used top view.

Expectation that boys will manage projection of space objects better was not confirmed. First stage pupils use in projection only experience. The precision, typical for girls at this age, led girls to better results. On the other hand, boys did not need the real model to express the space – draft figure was sufficient for them.

We confirmed the expectation, that there are differences between projection methods used in mathematics and in art lessons. Pupils Express space Berger in art lessons than in mathematics, were they probably feel obliged by presented rules and methods.

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TEACHING OF STATISTIC BY PROJECT METHOD

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Abstract. In this contribution we present our first experiences with teaching of statistics on 12-13 years old pupils in Košice. Fundamental statistical methods such as collection of data, classification and processing of data and making conclusions were presented to pupils by means of project teaching. The theme of the project was searching the answer to the question - Who will be the third Slovak Superstar?

ZDM Subject Classification: K40.

1. Introduction

Engorgement of our world with data leads to the need of creation of stochastic thinking in primary school. For ordinary people it is not easy to orient themself in this plenty of data. Maybe this is the reason why the word statistic has become very popular. We often find it in the titles of newspapers or in any advertisements, where the collected data are interpreted by various means. Probably the wrong interpretation of data gives the impetus to the following declaration: There are three kinds of lies: ordinary lies, brazen lies and statistics.

That is the reason, why we consider that data analysis is very useful for pupils in primary school.

In our opinion the project teaching is the most suitable method for the first touch with data analysis. Such project, which is in interest of pupils, gives the possibility to realize the whole procedure of statistical reasoning search. This procedure is divided into three stages:

- data collection,
- organization and displaying of relevant data,
- conclusion from data.

2. Methodology

Pupils of the age of 12-13 years from a primary school in Košice participated in this project. We note, that this pupils already had experiences with the concept of circle diagram and histogram.

Characteristic of project

This project was short-termed, was realized mostly in the school. Pupils worked in groups consisting of two or three persons. They collected and processed data. Results were presented on posters and discussed in the classroom.

Planning of the project.

This kind of project is suitable for younger pupils, too. They should have been able to collect data and also been able to visualize data in form of diagrams. The criterion for the evaluation of this project was chosen together with pupils as follows:

- Precision of the graphical visualization
- Aesthetics of the whole project
- Delivering of the project in prescribed time

Aims of the project.

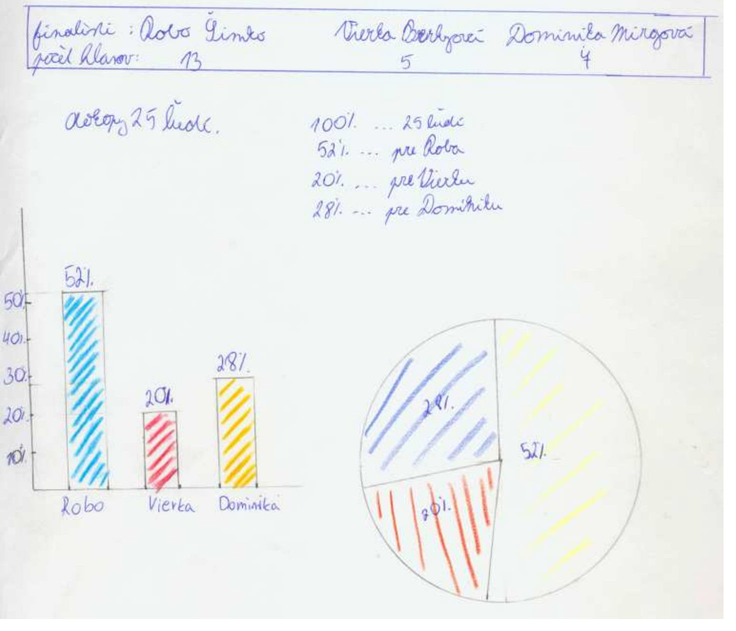
- Personal experience of pupils with data collection
- Reflexion on a notion of random choice
- Reflexion on possibilities of convenient question connected with processing of data and with conclusion from data.

During this project we wanted to know the answer to the following question:
Who will be the next Slovak superstar?

3. Experience with realization of project

Project,which is described in this paper, was a spontaneous and natural reaction on discussion between pupils in the classroom. This discussion originated during semifinal of the competition The Third Slovak Superstar, and in this time there remained three candidates for triumph. With enthusiasm pupils split up into groups containing two members, only one group had three members. On the lesson they arranged who will obtain information from various grades, and they also specified number of addressed respondents. One week later they already collected data and prepared an diagram for an easier orientation in data.

On a collective lesson we processed collected data and we made a record on a blackboard, which corresponded with the point of view of the school. Robert Šimko became the unambiguous winner between tree remaining candidates.



Example of data processing ...

After evening results in television we found out that "OUR" candidate was eliminated. Next day a conversation began in the classroom, in which pupils formulated interesting answers to a given questions. How is it possible, that Šimko was eliminated and he is not in the find? How to make an inquiry, in order that our results correspond to the reality? Between the first answers were the following:

- *Adam:* Ask all the people.
- *Boris:* Not all, it is sufficient to ask all noting people. But on the other side someone can send more than one vote ...

After this remark from one of the pupils we observed that they lost their trust that it is possible to predict the winner. For a restart of the discussion we gave them the following questions: *How does it work in the Statistical Office? Also, before an election day all people are asked? Only those people are asked, which are going to vote? Suppose that we are in the role of Statistical Office. How we choose a sample which we want to ask?*

In the following discussion pupils started to formulate their observations about a random choosing of sample with the emphasis on a group of voting people and also to the questions which should be asked with respect to the best possible correspondence of results of inquiry with reality:

- Will you vote on a superstar competition?
- Who will obtain your vote?
- How many votes do you want to send to your favorite?

4. Conclusion

In 1982 The International Statistical Institute (ISI) organized conference (one of the four International Conferences on Teaching Statistics). The main idea of this conference was to specify goals of teaching statistics for primary and secondary school. These goals were formulated in two items:

1. Children should become aware of and appreciate the role statistics plays in society. That is, they should know about the many and varied fields in which statistical ideas are used, including the place of statistical thinking in other academic subject.
2. Children should become aware of and appreciate the scope of statistics. That is, they should know the sort of questions that an intelligent user of statistics can answer, and understand the power and limitations of statical thought. [4]

In our opinion this draft of such project submit a proposal to teachers for accomplishment of the main goals mentioned above.

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HOW PUPILS WORK WITH MATHEMATICAL MODEL

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Abstract. This article deals with the problem of mathematical model interpretation. We were concerned about to what extent the students are able to formulate text of task when they are given solve of a particular mathematical model. We carried out the survey and that is the subject of this article.

ZDM Subject Classification: B10, C70, D70, M00.

1. Mathematical modelling

By mathematical model we understand an abstract model, which uses scientific notation to describe specific situations. That means that the model "represents" this particular situation. This expresses certain corresponding of objects, states, actions, processes, etc., which can be found in both, the situation and the model. This relation, however, is not always bilateral /4/. Thorough pondering of whole information specified in a problem and according to this information assembling of mathematical (abstract) model using the form of mathematical and logical relations and other presentations that reliably describe given situation represents the core of mathematical modeling. Modeling proces could be devided into three periods:

- Identification of ways out (results) of model situations,
- mathematical model cration,
- created model verification.

Identification of ways out (results) of model situations is the initial modeling period in which relations between various ways out are characterized mainly. The first and essential thing to do is to decide which input information is relevant in modeling process and it is necessary to include it into and on the other hand which information can be omitted. Competences related to working with information play very important role in this period. Following period is the period of mathematical model creation i.e. conversion of gained information in the first period into mathematical language - mathematization. The result of this action are various mathematical representations: various equations and non-equation, predicational functions, graphs, geometric figures. This is the most important period of modeling process and according to experience gained during realization of various surveys in this area it is considered to be the hardest one too.

The last period is the period of created model verification. In this period model adequacy is verified i.e. if it corresponds to given situation. Model must be clear, not contradictious, all logical principles must be kept valid and it must describe starting situaltion adequately. In this period where the mathematical model is interpreted retrospectively, dematematizaton is essential. It is also important by interpretation of solving of the model - explanation of gained solutions in the same language the original problem is formulated. We do not consider problem solution and interpretation to be a part of modeling process itself.

We hold the opinion that the basic feature of a mathematically literate individual is the competence to identify the problem in a simulation task, present it and make a mathematical situation model for solving the problem. He must be able to define starting points, find appropriate mathematical terms (structures, representations) related to the problem, gradually eliminate elements of reality (mathematization), solve mathematically formulated problem and transform the solution into the language of a real situation (demathematization). All these different kinds of competence are closely related to mathematical modelling and together they group into one of the most important categories of key competency that is the competence in problem solving.

The described way of viewing the core of mathematical education leads us to the conclusion that not the range of mathematical content, but sufficient development of cognitive activities is an essential part of mathematical education. That means a shift from cognitive and informative aims to functional and operational aims.

Word tasks, which reflect real situations can be an effective means of mediating operational knowledge. However, such tasks are scarcely used in schools. The students usually solve only these mathematical problems, which practise the knowledge presented by a teacher, and which are artificial and often abstract. The only aim is to find the solution. The discussion about the conditions for solving is rare. The solution of a quantitative task is rarely understood as mathematical modelling of a specific situation, and almost never is the solution repeated in order to find how the model changes under different conditions. What is more, the students never solve tasks where they are given the mathematical model to formulate text of task. Such this task was the tool of our survey.

2. Survey

For our survey we chose a group of 15-year-old students because they have a sufficient level of competence for solving the tasks presented. The survey took place in January 2007 in seven classes at Gymnasium.

Students solved this problem: *Formulate text of solved task.*

x ... green tea

y... black tea

$$x + y = 50$$

$$220x + 300y = 240.50$$

3. The outcome of the survey

In this part we list the student's formulations of the problem. 229 students took part in the survey, however, 37 of did not complete the task. There were also 7 students who formulated the task as follows: *"Solve system of linear equations."*

Based on the problem formulations of 126 students we can say that they are not familiar with the interpretation of a mathematical model of a system of linear equations with two unknowns. 43 of them formulate only the first half of the problem: *"John has green and black tea. He has got 50 packets of green and black tea altogether. Find out how many packets of green and black tea he has got."*

It seems that these students had forgotten about the second equation. Another 47 students work with both equations but forget the right side of the equation: *"A mother bought 50g of green and black tea. For the green tea she paid 220 crowns, for the black tea she paid 300 crowns. How much is 1g of black and 1g of green tea?"*

If we wanted to make a system of linear equations from this formulation, we would not be able to complete the second equation. The students probably were unable to formulate the part of the equation 240 x 50. It is possible that the number of such formulations would be lower if there had been number 1200 instead.

We did not expect a formulation with proportion (3 students): *"There are 50g of tea in a packet. How many grams of green and black tea are there? (We know that in 240 packets the teas are in proportion 220:300."*

These students were aware of the second equation and they tried to include the information in their formulations. They used proportion. We think that they cannot use the ratio interpretation correctly. 33 students used in their formulations all data from both equations. From the following formulations we are not able to make a mathematical model which we could use to solve this particular situation: *"We have green and black tea at home. They weigh 50g together. The green tea weighs 220g and the black tea 300g. How much do 240g of green and black tea weigh?"*

Fifty-nine students understood the given mathematical model: Twelve formulations are a little bit inaccurate. They would be good if we completed the first sentence with the information about a packet: *"Green and black tea cost together 50 crowns. 220 packets of green and 300 packets of black tea cost 240 times more than one packet of each. How much is green and black tea?"*

47 students decided not to waste words, their formulations are simple but clear. From this formulation we can form a mathematical model that is identical with the model given: *"If we buy a packet with one green and one black tea, we pay 50 crowns. If we buy a maxi pack with 220 packets of green and 300 packets of black tea, we pay 240 times more. How much is green tea and black tea?"*

The formulation of another student is as follows: *"New types of tea were bought for a local tea-house. The owner decided to sell it in a pack of 2 different types and so they weighed it. The pack of one black and one green tea weigh 50grams. If they wanted to weigh all 220 packets of green and 300 packets of black tea in such way, they would have to do so 240 times. How much do one packet of green and one packet of black tea weigh?"*

This formulation corresponds with the given mathematical model; however, it is not a real life situation. It makes us ask a question: *"Why would anyone do that?"* We were glad to find some original and creative formulations. For example: *"The shop ordered 220 packets of green and 300 packets of black tea. Jane bought 1 packet of green and 1 packet of black tea. She paid 50 crowns. How much did she pay for the black tea and how much for* *the green tea if we know that the shopkeeper paid for the ordered with 240 50-crown notes?"*

4. The survey's summary

>From the quantitative results' analysis we can see that only one quarter of the respondents was able to interpret the given mathematical model. We consider the three following factors to have influenced the survey's outcome.

- In Mathematics lessons, the emphasis is put on solving algorithm of problems, which is one-way orientated, as seen in this scheme: Text of problem, Notation (+ analysis), Creating of mathematical model, Solving of mathematical model,True-false test, Answer.
- Some students may understand a true-false test as a one-way verification of a problem solving.
- Insufficient motivation as the student's work was not graded.

Pupils have the greatest problem with the first period of mathematical modeling. Identification of ways out (results) of model situations is the most problematic competence. On the other side interpretation of model is underaverage developed too. We think that teachers should verify the model during lessons. It is the most unkept part of modeling.

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FROM RESEARCH ON DEVELOPING CONCEPTS OF DIFFERENT TYPES OF INTEGRALS IN STUDENTS OF PEDAGOGICAL STUDIES

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Abstract. During a lecture of the mathematical calculus we discuss the various kinds of integrals. It creates the image of these notions in consciousness of students in the result of this studying. The results of investigations over computability of these images with the notions of discussed integrals will be introduce in the work.

ZDM Subject Classification: I15, I55, D75.

1. Introduction

A lot of scientific centres which deal with didactics of mathematics, both in Poland and abroad, conduct research into understanding mathematical concepts on different levels of education. In their context researchers very often talk about concept images developed in students' minds during a cognitive process. A concept image is defined as patterns of thought, procedures, intuitions and facts assumed to be true as a result of a logical analysis or accepted as valid though not necessarily reflecting the intuition (Bugajska-Jaszczołt, 2001, Bugajska-Jaszczołt, Treliński, 2002, Gunčaga, Powązka, 2006, Major 2006a, 2006b, Powązka, 2006, Przeniosło, 2001, Sierpińska, 1985, Tall, Vinner, 1981). Matching images and their concepts appropriately depends on students' motivation and cognitive possibilities as well as teachers' didactic skills.

>From 2004 to 2006 in the Pedagogical University in Kraków I conducted classes in mathematical analysis during which I observed difficulties students had learning the basics of the lectures. This paper is devoted to concepts related to different types of integrals. The integral calculus has many important applications in both mathematics and related sciences, for

example physics. During lectures on mathematical analysis students learn first to integrate functions of one variable (in the first year of studies) and then to integrate functions of several variables (in the third year of studies). It is then that non-oriented and oriented multiple, curvilinear and surface integrals are covered. These are mostly Riemann integrals.

For a teacher of mathematics the concept of the Riemann integral is important due to its connection with the concept of the Jordan measure, which is covered in school at a predefinition level.

The tools used in the research included:

- a questionnaire on multiple, curvilinear and surface integrals,
- a set of questions and problems.

The research method consisted in analyzing products, i.e. the above–specified materials, with regard to the degree to which the concepts related to the integrals listed above were assimilated and understood. The research aimed at finding answers to the following three research questions:

- Do third year students of mathematics notice the process of generalizing and extending mathematical concepts which they have learnt before?
- What images of the different types of integrals covered during the lecture are developed in students' minds?
- What difficulties did students have using these concepts and what do these difficulties result from?

The answer to the first question can be found in a paper by Zbigniew Powązka and Lidia Zaręba (Powązka, Zaręba 2007). This paper presents answers to the second and third research questions.

2. Elements of concept images of different types of integrals revealed during the research

The literature (Bugajska-Jaszczołt, Treliński, 2002, Major 2006b) identifies different components of a concept image, which include: an intuition and association base, facts, performance tools, situational contexts, communication methods.

The research questionnaire asked the students to provide the following for each integral type:

- application of a given integral,
- examples of two problems related to a given integral,

- examples of problems related to a given integral with which students had difficulties.

Analyzing the research material revealed elements of the intuition and association base for each integral type and also made it possible to figure out situational contexts for each of these integrals. Thus a number of difficulties which students came across while studying the material emerged.

2.1 Revealed elements of a concept image of multiple integrals

Multiple integrals were defined first on a rectangle and then on regular and normal sets. In order to calculate them the Fubini's theorem and the change of variables theorem were applied. The lecture also covered applications of these integrals for calculating the volume of a solid, the surface area of a surface patch and the surface area of a flat figure. A number of physical applications were also specified.

For the multiple integral the research subjects indicated the following elements of the intuition and association base: the volume of a solid (13), the surface area (14), the lateral surface area (7), the surface area of a flat figure (5), coordinates of the centre of gravity (5), the moment of inertia (3), physical applications (2), the vector area (1), the density (1). The numbers given in brackets above show the frequency with which a given element was indicated by the research subjects.

By analyzing the answers given it can be concluded that although different applications of multiple integrals were provided calculating the volume of solids and calculating the surface areas of flat areas or surface patches were most common. Note also that similar to the research on the definite integral (Powązka 2007) some of the research subjects failed to specify exactly what applications they have in mind (e.g. calculating surface areas, physical applications).

Situational contexts include also problems which the study subjects associate with a given concept. For this reason the students were asked to formulate problems in which a multiple integral was to be used. To compose the problems the following methods were employed:

- a specific integral to be calculated was provided:
  - *Calculate the integral* $\iiint\limits_V dxdydz$, $x^2 + y^2 = b^2, x^2 + y^2 + z^2 = q^2$.
- a task was specified but no formulas of the figures in question were provided;
- a specific problem was formulated:
  - *Calculate the volume of a sphere described by the following equation* $x^2 + y^2 + z^2 = R^2$.

According to Krygowska's typology (Krygowska, 1977) students' problems can be ranked either as problems-exercises or problems-simple applications of a theory. Unfortunately not always were problems formulated correctly. The examples below show students' difficulties.

Example 1

• *Calculate the area created as a result of the function* $f(x, y) = x^2 + y^2$ *intersecting the axes of the system of coordinates.*

The figure described in the problem is an unlimited set with an infinite area. The axes of the system cross the set in the beginning of the system. The author of the problem used the concept of a function interchangeably with the concept of its graph.

• *Calculate the volume of a solid limited by a cylinder whose equation is* $x^2 + (y - 1)^2 = 5$ *and a sphere whose equation is* $x^2 + y^2 = 16$.

The author of this problem probably believed that the solid described with the equations above is limited. He unfortunately forgot what the equation for a sphere is and as a result considered two cylinder surfaces with parallel axes which do not form a limited solid.

One of the mistakes which presumably result from poor spatial imagination is to consider surfaces or solids which are not limited. The problems suggested by the study subjects included also such which do not require the integral calculus to be applied but only knowledge of geometry on junior high school or high school level.

Example 2

• *Calculate the volume of a solid limited by the surfaces of* $x^2 + y^2 = 3$ *and* $z = 1$.

The solid described in the problem is a cylinder which is not limited from the bottom. Probably the author of this problem assumed that the base of the cylinder is contained in the Oxy plane. This assumption seems quite obvious as cylinders were limited solids in the school education acquired so far. If the author had additionally specified in which plane the other base of the cylinder is contained, then solving the problem would not require the integral calculus to be applied as the author learnt the formula for the volume of a cylinder already in the junior high school.

• *Calculate the surface area of a figure defined by the following formulas:* $y = -x + 5$, *where* $x \geq 0$ *and* $y \geq 0$.

In this problem we have a right-angled triangle located in the first quarter of the coordinate system. The area of such a triangle also does not require the integral calculus to be applied and even if the integral calculus was used, then only a single integral would do.

• *Calculate the following area* $x + z = 3,\ x = 1,\ y = 1,\ z = 2.$
This area is unlimited. For example, if the plane $x = 0$ was added, we would have a prism with rectangular or triangular walls. Then the integral calculus would also be unnecessary to calculate the area.

The study subjects pointed also that for them the fundamental difficulty is to imagine the area of integration or the area whose surface area or volume is to be calculated.

2.2 Revealed elements of a concept image of curvilinear integrals

This section presents results obtained in connection with elements of concept images of a non-oriented and oriented curvilinear integral that exist in students' minds.

This section presents results obtained in connection with elements of concept images of a non-oriented and oriented curvilinear integral that exist in students' minds.

During the lectures attention was drawn to the fact that a non-oriented curvilinear integral, i.e. $\int\limits_k f dl$, where k is a curve contained within $\mathbb{R}^n$ and $f\colon k \to \mathbb{R}$ a given function interpreted as the density of mass in each point of the curve, can be used to calculate the mass distributed over the entire curve. On the basis of the theorem which changes this integral into a single integral it is known that if the function f is identically equal 1, then a non-oriented curvilinear integral is equal to the length of the curve. However, in the context of applications of the integral in question this is a border case.

The research showed that only this application was remembered by some of the study subjects and only one person indicated the possibility to use the integral to calculate the total mass on a curve. Other students did not answer this question.

While composing their problems that would require calculating a non-oriented curvilinear integral the students used either of the two methods:

- they provided a specific integral to be calculated,
- they formulated a problem which needs an integral in order to be solved.

Writing problems correctly turned out to be very difficult, which can be proved by the examples below.

Example 3
The difficulties with formulating problems correctly consisted in:

a) not differentiating between a figure and its edge;
   - *Calculate* $\int\limits_k (x^2y^2)ds$ *over the rectangle* $[0,2] \times [0,3]$.

b) omitting the symbol of a curve in the symbol of an integral;
   - *Calculate* $\int (xy)ds$ *where* $y = x^2$, $x \in [0,1]$.
   - *Calculate the non-oriented curvilinear integral* $\int (y)ds$, *where* S : $x^2 + y^2 = r$.

c) omitting the formula of a curve over which integration is to be performed;
   - *Calculate* $\int\limits_s xds$.

d) using a symbol of a different integral.
   - *Calculate* $\int (x+y)dx$, $y = x^2$, $x \in [0,3]$.

The author of the problem in point a) uses the symbol of a non-oriented curvilinear integral and probably means the perimeter of a rectangle and not the rectangle as such. Authors of problems in point b) forgot to place the symbol of a curve over which the integration was to be performed under the symbol of the integral. The problem in point c) does not specify over which curve integration is to be performed. In problem in point d) a symbol of an oriented curvilinear integral was used instead of the one for a non-oriented curvilinear integral.

The fewest mistakes were reported in the problems which were formulated with words without any integration symbols used. The difficulties connected with calculating this type of integrals included a complex formula of a subintegral function or a curve over which integration was to be performed.

Oriented curvilinear integrals were covered only for spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. The students were familiarized with its most important applications. As it is known, the integral $\int\limits_k Pdx + Qdy$ or $\int\limits_k Pdx + Qdy + Rdz$, where P, Q or P, O, R are given real functions of two or three variables and k is a curve contained in $\mathbb{R}^2$ or $\mathbb{R}^3$ respectively and connecting given points, is applied in physics to calculate work required to shift a material point in a vector area with components P, Q or P, O, R along the curve k between its beginning and end. Please note that for this shift what is important is its direction. In the space $\mathbb{R}^2$ if the curve k is closed and positively oriented and the functions P and Q fulfil the condition $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ on this curve, then in terms of its value the integral $\int\limits_k Pdx + Qdy$ is equal to the area limited by this curve. The research subjects showed that either they have no associations with the oriented curvilinear integral or these associations are wrong (three persons associated this integral with calculating the length of a curve).

In spite of this the students gave a number of interesting ideas for problems which use the integral in question. The following groups of the problems can be distinguished:

i) Problems which require the theorem that changes an oriented curvilinear integral into a single integral to be applied,
   - $\int\limits_k xdx + ydy,\ k : y = x^2,\ x \in [0,1]$.
   - $\int\limits_{ABCD} ydx + xdy + 2zdz$, where $A = (1,1,1)$, $B = (3,1,1)$, $C = (3,4,1)$, $D = (4,5,2)$.

ii) Problems which require the Green's theorem to be applied,
   - *Based on the Green's theorem calculate* $\int\limits_k ydx + 2xdy$, *where* k *is a positively oriented curve.*

iii) Problems which check independence of an integral from an integration path,
   - *Justify that the integral* $\int\limits_k dx+dy+dz$, *where* k *is a curve connecting the point* $A = (-2,-1,0)$ *with the point* $B = (2,1,0)$, *does not depend on the integration path.*
   - *Justify that the integral* $\int\limits_k (3x-y+1)dx + (x+4y+2)dy$, *where* k *is a curve connecting the point* $A = (-1,2)$ *with the point* $B = (0,1)$, *does not depend on the integration path.*

Similarly to non-oriented curvilinear integrals mistakes were made in this case as well. In addition to those enumerated in example 3 there were other mistakes connected with the orientation of a curve. This is shown by the examples below.

Example 4

a) Problems from group i) in general lacked the information about the orientation of a curve.
   - $\int\limits_k \frac{ydx-xdy}{x^2+y^2}$, *where* $k : x = a\cos\phi,\ y = a\sin\phi$.

   The author of this problem failed to indicate which part of a circle with its centre in the point $(0,0)$ and radius a should be taken into consideration while integrating and how to move around the circle.
   - *Calculate* $\iint\limits_S x^2dx + y^2dy,\ y = x^2,\ x \in [0,1]$.

   This problem does not define what the letter S means and confuses the symbol of an oriented curvilinear integral with a multiple integral.

b) The students made most mistakes in the problems from group ii).
   - *Based on the Green's theorem calculate* $\int\limits_k x dx + y dy$, *where* k *is a perimeter of a triangle.*

   The problem fails to define what triangle is meant and how its edge is oriented.
   - *Based on the Green's theorem calculate* $\iint\limits_D x dx + dy$, *where* $D : x = r\cos\phi,\ y = r\sin\phi,\ \phi \in [0, 2\pi)$.

   This problem confuses the denotation of an oriented curvilinear integral with a double integral and lacks information about the orientation of the curve.

The study students revealed the following difficulties related to the concept of an oriented curvilinear integral:

- determining the integration path,
- selecting the right parameterization of a curve,
- applying the Green's theorem to calculate oriented curvilinear integrals,
- calculating complicated single integrals,
- physical interpretation.

2.3 Revealed elements of a concept image of surface integrals

During the lectures attention was paid to the fact that a non-oriented surface integral is a generalization of a non-oriented curvilinear integral. The difference between them consists in the fact that instead of a smooth curve a regular surface patch with a continuous function of three variables specified on it is considered. This function was interpreted as density of mass distributed over a surface or as density of electrical charge. With this integral it is possible to calculate the total mass or charge on a given surface. Assuming that a subintegral function is equal to 1 for the entire patch, the value of the integral equals to the area of the surface patch.

It turned out that although 20% of the study subjects indicated these two applications, work was also calculated. Please note that half of the students did not provide any answer to this question.

Students' attempts to formulate their own problems related to the integral in question were slightly more successful. Similarly to the problems on curvilinear integrals, different ways of describing the problem were used and they included:

- providing a specific integral to be calculated,
- formulating a problem which needs an integral in order to be solved.

Half of the research subjects succeeded in formulating their own problems related to a non-oriented surface integral. In this case, like in the paragraphs above, there were problems in which an integral was to be calculated from a function specified on a given surface patch and there were also general problems without any details. The fact should also be noted that students quite often confused the symbol of a non-oriented surface integral with an oriented surface integral.

Below are difficulties that the students demonstrated to have:

- determining a patch when it belongs to two or three surfaces,
- making calculations after a non-oriented surface integral is changed to a double integral,
- applying the change of variables theorem,
- parameterizing a surface.

Moreover, the research showed that for the study subjects the concept of an oriented surface integral was the most difficult. None of them was able to provide any example of integral's application. It must be admitted that these issues were not thoroughly covered during the lectures and the students were asked to read the literature on the topic. However, I hoped that there would be a person to notice that by analogy to oriented curvilinear integrals and the Green's theorem it is possible to do the following. When a surface patch is positively oriented and limits the area V and the equality below occurs

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 1, \quad (x, y, z) \in V \cup S,$$

then on the strength of the Gauss-Ostrogradski theorem we have

$$\iint\limits_{S} P dydz + Q dxdz + R dxdy = |V|$$

where $|V|$ denotes the volume of the area V.

The problems formulated by the study subjects regarded only calculating a specific curvilinear integral oriented on a given surface patch. However, not always was the surface orientation specified.

The students also provided reasons of their difficulties related to the concept in question. These include:

- determining the orientation of a surface,
- calculating an oriented curvilinear integral when a surface patch is a sum of surfaces,
- determining directional cosines of a normal vector to a surface patch when an oriented surface integral is changed to a non-oriented one,
- imagining how a surface over which integration is to be performed looks like.

3. Summary

Based on the analysis of the research results the following conclusions can be drawn:

1. While studying different types of integrals the students could not always notice their analogy with concepts learnt before. For this reason their intuition and association base related to the concepts of these integrals does not contain many components. A guess can be made that the more abstract and the more distant from students' interests a concept was, the fewer associations they had.
2. Examples 2 and 4 show imprecise communication methods used by the research subjects.
3. For each integral type the students showed a number of possible types of problems related to these integrals. However, detailed problems contained different kinds of mistakes.
4. In the opinion of the study subjects spatial imagination has a significant influence on using the performance tools correctly.

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BRAIN-TEASERS AS AN INSTRUMENT FOR DEVELOPMENT THE CREATIVITY OF FUTURE TEACHERS

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Abstract. Usage brain-teasers in education mathematics especially then for evolution space imagination is devoted in this contribution. Attention is devoted to the brain-teasers above all, which exist in his tangible (mechanical) form.

ZDM Subject Classification: D50.

1. Introduction

The big attention to different mathematical game, brain-teasers and unusual exercises is devoted in big number of mathematical textbooks. We can find also considerable quantity publications with many unusual problems. It is not random but consequence of new era of computer techniques, cosmic years, atomic physics and with it contextual creative thinking. The space matrix imagination, which is necessary to evolve from tender age, takes an expressive role in this direction.

Already in early childhood children are leaded to speculation on the basis sufficient quantity inciting agents – different material, toys, building blocks whereby child teaches among others to perceive characteristics of space and understand regularities of movement. Brain - teasers we can rate among such inciting resources when in the mind of child complicated processes pass over through their solving. From the psychology point of view it means not only training imagination, memory, combination, logic and strategic judgments, relaxation and development creative and constructive, original thinking but also solving the brain-teasers strengthens e.g. possibility assembly, pertinacity and patience. Brain-teasers are however useful also otherwise. Watching child resolving brain - teaser we can do clear image of his imagination, originality, solution strategy etc.

Frame training program for education encloses know - how, skill, ability, postures and values important for personage development and exercise of each of member companies. Teaching mathematics has evolve except education, skill and attitudes also creativity of pupils.

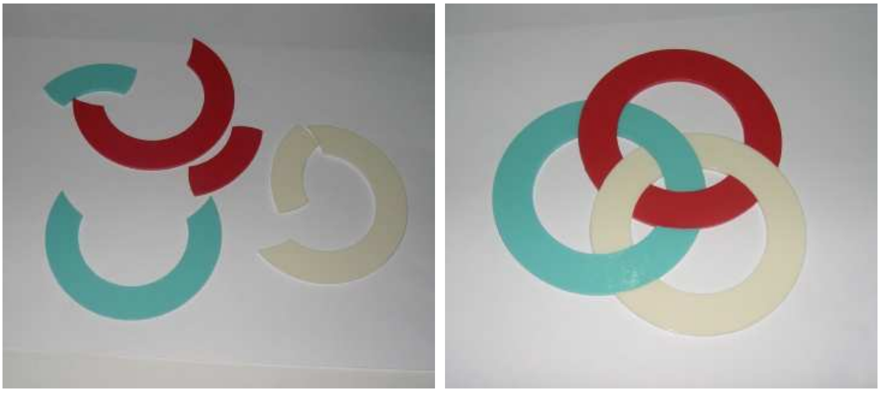
However creative pupil is able to bring up only by creative teacher. In contribution we have in mind the usage different types of brain-teasers for the development of creativity future teachers of mathematics.

1.1 Motivation of students-future teachers

Through seminars devoted to methods problem solving students were shown some topics for usage brain – teasers for development of space imagination. Subsequently they have ourselves get presentation solved engaged problem.

Jigsaw brain - teasers

Principle of jigsaw exercise (brain - teaser) is to put together from separated parts specified figure. Joggler it is possible produce from paper, plywood, plastic masses. At their production it is possible directly participate pupils. Formation joggler constructive imagination, combinational possibility and feeling for surface are practiced. Disjointed rings can be as an example.

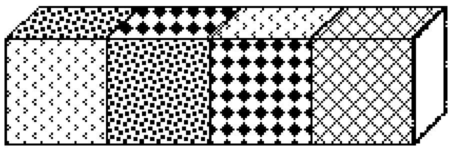


Mechanical brain - teasers

Mechanical - space - brain - teasers are often made from plastic, metal or from wood, which is pleasant on touch. Every time they surprise by its drawing, colour and subjects out of him made belongs to the favorite decorative subjects. Advantage of space brain - teasers is possibility to perceive the given formation from more views.

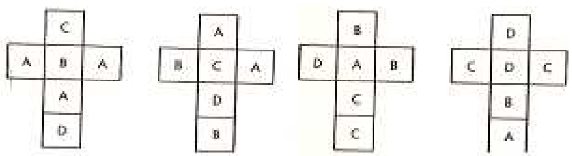
Colour cubes

Brain - teaser consists from four coloured cube. The colours on its sides are: red, blue, yellow, grey. Cube are coloured according to given rules.



The aim is to compare cube to the series and create ashlars 4x1x1 so all four colours on every rectangular side occur, [5].

Lay out the individual cube to the plains, we obtain next nets (the letters mean diggerent colour):



Solution is then intended according to keys:

Lower side:	DBCA
Top side:	BCAD
Near side prism:	ADCB
Reverse side prism:	CABD

Variant of this brain - teaser is using cube, where instead colours on sides ensigns of different countries may be placed, eventually we can use variant of six cube. However it is necessary to remark, that it does not mean classical dice-cube, but numbers are placed according to other keys, [1].

2. Students Topics

Manipulate activity at solving problem – development of space imagination.

In following parts topics students for development space imagination are submitted. For solving submitted problems we can characterize needed acquirements and motivate elements. All topics were processed in power point with usage interactive table. Personal presentation was accompanied handling created teaching aid.

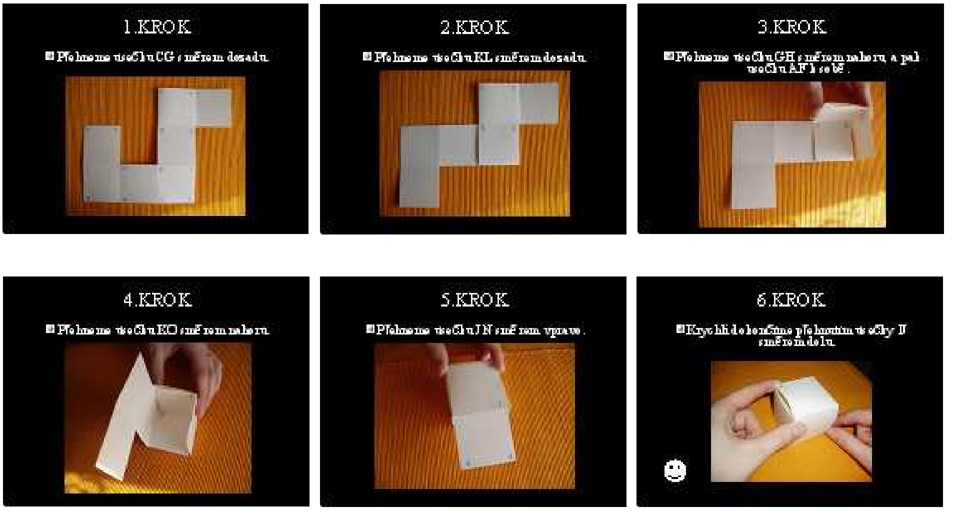
There are submitted the students topics for development of space imagination in the following part. No corrections in their paper are made.

Required skills:	***Motivation:***
– knowledge the concept of cube – hand-mindedness – imagination	– submitted picture – usage of modern technology – the way of processing given problem

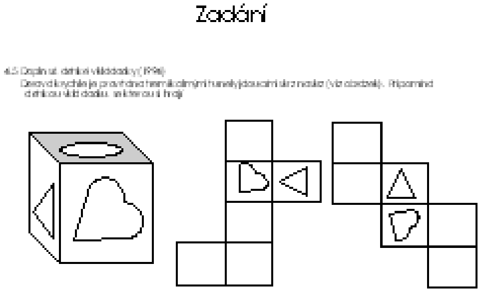
Example 1

Setting:
Draw and cut out square ADPM about side long 15 cm according to picture. Then cut through dash abscissa EF and cut out inside square. Could you from remaining eight squares plying only after marked abscissas put together mock - up cube about edge 5 cm?

Laying-out solving problem



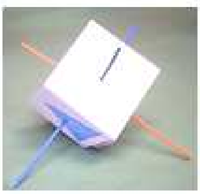
Example 2

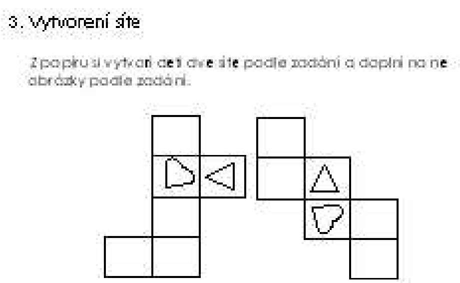


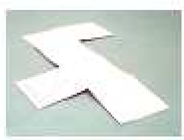
Setting:
Complete net child's inserting. Perforated cube is bored by three perpendicular tunnels going throughout.

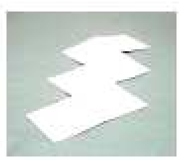
Laying-out solving problem:

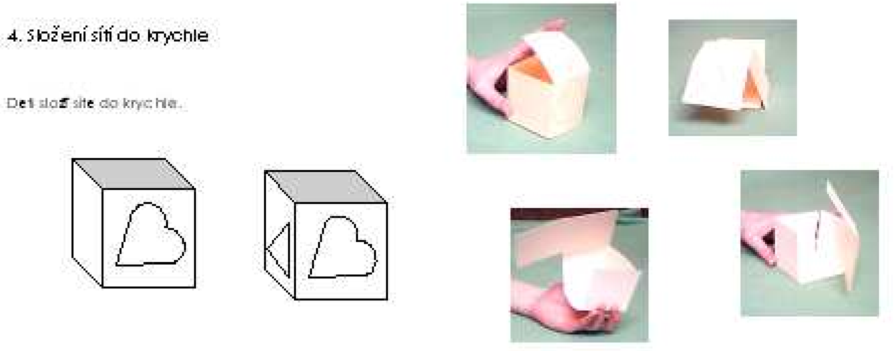
Remark: *We consider comparison hand - lettered picture with photo of the same situation as positive motivation for pupils.*











Remark:
Taking the photo of process composition (similarly as in the example 1) help the children to solve the given problem more easily instead of instruction only how to continue in making up the final product.

3. Conclusion

Only some topics for usage brain - teasers to development space imagination are submitted in contribution especially in initial motivational phase. Solving brain-teasers possibility class with how among manipulative so also mental activities, [4]. Some from solving strategies, which come across solving brain-teasers, is possible, mathematically describe then however it exceeds frame submitted benefit. Next topics it is possible draw from [6].

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PATTERNS, DISCOVERING THE REGULARITIES, GENERALIZATION AND ALGEBRAIC THINKING

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Abstract. Regularity is one of basic idea of mathematics. Rhythms and regularities can be practically found in every domain of mathematics: analysis, arithmetic, algebra, geometry and statistics. Solving the tasks concerning discovering and generalizing the regularity is one of ways to develop the student's algebraic thinking. In February 2008 I carried out research concerning discovering the regularity by 9-years old children. In this research I focused on mental process and interaction between the students.

ZDM Subject Classification: D52, H22.

1. Introduction and theoretical framework

In teaching mathematics in Polish primary school there is a little place for teaching of algebra and preparation of a student to algebraic reasoning. Curriculum for 4-6 grade students is geared to become familiar with arithmetics and good training of arithmetical skills. Meanwhile in world literature the thread of algebra teaching in the primary school is raised more and more frequently. Researches concerning algebraic reasoning of students on this level are conducted.

In order to both the teaching was effective and the knowledge was permanent the appearance of the web of interrelations among different elements of mathematical knowledge is very important (see [1]). Intuitive reasoning should be supported at once by the ordered knowledge.

The child learns/develops his/her mathematical knowledge through building its own cognitive structures, webs of interrelationships, and mental "maps" ([1], [3]). Accumulated experience enables to create so-called data set used by a child to build up its mathematical knowledge. The essential factors, which help a child to develop his/her mathematical knowledge, are interactions with his/her environment, particularly during the teaching-learning process (e.g. during mathematics classes). It is present during teacher-student interaction and student-student interaction, as well. Therefore it is very important that students should solve tasks together, and during this process they should communicate and exchange their own experiences.

Teachers approach to the subjects connected with algebra in a different manner. There are various ways of algebra teaching. One of the ways of teaching algebra in a primary school is ”superstructure of arithmetic”. The starting point is consideration in the domain of arithmetic. A student receives series of tasks, in which he/she has to discover occurring arithmetic dependences. And next he/she generalizes discovered rules. The last phase of solution of such task is giving the symbolic notation. In the behavior like this it is very difficult to separate arithmetic thinking from algebraic one. It is very hard to take hold of that moment when the student has stopped "to think in arithmetic way" and have started ”to think in algebraic way”. In some students the change of thinking manner appears earlier, in the others later. So the question arises: so if it is so hard to separate algebraic thinking from arithmetic one so maybe it is not worth doing it and it will be better to develop both in the same time?

Students usually have many difficulties with learning algebra in school. One of them is specific language of algebra, which usually is ”imposed” on students. Solving the task (para-algebraic or algebraic task) students first ”solve it for themselves”, during this process using ”the inner speech” - we think about the task, about the way of solving it; in thoughts we transform information and ”express” them for ourselves ([4]). When students have to present to others this ”their own” solution then they must cover from ”the inner speech” to ”the external speech” - in this moment the pronouncement of thoughts by means of words and verbalization of owner thinking process follows. Here the problem: how to call particular objects, which words to use often appears. At this stage students usually make use of the common language which they meet every day. Now the next stage follows, next cover, namely, the solution expressed in common language should be written in algebraic language by means of proper symbols. Students have difficulties with cover from common language to formal algebraic one. That language is so much difficult because the symbolic notations appear in various meaning. The letter in algebra on the one hand may be a variable and in the other hand it can be constant. On the one hand we treat it as a constant, on the other hand as an unknown. That diversity is very difficult for the student, particularly if algebraic language appears suddenly and one requires from the student to make proficient use of this language.

It seems that the good way to introduce students to algebraic thinking and algebraic issues are tasks concerning regularities. Why regularities? It is a basic issue for creation of mathematics, very interesting and it often appears in mathematics, but in Polish teaching of mathematics it is rather given a short shrift. The idea of the rhythm and the regularity are a tendency in mathematics education. We can practically find the rhythm and the regularity in every domain of mathematics: algebra, arithmetic, geometry, and statistics. For example the theory of sequences and mathematic induction are based on the rhythm and regularity.

Many researches show ([2], [5]) that just through patterns and regularities one could introduce the student in the algebra world. We can find references to the description of research concerning discovering and generalization of noticed rules.

In Polish practice of mathematics teaching children meet regularities and rhythm in their early stages of education that is pre-school and primary school. These are mainly geometrical regularities connected with drawing patterns. A child is supposed to finish a given pattern. It is not expected that he/she discovers any mathematical rule behind it. He/she just completes the task neatly. The primary school students sometimes encounter arithmetical regularities (e.g. magic squares, triangular numbers) or geometrical (mosaics). However, most teachers treat these tasks marginally and they underestimate their importance and usefulness.

Polish teachers sporadically refer to that kind of tasks, there is not clear recommendation for teaching regularities in the curriculum of mathematics. Some traces of that kind of activities we can find in the curriculum called "Matematyka 2001". So students do not very often have to do with tasks concerning regularities, and especially with those tasks in which certain rhythm and regularity should be found. Most often during school learning students meet the tasks depending on usage of concrete mathematical knowledge, certain algorithms and schemas. Most often those are tasks like this: "check up, solve" and very seldom tasks in the kind of: "what will be when, what will be next". In that case will students be able spontaneously come off from "schematic thinking" (i.e. what formula do I use, what algorithm should I use) and to discover occurring dependencies in the task, to show the existing regularities?

2. The aim of research

Presented here investigation is the part of the series of research concerning perception of regularities by students on different levels. The results of Polish students on PISA test and one of PISA tasks (called "Apple trees") were the inspiration to take up this subject.

The aim of my research was to get answer for the following questions:

- Will 9-10-years old students be able to perceive mathematical regularities and if yes - in what way do they ”think” about regularities and what are their thinking processes while solving the tasks in which they have to discover and use noticed earlier rules?
- Will they be able to cooperate while solving the task?
- To what degree this common work will have an effect on the way of solving the task and discovering the regularities as well as using them in the task?

3. Methodology

Presented here research was carried out in February 2008 among students from fourth grade of primary school. The research consisted of four following meetings during which students were solving successive tasks. All the meetings were recorded with the usage of video camera. After carrying out the research the report was made. Students worked in pairs. Researcher talked with every group of students while they were solving the task. Twelve students from fourth grade of primary school took part in this research (9-10 years old children). Students have work sheets, matches (black sticks), pens and calculator at their disposal. Before the students started to work they were informed that they can solve this task in any way they will recognize as suitable; their work will not be marked; teacher will be videotaping their work and they can write everything, what they recognize as an important, on the work sheets. The research material consists of work sheets filled by students as well as film recording of their work and stenographic record from it.

The research tool was four sheets and each of them consisted of two tasks. Tasks were as follow: students make a match pattern consisting of geometric figures - once there are triangles and another time there are squares with side length equal to one match. In the first two sheets the figures were arranged separately, in the two remaining - connected in one row. The question was: How many matches do you need to construct 1, 2, 3, 4, 5, 6, 7 such figures? Results should be written in the appropriate table. In the second task there was question about the number of matches which are needed to construct 10, 25 and 161 such figures.

Patterns, which were the subject of the next tasks, presented as follow:

1. Independent triangles
2. Independent squares
3. Connected squares
4. Connected triangles.

The choice of the tasks and the order of sheets were not random. The problem was to check up if students will use their earlier experiences while solving the new tasks; will once elaborated strategy of solution be applied during next task. This task and the way of its presentation (four following each other sessions) were something new for students. Till this time they did not solve the tasks concerning perception of appearing rules and generalization of noticed regularities during math lessons. It was new challenge for them.

4. The results

The first two sheets were solved by students very quickly. They did not need to construct the pattern consisting of proper figures, they at once began to fill the table and next they answered the second question. They were able to give the rule according to which the pattern was constructed in the perfect way. Some students arranged only one figure (one triangle in case of the first sheet and one square in case of the second sheet). It was rather designation what kind of figure was concerned in the task rather than aiding oneself while solving it. Difficulties appeared during the work with the third sheet. The first difficulty related the expression "connected in one row". The next obstacle appeared while moving from the table to the second question. Students did not have so many sticks in order to continue arranging the pattern. Besides they gave consecutive values in the table and in the second question the "jump" took place. At the beginning the problem for students was to fill this gap. In order to give an answer they started to analyze previous solution of the task and the way of constructing the pattern

Strategies of solutions were as follow:

The first sheet: generally students arranged one triangle. They filled the table automatically using the rule: add three to the previous value.

Justification for this rule was as follow: for each new triangle we need three matches so it is necessary to add three. When they moved to the second question they changed the strategy. They did not add successive threes but they multiplied number of triangles by three. As a justification for using this rule they claimed: I multiply by three because every triangle has three sides. Students were able to apply discovered earlier rule for any number of triangles. They were able to express it in general language. When they were asked to write down the discovered rule they did it using only words, they did not write it in symbolic language.

<u>The second sheet:</u> students perceived analogy to the previous task. Some of them used the rule "multiply by three". Other students seeing the task claimed: "what, once again the triangles? Oh, no, this time there are squares. Well we will multiply by four." Solving the task took much less time than in case of the first sheet. This time students applied only one rule for the whole task: multiply the number of squares by four. As a justification for that rule they claimed: because one square has got four sides.

<u>The third sheet</u> was the challenge for the students. Getting down to solving the task students expected "repetition" of the strategy from two previous sheets. Before they read carefully the contents of the task they asked: "what do we arrange this time? Pentagons, hexagons?" After reading the contents of the task they started to arrange suitable pattern. They arranged usually the first three elements and then they moved to filling the table. Some of the students "read" from arranged pattern the number of used matches and they wrote down achieved value in the table. In order to fill successive fields of the table they were adding following squares to the pattern. At the third square some students perceived that it is enough to add three to previous value from the table in order to get the number of matches they looked for. But there were such people that continued arranging and counting the matches for all squares from the table. Other students perceived the strategy of pattern formation and they used it while filling the table. They discovered the rule: the first square with four elements, every next with three ones, so in order to give the number of required matches it is necessary to add three to previous one. After filling the table there appeared two ways of proceeding: continuation "adding the threes" to 10 squares or looking for "components" in the table, using just obtained data. "Adding threes" worked only for 10 squares. In order to find the number of matches which are needed to build 25 squares students started to analyze already possessed data. In the first step they factored the number 25 into 10+10+5. They made use of data obtained in the previous subsection: for

10 squares 31 matches. In this case students' reasoning was as follows: I know the number of matches for 10 squares, so it is enough to multiply this number by two and still add the number of matches for 5 squares. Thanks to it I obtain the number of matches for 25 squares which we looked for. Curiously enough, giving the number of matches for 5 squares students have usually spoken: "I need 5 more squares of three matches", they did not appeal to the table, where the suitable result was given. Students tried to carry out very similar reasoning for 161 squares. But in this case they were trying to represent 161 as the sum of $25 + 25 + 25 + \ldots$ At the beginning students were convinced about correctness of their method. Only discussion with the teacher as well as verification of the used method for data from the table (i.e. if it would be like this for 7) caused change of the manner of thinking and discovering suitable dependence. And then students perceived that some matches in their solution "duplicate". Students corrected mistakes and gave out new dependences: "the number of matches is the number of squares multiplied by 3 and still add 1", "the number of matches is the number of squares lesser about 1 and multiplied by 3 and still add 4", "the number of matches is number of squares multiplied by 4 and afterwards subtract the number of squares lesser about 1" (the last rule related to concrete numbers).

<u>The fourth sheet</u> also constituted the challenge for the students. However, they used here the experiences which they acquired during work on the third sheet and therefore work with the task proceeded much efficiently. Furthermore, trained by the experience from the work with the second and the third sheets students, before they were starting to solve the task, ascertained what kind of figures would occur in the task. Some students transposed strategy from the previous task at once, they modified it for requirements of existing situation. Other students discovered rules anew. They arranged the triangle pattern, counted the matches and afterwards they wrote the following values into the table. All students justifying the values written in the table were saying: the first triangle consists of three matches so for the one there are three, and then we add only two, because there still is one side of triangle so we need only two matches. Moving from the table (7 triangles) to the subsection 2a) (10 triangles) some students perceived that they need still 3 times by 2 matches. They were calculating: $15+3*2$, they did not calculate successive twos as it took place in the third sheet. And this way of proceeding they continued while solving next tasks. So for 25 triangles students' activity was as follows: from the previous task (subsection a) I know the number of matches necessary for 10 triangles. That is 21. 15 elements are lacking for 25 triangles. Thus I must add to 21 matches 15 times 2 matches that are 30 ones. That is for 25 triangles I

need 51 matches. Answering to the question about 161 triangles students ”started” from the number of matches for 25 triangles, and then they were adding (161-25)*2 to it and in this way they received looked for value. Their activity was not oriented on decomposing the number of triangles to components for which values could be read off from the table. They looked upon last considered numbers of triangles as a basis for their calculations. Students "added" to the pattern/row consisting of previous considered number of triangles the new one which consisted of ”lacking” number of triangles. This time students have remembered about ”duplicating” one side of the triangle. In the ”added row” the first element, the first triangle consisted only from two matches, similarly as all following. This strategy was correct but it presented a lot of difficulties for the students, especially when the teacher asked about the number of matches for much bigger numbers of triangles (i.e. how many matches did you need for 1254 triangles? And how many do you need for 149756 ones?) Students during the conversation with the teacher noticed that the changing ”initial value” does not allow giving the general rule, which they would use for any number of triangles. They also guessed that just as it was in the case of previous three sheets, the teacher will ask them about general rule operating in the task (some students said: now we need to write it somehow, like in a text message?). Thus they were starting to look for the common rule for all cases. Rules noticed by the students are: multiply the number of triangles by 2 and add one; multiply the number of triangles lesser about one by two and next add 3. Justification for this rule was as follow: the first triangle is built with three matches and each following only with two ones. One student gave completely different justification. It did not refer to the way of the pattern creation. Namely he said: here the whole time the number of squares is multiplied by two and then one is added. That is in the table and here, in this second task. I do not know why it is like this, but it works. So in order to calculate how many matches are needed for any number of triangles we should multiply this number by two and still add one. The student analyzed only dependencies among numbers from the table, he did not connect them with arranged pattern.

5. Resume

Featured task was something new for students participating in my research. Up to this time they did not meet with this kind of tasks, concerning discovering connections, searching regularities. Nevertheless they were able to discover many interesting things. Although not all students’ solutions were correct, it often brings many interesting discoveries. Maybe if during their studies students meet that kind of problems much more frequently they will be more sensitive on them. And they will be able to obtain from that task a lot of existing dependencies and connections.

Students from the fourth grade of primary school are able to solve para-algebraic tasks. They are able to perceive dependencies appearing in the task and use them correctly, generalize. They are able to give the rule which is applicable "for every element". Algebraic-symbolic language is unfamiliar for students at this stage of education, hence certainly the symbolic notation did not appear for expressed rules. It is not important for using the algebraic thinking and reasoning whether student is "good" in arithmetic. Good arithmetic knowledge can help but it is not a guarantor of success. Sometimes attitude for number result (that is making correctly arithmetic calculations) disturbs in discovering general rule.

Algebraic thinking appeared both with students good in school mathematics or these a bit worse. Thus maybe it is worth to develop algebraic thinking already in primary school on equal terms with arithmetical one?

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CAN E-PLATFORM HELP WITH MATHEMATICAL PROBLEMS PUPILS WHO ARE INTERESTED IN MATH?

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Abstract. In nowadays many new medias came in to school. One of them is platform of distance learning. Some pupils have big troubles with solving math's problems in normal school. The question is "Can e-platform help with mathematical problems?" I concentrate on pupils who are interested in mathematic. Some results of investigations will be show in the work.

ZDM Subject Classification: U73, D40, D53.

The basis of the this world, how Stanisław Lem claims, there is the mathematics. *All what is and what takes pride the human world, stands on mathematics* [5]. We meet with mathematics in many moments of our life. Then we teach mathematics we should teach perceive it in the surrounding us world and value its values. So the question appears: How should we teach mathematics?

Then we teach mathematics, we can not restrain only to the realization of the lowest aim – the delivery of define mathematical contents. Very important is to practice essential postures which will *be useful in various domains of lives. The teaching based on problem solving makes possible the realization of this aim. Such way of teaching contribute to initiation of self-education, produce the ability of perception and solving problems and verification of get results* [4]. All these skills are necessary on all levels of the education and in various domains of it.

The didactic equipment play important role in this process. As R. Howe said: *the success of the traditional teaching contributed to the rise the leaning on mathematics technology, which in consequence created the situation the traditional teaching is not suitable any more* [2].Unfortunately, pupils perceive lesson as little attractive or dull in spite of curious theme and interesting tasks very often. Teacher, board and chalk do not excite as television, computer or multimedia telephone. The present-day world of the technology speaks to us by images, animations and sounds. It stimulates our senses and imagination in the fuller dimension. So the next question appears: How should we use the accessible technology to make teaching of mathematics more effective?

It is possible to notice that majority didactic investigations described in the literature conducted on this problem, for example T. Ratusiński (2003) [7], H. Kąkol (2004) [3], K. Parcia (2005) [6] – concentrate on the possibility of utilization of computer programmes at school (in situations approximate to school, or directly at the classroom). Investigation made by R. Wojtuś (2006) [8] goes beyond the classroom situation but concentrates on long-term works. However, there is investigative deficit concentrated on problem of work with able pupil beyond the school.

In present-day times access to the Internet is more and more convenient. Every pupil in *the home refuge* almost can freely use the blessing of the global net. From the other hand, the various type of informative media such e.g. graphic calculators or computers appear on lessons of mathematics more and more often. Pupils have rather positive attitude for this kind of technology, so it is proper to assign question: If is it possible to take advantage of these media to induce pupils to the expanding one's interests mathematics in the free-time? The idea of the Internet Mathematical League based on e-platform became in 2006.

The e-platform is the informative system equip in tools serving for organization of didactic process in very interactive way. With aid of them we can prepare, pile and render accessible didactic materials, moderate led discussions, organize work in groups as well as make full statistics and control of process of instructing and teaching [9].

Moodle platform is one of the most popular e-platforms at present. The word *Moodle* is the shortcut from *Modular Object–Oriented Dynamic Learning Environment.* It is one of the most useful tools for not only programmers but also theoreticians of the education at present. *Moodle* can also be considered a verb, which describes the improvisational process of doing things as it occurs to you to do them, an enjoyable tinkering that often leads to insight and creativity. As such it applies both to the way *Moodle* was developed, and to the way a student or teacher might approach studying or teaching an online course [10].

Technical factors of the *Moodle* platform give wide capabilities of communications among equal leading and participants as well as among participants directly. The mentor of the course has the possibility of assistance whole group, or severally individual persons. He has also capability of observation of whole process of pupil's work on platform – exactly what and when anyone did. These features of these investigative instruments, it belongs to regard as advantageous in didactic research.

However the leading question stayed:

Can the e-platform help with work over mathematical problems?

Range of smaller questions has appeared on base of this leading problem, among others:

- Will the new remote form of the project encourage pupils to the investigative posture, assigning of question, contacting between them, conceding of indicator, deployment of knowledge?
- Will the present-day youth dedicate the part of their free time on work over mathematical problems?
- Will be this form of work over such tasks and problems enough motivation for intensive work over them?
- Is the e-platform the effective tool for leadership of the mathematical league?
- Will be the League such popular as others accessible goods of Internet?

Figure 1.

First edition of ”League of fans of mathematical problem”* has started in 2007 (fig. 1). The second edition has taken place next year. Both have been located on http://www.e-dlaszkoly.pl.

The project has been divided onto modules, various types of mathematical problems have been presented in each. There has been also varied forms of accessible helps to offered tasks, depending on their content, for example: drawings, pictures, graphs, interactive animations serving to the simulation of described situations or dictionaries supplies defining more closer given question. There have been also available Java applets – small programms serving the deeper exploration of the given problem.

The main leader of ”League of fans of mathematical problem” has been T. Ratusiński, however additionally special person – teachers – has been attributed each unit. This person main task has been assistance within established problem.

To be able to participate in the work on the League the pupil had to communicate with leader, gave him personal data, and received individual *name user* and *password* enabling logging in to the system. The platform gives possibility of edition personal data for each user. The participant can not only change the login password, but also introduce several essential information about one’s and add one’s picture to the profile. None of participants of the League is anonymous thanks to this. If someone has login, he could not only find out something about others users, but also can see as they look.

The *Moodle* platform gives leading person possibility of access to user’s more detailed profile. There are possibility of controlling users activity and progresses on course, following of their work. Teacher can observe how often and how long each pupil works on platform (fig. 2).

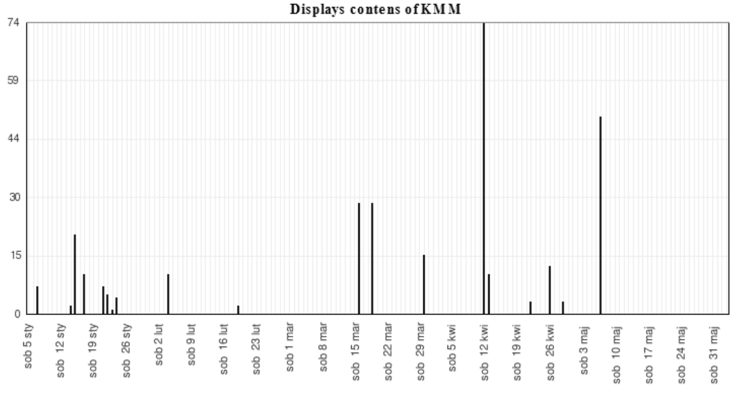
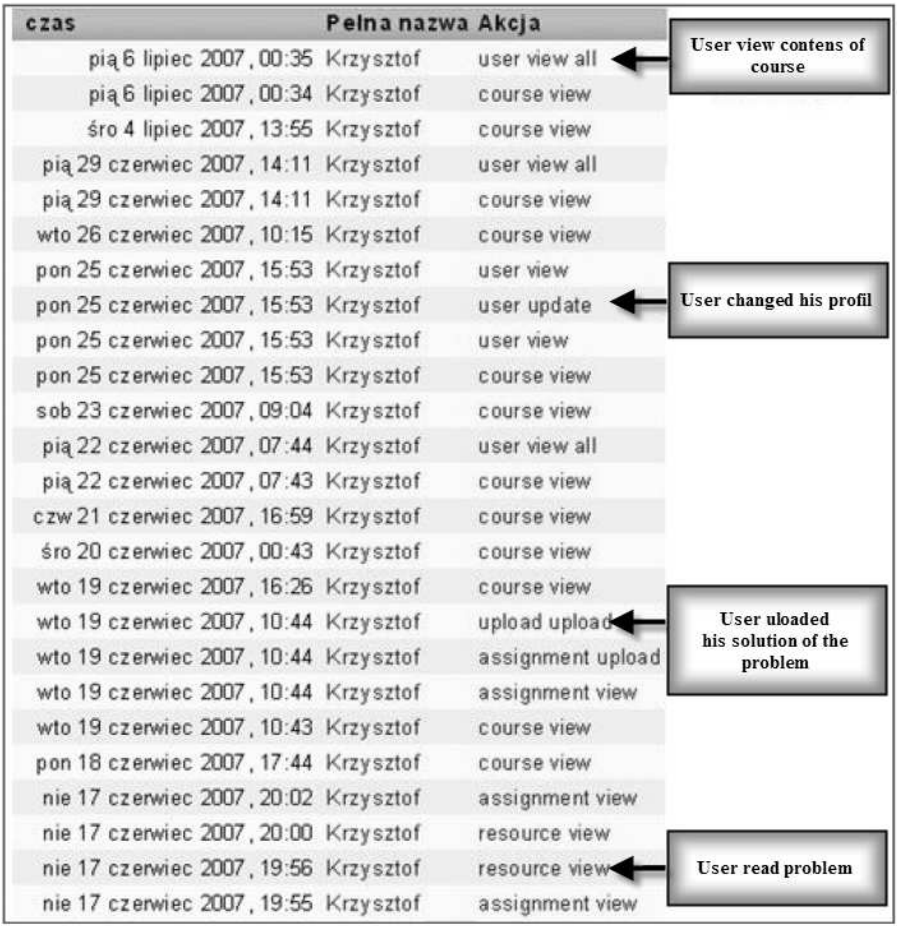


Figure 2.

*Oryginal polish name of the league is „*Koło Miłośników Matematycznych Problemów*”. Short id: *KMM.*

Leader has access to information about personal activity, can observe that tasks one's reviews, that forums interest him, which task has managed to solve, as well as how often concerns on established problem, even if he uses accessible assistance (fig. 3).



czas	Pelna nazwa	Akcja
pią 6 lipiec 2007, 00:35	Krzysztof	user view all
pią 6 lipiec 2007, 00:34	Krzysztof	course view
śro 4 lipiec 2007, 13:55	Krzysztof	course view
pią 29 czerwiec 2007, 14:11	Krzysztof	user view all
pią 29 czerwiec 2007, 14:11	Krzysztof	course view
wto 26 czerwiec 2007, 10:15	Krzysztof	course view
pon 25 czerwiec 2007, 15:53	Krzysztof	user view
pon 25 czerwiec 2007, 15:53	Krzysztof	user update
pon 25 czerwiec 2007, 15:53	Krzysztof	user view
pon 25 czerwiec 2007, 15:53	Krzysztof	course view
sob 23 czerwiec 2007, 09:04	Krzysztof	course view
pią 22 czerwiec 2007, 07:44	Krzysztof	user view all
pią 22 czerwiec 2007, 07:43	Krzysztof	course view
czw 21 czerwiec 2007, 16:59	Krzysztof	course view
śro 20 czerwiec 2007, 00:43	Krzysztof	course view
wto 19 czerwiec 2007, 16:26	Krzysztof	course view
wto 19 czerwiec 2007, 10:44	Krzysztof	upload upload
wto 19 czerwiec 2007, 10:44	Krzysztof	assignment upload
wto 19 czerwiec 2007, 10:44	Krzysztof	assignment view
wto 19 czerwiec 2007, 10:43	Krzysztof	course view
pon 18 czerwiec 2007, 17:44	Krzysztof	course view
nie 17 czerwiec 2007, 20:02	Krzysztof	assignment view
nie 17 czerwiec 2007, 20:00	Krzysztof	resource view
nie 17 czerwiec 2007, 19:56	Krzysztof	resource view
nie 17 czerwiec 2007, 19:55	Krzysztof	assignment view

Figure 3.

This tool delivers very important and valuable informations and spacious materials to the analysis.

There was thirty seven pupils who decided to take the part in both editions of "League of fans of mathematical problem". They were 15-18 years old, mainly from 3^{rd} class of gymnasium and 1^{st} or 2^{nd} class of secondary school. All persons have entered over mathematical problems voluntarily declaring wish work. Pupils came from the various sides of the Poland, mainly from Małopolska and Śląsk (Silesia) regions. There were persons from large cities such as Warszawa (Warsaw), Kraków (Cracow) or Bielsko-Biała, as well as from small villages such as Inwałd, Osiek or Wieliczka.

Main leader created thirty seven personal accounts on course. Only thirty five of them successfully logged in on platform, and thirty four persons took attempt of solution of any problem (fig. 4).

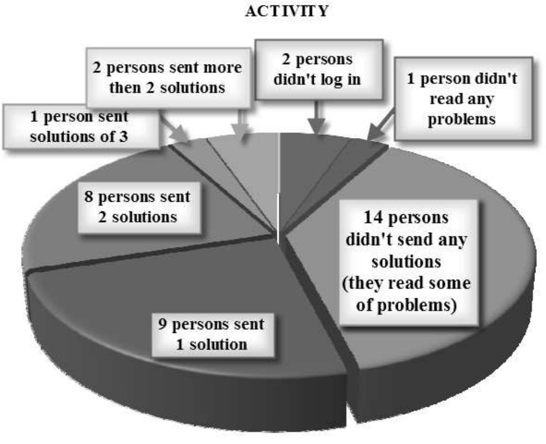


Figure 4.

Fourteen persons from them limited oneself to reading the content of problems being on platform. Some of them gave up, resigned from further work after acquainting with few first problems. They didn't appear on the platform any more. Others users appeared on it from time to time, but they didn't send any solution. The pupils from next group (nine persons) sent only one solution. Some of them appeared on the platform only few times but it was enough for them to solve one problem. The part of pupils have observed others problems, but they didn't send their solutions. Eight others persons sent solutions of two problems, one person solved three from all ten problems from the League. There were two of pupils who worked systematically and sent solutions for almost all problems – Christopher solved seven problems, Mark sent five solutions.

Every solution, attention, comment or question was noticed, analyzed and considered with attention formulating final conclusions.

What are results and observations after above the two-year work of the League?

Looking on presented above numbers relating to participants a doubt is born. The way of recruitment of the participants should assure that all persons have some kind of internal motivation for work with mathematical problems, all of them should to be interested in take part in the League. However activity on the platform doesn't confirm this suggestion. Students gave up very quickly. At the beginning they worked intensive, but in short time their activity stopped. What are the reasons for that situation? It seems that the essence of this problem does not lie in the form of transfer information directly. The work with e-platform requires from users a lot of work and takes a lot of time. This fact has been confirmed both by the observations of the activity on the platform and in my private conversations with pupils. Students logged in at weekends or late at night the most often. I think it is because current schooling was preemptive problem for them. They had to do their homework, learn new materials etc. Work with e-platform was unobligatory, uncompulsory, additional for them. I think that the lack of external motivation is next problem which has influence on decrease of activity on platform. Participation on the League was only voluntarily declared by pupils. They didn't receive for their work any opinions, praises or marks. It wasn't connected in any way with their school properties. This situation in connection with big amount of homework has discouraged forcefully students from further work in their *free time.* The selection of tasks could be next factor influenced on pupil's work. Presented problems could turn out difficult for pupils. Maybe they didn't have any intuitions or experiences connected with ways of solution such problems. However there is one confirmed fact in the investigation that shows situation when one of problems went beyond the range of the pupil's material. This boy didn't give up but started own searches concentrated on the problem. This difficulty motivated him and it has contributed to expansion of knowledge at him. He sent correct solution of this problem.

The next important aspect is the way of students work. First of all pupil after log in on the e-platform makes the choice interesting problem himself. Next he works over him in appropriate for oneself rate and time. However there are always possibility of contact both with the leader, or with others users. Such planning the work takes into account the presence of *social facilitation* (see [1]). This phenomenon is researched by social psychologists. It is the tendency for people to be aroused into better performance on *simple tasks* or tasks at which they are expert or that have become autonomous, when under the eye of others, rather than while they are alone. *Complex tasks* or tasks at which people are not skilled, however, are often performed in an inferior manner in such situations. So new things we learn in isolation more easily. Observers presence has influence on our emotions. If we show *simple task* under the eye of others the chance of success grows.

Planning education based on e-platform supported teaching we ought to pay attention at such kind of phenomenon, because we can organize materials in such way that we use advantages of it simultaneously eliminate its negative parts.

The work at "League of fans of mathematical problem" platform can confirm justness of this method. There were some pupils who read content of problems many times before they sent solution, however they usually didn't contact with others users in any way. Platform has turned out perfect instrument of investigation in this situation. Analyzing the concrete pupil long-term actions we can observe e.g. each returns to the task, even if he didn't make any discussion on forums or didn't send any solution – we can see long time process of solving problem.

However investigation showed next, quite serious technical nature shortcoming of this kind of education, connected with sending solutions of tasks by users. This operation requires fluency in equipment handling from students such as for example: scanning handprints, making digital pictures, editing texts with mathematical equations or creating computer graphics. Sometimes pupils doesn't have proper software or hardware so it is next problem for them. The result of this situation may be the barrier, which can make impossible sending solution for users.

The indirect consequence of this fact is that leaders don't receive the full description of solution's process, but only final result – the answer. Scanning all materials isn't always possible, and doesn't have the possibility access to full material of students work. It makes difficult estimate it. There were few persons who pay attention to send exact and detailed description of the whole conduct, also these stages of the solution which finally was unsuccessful. However such solutions are supplied documentary evidence. Perhaps, importance for this situation has pupil's *culture of mathematics*. They have convictions, that every step of the solution is essential to the opinion of the whole work and it should be written down.

The *Moodle* platform as a distance learning tool showed in my investigation many positive aspects of such work with pupils. Users *didn't have any objections to ask questions*, they not only took part in discussions actively but also asked leaders directly. It's seems that the access to forum has large influence on solving tasks and students work, and such form of contacts is sometimes necessary for them.

Pupil can not only ask questions in relation to met difficulties, but also discuss, receive hints, suggestions and motivations for further work. In my opinion pupils would give up more quickly without this communication tool. Seeing, that others colleagues have similar problems they discussed and helped mutually. Capability of argumentation and representation of hints from others is very important also. Next notes had to encourage pupil on purpose for making next step, which have to cause final result in consequence.

The possibility of joining discussion at any moment is the next important advantage of the e-platform. Every user has capability for secondary optional analyzing period of conversation, without the necessity of repetition or renewed description of talked over earlier facts, because every exchange of sentences is written down on the platform. It gives pupil possibility of multiple returns to the task - it's very important in process of problem solving and it's increases pupils chance to solve it.

The e-platform makes possible long-term problem solving, returns to bothering problems. The leader can also propose the prolongation of the problem at any moment of this process.

Recapitulating it seems that the e-learning platform creates large possibilities of development for students. It is more and more popular form which lets both earn knowledge and develop interests. However using this medium in teaching process requires from pupils skill of efficient using new technologies. So to say, it enforces students to broadening his knowledge sometimes. Pupils must show big engagement, reliability and self-discipline to take advantage of this tool completely. Work with this tool develops creativity and ingenuity, additionally. The pupil while solving tasks can not only use equipment placed on sides of the course by the teacher, but also can seek for indispensable to him information beyond the League, using for example supplies of the Internet in every moment. He can use mathematical programmes also (e.g. *C.a.R, Graphs 3*, *Cabri Geometry II* and many others), which can turn out helpful during work over problem.

At the end I will allude about slight inconvenience related with describing results of didactic investigation directly. There is a problem with appearance of idea of the structure of students work on *Moodle* platform. It's because each written work is written in linear match, but pupils were communicated on e-platform and many persons had influence in result of work of other users. Different threads, statements and thoughts were penetrated mutually. Problem appears with determination of correct structure during analysis materials of the single user.

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DEVELOPMENT OF MATHEMATICAL THINKING OF PUPILS AT ALL EDUCATIONAL LEVELS

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Abstract. We discuss mathematical games developing logical thinking of pupils by using exercises with the Möbius strip and spirals.

ZDM Subject Classification: D40.

1. Introduction

The primary school age is often spoken of as a "memory" age. The time which a pupil spends at school must be used most efficiently. Logical thinking formed at this period of time becomes an origin of mathematical adventure. Using attractive games–plays at school, it is possible to awake pupil's interest in mathematics, prompt him to reflect and to ascertain why it is so and what will be later. A search of striking solutions as well as mistakes develop logical thinking of young man and form intuitive understanding of mathematical notions. at one time. Example exercises and games teaching the notions of one-sided surface and mathematical spirals will be presented in the paper.

2. The Möbius strip

The Möbius strip or Möbius band is a surface with only one side and only one boundary component. It has the mathematical property of being non-orientable. It is also a ruled surface. It was discovered independently by the German mathematicians August Ferdinand Möbius and Johann Benedict Listing in 1858 [1].

Fig. 1. Möbius strip [1].

Exercise 1
Let us prepare two paper strips. The first is glued to obtain the traditional band. Before glueing the second, one end of a strip is rotated through the angle 180°. At one side of every strip we draw by a pencil a line along the whole length. It turns out that in the traditional band we obtain a continuos line at one side of a band: at interior or exterior depending on starting point of drawing. We can say that such an effect is expected. However, in the rotated and glued strip we obtain a continuous line running on both sides of a strip, whereas we have drawn the line on one side only. Such a property is typical for one-sided surfaces, and such a strip is known as the Möbius strip.

Exercise 2
Cut the traditional band and the Möbius strip along the median line. Observe what is obtained. We feel intuitively that from the traditional band two narrowed bands will be obtained. And what about the Möbius strip? It turns out that we still have the Möbius strip, but twice as long.

Exercise 3
Prepare wider paper strips and make bands from them as in Exercise 1. Cut every strip to three pieces (Fig. 2). What will be obtained? We leave this question to discretion of the reader.

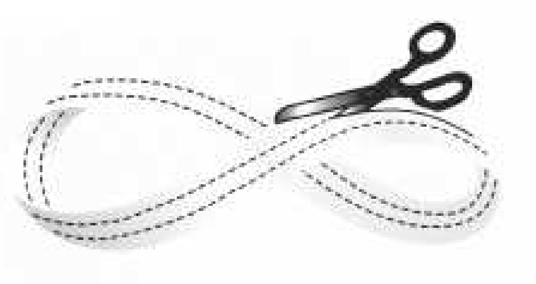

Fig. 2. The Möbius strip cut to three pieces [3].

Exercise 4
Glue the Möbius strip and the traditional band as it is shown in Fig. 3. What will be obtained when we cut both bands along the mediam lines?

Fig. 3. The glued band and Möbius strip [2].

To answer this question, conclusions from Exercise 2 should be used. The cuts cause that the Möbius strip (twice as long) will contain two separated bands.

The notion of the Möbius strip arises in topology. This division of mathematics is taught at high schools. Nevertheless, the described exercises can be carried out with pupils at the age of 12–13. Intuitively they keep in mind the notion of one-sided surface.

3. Mathematical spirals

In mathematics, a spiral is a curve which emanates from a central point, getting progressively farther away as it revolves around the point. The concise mathematical definition is "The locus of a point moving at constant speed whose distance from a fixed point increases at a specific rate" [3].

Exercise 5
On a checked sheet of paper we draw a line described by a sequence of numbers 1, 1, 2, 2, 3, 3, 4, 4, 5, ... (Fig. 4a).

Exercise 6
On a checked sheet of paper we draw a line described by a sequence of numbers 2, 1, 3, 2, 4, 3, 5, 4, 6, ... (Fig. 4b).

In both Exercises we ask about next numbers describing those spirals, verify the possibility of drawing other spirals, for example on a sheet with triangular net or spirals with loops (Fig. 4c).

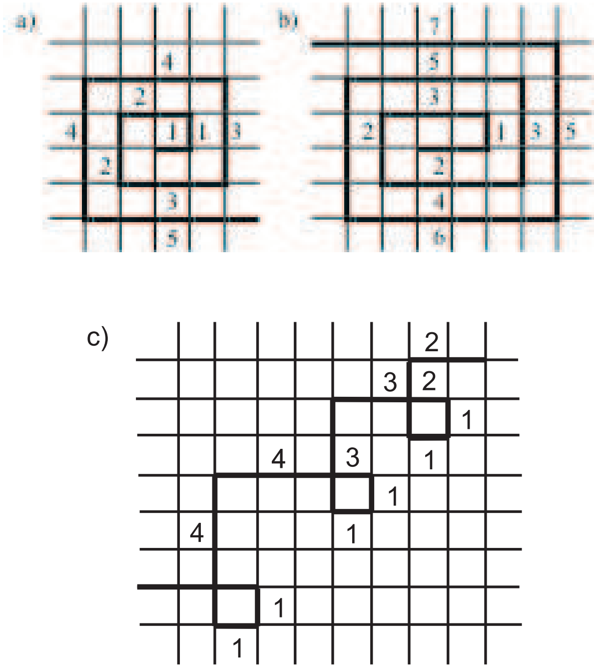

Fig. 4. Spirals obtained from a sequence of numbers [4]:
a) 1, 1, 2, 2, 3, 3, 4, 4, 5, ...,
b) 2, 1, 3, 2, 4, 3, 5, 4, 6, ... ,
c) 2, 2, 1, 1, 3, 3, 1, 1, 4, 4, ... (a spiral with loops).

Pupils can acquaint themselves with the Archimedean spiral by showing of rolled up carpet or vinyl gramophone record. We expect that pupils keep in mind the characteristic feature of this spiral: the distances between two turns of the spiral are the same. In the future, at another educational level, the Archimedean spiral will arise as a curve in polar coordinates (Fig. 5).

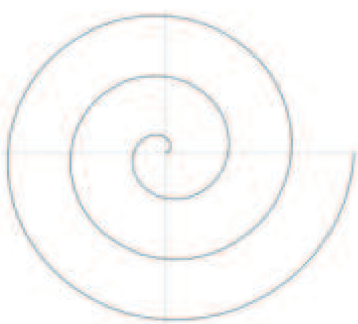

Fig 5. The Archimedean spiral [5].

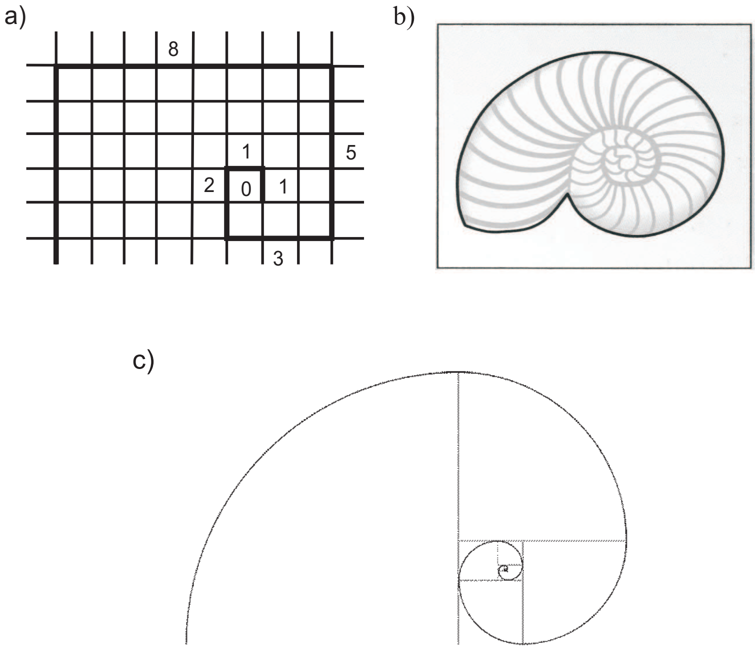

Fig. 6. A spiral obtained from a sequence of numbers
0, 1, 1, 2, 3, 5, 8, 13, ... , a nautilus shell and the Fibonacci spiral.

Exercise 7

Draw a spiral which is characterized by a sequence of numbers 0, 1, 1, 2, 3, 5, 8, 13, 21, ... (Fig. 6a).

Which common properties have a spiral in Fig. 6a and a shell in Fig. 6b? They are models of the Fibonacci spirals (Fig. 6c). We expect that a pupil observes that a nautilus shell is also built as a spiral and has a sequence of cells. Every next cell is a sum of two previous ones.

Exercise 8

On a checked sheet of paper we draw a spiral described by a sequence of natural numbers 1, 2, 3, ... (Fig. 7). Let us mark prime numbers at this spiral and observe how they are arranged. For a pupil it will be interesting amusements, but in the future he will get to know that the obtained spiral is the Ulam spiral.

37	36	35	34	33	32	31	56
38	17	16	15	14	13	30	55
39	18	5	4	3	12	29	54
40	19	6	1	2	11	28	53
41	20	7	8	9	10	27	52
42	21	22	23	24	25	26	51
43	44	45	46	47	48	49	50

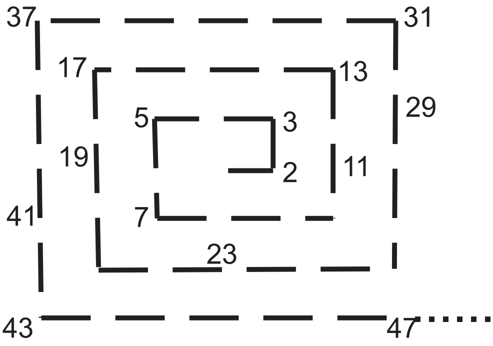

Fig. 7. The Ulam spiral [6].

4. Conclusions

The use of games developing logical thinking at mathematics lessons prompts pupils to activity, interest in this subject and shows beauty of mathematics. Today we observe small interest of young people in choice of fields of studies connected with mathematics, hence, it is necessary to modify the style of teaching mathematics (see also [7]). Exercises proposed in the paper can inspire a teacher to make a mathematics lesson more attractive and to facilitate understanding of mathematical notions.

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MATRIX STRUCTURE OF COMPLEX NUMBERS

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Abstract. Complex numbers can be considered as a fragment of matrix algebra. The complex field is isomorphic to a set of squarte matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a, b \in \mathbb{R}$, with operations of addition and multiplication of matrices. Such a consideration of complex numbers can serve as an interesting didactic solution for students.

ZDM Subject Classification: F50.

It is known from higher algebra that the complex field (considered as an ordered set of real numbers with correspondingly defined operations of addition and multiplication) is isomorphic to a set of matrices $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $(a, b \in \mathbb{R})$ with operations of addition and multiplication of matrices.

Hence, complex numbers can be treated as a fragment of matrix algebra. Such an interpretation of complex numbers can be found in some French textbooks [1].

Consider squarte matrices of the second order with real elements. The identity element for operation of addition of such matrices is the null matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, whereas the unitary matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity element for multiplication operation. The element opposite to a matrix A (denoted as $-A$) is an element $(-A)$ with the property $A + (-A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. The element inverse to nonsingular* matrix A (denoted as A^{-1}) is an element (A^{-1}) with the property $A \cdot A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The matrices $-A$ and A^{-1} have the following properties:

a) if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $-A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$,

b) if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det A \neq 0$, then $A^{-1} = \dfrac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

A set $\mathbb{C}$ of matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $(a, b \in \mathbb{R})$ with operations of addition and multiplication of matrices will be called the set of complex numbers (the complex field)*.

Subtraction and division of complex numbers are defined as follows:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} - \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \left(- \begin{bmatrix} c & -d \\ -d & c \end{bmatrix} \right) =$$

$$= \begin{bmatrix} a-c & -(b-d) \\ b-d & a-c \end{bmatrix};$$

if $c^2 + d^2 \neq 0$, then

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} : \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \cdot \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \cdot \frac{1}{c^2+d^2} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \frac{1}{c^2+d^2} \begin{bmatrix} ac+bd & -(bc-ad) \\ bc-ad & ac+bd \end{bmatrix}.$$

A set $\mathbb{C}_1$ of diagonal matrices of the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $a \in \mathbb{R}$, can be identified with a set of real numbers $\mathbb{R}$, since the algebraic structures $\langle \mathbb{C}_1, +, \cdot \rangle$, $\langle R, +, \cdot \rangle$ are isomorphic.

Such an isomorphism is established by the function:

$$\varphi : R \to \mathbb{C}_1, \quad \varphi(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

*A matrix A is nonsigular, if $\det A \neq 0$.

*A set of matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a, b \in \mathbb{R}$, is only a noncommutative ring (with respect to addition and multiplication of matrices), while the set $\mathbb{C}$ is a field.

The algebraic (Cartesian) form of a complex number $z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ can be obtained as follows [2]:

Let $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = i$. Then:

$$z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = a + bi.$$

The imaginary unit i has the following property:

$$i^2 = i \cdot i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -1.$$

Based on the algebraic form $z = a + bi$, it is possible to obtain the trigonometric form of a complex number $z = |z|(\cos\varphi + i\sin\varphi)$ and, using the Euler formula $e^{i\varphi} = \cos\varphi + i\sin\varphi$, the exponential form $z = |z| \cdot e^{i\varphi}$.

On basis of accepted definition we have

$$(a + bi) \pm (c + di) = \begin{bmatrix} a \pm c & -(b \pm d) \\ b \pm d & a \pm c \end{bmatrix} = (a \pm c) + (b \pm d)i,$$

$$(a + bi) \cdot (c + di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \cdot \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{bmatrix} =$$

$$= (ac - bd) + (ad + bc)i.$$

If $c^2 + d^2 \neq 0$, then:

$$(a + bi) : (c + di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : \begin{bmatrix} c & -d \\ d & c \end{bmatrix} =$$

$$= \frac{1}{c^2 + d^2} \begin{bmatrix} ac + bd & -(bc - ad) \\ bc - ad & ac + bd \end{bmatrix} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i.$$

The tradition approach to complex numbers accepts the following definitions of addition and multiplication operations:

$$\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle,$$

$$\langle a, b\rangle \cdot \langle c, d\rangle = \langle ac - bd, ad + bc\rangle.$$

In the matrix approach, addition and multiplication of complex numbers is a natural consequence of carrying out of the corresponding operations on matrices.

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TRIOMINOES: A SOURCE OF BUILDING-UP OF COMBINATORIAL THINKING

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Abstract. The aim of this paper is to present the game Triominoes as a setting for a mathematical investigation that leads students to creating representations and engaging them in a mathematical reasoning, abstraction and generalization. We show that triominoe tiles can be a vehicle conveying different combinatorial ideas. The use of the game can foster independent thinking, encourage flexibility, solution sharing, and increase the interest of students in mathematics.

ZDM Subject Classification: K23.

1. Introduction

In this paper, we describe our attempt to implement combinatorial problem solving as a teacher of 7th class of primary school. We choose problem solving situations related to the game Triominoes as a setting for a mathematical investigation*. Our goal is to use Triominoes to help students think mathematically, that is, to engage in mathematical reasoning, to develop systematic thinking, to construct representations, and to recognize and to apply mathematics in contexts outside of mathematics.

The game Triominoes, similarly as Dominoes (see [2]), is a rich source of diversified problem solving experience. In this article we provide a set of problems concerning Triominoes. The main part of the paper is divided into four sections. The first section is devoted to activities related to the process of making triominoe tiles. The second section presents a collection of various

*Triominoes is a variant of Dominoes using triangular tiles (see figure 1). Each vertex of a triangle is labeled with a number from the set $\{0, 1, 2, 3, 4, 5\}$. The game includes only 56 out of 76 possible tiles, omitting those tiles in which the numbers appear in clockwise descending order.

problems that can be treated in connection with the set of triominoe tiles. In the third section we provide steps that lead students to a development of a strategy of the game Triominoe. In the fourth section we deal with some other combinatorial problems related to Triominoes. The problem from the first section and some of the problems from the following sections were solved by 57 students of 7th class of primary school.

2. Making triominoe tiles

In this section we describe the process of game preparation. The substantial activity at this stage is not the physical creation of the tiles, but the enumeration of all possible triples (labels of the tiles) and a selection of suitable ones.

First of all, students were given the following task: *"Last week I found a new game called Triominoes. This game is similar to Dominoes. There are different triangular tiles in the game labeled with one of numbers* 0, 1, 2, 3, 4, 5 *at each their vertex. To place a tile, two of the three numbers must align with the adjacent tile (see figure 1). I shall provide you complete rules of the game, but firstly you have to create the triominoe tiles."*

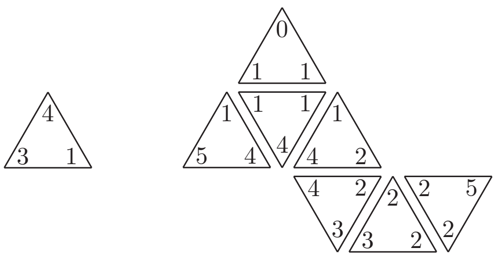

Figure 1. Triominoe tile

At the beginning of this stage, the most of the students were listing the triominoe tiles randomly. After a while, some of them started to look for a suitable representation and arrangement. Usually they represent a triominoe tile as a triple of numbers e.g. $0, 0, 2$ or $1, 3, 4$.

The strategies used by the students were of the following types:

1. Students listed the triple containing three zeros, then the triples with two zeros and one different number, and with one zero and two other numbers. Analogously, they listed the possibilities with number one and without number zero, with number two and without number zero and one etc. In such a way they found out $21+15+10+6+3+1 = 56$ possibilities. This strategy could be later generalized for triominoes with n different labels using the following formula: $T(0) = 0$, $T(n) = T(n-1) + \frac{(n+1)\cdot n}{2}$.

2. Students firstly listed all possibilities with three identical numbers, then with two identical numbers, and finally with three different numbers (some students grouped the possibilities with three and two same numbers). In this case, the number of triominoe tiles with n labels can be obtained using the following formula: $T(n) = \binom{n}{1} + 2 \cdot \binom{n}{2} + \binom{n}{3}$.
3. Students started with the triple $0, 0, 0$. Then they changed repeatedly the third number in ascending order. When they had examined all triples of this pattern they changed the second number and so on. Students who used this strategy have been looking for the number of combinations with repetitions. The number of triominoe tiles with n labels is $T(n) = \binom{(n-1)+3}{3}$.
4. Students listed $6 \cdot 6 \cdot 6 = 216$ possibilities. Then they started to cross out the possibilities which they consider to be the same.

The first two types of strategies were the most frequent. It is worth to emphasize that only one student noticed that the triples a, b, c and a, c, b are not the same with respect to Triominoes game. He used a strategy of type four. Some other recognised the same fact but only when they started to place tiles according to the rules of the game.

3. Problems concerning triominoe tiles

At this section we develop a series of activities and tasks that can help the students to understand the substantial relationship among different triples of numbers.

At the beginning of this part, we recommend that students share their solutions with their peers. In doing so, they should explain how they arrived at their solution and why they consider their solutions to be successful. When students share their solutions, they give the teacher an opportunity to understand their way of combinatorial thinking and provide a possibility for their peers to give constructive feedback.

After sharing students' solutions they tried to answer the following questions:

1. How many different triominoe tiles do you have
   a) with three identical numbers at vertices?
   b) with exactly two identical numbers at vertices?
   c) with number 0 (3, 5) at some vertices of the tile?
   d) with different numbers at vertices?
2. Consider that each tile has one marked vertex. How many different tiles you can create in such a case?

3. Consider that you can flip the triominoe tile. How many different tiles you need in such a case?

After answering these questions students were told that the standard game includes only 56 out of 76 possible tiles, omitting those tiles in which the numbers appear in clockwise descending order. We discuss the possible reasons for omitting these tiles.

4. Strategy of the game

In the section we describe the process of the development of a strategy of the game.

Students were given the rules of the game (see [4]). They had about 15 minutes to play the game in order to obtain a flavour of the game. Then we formulated the following problems:

1. How many points can the starting player obtain at the beginning of the game? Why it is convenient to begin the game with the tile with three same numbers?
2. Is it possible that no player will have the tile with three identical numbers, with two identical numbers?
3. Determine all triominoe tiles that can be placed to the marked positions in the figure 2.

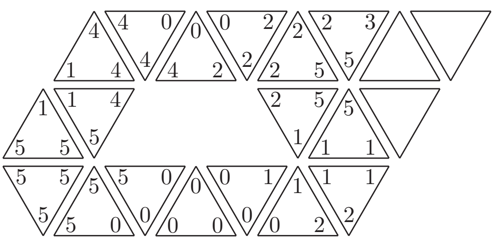

Figure 2. Possible triominoe tiles

4. In the figure 3, there is depicted a configuration during a game. What triominoe tiles can you play in the next round? Is is possible to create hexagon or bridge?

The goal of the presented problems is to stimulate students to think about suitable strategy of the game. Students should be aware that they hold a matching tile or tiles and need to decide whether or not to play them or in which order to play them for their best advantage.

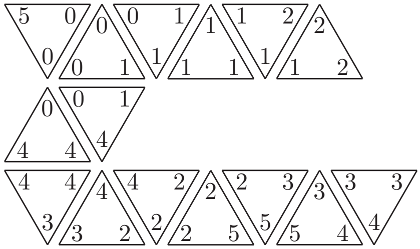

Figure 3. What we can play in the next round?

5. Combinatorial problems related to triominoes

In this part, we present a few combinatorial problems concerning triominoes. The last problem called for generalization of the pattern they had discovered during game preparation stage.

1. Label the triominoe tiles in the figure 4 by appropriate numbers.
2. Label the triominoe tiles in the figure 4 by appropriate numbers so that the sum of the numbers will be the biggest (least) possible.
3. How many different tiles will you have if you can use the labels from 0 to 6 (from 0 to $7, 8, \cdots, n$) for a tile?

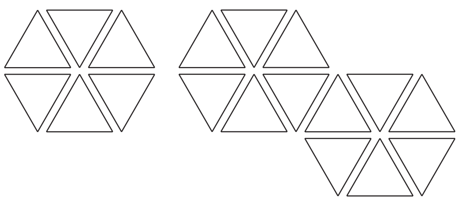

Figure 4. Combinatorial problems 1 and 2

6. Conclusion

Many of our students became articulate in their description of the strategies during the solution of the given problems. They found a suitable representation for triominoe tiles. Some students discovered the mathematical structure of the problem and they were able to extend and generalize their strategy for finding the number of triominoe tiles with more than 6 labels.

We think that a classroom teacher can use such combinatorial problems to help students to systematize their thinking and to think independently. A teacher can facilitate discovery process by asking students to explain and justify their solutions. When students are presented with novel combinatorial problems, they naturally use a number of different solution approaches, as was presented above.

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INTERDISCIPLINARY RELATIONS BETWEEN MATHEMATICS AND PHYSICS AT HIGH SCHOOLS

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Abstract. Mathematics is the most important tool of physics. The paper deals with interdisciplinary relations between mathematics and physics at high schools. We describe possibilities of utilization and application of mathematics and mathematical knowledge in physics.

ZDM Subject Classification: M10.

1. Introduction

One of the aims of high school education in the field of mathematics is that students be able to apply the acquired knowledge when solving different problems not only from physics, but also during studies of other natural science subjects or technical subjects; in addition, they can apply this knowledge when modelling simple physical, chemical, biological, but also economic situations, while effectively using computers. Education in mathematics is necessary not only for physicists and engineers, but also for doctors, biologists, psychologists, lawyers, politicians, linguists, and many others. Mathematics not only gives useful information; it also significantly forms the thinking and other personality characteristics.

Even though mathematics itself is sometimes not considered to be a purely natural science, the problems it studies and describes most often have their origin in natural sciences, especially in physics. Very often it also deals with problems, the origin of which is not purely mathematical, through which it provides generalizations linking several scientific fields, or considerably simplifies solutions.

Mathematics contributes to realization of the systematic principle in other school subjects. Such systematic approach is ensured in teaching every subject, as well as in the interdisciplinary relations. This systematic link among the individual subjects teaches the students that there are no sharp edges between the different kinds of knowledge, and that different spheres of science and technology are not separated, but rather closely interconnected.

2. Interdisciplinary relations in education

Phenomena and processes, which are studied by physics, are closely related. Thus, there exist certain mutual dependencies. Examination of these dependencies in physics is possible through utilization of mathematics and its individual disciplines (Hanisko, 2007b).

Mathematics and physics are included among the compulsory subjects at high schools. They are taught in various forms, whether it is in a form of teaching compulsory class of mathematics and physics, which all students must take, or in a form of optional seminars, which are taught for those students, who are more deeply interested in these subjects. The contents of the subjects of mathematics and physics, which is compulsory for all students, is determined by curricula of mathematics and physics (MŠ SR, 1997a; MŠ SR, 1997b). Mastering the subject contents as given by curricula requires systematic studying of both subjects, on the side of the students, and a understandable explanation of the given subject matter on the side of the teacher. It is important that students are not given too much studying material, which is too difficult for high school studying and requires knowledge of higher mathematics. The subject contents should be selected and formulated so that one theme ties on the previous one. The reality often shows, that students at physics classes are required to understand a certain subject matter, for which they need a mathematical apparatus, which was, on the other hand, not covered at mathematics classes.

Such situations, however, often occur at high schools, and that creates quite big problems in the learning process not only for the teachers, but mainly for the students, for those have difficulties to understand such subject matter. The problem of lack of coordination of curricula for mathematics and physics is a serious one in teaching both of these classes, and it appears again and again and causes problems to the students as well as the teachers from the first year of studies. The problem appears mainly in organization of particular thematic units of both subjects and in their integration into the individual years' studies (Hanisko, 2006).

Each natural science is a set of internally logically organized information, which, through their contents makes individual scientific fields. Today, for development of natural sciences it is typical, that the findings of the individual sciences, but also of the individual science fields, are not separated and isolated, but they rather blend together and relate to one another. That leads to their integration. It is possible to divide the relations among the findings of the individual scientific fields into two categories (Hanisko, 2007a).

- **Interdisciplinary relations** - these are the relations among the findings of individual scientific fields, disciplines of different sciences.
- **Intradisciplinary relations** - these are the relations among the findings of individual scientific fields of the same kind of science.

The interdisciplinary and the intradisciplinary relations are together termed as **inter-scientific relations** and reflect the connectivity and dependency of natural phenomena. Respecting and utilization of inter-scientific relations among the individual natural sciences enables solving numerous problems, leads to understanding the essence of phenomena and processes occurring in the nature, and helps in creation of a simplified picture of the world.

Interdisciplinary relations are conditioned by the existence of individual school subjects in the school system, and reflect objectively existing interdisciplinary relations. In high school education the interdisciplinary relations can be carried out in two ways (Reznikov, 1972, p. 24).

- Through time coordination in teaching different school subjects.
- Through consistent explanation of the scientific terminology covered in mathematics, physics, and other school subjects.

The connectivity among the individual school subjects has a principled pedagogical significance. The durability and applicability of the knowledge of students, and their thinking development depend on its implementation, and it also contributes to systematic education. When teaching mathematics at high schools, there are important interdisciplinary relations mainly with physics and other natural science subjects. Utilization of these relations helps to overcome having isolated knowledge from individual subjects. Through this it is possible to better contribute to deeper knowledge of the students, and to increasing quality of their thinking processes, which supports the individual problems solution skills. When teaching physics at high schools, it is possible to build not only on previous knowledge in physics, but also on certain knowledge from other natural science subjects and humanities.

3. Interdisciplinary relations of mathematics and other school subjects

The issue of interdisciplinary relations is connected mainly with natural science subjects, such as physics, mathematics, chemistry, biology and geography. Therefore, it is very important that, when teaching these subjects, there are reflected their mutual effects and overlapping of contents of their findings. Natural science subjects use a lot of common terminology, they study the same objects and systems (even though from different perspectives) according to their own subject of observation, and that is exactly where the core of their cooperation lies. The essence of the interdisciplinary relations implementation in natural science subject is in the fact that it is not only about learning natural reality, but mainly about comprehensive development of the student's personality.

Mathematics is used in teaching almost all subjects at high schools. The subjects, where mathematics is used can be divided into three categories (Hanisko, 2007a).

- Subject that cannot do without mathematics (physics, chemistry, informatics, basics of economy).
- Subjects in which mathematics is used to a lesser degree (biology, geography, geology).
- Subjects in which mathematics is not used directly for calculations, but indirectly, e.g., in connection with historical events which are connected with mathematics (history).

4. Interdisciplinary relations of mathematics and physics

The subject matter of physics is closely related to mathematics because in physics, besides experimental method, in most cases mathematical methods are used. In physics taught at high schools there is probably no sphere in which the students would not encounter mathematics when formulating physical terms and laws. Mathematics terminology and notions are used in defining different physical parameters and their units.

Physical dependencies in high school physics are usually expressed analytically, through equations. There is a great danger that students could understand these equations as algebraic expressions. Therefore, it is necessary to pay great attention to the right understanding of the functional dependence. The functional dependence can be expressed in three ways - *analytically, in a chart*, and *graphically*. In teaching physics at high school, using graphs is preferred. There are used three mathematical terms – coordinate system, variable quantity and function. The graph expresses the quantitative course of physical dependency. The dynamics of the physical phenomenon or process itself, which is expressed in an analytical form, will be understood only by those students, who are at a sufficiently high level in mathematics.

Determination of graphical dependencies among physical quantities based on experiments helps students imagine mainly change in physical quantities depending on time (velocity of movement, velocity of the current velocity change, etc.) and on *location* (change of temperature along the metal rod, potential in electric circuit, etc.). Graphical method is used in solving physical problems, processing and analysis of the outcomes of laboratory works.

In some cases, students must be familiar also with the term *derivation*, which is, however, taught at high schools only marginally, and in elective or optional subjects. This applies mainly to the cases of introduction of the term point derivative (e.g. when implementing physical terms like velocity, acceleration, etc.). It is a disadvantage that derivations are taught at the end of the studies in the fourth year, when all the material of at the physics subject, where derivations are used) is already covered. Therefore, when implementing some physical terms, where derivation is necessary, there is used a ratio $\Delta x/\Delta t$, where the quantities Δx and Δt are defined as the lowest values measurable by instruments. In high schools it is possible to solve this problem by using the term derivation during review of the physics, so that the physical terms introduced before are more closely specified, and also the most important equations in physics are derived, which were before derived in a simplified form or they were formulated without reasoning.

In teaching physics at high schools there is often used mostly approximated determination of values. When processing the results of different physical experiments, it makes no sense to determine the result with accuracy higher than the accuracy of the measuring itself, since the measuring accuracy depends on accuracy of the measurement devices. When determining the results in physics it is therefore very important to round the results of the mathematical operations and determine the errors of the measurement results correctly. Therefore, in solving physical problems it is necessary to apply the rules of approximate solving and rounding consistently. It is because it is not always necessary to make accurate calculations. In most cases it is sufficient to know the approximate value of the physical parameter.

The significance of the mathematics in teaching of physics lies mostly in the fact that there are mathematically formulated conditions and initial data, and based on those, with the help of mathematical operations, the conclusions are derived, and those are afterwards theoretically or experimentally verified.

5. Conclusion

Mathematics and other subjects where mathematics is used comprise a unite entity, which is daily used in solving problems of various kinds. Creation of mathematical models is, however, always connected with certain simplification assumptions, which make solution of the problem easier and sometimes even condition it.

When studying any subject it is necessary to be familiar with the methods and processes used in this study. It is particularly important in studying physics, where it is not possible to do without mathematics and its apparatus. At the first sight it may seem that in teaching physics in high schools no such difficult mathematical apparatus is necessary. However, the opposite is true. Without the accurate and mathematically correct explanation if physical laws the teaching of physics loses its meaning, since just a superficial explanation and an encyclopaedic description of physical phenomena without deeper mathematical justification does not have any significance for the students.

Mathematics and physics, and with some delay also their teaching have always been able to adjust to the requirements of the real world. Today's mathematics and physics are the evidence of that. All those who are familiar with educational process and its preparation (creators of pedagogical documentation, teachers, students themselves) should do their best to achieve that teaching of mathematics and physics does not lag behind the reality of today's world either.

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GEOMETRICAL COMPETENCES OF ELEMENTARY SCHOOL PROSPECTIVE TEACHERS

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Abstract. The paper deals with competency of new primary school teachers mainly within the topic of geometry. This part of mathematics belongs to the most difficult and therefore it is necessary to focus on it during their preparation and evaluation.

ZDM Subject Classification: B50.

1. Introduction

Current education is strongly influenced by an ongoing curricular reform. The curriculum consists of a general Rámcový vzdělávací program (abbr. as RVP) – at the central level – and Školní vzdělávací program (abbr. as SVP) at school levels. Even though universities are not subject to this reform, pedagogical faculties aim at creating such an environment that their graduates acquire all competences required of teachers by the reform.

Since they themselves have not experienced the SVP's, current prospective teachers are not aware of the ways of implementing the SVP's, which are supposed to guarantee the pupils' competences. Prospective teachers learn about them at universities only. Therefore they must be directed to acquire the necessary professional competences. Having them they will be able to work as truly competent primary school teachers (under the curricular reform) as they will be able to create and shape pupils' competences.

2. Geometry: an integral part of school mathematics

The geometrical component of mathematical education has had the probably most complicated evolution out of all areas of mathematical education in the recent years. Based on checking the state of geometrical knowledge and skills and on expressing attitudes to the subject among primary school pupils and prospective primary school teachers we have found out that the results of geometrical teaching are not satisfactory. This may be caused by the fact that the number of classes allocated for geometry has been declining (sketching has been cancelled as a subject at elementary schools, descriptive geometry is taught at grammar schools only – and only as an optional subject, etc.)

Teachers often suggest that time for practicing the subject matter and application of practical pupils' experience is limited. Teachers sometimes schedule geometry in the less intensive parts of the school year. Pupils are not sufficiently motivated for exact sciences. However children – unlike teachers – in general like geometry. Teachers should not forget that pupils' interest in geometry may be used as a means of raising interest in mathematics in general.

Geometry is a means of communication between people, a means of organizing tools or work; it is even a means of such things as traffic management. Geometrical components of orientation in plane or space are used when reading (orientation in text), at school, library or when walking in town or telling the way. In most cases all of us use geometrical knowledge at least subconsciously.

Geometry helps pupils with reading difficulties (by means of the reading window), helps the blind with space orientation or gives pictograms to the deaf. The immobile make use of architectural features such as sloped plain, horizontal plain or elevators.

These examples support the idea that geometry is everywhere around us, that it is the natural part of school education of children as well as real life situations of adults. Therefore knowledge of geometry concepts may not be excluded from our lives.

Mathematical competences required by modern times dominated by information technologies are different from those required a few years ago. They are: observation, analysis, prediction, verification, decision making, choosing best (however defined) strategy.

Hejný in (Hejný, 2001) discusses ten principles of constructivism. One of the principles – educational process – must evaluated from at least three points of view: *understanding mathematics, mastering mathematical drill* *and applying mathematics. Creating ideas, concepts and strategies, and realizing mutual relationships has profound importance understanding mathematics. Developing mathematical drill requires training and memorizing certain laws, algorithms and definitions. Applying mathematics need not be the climax of the educational process – it may have a motivating role as well. We learn mathematics by doing it.* (Hejný, 2001)

Geometry originated and was shaped by the needs of lives of ordinary people. It is naturally integrated in everybody's life and cannot be separated from it. From the very same reason it may not disappear from primary school subjects, which are supposed to help pupils in their everyday lives. Pupils learn means for handling real life situations, prepare for further education and acquires a very important skill – space imagination.

Basic mathematical concepts are shaped very slowly and are précised and polished by everyday experience. They start as specific notions, often connected with specific objects; by a slow process of generalization and abstraction they evolve into concepts. The abstraction in its first phase is generalization of properties observed in common objects; first ideas are introduced. They are later followed by concepts of elementary mathematics such as a number, circle, etc. Each concept is symbolized by a specific object or a group of objects in children's mind. Creating mathematical concepts should be closely related with real life and their correctness should be confronted with practical situations.

In order to correctly teach mathematical subject matter:

- the teacher's own geometrical ideas must be correct,
- the teacher should know geometrical terminology, which is sometimes a foreign language for students; the teacher teaches this foreign language as well,
- the teacher should be able to solve problems of geometry using means of arithmetic and algebra,
- the teacher should teach other aspects such as space imagination as well.

Despite all effort of mathematics didactics and elementary geometry teachers a negative approach towards teaching geometry at primary schools prevails. When teaching geometry we try to make geometry attractive for our students.

Some test results of elementary school prospective teachers show incapacity in elementary *understanding of geometry*. I would like to present some examples I have met in teaching elementary geometry. The subject *Elementary geometry and didactics* is scheduled for the second year of studies and its aim is to standardise students' knowledge of geometry and to make them able to use basic knowledge from their previous studies of geometry (introduction to set theory, predicate calculus, binary relations and their properties, equivalences, orderings, mappings, binary operations and their properties, etc.).

In order to reveal students' most frequent mistakes in solving tasks we use didactical multiple choice tests or solving tasks without predefined answers. One of the most frequent mistakes is the fact that students cannot use their knowledge in different contexts.

3. Examples

In the following examples we show different students' results when following educational aims in cognitive learning. We follow the requirements for regulated students' cognitive activity in teaching which is aimed at a single part of student's personality. According to Bloom, the personality consists of the following categories: knowledge, understanding, application, analysis, synthesis, evaluation.

Task 1a): There are collinear points A, B, C, D (in this order, starting from the left) on a line – see figure. Find the intersection of segments AC and BD.
Solution: $AC \cap BD = BC$. Almost all students were correct, but there were others who wrote that it was points B and C. If we change the task wording, we get a different level of difficulty, which is not only a matter of knowledge.

Task 1b): Which geometrical shape may be an intersection of two segments of the same line?
Solution: The following options are included: segment of a line, line, point, half-line. Students are to choose correct answers (not all options are correct). Student certainly choose some answers.

Task 1c): Which geometrical shape may be an intersection of two segments of the same line?
Solution: Students have no options to choose from. Some students are at a loss.

Task 2a): When are two lines concurrent?
Solution: Students write the answer using memory only. The task changes into a more difficult one if we ask:

Task 2b): Which property is the same (different) for pairs of parallel and concurrent lines?
Solution: Students often do not include the common property, they only state when lines are parallel or concurrent. Thus they do not answer the question.

Task 3a): What is a plane uniquely identified by? *Solution:* By three non-collinear points.

Task 3b): Can a plain be uniquely identified by a pair of parallel lines / a pair of concurrent lines?

More examples like the above ones could be invented. They would all show that even though students know such terms as segment of a line, half-line, line, plane, etc., once the terms occur in a certain context, they are at a loss. The above tasks are included in elementary school subject matter. We studied student reactions and their solutions.

Most students (mostly women) studied at elementary and secondary schools aimed at transmission of part of a ”ready science” from the world of culture into students' memory. However, this is not an ideal solution, as it teaches facts and results, but it does not involve understanding (Hejný, 2001). It is thus necessary to develop key competences represented by the complex of knowledge, skills, abilities, attitudes and values necessary for personal and social development. The competences require activity and processing. It is necessary to change the content and the way of teaching, teaching methods and strategies in order to make learning interactive and experiencing and which makes learning based on practical experience and linked to real life. In the process of elementary education the key competences involve the ability to learn, to solve problems of different character, such as communicational, social, personal, civic and working (RVP, 2007).

4. Conclusion

It is necessary to apply constructive approaches to teaching of geometry at all levels and type of schools. It is also essential to stress the activity character in teaching geometry at universities for prospective teachers and look for effective means of developing space imagination as a useful skill.

Focus on good graphical skills is an important part of teaching geometry. Communication via graphics is most effective (e.g. illustrations in books, scientific works, schemes, layouts, etc.).

Memory training, readiness training and working on abilities to comprehend simple shapes, layout of objects in plane and space and their mutual positions and relationships is equally important for building foundations of geometry. Young children at this age already have certain manual skills, which they had acquired when playing with toys such as LEGO or PUZZLE. Such toys should not be neglected – I believe they still have their importance.

Teachers who themselves have problems with space imagination are not likely to include many tasks training it or requiring it. The same holds for teachers who underestimate their pupils.

Prospective teachers themselves should fully comprehend geometrical concepts and (thus) be able to explain them to their pupils. They should also be familiar with mathematical terminology and with the way – both mathematically and psychologically – geometrical notions, ideas and concepts develop in children's minds. When so they will be able to apply the attitude towards geometry when working with pupils; thus the class will become an environment for developing competences and personalities of pupils.

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SOME ASPECTS OF BLENDED-LEARNING EDUCATION

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Abstract. Mathematics is the language of technology, and e-learning techniques make math and science exciting through directly engaging students. The directive of the Higher Education Law in Poland (from 27 July 2005) allows introducing e-learning in Poland. Universities can not, however, legally organize pure virtual studies, because the latest directive does not concern classes run solely via e-learning methods. For these reasons technical universities in Poland keeps trying to formulate clear regulations for blended-learning courses and trying to find best "e-methods" for improving mathematical education.

ZDM Subject Classification: D20.

1. The directive in the Higher Education Law

Virtual reality of computer networks is becoming more and more omnipresent in our lives. The internet is playing a more and more important role in education, recruitment, trade, exercising power or in entertainment. For these reasons it is important to consider the problems which encounter both the lecturers preparing material that would be accessible on the net, and the students while using new technologies in obtaining information and knowledge. One should remember that the internet ensures not only an unprecedented access to information, but it also offers the possibility of interacting to the handicapped people. One thing is certain: the new medium, which is internet teaching, requires serious empirical and scientific research. Internet teaching creates the possibilities of assimilating knowledge at an individual pace, suited for the real psychophysical abilities of a student. It is an element that aids perfectly well the traditional didactic process.

In this paper we attempt to give an answer to key questions: What can we change or how can we find new improved methods in mathematics education?

The directive in the Higher Education Law from 25 September 2005 allows introducing e-learning in Poland. It should be noticed, however, that this version of the directive does not concern classes run solely via e-learning methods. This is a ministerial regulation concerning so-called blended learning. This is complementary (hybrid) learning, combining traditional and internet classes.

Here is a fragment from this directive:

§5 *The number of teaching hours both at full-time and part-time studies held with the use of e-learning techniques can not be greater than:*

1. *80% - in the case of units at Universities having the right to confer post-doctoral degrees,*
2. *60% - in the case of units at Universities having the right to confer doctoral degrees,*
3. *40% - in the case of all other units at a given University of the general number of hours of classes stated in the educational standards for specific fields and subjects of studies and their levels, excluding practical and laboratory classes.*

and directive from 31 October 2007

§5*a* *The number of teaching hours both at full-time and part-time studies held with the use of e-learning techniques can not be greater than 70% (...) for students, whose educate in not European countries.*

2. Blended-learning at a good level

It should first of all be considered whether all the sides of a university didactic process are ready for e-learning. An obvious requirement is the necessity to possess a well prepared hardware basis and teaching staff - not only to conduct such education, but also to administer the e-learning platform. It is sure that students treat the internet and its reserves as a natural source of information, knowledge, contacting friends - they spend a lot of their time writing blogs, chatting in discussion groups or playing games on the net. The teachers on the other hand lack the necessary skills and hence they are afraid to loose their position in the didactic process.

Using an e-learning platform can be a two-way action - realizing whole subjects solely with the use of the platform, or a support for traditional teaching. This way of organizing work brings in new possibilities of organizing the educational process, such as having the forum (that is having subject-oriented discussions "outside the classroom"), the possibility of exercising an additional system of control over the realized curriculum, the possibility of preparing exercises or solving problems at home (revision of material, catching up when students are absent), presenting interesting problems outside the curriculum, which allows students to develop their interests and makes it possible to work better with the most able students.

E-learning at a good level is neither cheap nor easy. Electronic courses are different from the traditional ones so the content of such courses and the way of presenting it have to be different. That is why specially qualified people are needed to prepare the content of e-courses - they not only need the knowledge, but also the ability to think in "hypertext" and "windows", and they have to prepare the subsequent elements of the electronic course along these lines. Such people are sometimes called Content Managers or Content Editors. It is a new job placed by the job market specialists among the jobs of the future. It should also be remembered that introducing e-learning is not a closed process. One should analyze the effects and introduce changes - otherwise the existence of a dynamic platform and the courses will not be possible. In the big companies it is a common practice that the e-courses are prepared by the whole teams of teachers:

- An expert responsible for the outline of the subject prepares source materials and exercises, etc.,
- A methodologist - e-learning specialist - decides about the spectrum and the kind of information technologies to be used,
- A team of computer scientists prepares electronic versions of the educational materials.

All these actions are supervised by a coordinator and are used through active cooperation (the experts assesses and verifies the content prepare by the computer scientists, the methodologist checks their accordance with the e-learning techniques to ensure the highest efficiency, etc.). As we can see, to prepare a professional e-learning course, we need to allocate special funds for that.

3. Blended-learning at technical universities

Gdańsk University of Technology, like other universities in Poland, keeps trying to formulate clear regulations for blended-learning courses. At present at our university we have e-learning courses basing on the university moodle platform. It is possible to use the e-learning course in different ways as additional help (as additional material for students, as obligatory material, as e-tests). Depending on the requirements of the teacher holding a given course, the students' participation in the e-courses can be either obligatory (the teacher has to have some statistics) or optional.

At our university we still have the problem of the role of e-materials in the academic teacher's assessment and there is also a problem with formulating the criteria for including the hours devoted to e-learning into the teachers' syllabus.

At present we have three kinds of courses at the Mathematics Teaching Department:

- General ones - such as "Math Compensation Classes", "Math Forum" (at which students can ask questions about any field of mathematics), "Revision of the secondary school material in Mathematics";
- Specialistic ones - concerning a given field of math, e.g. "Linear Algebra -Informatics", "Ordinary Differential Equations", "Complex Numbers";
- One semester long courses - for a given faulty - e.g. "Math for the Mechanical Faculty IV semester, "Math for Material Engineering II semester".

Statistical data concerning courses provided by the Math Teaching Department at the Gdańsk University o f Technology:

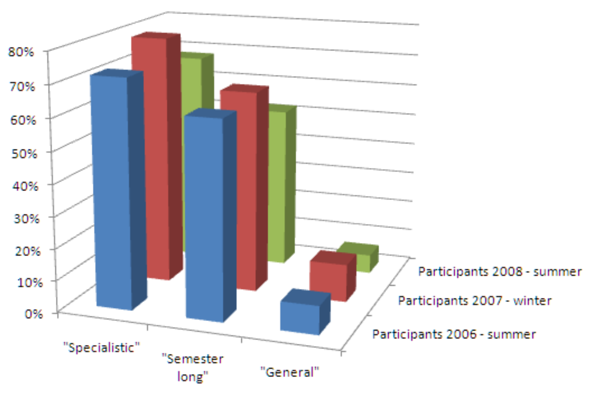

Figure 1.

4. Blended-learning as an element of an integrated information system

One has to admit that organizing such a course well requires a lot of time and energy - one should be prepared well both technically and factually. First of all it requires preparing one's own materials and browsing the net in search for good sites. Secondly, such a course calls for good maintenance and constant modernization of materials on the Moodle platform (such courses are only valuable if they are actively created and moderated, and the students' work is controlled on a regular basis). This is the kind of job that brings positive didactic effects, yet is not easy and very time-consuming, and the level of a student's engagement and satisfaction from such a course depends on many factors. It can not be denied that using multimedia and the Internet makes it possible to add valuable, from the didactic point of view, educational components creating professional and academic competence. Yet it should be remembered that e-learning should be an element of an integrated information system at a given university, which should also comprise recruitment, deans' offices service, planning the hourly room usage, or administering the university.

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SOME POSSIBILITIES OF A PLAYER BEHAVIOUR IN A STOCHASTIC GAME

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Abstract. The article discusses particular possibilities of the behaviour of a player in a stochastic game. Ways leading to the optimal bet amounts with respect to the initial capital and the probability of the win are calculated using an extensive simulation study.

ZDM Subject Classification: K50.

1. Introduction

Randomness and unpredictable situations play in our lives a significant role. On a few concrete examples, we can illustrate how randomness influences the result of a particular process. We will also show how to take into account random influences and subsequently how to make optimal decisions.

2. Certainty and Uncertainty

In every-day life (in a shop, services, in a game, ...), we can encounter the fact that the expected profit is for example 20%. What does it mean? The expected profit of 20% represent for example the situation when an investor (player) gets CZK 120 for every invested CZK 100. Another possibility is that an investor gets CZK 200 for every CZK 100 in every six cases out of ten (the probability of the success is 0.6). In the other four cases, the investor loses the invested CZK 100 and gets nothing. We compare from the mathematical point of view how particular situations are realistic and we will deal with the optimal behaviour of the investor (player).

3. Situation One

If we can be sure that there is no danger and all the invested financial resources will be profitable, it is optimal to invest the maximum possible amount. Let us imagine that we have at the beginning CZK 1 000 and we will invest (play) repeatedly a hundred times the amount of

$$1\,000 \cdot 1.2^{100} = 69\,014\,978\,768$$

This amount is not realistic.

4. Situation Two

At the beginning of the game, we have CZK 1 000 and we assume that we will bet a hundred times (if we have something to bet). The invested amount will double with the probability of 0.6 and with the probability of 0.4 we lose it. If we invested the maximum amount and won every time, we would get

$$1\,000 \cdot 2^{100} \doteq 1.268 \cdot 10^{33}.$$

This situation is very unlike and it happens with the probability of

$$0.6^{100} \doteq 6.533186235 \cdot 10^{-23}.$$

It is obvious that it is better to use a different strategy. To find it, we use a simulation study.

5. Simulation Study

We are interested how the expected profit changes at the end of the game in accordance with the invested amount. We have gradually changed the invested money from CZK 10 to CZK 1 000 (with a step of CZK 10) and we have simulated every case 1 000 000 times. The following graph shows the results of the simulation study.

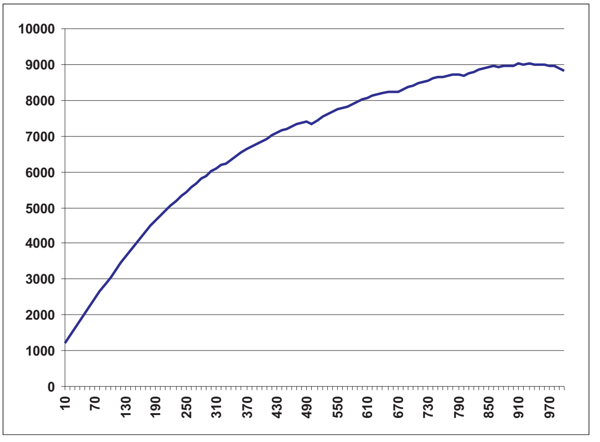

Fig. 1a): Average profit regarding the bet amount

It is obvious from the figure that the optimal way is to bet about CZK 900. For this reason, we display in detail the given area (see Fig. 1b).

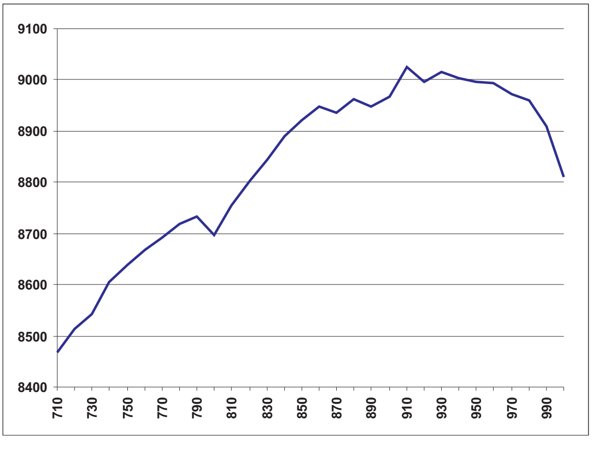

Fig. 1b): Average profit

In many cases, the criterion of the maximum profit need not be acceptable for players. It is not uninteresting to deal with the opposite situation when we are interested in how many cases the player loses everything. The following graph shows how many (out of 1 000 000) players ended with an empty account.

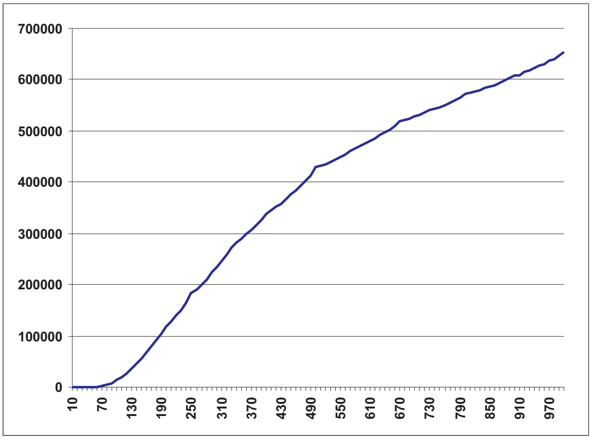

Fig. 2: Number of games ending with an empty account.

It is obvious that we could achieve even better results in case we would change the amount of the deposit in accordance with the remaining amount of the player. The model of such a situation is significantly more difficult.

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SYSTEMS OF LINEAR EQUATIONS AND THE THEORY OF VECTOR SPACES

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Abstract. Connection between systems of linear equations and theory of vector spaces by means of geometric interpretation is studied. The program Cabri 3D is applied.

ZDM Subject Classification: B40.

This contribution has arisen thanks to the grant from the European Social Fund (in frame of the Provision 3.2 Support of the Tertiary Education, Research and Development) to the project issue CZ.04.1.03/3.2.15.3/ 0434 *"Increasing of the Levels of the Education at Mathematics"*. The objective of this project is to raise the level of teacher's readiness in the area of the mathematics and didactics in order they could expertly perform in their profession according to in the Czech Republic newly implementing Frame Education Program; the objective of this contribution is to show the connection between systems of linear equations and the theory of vector spaces by means of the geometric interpretation.

To speak about a geometric interpretation means to use suitable figures and we will use the program Cabri 3D for them. In general, because the use of ICT technologies the research in mathematics education has been changed. This article tries to show how could be understood these changes within the problem of a set of solutions of a system of linear equations. It is obvious that by means of suitable mathematics software one can find this set easily. On the other hand there is the theory of vector spaces as the theoretical base of solutions of system of linear equations here and one can analyze the different languages and associated ways of representing constructs in the subject of linear algebra, as well as their modes of interaction. On principal there are three of them: the language of the general theory, the language of $\mathbb{R}^n$, and the geometric language of space in two and three dimensions.

We will show this reasoning by the systems of two linear equations in two unknowns. Three such systems will be solved, more precisely their solutions will be shown as the figures in the program Cabri Geometry 3D. Unfortunately, we are restricted by just 3D space, thus there is not possible to use this procedure for systems of linear equations in three (and more) unknowns.

Fundamental idea common for all three systems is very simple and consists in what follows:

To solve the system

$$a_{11}x + a_{12}y + a_{13} = 0$$

$$a_{21}x + a_{22}y + a_{23} = 0$$

alias

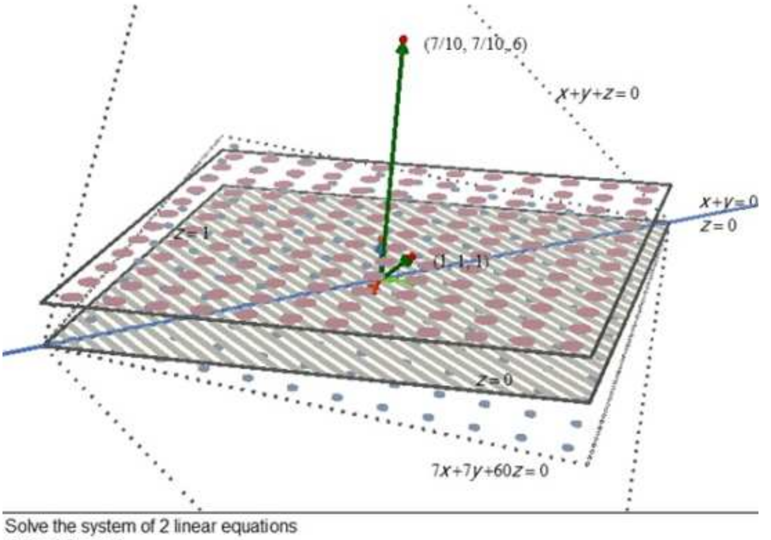
$$(a_{11}, a_{12}, a_{13})(x, y, 1) = 0$$

$$(a_{21}, a_{22}, a_{23})(x, y, 1) = 0$$

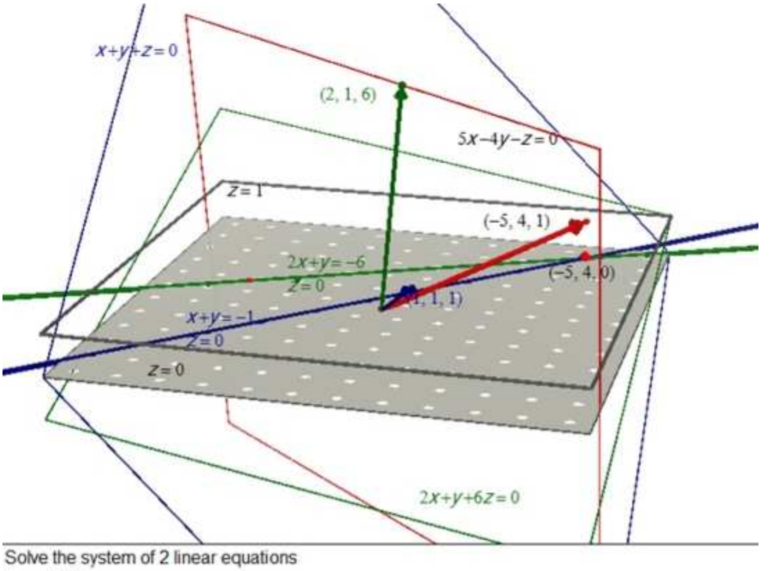
is equivalent to the problem of finding of a vector $(x, y, 1)$ that is perpendicular both to the vector (a_{11}, a_{12}, a_{13}) and the vector (a_{21}, a_{22}, a_{23}). And this is all!

It has been already said the purpose of this contribution is only a visualization of the relation between system of two linear equations in two unknowns and theory of vector spaces (with the scalar product naturally) and therefore there is no need of other words - all substantial one can see in the following figures constructed in the Cabri Geometry 3D.

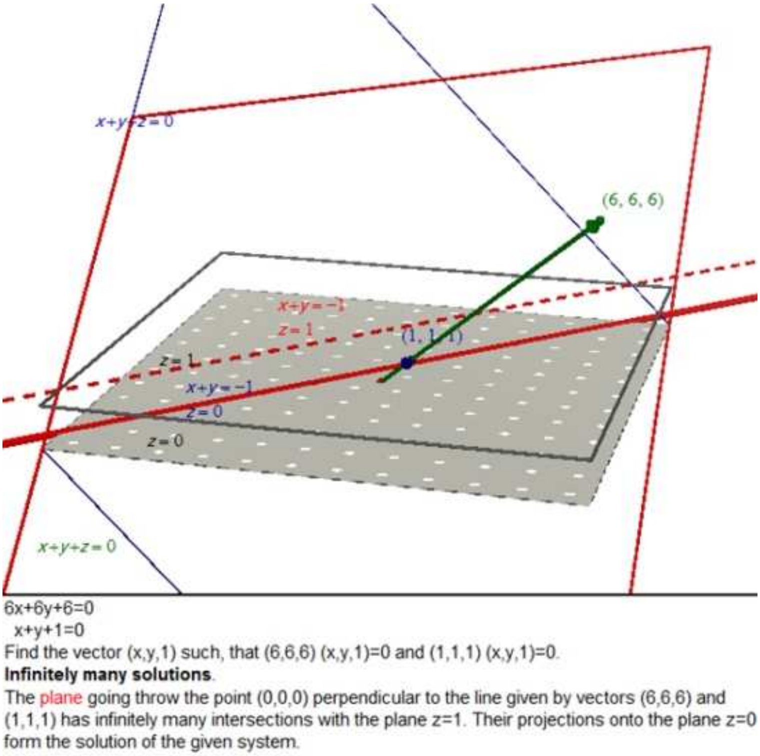
The last note on the end: The best way for this reasoning is when one has the possibility of using the Cabri 3D and Replay Construction command on following figures...



Solve the system of 2 linear equations
0,7x+0,7y+6=0
x+y+1=0
Find the vector (x,y,1) such, that (0.7,0.7,6) (x,y,1)=0 and (1,1,1) (x,y,1)=0.
No solution.
The vector going throw the point (0,0,0) perpendicular to the plane given by vectors (0.7,0.7,6) and (1,1,1) has no intersection with the plane z=1.



Solve the system of 2 linear equations
2x+y+6=0
x+y+1=0.
Find the vector (x,y,1) such, that (2,1,6) (x,y,1)=0 and (1,1,1) (x,y,1)=0.
1 solution.
This vector is perpendicular to the plane given by vectors (2,1,6) and (1,1,1) and the projection of its ending point (-5,4,1) onto the plane z=0 is the solution of the given system.



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REPRESENTATION OF MATHEMATICAL PROBLEMS BY ANIMATIONS

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Abstract. Production of computer animations with mathematical content can be based on various platforms and can serve different functions in mathematics education. The contribution gives ideas and instructions on how to create mathematical interactive animations. Examples include animations based on current interactive geometrical systems, graphical systems, animations in combination with virtual and real video. Educational use of animations is focused on visualization of mathematical concepts including definitions and clarifying various algorithms in mathematics. Simple animations can be used in electronic textbooks and e-learning courses. They also fit into a traditional mathematics instruction enriched by the use of ICT.

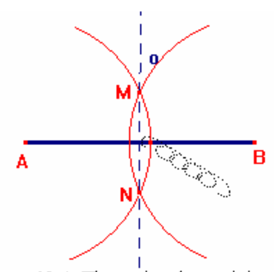
ZDM Subject Classification: U70.

1. Animations based on platforms of interactive dynamic systems

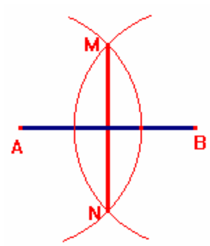
Animations based on a quick exchange of static pictures in computer setting operate on the principle of generated pictures by a computer. Their speedy projection impresses human perception as a continuous move by which dynamics of an observed action is caught in certain time and semantic connections.

While creating multimedia instructive tools it is necessary to take into account the correct didactic transposition of traditionally elaborated curriculum contents into the form of electronic materials. Processing and creating computer animations with mathematical content can be based on various platforms and can serve different functions in mathematics education. Solving most school geometrical problems is based on finding out characteristics (position and metric) of geometrical figures and relations between them. Dynamic geometrical systems (Cabri IIPlus, Compass and Ruler, Cabri3D) serve as means of creating interactive constructions. They play an important role in mathematics teaching. Mentioned programme applications dispose of options to set some attributes that enable simple animations:

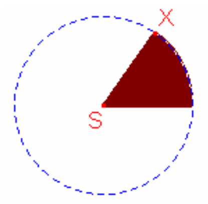
- *Point animation* (e.g. in Cabri it is realised by "stretching a spring", that determines the direction and the speed of an animation of marked figures, the system C.a.R. animates one point along a line segment or along a circle).
- *Watching traces of points* (switch on/switch off the trace) - in Cabri geometry it is related to drawing traces of marked figures after changing their position; in C.a.R. there is an option of manual (drawing traces of moving points and lines) and automatic tracing (tracing traces of dependent points and lines by animating the movement of one independent point, for example, along a circle or a line).



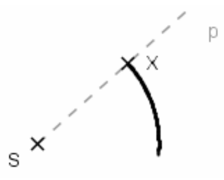
Picture No1: The spring determining the direction and the speed of the point animation that defines the circle radius.



Picture No 2: Switching on watching the trace of points M, N. Drawing the axis of the line segment AB.

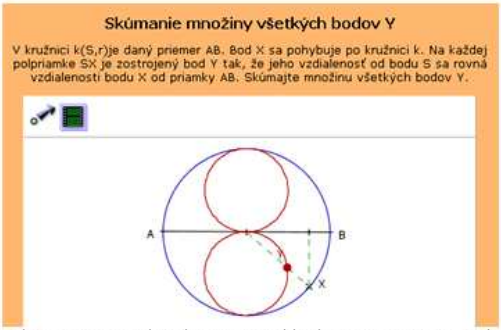


Picture No 3: Watching the trace of the line segment SX while animating the point X (Cabri geometry).



Picture No 4: Manual watching the trace of the point X while changing the position of the semi-line SX (C.a.R.).

If the drawing created in the system C.a.R. is saved during actual animation, the animation will be saved together with the file. In the combination with animation saving in the html format it means that application will be animated after its opening in a browser [6]. If we provide and depict other tools to a user of animated application to make experiments with characteristics of the circle elements, then we speak about an animation with limited interactivity.

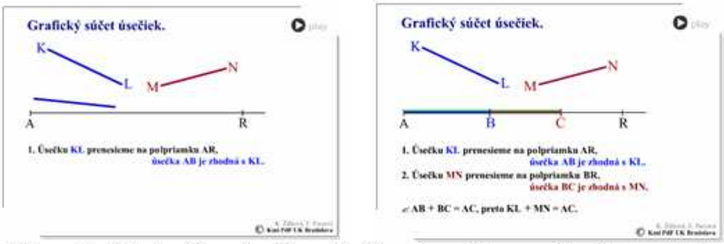


Picture No 5: Animation prepared in the system C.a.R. and arranged according to the need of web publication.

One of fully interactive applications is Cabri application published on the Web sides and run by Cabri Java that is seen in a standard window with the options to switch on and off the trace and define an animation. A user (a teacher or a pupil) has to create the commands on his/her own, because if he is not the author of the created applet and does not know which construction elements are dependent and which independent, so animation commands can make problems. Purposefulness and significance of mentioned tools are obvious in mathematics teaching. Point trace visualization and figure animation of contribute to clearness in teaching and to a possibility to find out mutual causally connections, hence to a possibility to integrate observed phenomenon, possibly new emerged figures, into a cognitive structure of mathematical knowledge.

2. Animations created on the base of professional graphic systems

There exist plenty of types of professional tools to create animations based on vector graphics. Macromedia Flash belongs to one of the most famous, long-time used programme products. Two methods can be used to create animation sequences. One of them is based on gradual production of all images that will be animated. The second method is more comfortable for a user, because so-called "key frames" (the first and the last ones) are sufficient to produce and Flash makes up animated transition between them automatically. Flash applications with mathematical content are applicable very easily not only in electronic textbooks, e-learning courses, mainly in distant mathematical education, but also as a complementary material in traditional mathematics instruction enriched by the active use of ICT means. From the viewpoint of didactics of mathematics we see the great importance in creating minor applications containing the explanation of fundamental and derived concepts including some definitions as well as in clarifying various algorithmic methods.

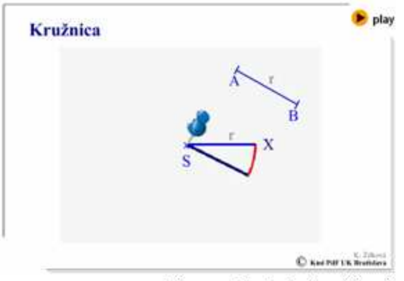


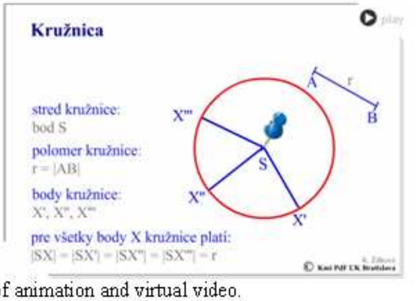
Pictures No 6, 7: Depicting algorithm of finding out graphic sum of two line segments.

Visualization and geometric interpretation of some mathematical relations belong to further, but not less important functions of such processed mathematical curriculum. Though to clarify (sometimes also to prove) them is not an immediate teaching goal, but to bring them closer and remember them easier their representations by animation technique are useful. Mentioned examples of mathematical animations created in Macromedia Flash environment are not interactive or their interactivity is restricted only to the possibility to start or finish the animation. We put such animations into the group of *demonstrable animations*. The aim and the role of demonstrable animations according to V. Stoffová are: "to motivate learners, attract their attention, increase the clearness of teaching, demonstrate principles of activities and working things, demonstrate technological procedures, draw dynamic phenomena, processes, changes (time and space, possibly time-space), ...", and they have many other functions that follow from subject, professional and content viewpoints.

3. Animation combined with virtual video

Teaching mathematics in ICT environment often requires performing some users' procedures in various mathematical educational programmes. The demonstrable procedures are important not only for a teacher when he creates didactic materials for teaching, but also for pupils as a manual for active use of mathematical programme applications. In the same way we also consider tools that enable to create educational video materials to be useful means. The summary of accessible programmes for creating simple video sequences and a manual for preparing visual electronic materials are provided by D. Brigán [1] who finds video sequences accompanied by text helpful in key moments and able to simplify the learning process. Video record of events on the screen caught by some of accessible software is called *virtual video*. In this context any programmes or micro-spaces aimed at supporting mathematics education, including mentioned interactive geometric systems, can be used for creating mathematical educational materials. The animation process in Cabri geometry or in the C.a.R. system can be recorded with the target to be used single-handed or in the combination with other animations and supplemented text information in Macromedia Flash environment.





Pictures No 8, 9: Combination of animation and virtual video.

4. Animation combined with real video

The important supplement of mathematics education is various manipulative activities (work with building sets, puzzles, strategic plays, "proof" techniques by paper folding and etc.). According to O. Židek "in didactics of mathematics we perceive manipulations as multi-sensory tools helping pupils´ learning through their own experience acquired not only by eyes, but mainly by sense of touch" [5]. At the same time the author of the quote put the accent on the importance of the combination of manipulative teaching method with virtual cognition [4]. As mentioned manipulative activities are difficult to formulate and present in verbal or written forms there is a possibility to create so-called "real video" which means to catch events in real world by a camcorder or possibly digital camera. Useful educational material can arise in combination with other methods mentioned in the contribution.

Picture No 10: Combination of animation and real video.

5. Conclusion

The need of visualization as the most appropriate and effective sense preference that "plays an important role in finding out relations and connections among mathematical figures and in transmission and communication in mathematics" is emphasized by D. Malá [2]. She also draws attention to "the proved relation between preference of visualization and successfulness in teaching mathematics to 11–15 aged pupils of elementary schools". This is one of the reasons why we consider drawing and visualizing mathematical abstract concepts and relations within increasing efficiency of instruction process to be the basic aims of using animations in mathematics instructions. Playing sound records during animations, possibly verbal teacher's comments during playing animation in mathematics instruction can address also auditory pupils.

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