Pedagogical Faculty
Catholic University in Ružomberok

Mathematica

Proceedings of the

XIth Slovak - Czech - Polish
Mathematical School

Ružomberok
June 2nd - 5th, 2004

Scientific Committee

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ISBN 80-8084-004-0

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Preface

The fruitful cooperation between the Institute of Mathematics and Computer Science of Czestochowa Pedagogical University (Poland) and the Faculty of Education of Jan Evangelista Purkyně in Ústí nad Labe (Czech Republic) started a long time ago. One form of the joint effort were the Mathematical Schools organized in succession in Poland and Czech Republic. Later on the Institute of Mathematics of Pedagogical University in Krakow (Poland) and the Pedagogical Faculty of Catholic University in Ružomberok (Slovakia) joined ranks. In 2004 the Mathematical School for the first time took place in Slovakia. The hills of Veľká Fatra provided a nice background for the deepening collaboration.

The XI Slovak-Czech-Polish Mathematical School (Ružomberok, June 2nd - 5th, 2004) served as a forum for the exchange of scientific achievements and opinions between educational and research workers, represented a broad cross-section of interests and results, and brought fresh viewpoints and methods to the participants. The present volume contains extended versions of the contributions presented at the conference. The lectures have been grouped into two sections: Mathematics Teaching and Mathematics and Application. Accordingly, the papers are divided into two parts. The papers contained in the first part address the advances in modern methods and technologies used in teaching mathematics. Two contributions represented at the conference Mathematics in School Today and Tomorrow (Ružomberok, September 8th - 2003) are also included. The second part represents research results of participants carried out in various fields of mathematics.

The editor of this issue is indebted to the authors for the effort in preparing their contributions, and to the reviewers for their useful comments and suggestions. It is our earnest hope that further research in the fields covered in this volume, as well as the cooperation between Czech, Polish, and Slovak colleagues will broaden.

The organizers

PART I

MATHEMATICS AND APPLICATIONS

On some algebras of Ex compatible varieties

Teresa Biegańska

For every variety V there exists an algebra $\mathcal{A}$ generating V by means of direct products, subalgebras and homomorphic images, i.e. $V = HSP(\mathcal{A})$. Such algebras are called generics of V. Obviously, a free algebra over V with ω generators is a generic of V. A generic $\mathcal{A}$ of V is called minimal generic of V if for every generic $\mathcal{B}$ of V we have $card(\mathcal{B}) \geq card(\mathcal{A})$. The aim of this paper is to find minimal generics for some P – compatible varieties.

In this paper we consider algebras of a given type $\tau : F \to N$, where F is a set of fundamental operation symbols and N is the set of positive integers.

Let P be a partition of F. For any $f \in F$ we denote by $[f]_P$ the block of the partition P containing f. An identity $\varphi \approx \psi$ of type τ is called P–compatible (see [9]) if it is of the form $x \approx x$ or of the form

$$f\left(\varphi_1, \ldots, \varphi_{\tau(f)}\right) \approx g(\psi_1, \ldots, \psi_{\tau(g)}), \text{ where } [f]_P = [g]_P,$$

$\varphi_1, \ldots, \varphi_{\tau(f)}, \psi_1, \ldots, \psi_{\tau(g)}$ are terms of type τ.

Let Ex denote the partition of F consisting of singletons only. Ex–compatible identities are called externally compatible. It means that an indentity $\varphi \approx \psi$ is externally compatible if it is of the form $x \approx x$ or the most external fundamental operation symbols in φ and ψ are identical (see [4]).

Let V be a variety of type τ. We denote by V_P the variety defined by all P–compatible identities of type τ satisfied in V and by $P(\tau)$ the set of all P–compatible identities of type τ. By this definition $Id(V_P) = Id(V) \cap P(\tau)$, where $Id(V_P)$ and $Id(V)$ denote the sets of all identities satisfied in the classes V_P and V respectively. A variety V is called P–compatible if the identities satisfied in V are P–compatible.

To represent algebras from V_P by means of algebras from V, J. Płonka defined in [9] a construction called a P–dispersion of an algebra as follows:
Let $\mathcal{D} = \left(P, \mathcal{I}, \{A_i\}_{i\in I}, \{o_{[f]_P}\}_{f\in F}\right)$ be a quadruple satisfying the following condition (1)–(4):

(1) P is a partition of F,

(2) $\mathcal{I}$ is an algebra of type τ and $\mathcal{I} = (I, F)$;

(3) $\{A_i\}_{i\in I}$ is a family of non-empty pairwise disjoint sets,

(4) $\{o_{[f]_P}\}_{f\in F}$ is a family of mappings $o_{[f]_P} : I \to \bigcup\limits_{i\in I} A_i$ such that for every $i \in I$ we have $o_{[f]_P}(i) \in A_i$ and $o_{[f]_P} = o_{[g]_P}$ if $g \in [f]_P$.

The quadruple $\mathcal{D}$ is called a P–dispersing system.
For a P–dispersing system $\mathcal{D}$ an algebra $\mathcal{A}$ of type τ is defined as follows: $\mathcal{A} = (A, F)$, where
$A = \bigcup_{i\in I} A_i$, and for each $f \in F, a_k \in A_{i_k} (k = 1, 2, \ldots, \tau(f))$ we put:

$$f^{\mathcal{A}}(a_1, \ldots, a_{\tau(f)}) = o_{[f]_P}\left(f^{\mathcal{I}}(i_1, \ldots, i_{\tau(f)})\right).$$

The algebra $\mathcal{A}$ is called a P–dispersion of $\mathcal{I}$ by the P–dispersing system $\mathcal{D}$ or briefly the P–dispersion of $\mathcal{I}$.

Let $Id(\mathcal{A})$ and $Id(V)$ denote the sets of all identities of type τ satisfied in an algebra $\mathcal{A}$ and in a variety V, respectively. Denote by $ex(q)$ the most external fundamental operation symbol of a term q (if it exists). A block $[f]_P$ will be called nullary if $\tau(g) = 0$ for all $g \in [f]_P$. A term q different from a variable will be called non-trivial.

Let us consider the following conditions for a variety V and a partition P of F :

(i) If $[f]_P$ is a non-nullary block of P and $g, h \in [f]_P$, then there exists a non-trivial unary term $q_{g,h}$ such that $ex(q_{g,h}) \in [f]_P$ and the identities:

$$g(x_1, \ldots, x_{\tau(g)}) \approx q_{g,h}(g(x_1, \ldots, x_{\tau(g)})),$$

$$h(x_1, \ldots, x_{\tau(h)}) \approx q_{g,h}(h(x_1, \ldots, x_{\tau(h)}))$$

belong to $Id(V)$;

(ii) If $[f]_P$ is a nullary block of P, then for each $g \in [f]_P$ the identity $f \approx g$ belongs to $Id(V)$.

In [5] K. Hałkowska proved the following:

Theorem 1. If P is a partition of F and V is a variety of type τ satisfying conditions (i) and (ii), then an algebra $\mathcal{A}$ belongs to V_P if and only if $\mathcal{A}$ is a P–dispersion of some algebra from the variety V.

The aim of this paper is to find minimal generics for some P–compatible varieties. Given an algebra $\mathcal{A} = (A, F)$ of type τ and $e \in A$, we construct an algebra $\mathcal{A}_P^e = (A\backslash\{e\} \cup F'_e, F)$ called the P_e–extension of $\mathcal{A}$.
Let F' be the set of all classes $[f]_P \in P$ such that for some terms φ, ψ of type τ we have $\varphi \approx \psi \in Id(\mathcal{A}), ex(\varphi) = f, ex(\psi) = g$ and $[f]_P \neq [g]_P$.

We take

$$F'_e = \{e_{[f]_P} : [f]_P \in F'\},$$

where $F'_e \cap A = \emptyset$ and for any $f, g \in F$ we have $[f]_P = [g]_P$ iff $e_{[f]_P} = e_{[g]_P}$.

For any $a_1, \ldots, a_n \in (A\backslash\{e\}) \cup F'_e$ define the elements $a'_1, \ldots, a'_n$ as follows. For each $i \in \{1, \ldots, n\}$,

$$a'_i = \begin{cases} a_i & \text{if} \quad a_i \in A, \\ e & \text{otherwise.} \end{cases}$$

For any $f \in F$ and $a_1, \ldots, a_n \in (A\backslash\{e\}) \cup F'_e$ we define

$$f^{\mathcal{A}_P^e}(a_1, \ldots, a_n) = \begin{cases} f^{\mathcal{A}}(a'_1, \ldots, a'_n) & \text{if} \quad f^{\mathcal{A}}(a'_1, \ldots, a'_n) \neq e, \\ e_{[f]_P} & \text{otherwise.} \end{cases}$$

We recall that an element e of A is called an idempotent element of the algebra $\mathcal{A}$ if for any $f \in F$ we have $f(e, \dots, e) = e$.

It is known from the paper [3] that

THEOREM 2. If $\mathcal{A}$ is a generic of a variety V, $e \in A$ is an idempotent element of the algebra $\mathcal{A}$, P is a partition of F and $F' \neq \emptyset$, then $\mathcal{A}_P^e$ is a generic of V_P.

THEOREM 3. If V is a variety of algebras of type τ satysfying conditions (i), (ii) and $\mathcal{A} = (\{e, a_1, \dots a_n\}, F)$ is a minimal generic of a variety V, e is an idempotent element of $\mathcal{A}$ and

$$\text{(iii)} \quad \bigwedge_{f \in F} \bigwedge_{x,y \in \{e, a_1, \dots, a_n\}} [x \neq y \Rightarrow f(x, \dots, x) \neq f(y, \dots, y)],$$

then $\mathcal{A}_{Ex}^e$ is a minimal generic of V_{Ex}.

PROOF.
It follows from theorem 2. that the algebra $\mathcal{A}_{Ex}^e$ is a generic of V_{Ex}.

Let an algebra $\mathcal{D} = (\{b_1, \dots, b_s\}, F)$ be a generic of V_{Ex} such that $s < |F_e'| + n$.

From the fact that V is a variety satysfying condition (i), it follows, by theorem 2, that there exists an algebra $\mathfrak{C} = (C, F) \in V$ such that $\mathcal{D}$ is a dispersion of this algebra. It is an essential case when $V \neq V_{Ex}$. Thus, the set F' contains more than one element.

Let us consider the following algebra $\mathcal{A}_{Ex}^e$ as a dispersion of the algebra $\mathcal{A}$:

First, a family of non-empty pairwise disjoint sets of the set $F_e' \cup \{a_1, \dots, a_n\}$ is defined as follows:

$\{A_i\}_{i \in \{e, a_1, \dots, a_n\}}$, where $A_i = \{a_i\}$, if $1 \leq i \leq n$ and $A_e = F_e'$.

We then define the family of mappings $o_f : \{e, a_1, \dots, a_n\} \to F_e' \cup \{a_1, \dots, a_n\}$, for $f \in F$ as follows:

$$o_f(i) = \begin{cases} i & \text{if} \quad i \in \{a_1, \dots, a_n\}, \\ e_f & \text{if} \quad i = e, \ f \in F', \\ e_{f_o} & \text{if} \quad i = e, \quad f \in F \backslash F', \end{cases}$$

where f_o is an established element of the set F'.
The algebra $\mathcal{A}_{Ex}^e$ is then a dispersion of the algebra $\mathcal{A}$.

Let $f \in F$ and $x_1, \dots, x_{\tau(f)} \in F_e' \cup \{a_1. \dots, a_n\}$, where $x_i \in A_{j_i}$, for $1 \leq i \leq \tau(f)$ and $j_i \in \{e, a_1, \dots, a_n\}$. According to the definition of $f^{\mathcal{D}}$ in a dispersion $\mathcal{D}$, we have:

$$f^{\mathcal{D}}(x_1, \dots, x_{\tau(f)}) = o_f(f^{\mathcal{A}}(j_1, \dots, j_{\tau(f)}))$$

where

$$o_f(f^{\mathcal{A}}(j_1, \dots, j_{\tau(f)})) = \begin{cases} f^{\mathcal{A}}(j_1, \dots, j_{\tau(f)}) & \text{if } f^{\mathcal{A}}(j_1, \dots, j_{\tau(f)}) \neq e, \\ e_f & \text{if } f^{\mathcal{A}}(j_1, \dots, j_{\tau(f)}) = e, f \in F', \\ e_{f_0} & \text{if } f^{\mathcal{A}}(j_1, \dots, j_{\tau(f)}) = e, f \in F \backslash F'. \end{cases}$$

If $f^{\mathcal{A}}(j_1, \dots, j_{\tau(f)}) \neq e$, then $A_{f^{\mathcal{A}}(j_1, \dots, j_{\tau(f)})} = \{f^{\mathcal{A}}(j_1, \dots, j_{\tau(f)})\}$,
so $o_f(f^{\mathcal{A}}(j_1, \dots, j_{\tau(f)})) = e_f$.
If $f^{\mathcal{A}}(j_1, \dots, j_{\tau(f)}) = e$, then $A_e = F_e'$ and $o_f(e) = e_f$,
so $f^{\mathcal{D}}(x_1, \dots, x_{\tau(f)}) = f^{\mathcal{A}_{Ex}^e}(x_1, \dots, x_{\tau(f)})$.

The algebra $\mathcal{D}$ is a dispersion of the algebra $\mathfrak{C}$. If each class would consist of singletons only, we would have the dispersion with the smallest number of elements. Then $Id(\mathfrak{C}) = Id(\mathcal{D})$. Therefore some class A_x must contain at least two elements. In this dispersion there must be at least $|F'_e| - 1$ new elements, because this number is the infimum of the set containing numbers $\binom{|F'_e|}{2}$ of two-element subsets of the set F'_e, for $|F'_e| \geq 2$. Thus $|F'_e| + n > s \geq |C| + |F'_e| - 1$, so $|C| < n + 1$.

Since $\mathcal{A}$ is a minimal generic of the variety V, the algebra $\mathfrak{C}$ is not a generic of V.

Thus $Id(\mathfrak{C}) \supsetneq Id(V)$. Therefore, there exists an identity $\varphi \approx \psi$ belonging to $Id(\mathfrak{C})$, which is not satisfied in V. This identity cannot be externally compatible, because in that case it would be satisfied in the dispersion $\mathcal{D}$, and consequently in V_{Ex}. That is impossible, because $Id(V_{Ex}) \subseteq Id(V)$. Therefore, that is an identity such that $ex(\varphi) \neq ex(\psi)$. However, in this case the identity $f(\varphi, \ldots, \varphi) = f(\psi, \ldots, \psi)$ is externally compatible and satisfied in $\mathfrak{C}$, and so in the dispersion $\mathcal{D}$ as well. Since $\mathcal{D}$ is a generic of V_{Ex}, the identity is satisfied in V_{Ex}, so in $\mathcal{A}^e_{Ex}$ as well. We thus arrived at a contradiction. Indeed, let $f(\varphi_1, \ldots, \varphi_{\tau(f)})(x_1, \ldots, x_n) = g(\psi_1, \ldots, \psi_{\tau(g)}))(x_1, \ldots, x_n)$, where $f \neq g$, be an identity, which is not satisfied in the algebra $\mathcal{A}$. Thus there exist $b_1, \ldots, b_n \in \{e, a_1, \ldots, a_n\}$ such that

$$f(\varphi_1, \ldots, \varphi_{\tau(f)})(b_1, \ldots, b_n) \neq g(\psi_1, \ldots, \psi_{\tau(g)})(b_1, \ldots, b_n).$$

Let $f(\varphi_1, \ldots, \varphi_{\tau(f)})(b_1, \ldots, b_n) = a$ and $g(\psi_1, \ldots, \psi_{\tau(g)})(b_1, \ldots, b_n) = b$.

The identity

$$(*) \qquad f(f(\varphi_1, \ldots, \varphi_{\tau(f)}), \ldots, f(\varphi_1, \ldots, \varphi_{\tau(f)}))(x_1, \ldots, x_n) =$$

$$= f(g(\psi_1, \ldots, \psi_{\tau(g)}), \ldots, g(\psi_1, \ldots, \psi_{\tau(g)})(x_1, \ldots, x_n)$$

is externally compatible, but it is not satisfied in $\mathcal{A}^e_{Ex}$, because

$$f^{\mathcal{A}^e_{Ex}}(f(\varphi_1, \ldots, \varphi_{\tau(f)}), \ldots, f(\varphi_1, \ldots, \varphi_{\tau(f)}))(b_1, \ldots, b_n) = f^{\mathcal{A}^e_{Ex}}(a, \ldots, a),$$

$$f^{\mathcal{A}^e_{Ex}}(g(\psi_1, \ldots, \psi_{\tau(g)}), \ldots, g(\psi_1, \ldots, \psi_{\tau(g)}))(b_1, \ldots, b_n) = f^{\mathcal{A}^e_{Ex}}(b, \ldots, b)$$

and $f^{\mathcal{A}^e_{Ex}}(a, \ldots, a) \neq f^{\mathcal{A}^e_{Ex}}(b, \ldots, b)$.

Indeed, if $a \in A$, then $b \notin A$ and

$$f^{\mathcal{A}^e_{Ex}}(a, \ldots, a) = o_f(f^{\mathcal{A}}(e, \ldots, e)) = o_f(e) = e_f,$$

$$f^{\mathcal{A}^e_{Ex}}(b, \ldots, b) = o_f(f^{\mathcal{A}}(b, \ldots, b)) = o_f(b) = b \text{ and } e_f \neq b,$$

while if $a, b \in \{a_1, \ldots, a_n\}$, then $f^{\mathcal{A}^e_{Ex}}(a, \ldots, a) = a$ and $f^{\mathcal{A}^e_{Ex}}(b, \ldots, b) = b$ and $a \neq b$.

The identity $(*)$ is not satisfied in $\mathcal{A}^e_{Ex}$, so it is not satisfied in V_{Ex} as well. Thus, $\mathcal{D}$ is not a generic of V_{Ex}.

The theorem is proved.

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The Use of the Sheaf of Planes and Spheres at the Solving of Tasks

Martin Billich

In the present paper possibility of using the sheaf of the planes and sheaf of the spheres to solve some problems of analytic geometry are discussed. We describe some facts from the theory of the spheres in relation to linear combination of their equations. There are shown several examples of this part of geometry in this article.

Sheaf of the Planes

We don't meet with the term of a linear combination in the solving of tasks of the analytic geometry. The most often use of a linear combination be related to the bunch of the lines and circles in a Euclidean plane (see [1]).

Let us consider two concurrent planes $\boldsymbol{p_1}$, $\boldsymbol{p_2}$ with the equations

$$\boldsymbol{p_1}\colon a_1x + b_1y + c_1z + d_1 = 0 \tag{1}$$

$$\boldsymbol{p_2}\colon a_2x + b_2y + c_2z + d_2 = 0 \tag{2}$$

If we multiply an equation (1) by a real number α and an equation (2) by a real number β and sum both equations, we obtain

$$\alpha\,(a_1x + b_1y + c_1z + d_1) + \beta\,(a_2x + b_2y + c_2z + d_2) = 0 \tag{3}$$

which is a *linear combination* of the equations (1) and (2). From the concurrent of the planes $\boldsymbol{p_1}$, $\boldsymbol{p_2}$ follows a linearly independent of their normal vectors (a_1, b_1, c_1) and (a_2, b_2, c_2). The set of all points of 3-dimensional Euclidean space which coordinates satisfy an equation (3) is a plane containing common points of both plane $\boldsymbol{p_1}$, $\boldsymbol{p_2}$ (their intersection), because their coordinates satisfy both equations (1) and (2).

Task 1. Let us introduce the line $\boldsymbol{l}$ by intersection of two planes

$$x + y - 3 = 0, x - 2y + z = 0$$

and the plane $\boldsymbol{p}$: $2x + z + 1 = 0$. Write the equation of plane, which contains the line $\boldsymbol{l}$ and which is perpendicular to the plane $\boldsymbol{p}$.

Solution. A searched plane will has an equation (3) in the form

$$\alpha\,(x+y-3)+\beta\,(x-2y+z)=0$$

and then

$$(\alpha+\beta)\,x+(\alpha-2\beta)\,y+\beta z-3\alpha=0$$

If a searched plane is perpendicular to the plane $\boldsymbol{p}$, then their normal vectors are perpendicular too. We get

$$(\alpha+\beta)\,2+(\alpha-2\beta)\,0+\beta 1=0$$

and after adjustment we have

$$\frac{\alpha}{\beta}=-\frac{3}{2}$$

Now, for $\alpha=3$ we get $\beta=-2$ and the equation of searched plane will be

$$x+7y-2z-9=0$$

Sheaf of the Spheres

Let us look at the next application of the linear combination not only of the linear but also of nonlinear geometric objects. We deal with the sheaf of the spheres and their applications at the solving of the tasks of analytic geometry in this section.

Let us now consider a sphere $\boldsymbol{g}$ and a plane $\boldsymbol{p}$ with equations

$$\boldsymbol{g}\colon (x-s_1)^2+(y-s_2)^2+(z-s_3)^2=r^2 \tag{4}$$

$$\boldsymbol{p}\colon ax+by+cz+d=0 \tag{5}$$

If we multiply an equation (5) by a real number α and sum both equations, we obtain

$$[(x-s_1)^2+(y-s_2)^2+(z-s_3)^2-r^2]+\alpha\,(ax+by+cz+d)=0 \tag{6}$$

We could write this equation in the form

$$(x-A)^2+(y-B)^2+(z-C)^2=A^2+B^2+C^2-D \tag{7}$$

where

$$A=s_1-\frac{\alpha}{2}a,\ B=s_2-\frac{\alpha}{2}b,\ C=s_3-\frac{\alpha}{2}c,\ D=s_1^2+s_2^2+s_3^2-r^2+\alpha d$$

What can we say about the set of all point of 3-dimensional Euclidean space, which coordinates satisfy the equation (7) ?

If $A^2+B^2+C^2-D<0$ then there doesn't exist a point which coordinates satisfy the equation (7). Otherwise it can be a point or a sphere. If the sphere $\boldsymbol{g}$ and the plane $\boldsymbol{p}$ have at least two common points then (7) is the equation of a sphere which contains all of their common points, because their coordinates satisfy both equations (4) and (5).

Task 2. Let $\boldsymbol{k}$ be a circle given as the intersection of sphere $\boldsymbol{g}$ and plane $\boldsymbol{p}$, which equations are

$$\boldsymbol{g}\colon (x-1)^2 + (y+2)^2 + z^2 = 16, \quad \boldsymbol{p} : x - y - z + 2 = 0$$

Write the equation of a sphere which contains the circle $\boldsymbol{k}$ and which goes through the point $M[2, -3, 2]$.

Solution. The equation of the searched sphere has the form

$$[(x-1)^2 + (y+2)^2 + z^2 - 16] + \alpha\,(x - y - z + 2) = 0$$

After the substitution of point M to this equation we obtain $\alpha = 2$. If we substitute this value to the equation above, then the equation of the searched sphere will be

$$x^2 + (y+1)^2 + (z-1)^2 = 9.$$

Task 3. Let us consider a circle $\boldsymbol{k}$ as the intersection of sphere $\boldsymbol{g}$ and plane $\boldsymbol{p}$, which equations are

$$\boldsymbol{g}\colon (x-1)^2 + (y+2)^2 + z^2 = 18, \quad \boldsymbol{p} : x - y - z + 2 = 0$$

and the line $\boldsymbol{l}$: $y - 3 = 0,\ 2y + z - 5 = 0$. Write the equation of a sphere which contains the circle $\boldsymbol{k}$ and which tangent to the line $\boldsymbol{l}$.

Solution. The equation of the searched sphere has the form

$$[(x-1)^2 + (y+2)^2 + z^2 - 18] + \alpha\,(x - y - z + 2) = 0$$

Putting in the previous equation an equation of the line $\boldsymbol{l}$, we get the quadratic equation with parameter α in the form

$$x^2 + (\alpha - 2)x + 9 = 0$$

The solution of this equation are x-coordinates of the common points of searched sphere and given line $\boldsymbol{l}$. If the line $\boldsymbol{l}$ will be a tangent of this sphere thus the discriminant of quadratic equation above must be equal to null, i.e.

$$D = (\alpha - 2)^2 - 4.1.9 = 0$$

From this is obvious that $\alpha_1 = 8,\ \alpha_2 = -4$. Then the equations of searched spheres we could write in the form

$$(x+3)^2 + (y-2)^2 + (z-4)^2 = 16, \quad (x-3)^2 + (y+4)^2 + (z+2)^2 = 16.$$

Consider now two nonconcentric spheres with the equations

$$\boldsymbol{g_1}\colon (x - s_1)^2 + (y - s_2)^2 + (z - s_3)^2 = r_1^2 \tag{8}$$

$$\boldsymbol{g_2}\colon\ (x - t_1)^2 + (y - t_2)^2 + (z - t_3)^2 = r_2^2 \tag{9}$$

If we multiply the first equation by a real number α and the second equation by a real number β and sum both equations, we obtain

$$(\alpha + \beta)(x^2 + y^2 + z^2) - 2[(\alpha s_1 + \beta t_1)x + (\alpha s_2 + \beta t_2)y + (\alpha s_3 + \beta t_3)z] +$$

$$+\alpha s_1^2 + \beta t_1^2 + \alpha s_2^2 + \beta t_2^2 + \alpha s_3^2 + \beta t_3^2 - \alpha r_1^2 - \beta r_2^2 = 0 \tag{10}$$

What can we say about the set of all point of 3-dimensional Euclidean space, which coordinates (10) holds?

If $\alpha + \beta = 0$, then without of the generality, we can put $\alpha = 1, \beta = -1$ and (10) is the equation of the plane $\boldsymbol{p}$ in the form

$$2(t_1 - s_1)x + 2(t_2 - s_2)y + 2(t_3 - s_3)z + (s_1^2 + s_2^2 + s_3^2 - r_1^2) -$$
$$-(t_1^2 + t_2^2 + t_3^2 - r_2^2) = 0 \tag{11}$$

Because given spheres are nonconcentric so at least one of the coefficients $(t_i - s_i), i = 1, 2, 3$ is not equal to null. This plane is perpendicular to a line contained the centers of these spheres and it's called a *chordal's plane* of the given spheres. Furthermore, if $\boldsymbol{g_1}$ and $\boldsymbol{g_2}$ have nonempty intersection so coordinates of the all common points satisfy the equation (10).

If $\alpha + \beta \neq 0$, then we could write the equation (10) in the form

$$(x - A)^2 + (y - B)^2 + (z - C)^2 = A^2 + B^2 + C^2 - D \tag{12}$$

where

$$A = \frac{\alpha s_1 + \beta t_1}{\alpha + \beta}, \ B = \frac{\alpha s_2 + \beta t_2}{\alpha + \beta}, \ C = \frac{\alpha s_3 + \beta t_3}{\alpha + \beta},$$
$$D = \frac{\alpha(s_1^2 + s_2^2 + s_3^2) + \beta(t_1^2 + t_2^2 + t_3^2)}{\alpha + \beta}$$

A number of the common points of these spheres depend on an expression from the right-hand equation of (12). Denote now $R = A^2 + B^2 + C^2 - D$.

(i) If $R < 0$ then there doesn't exist a point satisfying the equation (12),

(ii) If $R = 0$ then the point $[A, B, C]$ only satisfy the equation (12),

(iii) If $R > 0$ then the equation (12) define the sphere with a center $[A, B, C]$ and a radius $\sqrt{R}$

The sheaf of the spheres assigned to $\boldsymbol{g_1}$ and $\boldsymbol{g_2}$ is the set of all spheres, which equations can be represented in the form (10).

The intersection of both spheres is the intersection of one of the given spheres and chordal's plane $\boldsymbol{p}$. We know that the intersection of the sphere and a plane can be a circle or an empty set or a point according to distance from a center of the sphere to a plane.

A distance from the center $S[s_1, s_2, s_3]$ of the sphere $\boldsymbol{g_1}$ with a radius r_1 to the chordal's plane of the spheres $\boldsymbol{g_1}$ and $\boldsymbol{g_2}$ (11) is given by

$$d = \frac{|2(t_1 - s_1)s_1 + 2(t_2 - s_2)s_2 + 2(t_3 - s_3)s_3 + (s_1^2 + s_2^2 + s_3^2 - r_1^2) - (t_1^2 + t_2^2 + t_3^2 - r_2^2)|}{\sqrt{[2(t_1 - s_1)]^2 + [2(t_2 - s_2)]^2 + [2(t_3 - s_3)]^2}}$$

consequently, we have

$$d = \frac{|(t_1-s_1)^2+(t_2-s_2)^2+(t_3-s_3)^2+r_1^2-r_2^2|}{2\sqrt{(t_1-s_1)^2+(t_2-s_2)^2+(t_3-s_3)^2}} \tag{13}$$

(a) If $d > r_1$ then the intersection of the spheres $\boldsymbol{g_1}$ and $\boldsymbol{g_2}$ be an empty set.

(b) If $d = r_1$ then the intersection of these spheres be only the point P (the point of intersection of a line containing the centers of spheres $\boldsymbol{g_1}$ and $\boldsymbol{g_2}$ with the chordal's plane $\boldsymbol{p}$).

(c) If $d < r_1$ then the intersection be a circle lying in a plane $\boldsymbol{p}$ with a radius $\sqrt{r_1^2 - d^2}$ and a center P.

Task 4. Find the intersection of the spheres $\boldsymbol{g_1}$ and $\boldsymbol{g_2}$, where

$$\boldsymbol{g_1}\colon x^2+y^2+z^2=25, \quad \boldsymbol{g_2} : (x+1)^2+(y-2)^2+(z-3)^2=25.$$

Solution. The chordal's plane of the given spheres has an equation (11) in the form

$$2(-1)x+2(2y)+2(3z)+(-1-4-9-25+25)=0$$

or

$$x-2y-3z+7=0$$

The distance d from the center $S_1[0,0,0]$ of sphere $\boldsymbol{g_1}$ to the chordal's plane be

$$d=\frac{|1+4+9+25-25|}{2\sqrt{1+4+9}}=\frac{\sqrt{14}}{2}$$

Considering $d < 5$ (radius $\boldsymbol{g_1}$), the intersection of chordal's plane and sphere $\boldsymbol{g_1}$ will be a circle with the center $S[-\frac{1}{2},1,\frac{3}{2}]$ and the radius

$$r=\sqrt{25-\left(\frac{\sqrt{14}}{2}\right)^2}=\frac{\sqrt{86}}{2}$$

Thus the intersection of the given spheres be a circle with the center S and the radius r lying in the plane $x-2y-3z+7=0$.

Task 5. Write the equation of the sphere which contains all of the common points of the spheres $\boldsymbol{g_1}, \boldsymbol{g_2}$ and which tangent to the plane $\boldsymbol{p}$, where

$$\boldsymbol{g_1}\colon x^2+y^2+z^2=9, \quad \boldsymbol{g_2} : (x-1)^2+(y+1)^2+(z-2)^2=7, \quad \boldsymbol{p}\colon x-5=0$$

Solution. A searched sphere (if there exists) has an equation

$$\alpha\left(x^2+y^2+z^2-9\right)+\beta\left[(x-1)^2+(y+1)^2+(z-2)^2-7\right]=0$$

on condition that $\alpha+\beta\neq 0$. Substitute the equation of plane $\boldsymbol{p}$ ($x=5$) to this equation, we obtain

$$(\alpha+\beta)\,y^2+2\beta y+(\alpha+\beta)\,z^2-4\beta z+16\alpha+14\beta=0$$

We could write this equation in the form

$$(y-A)^2+(z-B)^2=A^2+B^2-C$$

where

$$A=\frac{\beta}{\alpha+\beta},\quad B=\frac{2\beta}{\alpha+\beta},\quad C=\frac{16\alpha+14\beta}{\alpha+\beta}$$

If the plane $\boldsymbol{p}$ is tangent to searched sphere then A^2+B^2-C must be equal to null. Step by step we arrive at the following equation

$$5\beta^2=(\alpha+\beta)(16\alpha+14\beta)$$

$$\left(\frac{8\alpha}{\beta}+3\right)\left(\frac{2\alpha}{\beta}+3\right)=0$$

If we put $\alpha=-3$ or $\alpha=3$ then we get $\beta=8$ or $\beta=-2$ and searched spheres have the equations

$$\left(x-\frac{8}{5}\right)^2+\left(y+\frac{8}{5}\right)^2+\left(z-\frac{16}{5}\right)^2=\frac{289}{25}$$

$$(x+2)^2+(y-2)^2+(z+4)^2=49.$$

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Volumes of Closed Bounded Solids in n-dimensional Space

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In this paper, the formula for calculation of volumes of bounded closed solids in Euclidean space is given, if we know the parametric descriptions of the surface areas of these solids. These surface areas are the smooth areas in Euclidean space of the corresponding dimension. The article also contains a concrete example of this calculation.

Introduction

One of known application of the multidimensional real integrals is a calculation of volumes of solids generally in n-dimensional space $\mathbf{R_n}$ where the theory of measure is used. The exact theory of measurability of geometric figures originates from the beginning of the 20th century. The contemporary mathematics deals with measurable structures whose measures are described as the set functions of certain properties. Let us look at this problem as a topological problem now. We are going to use some important topological terms and their properties to it. On the base of the theory as follows we can calculate areas of n - dimensional solids in Euclidean space $\mathbf{E_n}$ whose areas are possible to describe parametrically in the certain presumptions as the manifolds in $\mathbf{E_n}$. The question is the existence of this mapping.

Basic Terms and Introduction Notations

Let us consider Euclidean n - dimensional space $\mathbf{E_n}$ with points $\mathbf{x}, n \geq 2$. Let us denote
x^α, $\alpha = 1, \ldots, n$, the Cartesian co-ordinates of the point $\mathbf{x} \in \mathbf{E_n}$,
u^a, $a = 1, \ldots, n-1$, the Cartesian co-ordinates of the point $\mathbf{u} \in \mathbf{E_{n-1}}$.
Let Ω be the bounded closed domain in $\mathbf{E_{n-1}}$ and $x^\alpha(u^1, \ldots, u^{n-1})$, where $\alpha = 1, \ldots, n$, given functions defined on some domain $O \subset \mathbf{E_{n-1}}, \Omega \subset O$. Let us suppose that

a) The vector function $\mathbf{x}(\mathbf{u}) = \{x^\alpha(u^a)\}$ has almost everywhere in Ω the continuous partial derivatives

$$B^\alpha_a := \frac{\partial x^\alpha}{\partial u^a} \text{ for } \alpha = 1, \ldots, n, a = 1, \ldots, n-1;$$

b) The rank of the matrix

$$\left(B_a^\alpha\right)_{n\times(n-1)} = \begin{pmatrix} B_1^1 & B_1^2 & \cdots & B_1^n \\ B_2^1 & B_2^2 & \cdots & B_2^n \\ \vdots & \vdots & \ddots & \vdots \\ B_{n-1}^1 & B_{n-1}^2 & \cdots & B_{n-1}^n \end{pmatrix}$$

is maximal in Ω, i.e. it is equal to n - 1 almost everywhere in Ω;

c) The subset

$$P^0 := \{\mathbf{x} \in \mathbf{E_n} | \mathbf{x} = \mathbf{x}(\mathbf{u}),\, \mathbf{u} \in \mathbf{int}\Omega\}$$

of the matrix

$$P := \{\mathbf{x} \in \mathbf{E_n} | \mathbf{x} = \mathbf{x}(\mathbf{u}),\, \mathbf{u} \in \Omega\} \tag{1}$$

is a homeomorphic range of the set $\mathbf{int}\,\Omega$ (of all interior points of the set Ω) in $\mathbf{E_n}$.

It follows from the given assumptions, that the set P is bounded and it is a smooth hypersurface by parts in the space $\mathbf{E_n}$. This manifold does not intersect by itself and that divides the space $\mathbf{E_n}$ to two disjoint regions in $\mathbf{E_n}$, in which one is bounded and the second one is unbounded.

The closure of the relevant bounded region is called the n-dimensional solid in space $\mathbf{E_n}$. Let us denote it by W.

Remark:
Especially,
for $n = 3$, let the closure be **the solid** in $\mathbf{E_3}$,
for $n = 2$ **the closed area** in $\mathbf{E_2}$.

If we denote the set of all internal points of the set W by $\mathbf{int}\,W$, and the boundary of it by ∂W, it is obvious that

$$W = P \cup \mathbf{int}\,W, \quad \text{where} \quad \partial W = P.$$

Let us consider the hypersurface P as the set of points immersed to Euclidean space $\mathbf{E_{n+1}} = \mathbf{E_n} \times \mathbf{E_1}$. Then this hypersurface P can be expressed in a form

$$P := \{X = (\mathbf{x}; x^{n+1}) \in \mathbf{E_{n+1}} | \mathbf{x} = \mathbf{x}(\mathbf{u}),\, x^{n+1} = 0,\, \mathbf{u} \in \Omega\}. \tag{2}$$

Let us choose the point $V = (\mathbf{x_0}; x_0^{n+1})$ in the space $\mathbf{E_{n+1}}$ where $x_0^{n+1} > 0$, and the set of all half-lines

$$\mathbf{x} = \mathbf{x_0} + [\mathbf{x}(\mathbf{u}) - \mathbf{x_0}] \cdot t, \tag{3}$$

$$x^{n+1} = x_0^{n+1}(1-t),\, \mathbf{u} \in \Omega,\, t \in \langle 0; +\infty).$$

This set represents the conical hypersurface in $\mathbf{E_{n+1}}$ with the vertex V. With respect to the condition

$$x^{n+1} \geq 0, \tag{4}$$

we obtain the smooth by parts and closed bounded (compacted) hypersurface in $\mathbf{E_{n+1}}$ from it. This hypersurface divides the space $\mathbf{E_{n+1}}$ into two disjuct regions where one is bounded and the second one is unbounded. Let us denote the closure of the relevant bounded region by K. Then the set K be called the hypercone in the space $\mathbf{E_{n+1}}$ with the vertex $V = (\mathbf{x_0}; x_0^{n+1})$, the set W be called the base of this hypercone, and the number $x^{n+1} > 0$ the height of it.

<u>Remark:</u> Under the term "the closed hypersurface in $\mathbf{E_n}$ ", we understand the manifold in $\mathbf{E_n}$ in the topological sense which is a homeomorphic range of the hypersphere in $\mathbf{E_n}$, if this mapping exists.

The Measure of the Hypercone

We would like to determine the volume of the hypercone K, i.e. to determine the measure of the set K. The measure of the set K is described by the relation

$$I = \overbrace{\iint \ldots \int_K} \mathrm{d}x^1\, \mathrm{d}x^2\, \ldots\, \mathrm{d}x^{n+1}. \tag{5}$$

To calculate this integral, we introduce the generalized coordinates u^a, t, v, in the space $\mathbf{E_{n+1}}$, where $a = 1, \ldots, n-1$. The relations

$$\mathbf{x} = \mathbf{x_0} + [\mathbf{x}(\mathbf{u}) - \mathbf{x_0}] \cdot t, x^{n+1} = v(1-t),\ \mathbf{u} \in \Omega,\ t \in \langle 0; 1\rangle,\ v \in \langle 0; x^{n+1}\rangle \tag{6}$$

define these coordinates. With respect to the given goal, we can suppose that $V = (0; x_0^{n+1})$, i. e. $x_0^\alpha = 0$ $\alpha = 1, \ldots, n$, without loss of generality. Then the relation (6) can be simplified to the form

$$x^\alpha = x^\alpha(u^a) \cdot t,\ \alpha = 1, \ldots, n \tag{7}$$

$$x^{n+1} = v(1-t).$$

If we use the coordinates determined by the relations (7) we obtain the integral of the form

$$I = \overbrace{\iint \ldots \int_{K'}} J\, \mathrm{d}u^1\, \mathrm{d}u^2\, \ldots\, \mathrm{d}u^{n-1} \mathrm{d}t\, \mathrm{d}v, \tag{8}$$

where the symbol J is the Jacobian associated to the transformation (7) and the set $K^{'} = \Omega \times \langle 0; 1\rangle \times \langle 0; x_0^{n+1}\rangle$. It follows

$$J = \begin{vmatrix} \frac{\partial x^1}{\partial t} & \frac{\partial x^2}{\partial t} & \cdots & \frac{\partial x^n}{\partial t} & \frac{\partial x^{n+1}}{\partial t} \\ t\cdot\frac{\partial x^1}{\partial u^1} & t\cdot\frac{\partial x^2}{\partial u^1} & \cdots & t\cdot\frac{\partial x^n}{\partial u^1} & t\cdot\frac{\partial x^{n+1}}{\partial u^1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t\cdot\frac{\partial x^1}{\partial u^{n-1}} & t\cdot\frac{\partial x^2}{\partial u^{n-1}} & \cdots & t\cdot\frac{\partial x^n}{\partial u^{n-1}} & t\cdot\frac{\partial x^{n+1}}{\partial u^{n-1}} \\ t\cdot\frac{\partial x^1}{\partial v} & t\cdot\frac{\partial x^2}{\partial v} & \cdots & t\cdot\frac{\partial x^n}{\partial v} & t\cdot\frac{\partial x^{n+1}}{\partial v} \end{vmatrix} = \tag{9}$$

$$= \begin{vmatrix} x^1(\mathbf{u}) & x^2(\mathbf{u}) & \cdots & x^n(\mathbf{u}) & -v \\ B_1^1 & B_1^2 & \cdots & B_1^n & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B_{n-1}^1 & B_{n-1}^2 & \cdots & B_{n-1}^n & 0 \\ 0 & 0 & \cdots & 0 & 1-t \end{vmatrix} \cdot t^{n-1}$$

for the Jacobian from (8).

Let us denote

$$\triangle(\mathbf{u}) := \begin{vmatrix} x^1(\mathbf{u}) & x^2(\mathbf{u}) & \cdots & x^n(\mathbf{u}) \\ B_1^1 & B_1^2 & \cdots & B_1^n \\ \vdots & \vdots & \vdots & \vdots \\ B_{n-1}^1 & B_{n-1}^2 & \cdots & B_{n-1}^n \end{vmatrix} \tag{10}$$

Then the absolute value of the Jacobian can be expressed shortly in the form

$$|J| = (t^{n-1} - t^n)\cdot|\triangle(\mathbf{u})|.$$

We obtain for the integral from (9)

$$I = \overbrace{\iint \ldots \int_\Omega} |\triangle(\mathbf{u})| \, \mathrm{d}u^1 \, \mathrm{d}u^2 \ldots \mathrm{d}u^{n-1} \cdot \int_0^1 (t^{n-1} - t^n) \, \mathrm{d}t \cdot \int_0^{x_0^{n+1}} \mathrm{d}v =$$

$$= \frac{1}{n(n+1)} \cdot x_0^{n+1} \overbrace{\iint \ldots \int_\Omega} |\triangle(\mathbf{u})| \, \mathrm{d}u^1 \, \mathrm{d}u^2 \ldots \mathrm{d}u^{n-1}.$$

Thus

$$I = \frac{x_0^{n+1}}{n+1} \overbrace{\iint \ldots \int_\Omega} \frac{1}{n} |\triangle(\mathbf{u})| \, \mathrm{d}u^1 \, \mathrm{d}u^2 \ldots \mathrm{d}u^{n-1}. \tag{11}$$

The number

$$\mu_W := \frac{1}{n} \overbrace{\iint \ldots \int_\Omega} |\triangle(\mathbf{u})| \, \mathrm{d}u^1 \, \mathrm{d}u^2 \, \ldots \, \mathrm{d}u^{n-1} \tag{12}$$

be called **the volume** of the n-dimensionalsolid W in the space $\mathbf{E_n}$.

Especially for
$n = 3$ this number is called **the volume of the solid** W in $\mathbf{E_3}$
$n = 2$ **the area of the bounded figure** W in $\mathbf{E_2}$.

Example

Calculate the volume of the ellipsoid determined by the set

$$P := \{\mathbf{x} \in \mathbf{E_3} | \sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} + \sqrt{\frac{z}{c}} = 1\}, \tag{13}$$

$x \geq 0,\ y \geq 0,\ z \geq 0,\ \ a,\ b,\ c \in \mathbf{R}^+$.

Solution: The set P is a smooth closed area in the space $\mathbf{E_3}$ (in the sense above). The projection of this area P to the (x, y) - plane is the closure of the interior of the ellipse whose implicit description is of the form

$$K := \{(\mathbf{x}, \mathbf{y}) \in \mathbf{E_2} | \sqrt{\frac{x}{a}} + \sqrt{\frac{x}{b}} = 1\}, x \geq 0,\ y \geq 0,\ a,\ b \in \mathbf{R}^+.$$

The area P is obviously described by these parametric equations

$$x = a \cos^4 \varphi \sin^4 \nu,\ y = b \sin^4 \varphi \sin^4 \nu,\ z = c \cos^4 \nu,$$

$$\varphi \in \langle 0; \frac{\pi}{2} \rangle, \nu \in \langle 0;\ \frac{\pi}{2} \rangle.$$

According to the notation given above, where $u^1 = \varphi, u^2 = \nu,\ n = 3$, there is

$$x^1(u^1; u^2) = x(\varphi; \nu) = a \cos^4 \varphi \sin^4 \nu,$$

$$x^2(u^1; u^2) = y(\varphi; \nu) = b \sin^4 \varphi \sin^4 \nu,$$

$$x^3(u^1; u^2) = z(\varphi; \nu) = c \cos^4 \nu,$$

$$B_1^1 = \frac{\partial x}{\partial \varphi} = -4a \cos^3 \varphi \ \sin \varphi \ \sin^4 \nu, \qquad B_2^1 = \frac{\partial x}{\partial \nu} = 4a \ \cos^4 \varphi \ \sin^3 \nu \ \cos \nu,$$

$$B_1^2 = \frac{\partial y}{\partial \varphi} = 4b \ \sin^3 \varphi \ \cos \varphi \ \sin^4 \nu, \qquad B_2^2 = \frac{\partial y}{\partial \nu} = 4b \ \sin^4 \varphi \ \sin^3 \nu \ \cos \nu,$$

$$B_1^3 = \frac{\partial z}{\partial \varphi} = 0, B_2^3 = \frac{\partial z}{\partial \nu} = -4c \ \cos^3 \nu \ \sin \nu.$$

In this case, the determinant defined in (10) is equal to

$$\triangle(\mathbf{u}) := \triangle(\varphi;\nu) =$$

$$= \begin{vmatrix} a\cos^4\varphi\,\sin^4\nu & b\sin^4\varphi\,\sin^4\nu & c\cos^4\nu \\ -4a\cos^3\varphi\,\sin\varphi\,\sin^4\nu & 4b\,\sin^3\varphi\,\cos\varphi\,\sin^4\nu & 0 \\ 4a\,\cos^4\varphi\,\sin^3\nu\,\cos\nu & 4b\,\sin^4\varphi\,\sin^3\nu\,\cos\nu & -4c\,\cos^3\nu\,\sin\nu \end{vmatrix} =$$

$$= ... = -16abc\,\sin^3\varphi\,\cos^3\varphi\,\sin^7\nu\,\cos^3\nu.$$

The whole volume is equal to

$$\mu_W = \frac{1}{3}\int\int_\Omega |\triangle(\varphi;\nu)|\,\mathrm{d}\varphi\,\mathrm{d}\nu$$

$$= \frac{16}{3}\int\int_\Omega abc\sin^3\varphi\,\cos^3\varphi\,\sin^7\nu\,\cos^3\nu)\,\mathrm{d}\varphi\,\mathrm{d}\nu = \cdots = \frac{abc}{90}.$$

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Remark: The Ministry of Education of Czech Republic under grant MSM 243100302 supports this work.

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Axiomatic definition of the cross product

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In this paper we prove that a notion of the cross product can be introduced on the basis of axioms. Using in part the results of the paper [3] three different models of the cross product are proposed and several important properties of the cross product are proved based on the suggested axiomatic system.

Axiomatics for the cross product was proposed by T.M. Apostol [1] who assumed the following axioms:

A1 $\quad \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$,

A2 $\quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$,

A3 $\quad \lambda (\mathbf{a} \times \mathbf{b}) = (\lambda \mathbf{a}) \times \mathbf{b}$

A4 $\quad \mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$,

A5 $\quad |\mathbf{a} \times \mathbf{b}| = \left[|\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \right]^{1/2}$,

where

"+" is the operation of vector addition,

"$\cdot$" is the functional of dot product,

$|\ |$ is the norm functional,

λ is a scalar ($\lambda \in R$).

The norm functional is defined by

$$|\mathbf{a}| = (\mathbf{a} \cdot \mathbf{a})^{1/2}. \tag{1}$$

It was shown in the paper [3] that if the axioms given above are fulfilled in a linear space R^n, then $n = 1$ or $n = 3$ or $n = 7$. In spaces R^1, R^3 and R^7 the axioms (A1)–(A5) are essentially fulfilled as demonstrated by the following models.

Model 1. It is known that in a linear space R^3 in the orthonormal basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ the cross product can be defined by the following symbolic determinant:

if $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \tag{2}$$

Therefore,

$$\mathbf{a}\times\mathbf{b} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right). \tag{3}$$

Often in a space R^3 the cross product is also defined geometrically.

Based on the abovementioned definition it can be verified that axioms (A1)–(A5) are fulfilled. For this purpose it is sufficient to use the properties of the determinant, equation (1) and the following formula

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Model 2. If in a linear space R^1 we assume that

$$\mathbf{a}\times\mathbf{b} = \mathbf{0}, \tag{4}$$

then checking the fulfilment of the axioms (A1)–(A5) is almost immediate. For example, the validity of axiom (A5) is proved in the following way.

Let $\mathbf{a} = (a_1)$, $\mathbf{b} = (b_1)$, then

$$|\mathbf{a}\times\mathbf{b}| = [(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})]^{1/2} = (\mathbf{0}\cdot\mathbf{0})^{1/2} = 0,$$

$$\left[|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2\right]^{1/2} = \left[(\mathbf{a}\cdot\mathbf{a})\,(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})^2\right]^{1/2} =$$

$$= \left[a_1^2b_1^2 - (a_1b_1)^2\right]^{1/2} = 0.$$

Model 3. If a vector $\mathbf{v}\in R^7$ in the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_7)$ has the form

$$\mathbf{v} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_7\mathbf{e}_7,$$

then this vector can be considered as an ordered triple $(\mathbf{a}, \lambda, \mathbf{b})$, where $\mathbf{a}, \mathbf{b}\in R^3$, $\lambda\in R$,

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \qquad \mathbf{b} = a_5\mathbf{i} + a_6\mathbf{j} + a_7\mathbf{k}.$$

Thus, we have one-to-one mapping between the vector $\mathbf{v}$ and the vector

$$\mathbf{v}^* = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, a_4, a_5\mathbf{i} + a_6\mathbf{j} + a_7\mathbf{k}). \tag{5}$$

Vector addition, multiplication by scalar, the dot product and the cross product are defined as follows:

if $\mathbf{v}^* = (\mathbf{a}_1, \lambda_1, \mathbf{b}_1)$, $\mathbf{w}^* = (\mathbf{a}_2, \lambda_2, \mathbf{b}_2)$, then

$$\begin{aligned}
\mathbf{v}^* + \mathbf{w}^* &= (\mathbf{a}_1 + \mathbf{a}_2, \lambda_1 + \lambda_2, \mathbf{b}_1 + \mathbf{b}_2),\\
\lambda\mathbf{v}^* &= (\lambda\mathbf{a}_1, \lambda\lambda_1, \lambda\mathbf{b}_1),\\
\mathbf{v}^*\cdot\mathbf{w}^* &= (\mathbf{a}_1\cdot\mathbf{a}_2, \lambda_1\lambda_2, \mathbf{b}_1\cdot\mathbf{b}_2),\\
\mathbf{v}^*\times\mathbf{w}^* &= \left(\left[\begin{vmatrix} \lambda_1 & \lambda_2 \\ \mathbf{b}_1 & \mathbf{b}_2 \end{vmatrix} + (\mathbf{a}_1\times\mathbf{a}_2) - (\mathbf{b}_1\times\mathbf{b}_2) \right], \right.\\
&\qquad \left. \left[-(\mathbf{a}_1\cdot\mathbf{b}_2) + (\mathbf{b}_1\cdot\mathbf{a}_2) \right], \left[\begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 \\ \lambda_1 & \lambda_2 \end{vmatrix} - (\mathbf{a}_1\times\mathbf{b}_2) - (\mathbf{b}_1\times\mathbf{a}_2) \right] \right).
\end{aligned}$$

The symbols $+$, $-$, $\cdot$, $\times$ in the right-hand side of the above equalities are matched by the corresponding operations in R^3.

In the model under consideration axioms (A1)–(A5) are fulfilled. For example, we verify axiom (A4), i.e. formula $\mathbf{v}^* \cdot (\mathbf{v}^* \times \mathbf{w}^*) = 0$.

Based on definitions of operations in R^7 given above we obtain:

$$\mathbf{v}^* \cdot (\mathbf{v}^* \times \mathbf{w}^*) = (\mathbf{a_1}, \lambda_1, \mathbf{b_1}) \cdot \big[(\mathbf{a_1}, \lambda_1, \mathbf{b_1}) \times (\mathbf{a_2}, \lambda_2, \mathbf{b_2})\big] =$$

$$= (\mathbf{a_1}, \lambda_1, \mathbf{b_1}) \cdot \left(\left[\begin{vmatrix} \lambda_1 & \lambda_2 \\ \mathbf{b}_1 & \mathbf{b}_2 \end{vmatrix} + (\mathbf{a}_1 \times \mathbf{a}_2) - (\mathbf{b}_1 \times \mathbf{b}_2) \right], \right.$$

$$\left. \Big[-(\mathbf{a}_1 \cdot \mathbf{b}_2) + (\mathbf{b}_1 \cdot \mathbf{a}_2) \Big], \left[- \begin{vmatrix} \lambda_1 & \lambda_2 \\ \mathbf{a}_1 & \mathbf{a}_2 \end{vmatrix} - (\mathbf{a}_1 \times \mathbf{b}_2) - (\mathbf{b}_1 \times \mathbf{a}_2) \right] \right) =$$

$$= \mathbf{a}_1 \cdot \left[\begin{vmatrix} \lambda_1 & \lambda_2 \\ \mathbf{b}_1 & \mathbf{b}_2 \end{vmatrix} + (\mathbf{a}_1 \times \mathbf{a}_2) - (\mathbf{b}_1 \times \mathbf{b}_2) \right] + \lambda_1 \Big[-(\mathbf{a}_1 \cdot \mathbf{b}_2) + (\mathbf{b}_1 \cdot \mathbf{a}_2) \Big] +$$

$$+ \mathbf{b}_1 \cdot \left[- \begin{vmatrix} \lambda_1 & \lambda_2 \\ \mathbf{a}_1 & \mathbf{a}_2 \end{vmatrix} - (\mathbf{a}_1 \times \mathbf{b}_2) - (\mathbf{b}_1 \times \mathbf{a}_2) \right] =$$

$$= \lambda_1 (\mathbf{a}_1 \cdot \mathbf{b}_2) - \lambda_2 (\mathbf{a}_1 \cdot \mathbf{b}_1) + \mathbf{a}_1 \cdot (\mathbf{a}_1 \times \mathbf{a}_2) - \mathbf{a}_1 \cdot (\mathbf{b}_1 \times \mathbf{b}_2) - \lambda_1 (\mathbf{a}_1 \cdot \mathbf{b}_2) +$$

$$+ \lambda_1 (\mathbf{b}_1 \cdot \mathbf{a}_2) - \lambda_1 (\mathbf{b}_1 \cdot \mathbf{a}_2) + \lambda_2 (\mathbf{a}_1 \cdot \mathbf{b}_1) - \mathbf{b}_1 \cdot (\mathbf{a}_1 \times \mathbf{b}_2) - \mathbf{b}_1 \cdot (\mathbf{b}_1 \times \mathbf{a}_2) = 0,$$

since in a space R^3 (model 1) the following equalities

$$\mathbf{a}_1 \cdot (\mathbf{a}_1 \times \mathbf{a}_2) = 0, \quad \mathbf{b}_1 \cdot (\mathbf{b}_1 \times \mathbf{a}_2) = 0, \quad -\mathbf{b}_1 \cdot (\mathbf{a}_1 \times \mathbf{b}_2) = \mathbf{a}_1 \cdot (\mathbf{b}_1 \times \mathbf{b}_2)$$

are fulfilled.

Based on axioms (A1)–(A5) several important formulae can be proved.

Theorem 1a. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})$,

b. $\lambda (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\lambda \mathbf{b})$.

Proof of Theorem 1a:

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} \underset{(A1)}{=} -\big[\mathbf{c} \times (\mathbf{a} + \mathbf{b})\big] = -\big[(\mathbf{c} \times \mathbf{a}) + (\mathbf{c} \times \mathbf{b})\big] =$$

$$= -(\mathbf{c} \times \mathbf{a}) - (\mathbf{c} \times \mathbf{b}) \underset{(A1)}{=} (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}).$$

Proof of Theorem 1b:

$$\mathbf{a} \times (\lambda \mathbf{b}) \underset{(A1)}{=} -\big[(\lambda \mathbf{b}) \times \mathbf{a}\big] \underset{(A3)}{=} -\big[\lambda (\mathbf{b} \times \mathbf{a})\big] = \lambda \big[-(\mathbf{b} \times \mathbf{a})\big] \underset{(A1)}{=} \lambda (\mathbf{a} \times \mathbf{b}).$$

Theorem 2. $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -\big[\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})\big]$.

Proof:

1. $(\mathbf{a} + \mathbf{c}) \cdot \big[(\mathbf{a} + \mathbf{c}) \times \mathbf{b}\big] = 0$ $\{A4: \ \mathbf{a}/\mathbf{a} + \mathbf{c}\}$
2. $(\mathbf{a} + \mathbf{c}) \cdot \big[(\mathbf{a} \times \mathbf{b}) + (\mathbf{c} \times \mathbf{b})\big] = 0$ $\{1, \text{T1a}\}$
3. $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) + \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{c} \cdot (\mathbf{c} \times \mathbf{b}) = 0$ $\{2\}$
4. $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$
5. $\mathbf{c} \cdot (\mathbf{c} \times \mathbf{b}) = 0$ $\{A4\}$
6. $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = -\big[\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})\big]$ $\{3,4,5\}$

$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -\big[\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})\big]$ $\{6\}$

Theorem 3. $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = -\big[\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})\big]$.

Proof:

$$-\big[\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})\big] = \mathbf{b}\big[-(\mathbf{a}\times\mathbf{c})\big] \underset{(A1)}{=} \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b} \underset{(T2)}{=}$$

$$= -\big[\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a})\big] = \mathbf{c}\cdot\big[-(\mathbf{b}\times\mathbf{a})\big] \underset{(A1)}{=} \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}.$$

Theorem 4a. $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}$,

b. $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$.

Proof of Theorem 4a:

1. $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \underset{(T3)}{=} -\big[\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})\big] = \mathbf{b}\cdot\big[-(\mathbf{a}\times\mathbf{c})\big] \underset{(A1)}{=}$
   $= \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}$
2. $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}$ $\{1\}$
3. $(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ $\{2, \mathbf{a}/\mathbf{b}, \mathbf{b}/\mathbf{c}, \mathbf{c}/\mathbf{a}\}$

   Statement $\{2, 3\}$

Theorem 4b follows immediately from Theorem 4a and the commutative law for the dot product and is known as the interchangeable law.

Theorem 5a. $\mathbf{b}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{b}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{b})\,\mathbf{c}$,

b. $\mathbf{c}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\,\mathbf{c}$.

Proof of Theorem 5a.

1. $\big[\mathbf{b}\times(\mathbf{c}+\mathbf{d})\big]\cdot\big[\mathbf{b}\times(\mathbf{c}+\mathbf{d})\big] \underset{(A5)}{=} |\mathbf{b}|^2|\mathbf{c}+\mathbf{d}|^2 - \big[\mathbf{b}\cdot(\mathbf{c}+\mathbf{d})\big]^2 =$
   $= |\mathbf{b}|^2\big(|\mathbf{c}|^2 + 2(\mathbf{c}\cdot\mathbf{d}) + |\mathbf{d}|^2\big) - (\mathbf{b}\cdot\mathbf{c})^2 - 2(\mathbf{b}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{d})^2$
2. $\big[\mathbf{b}\times(\mathbf{c}+\mathbf{d})\big]\cdot\big[\mathbf{b}\times(\mathbf{c}+\mathbf{d})\big] \underset{(A2)}{=}$
   $= \big[(\mathbf{b}\times\mathbf{c}) + (\mathbf{b}\times\mathbf{d})\big]\cdot\big[(\mathbf{b}\times\mathbf{c}) + (\mathbf{b}\times\mathbf{d})\big] =$
   $= |\mathbf{b}\times\mathbf{c}|^2 + 2(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d}) + |\mathbf{b}\times\mathbf{d}|^2 =$
   $\underset{(A5)}{=} |\mathbf{b}|^2|\mathbf{c}|^2 - (\mathbf{b}\cdot\mathbf{c})^2 + 2(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d}) + |\mathbf{b}|^2|\mathbf{d}|^2 - (\mathbf{b}\cdot\mathbf{d})^2$
3. $(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d}) = (\mathbf{b}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) =$ $\{1, 2\}$
   $= \big[(\mathbf{b}\cdot\mathbf{b})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{b}\big]\cdot\mathbf{d}$
4. $(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d}) = (\mathbf{b}\times\mathbf{d})\cdot(\mathbf{b}\times\mathbf{c}) \underset{(T2)}{=}$
   $= -\big\{\mathbf{d}\cdot\big[\mathbf{b}\times(\mathbf{b}\times\mathbf{c})\big]\big\} = \big[-\big(\mathbf{b}\times(\mathbf{b}\times\mathbf{c})\big)\big]\cdot\mathbf{d}$
5. $\big[-\big(\mathbf{b}\times(\mathbf{b}\times\mathbf{c})\big)\big]\cdot\mathbf{d} = \big[(\mathbf{b}\cdot\mathbf{b})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{b}\big]\cdot\mathbf{d}$ $\{3, 4\}$

Because a vector $\mathbf{d}$ is arbitrary, then

$$\mathbf{b} \times (\mathbf{b} \times \mathbf{c}) == (\mathbf{b} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{b} \cdot \mathbf{b})\,\mathbf{c}.$$

Proof of Theorem 5b.

$$\mathbf{c} \times (\mathbf{b} \times \mathbf{c}) \underset{(A1)}{=} \mathbf{c} \times \big[-(\mathbf{c} \times \mathbf{b})\big] = -\big[\mathbf{c} \times (\mathbf{c} \times \mathbf{b})\big] \underset{(T5a)}{=}$$

$$= -\big[(\mathbf{c} \cdot \mathbf{b})\,\mathbf{c} - (\mathbf{c} \cdot \mathbf{c})\,\mathbf{b}\big] = (\mathbf{c} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\,\mathbf{c}.$$

Theorem 6a. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$,

b. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a}$.

Formulas given above are known as the development laws.

Proof of Theorem 6a.

Because on the basis of (A4) the pairs of vectors $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, $\mathbf{b} \times \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, $\mathbf{a}$ are orthogonal, then a vector $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ can be represented as a linear combination of vectors $\mathbf{b}$ and $\mathbf{c}$. Thus, where exist such real numbers β_1 and γ_1 that

1. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \beta_1 \mathbf{b} + \gamma_1 \mathbf{c}$

2. $(\beta_1 \mathbf{b} + \gamma_1 \mathbf{c}) \cdot \mathbf{b} = \beta_1(\mathbf{b} \cdot \mathbf{b}) + \gamma_1(\mathbf{c} \cdot \mathbf{b}) \underset{(1)}{=} \big[\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\big] \cdot \mathbf{b} \underset{(T4b)}{=}$

$$= \mathbf{a} \cdot \big[(\mathbf{b} \times \mathbf{c}) \times \mathbf{b}\big] = \mathbf{a}\big(-\big[\mathbf{b} \times (\mathbf{b} \times \mathbf{c})\big]\big) \underset{(T5a)}{=}$$

$$= \mathbf{a} \cdot \big[(\mathbf{b} \cdot \mathbf{b})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{b}\big] = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{c})$$

3. $(\beta_1 \mathbf{b} + \gamma_1 \mathbf{c}) \cdot \mathbf{c} = \beta_1(\mathbf{b} \cdot \mathbf{c}) + \gamma_1(\mathbf{c} \cdot \mathbf{c}) \underset{(1)}{=} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{c} \underset{(T4b)}{=}$

$$= \mathbf{a} \cdot \big[(\mathbf{b} \times \mathbf{c}) \times \mathbf{c}\big] = \mathbf{a}\big(-\big[\mathbf{c} \times (\mathbf{b} \times \mathbf{c})\big]\big) \underset{(T5a)}{=}$$

$$= \mathbf{a} \cdot \big[(\mathbf{b} \cdot \mathbf{c})\mathbf{c} - (\mathbf{c} \cdot \mathbf{c})\mathbf{b}\big] = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{c})$$

From 2. and 3. we obtain a system of equations

$$4. \quad \begin{cases} \beta_1(\mathbf{b} \cdot \mathbf{b}) + \gamma_1(\mathbf{b} \cdot \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{c}) \\ \beta_1(\mathbf{b} \cdot \mathbf{c}) + \gamma_1(\mathbf{c} \cdot \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{c}) \end{cases}$$

System (4) has the following solution:

5. $\beta_1 = \mathbf{a} \cdot \mathbf{c}, \qquad \gamma_1 = -\mathbf{a} \cdot \mathbf{b}.$

Inserting the found values of β_1 and γ_1 into item 1. of the proof we finally obtain

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}.$$

Proof of Theorem 6b.

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \underset{(A1)}{=} -\big[\mathbf{c} \times (\mathbf{a} \times \mathbf{b})\big] \underset{(T6a)}{=}$$

$$= -\big[(\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}\big] = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}.$$

Corollary 1.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 2(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

Proof.

$$\left.\begin{array}{ll} 1. & \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \\ 2. & \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = (\mathbf{c}\cdot\mathbf{b})\mathbf{a} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b} \end{array}\right\} \qquad \{\text{T6a}\}$$

Subtracting the corresponding sides of the above equalities we obtain the required formula.

Corollary 2. If vectors **a**, **b**, **c** are pairwise orthogonal, then

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{c}\times(\mathbf{a}\times\mathbf{b}).$$

This formula follows immediately from corollary 1, inasmuch as

$$\mathbf{a}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{b} = 0.$$

Theorem 7.

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}.$$

Proof.

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) \underset{(\text{T4b})}{=} \mathbf{a}\cdot\left[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\right] \underset{(\text{T6a})}{=}$$

$$= \mathbf{a}\cdot\left[(\mathbf{b}\cdot\mathbf{d})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{d}\right] =$$

$$= (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}.$$

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Remarks about the replacement of school induction definitions by normal definitions

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Pupils and teachers often ask themselves a question: can induction definitions be replaced in an equivalent way by normal definitions? In this paper we present a method of replacement of induction definitions by normal definitions illustrating the given theorems by a few examples.

From the viewpoint of the set theory operations and relations can be treated as certain sets. We discuss a method of replacement of an induction definition of the given set by a normal definition of this set. An induction definition of a set A has in general the following form (compare with [2]):

D1. A set A is the least one from among the sets X satisfying the conditions:

$$\mathrm{W}_1(X):\ a_1,\dots,a_n \in X \qquad \text{(the starting condition)},$$

$$\mathrm{W}_2(X):\ x_1,\dots,x_n \in X \Rightarrow f(x_1,\dots,x_n) \in X \quad \text{(the induction condition)}.$$

For example, the following definition of a set of all sensible expressions of implication-negation propositional calculus has such a typical form:

D2. The set S of sensible expressions is the least one from among the sets X satisfying the conditions:

$$\mathrm{W}_1(X):\ p_1, p_2, \dots \in X,$$

$$\mathrm{W}_2(X):\ \alpha, \beta \in X \Rightarrow \sim\alpha,\ \alpha \to \beta \in X$$

($p_1, p_2, \dots$ are the sentential variables).

Similarly, the induction definition of a family Fin of all finite subsets of an infinite set V has the following form (compare with [2]):

D3. A family Fin is the least one from among the families $\mathfrak{A}$ ($\mathfrak{A} \subseteq 2^{\mathrm{V}}$) satisfying the conditions:

$$\mathrm{W}_1(\mathfrak{A}):\ \emptyset \in \mathfrak{A},$$

$$\mathrm{W}_2(\mathfrak{A}):\ \underset{X}{\forall}\ \underset{a}{\forall} \Big[\mathrm{X} \in \mathfrak{A} \wedge \mathrm{a} \in \mathrm{V} \Rightarrow \mathrm{X} \cup \{\mathrm{a}\} \in \mathfrak{A}\Big].$$

The normal definition of the set A discussed in definition 1 has the form:

D4. $x \in A \Leftrightarrow \underset{X}{\forall}\Big[\mathrm{W}_1(X) \wedge \mathrm{W}_2(X) \Rightarrow x \in X\Big].$

It can be shown that (compare with [2]):

Theorem I. The expression D4 is equivalent to conjunction of three conditions:

(a) $\mathrm{W}_1(A)$,

(b) $\mathrm{W}_2(A)$,

(c) $\underset{X}{\forall}\Big[\mathrm{W}_1(X) \wedge \mathrm{W}_2(X) \Rightarrow A \subseteq X\Big].$ (the minimization condition)

To verify that the set A specified by definition D4 is a function the existence and uniqueness conditions should be proved. For three-argument relation A being a function of two variables the existence and uniqueness conditions take the form:

(i) $\underset{a,b}{\forall}\ \underset{c}{\exists}\ (a,b,c) \in A,$ (the existence condition)

(j) $\underset{a,b,c,d}{\forall}\ \big[(a,b,c),(a,b,d) \in A \Rightarrow c = d\big].$ (the uniqueness condition)

Example 1. In a set of natural numbers $\mathbb{N}$ the induction definition of the relation less than $<$ ($< \subseteq \mathbb{N} \times \mathbb{N}$) is as follows:

D5a. $0 < x$,

b. $x < y \Rightarrow s(x) < s(y)$,

where s is a function of a successor ($s(n) = n + 1$).
Then, the conditions W_1 and W_2 a take the form

$\mathrm{W}_1(<):$ $(0,x) \in <$,

$\mathrm{W}_2(<):$ $(x,y) \in < \Rightarrow (s(x), s(y) \in <$.

Using the normal definition the relation "<" can be defined in the following way:

D6. $(x,y) \in < \Leftrightarrow \underset{X}{\forall}\Big[\mathrm{W}_1(X) \wedge \mathrm{W}_2(X) \Rightarrow (x,y) \in X\Big]$

or

D6′. $$x < y \Leftrightarrow \underset{X}{\forall}\Big[\underset{x}{\forall}(0,x) \in X \wedge \underset{x,\,y}{\forall}\Big((x,y) \in X \Rightarrow (s(x), s(y)) \in X\Big) \Rightarrow$$
$$\Rightarrow (x,y) \in X\Big].$$

It should be noted that in the natural number arithmetic the relation $<$ is generally defined using the normal definition:

D7. $(x,y) \in < \Leftrightarrow \underset{z \neq 0}{\exists}\ (x + z = y).$

In the work [3] (pp. 26–28) the following theorem is proved using the complete induction:

Theorem II. If $g(n_2, \dots, n_k)$ and $h(n_1, n_2, \dots, n_k, n_{k+1})$ are operations in a set of natural numbers, then there exists one and only operation $f(n_1, \dots, n_k)$ satisfying the conditions:

$$\mathrm{W}_1(f): \qquad f(0, n_2, \ldots, n_k) = g(n_2, \ldots, n_k),$$

$$\mathrm{W}_2(f): \qquad f(s(n), n_2, \ldots, n_k) = h(n, n_2, \ldots, n_k, f(n, n_2, \ldots, n_k)).$$

Introducing the notation $\mathbf{a} = (n_2, \ldots, n_k)$ these conditions can be written as follows:

$$\mathrm{W}_1'(f): \qquad f(0, \mathbf{a}) = g(\mathbf{a}),$$

$$\mathrm{W}_2'(f): \qquad f(s(n), \mathbf{a}) = h(n, \mathbf{a}, f(n, \mathbf{a})),$$

where $s(n) = n + 1$.

The normal definition of an operation f satisfying the conditions $\mathrm{W}_1'(f)$, $\mathrm{W}_2'(f)$ has the form:

D8. $$y = f(n, \mathbf{a}) \ \Leftrightarrow\ \forall_z \Big[\mathrm{W}_1' \wedge \mathrm{W}_2' \Rightarrow y = z(n, \mathbf{a}) \Big].$$

If the k-ary operation is treated as a relation T between $k+1$ independent variables, then the conditions $\mathrm{W}_1'(f)$, $\mathrm{W}_2'(f)$ can be written in the form

$$\mathrm{W}_1''(f): \qquad (0, \mathbf{a}, g(\mathbf{a})) \in T,$$

$$\mathrm{W}_2''(f): \qquad \forall_{u,v} \Big[(u, \mathbf{a}, v) \in T \Rightarrow (s(u), \mathbf{a}, h(u, \mathbf{a}, v)) \in T \Big].$$

The normal definition of the relation T (or the operation f) takes the form:

D8′. $$(x, \mathbf{a}, y) \in T \ \Leftrightarrow\ \forall_X \Big[\mathrm{W}_1'' \wedge \mathrm{W}_2'' \Rightarrow (x, \mathbf{a}, y) \in X \Big].$$

Example 2. Using induction the operation of addition in a set of natural numbers $\mathbb{N}$ can be defined by a function of a successor$s$ in the following manner:

D9 a. $m + 0 = m,$ $\qquad (\mathrm{W}_1(+))$

b. $m + s(n) = s(m + n).$ $\qquad (\mathrm{W}_2(+))$

The normal definition of the operation "+" is as follows:

D10. $$y = u + v \ \Leftrightarrow\ \forall_z \Big[\mathrm{W}_1(z) \wedge \mathrm{W}_2(z) \Rightarrow y = z(u, v) \Big],$$

where

$$\mathrm{W}_1(z): \qquad \forall_m \Big[z(m, 0) = m \Big],$$

$$\mathrm{W}_2(z): \qquad \forall_{m,n} \Big[z(m, s(n)) = s(z(m, n)) \Big].$$

If the operation "+" is treated as a relation between three independent variables, then the conditions (a) and (b) of the definition D9 take the form:

$$\mathrm{W}_1'(+): \qquad (m, 0, m) \in +,$$

$$\mathrm{W}_2'(+): \qquad \forall_{u,v} \Big[(m, u, v) \in + \Rightarrow (m, s(u), s(v)) \in + \Big].$$

The normal definition of the relation + is as follows:

D10′. $$(m, x, y) \in + \ \Leftrightarrow\ \forall_X \Big[\mathrm{W}_1' \wedge \mathrm{W}_2' \Rightarrow (m, x, y) \in X \Big].$$

A proof that the relation "+" due to definition D10’ is a function can be found in the work [1, pp. 339-344] (see also [4]).

Example 3. The induction definition of the multiplication operation in a set of natural numbers using the addition operation and the operation of a successor has the form:

D11 a. $m \cdot 0 = m,$

b. $m \cdot s(n) = (m \cdot n) + m.$

The normal definition is as follows:

D12. $$y = u \cdot v \;\Leftrightarrow\; \underset{z}{\forall}\Big[\mathrm{W}_1(z) \wedge \mathrm{W}_2(z) \Rightarrow y = z(u,v)\Big],$$

where

$$\mathrm{W}_1(z): \quad \underset{m}{\forall}\Big[z(m,0) = 0\Big],$$

$$\mathrm{W}_2(z): \quad \underset{m,n}{\forall}\Big[z(m,s(n)) = z(m,n) + m\Big].$$

Treating the binary operation "$\bullet$" as a relation between three independent variables the starting condition and the induction condition can be written as:

$$\mathrm{W}_1'(\bullet): \quad (m,0,0) \in \bullet\,,$$

$$\mathrm{W}_2'(\bullet): \quad \underset{u,v}{\forall}\Big[(m,u,v) \in \bullet \Rightarrow (m,s(u),v+m) \in \bullet\Big].$$

Then the normal definition of the relation "$\bullet$" takes the form:

D12′. $$(m,x,y) \in \bullet \;\Leftrightarrow\; \underset{X}{\forall}\Big[\mathrm{W}_1' \wedge \mathrm{W}_2' \Rightarrow (m,x,y) \in X\Big].$$

Example 4. The factorial function has the following induction definition:

$$0! = 1; \qquad s(n)! = n! \cdot s(n), \qquad (n \in \mathbb{N}).$$

The normal definition is as follows:

$$y = n! \;\Leftrightarrow\; \underset{z}{\forall}\Big[\mathrm{W}_1(z) \wedge \mathrm{W}_2(z) \Rightarrow y = z(n)\Big],$$

where

$$\mathrm{W}_1(z): \quad z(0) = 1,$$

$$\mathrm{W}_2(z): \quad \underset{n \in \mathbb{N}}{\forall}\Big[z(s(n)) = z(n) \cdot s(n)\Big].$$

Example 5. Exponentiation with natural power is defined by induction:

$$a^0 = 1; \qquad a^{s(n)} = a^n \cdot a, \qquad (n \in \mathbb{N},\; a > 0,\; a \in R).$$

The normal definition has the form:

$$y = a^n \;\Leftrightarrow\; \underset{z}{\forall}\Big[\mathrm{W}_1(z) \wedge \mathrm{W}_2(z) \Rightarrow y = z(n)\Big],$$

where

$$\mathrm{W}_1(z): \quad z(0) = 1,$$

$$\mathrm{W}_2(z): \quad \underset{n \in \mathbb{N}}{\forall}\Big[z(s(n)) = z(n) \cdot a\Big].$$

Example 6. The predecessor operation P in a set of natural numbers has the following induction definition:

$$\begin{cases} P(0) = 0, \\ P(s(n)) = n, \qquad n \in \mathbb{N}. \end{cases}$$

The normal definition is:

$$y = P(n) \;\Leftrightarrow\; \underset{z}{\forall}\Big[\mathrm{W}_1(z) \wedge \mathrm{W}_2(z) \Rightarrow y = z(n)\Big],$$

where

$$\mathrm{W}_1(z): \qquad z(0) = 0,$$

$$\mathrm{W}_2(z): \qquad \underset{n\in\mathbb{N}}{\forall}\Big[z(s(n)) = n\Big].$$

If the predecessor operation is defined by the definition

$$P(n) = \begin{cases} 0, & \text{gdy } \; n = 0, \\ n-1, & \text{gdy } \; n > 0, \end{cases}$$

then the normal definition of this operation takes the form:

$$y = P(n) \;\Leftrightarrow\; \Big[(n = 0 \wedge y = 0) \vee (n > 0 \wedge y = n-1)\Big].$$

Example 7. The operation of bounded subtraction "$\dot{-}$" in a set $\mathbb{N}$ can be defined by induction due to the following formulae:

$$\begin{cases} m \dot{-} 0 = m, \\ m \dot{-} s(n) = P(m \dot{-} n), \qquad n \in \mathbb{N}, \end{cases}$$

where P is the predecessor operation.

The normal definition has the form:

$$y = u \dot{-} v \;\Leftrightarrow\; \underset{z}{\forall}\Big[\mathrm{W}_1(z) \wedge \mathrm{W}_2(z) \Rightarrow y = z(u,v)\Big],$$

where

$$\mathrm{W}_1(z): \qquad \underset{m\in\mathbb{N}}{\forall}\Big[z(m,0) = m\Big],$$

$$\mathrm{W}_2(z): \qquad \underset{m,n}{\forall}\Big[z(m,s(n)) = P(z(m,n))\Big].$$

The operation "$\dot{-}$" can also be defined without induction:

$$x \dot{-} y = \begin{cases} 0, & \text{gdy } \; x \le y, \\ x-y, & \text{gdy } \; x > y, \end{cases}$$

with the operation "$-$" defined as follows:

$$x > y \Rightarrow \Big[u = x - y \;\Leftrightarrow\; u + y = x\Big].$$

Then the normal definition of the operation "$\dot{-}$" takes the form:

$$u = x \dot{-} y \;\Leftrightarrow\; \Big[(x \le y \wedge u = 0) \vee (x > y \wedge u = x - y)\Big].$$

Example 8. The induction definitions of arithmetic and geometric sequences have the form:
An arithmetic sequence with the first term a and a common difference r:

$$\begin{cases} f(0) = a, \\ f(s(n)) = f(n) + r. \end{cases}$$

A geometric sequence with the first term a and a common ratio q:

$$\begin{cases} g(0) = a, \\ g(s(n)) = g(n) \cdot q. \end{cases}$$

The normal definitions of these sequences are the following:

$$y = f(n) \;\Leftrightarrow\; \forall_{z} \Big[z(0) = a \wedge \forall_{n \in \mathbb{N}} \Big(z(s(n)) = z(n) + r \Big) \Rightarrow y = z(n) \Big],$$
$$y = g(n) \;\Leftrightarrow\; \forall_{z} \Big[z(0) = a \wedge \forall_{n \in \mathbb{N}} \Big(z(s(n)) = z(n) \cdot q \Big) \Rightarrow y = z(n) \Big].$$

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Symmetry Reduction of Some Classes of the First-Order Differential Equations in the Space $M(1,4) \times R(u)$

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We consider classes of the first-order differential equations in the space $M(1,4) \times R(u)$, which are invariant under splitting subgroups of the Poincaré group $P(1,4)$. To study these classes we used their symmetry properties. Using the functional bases of the invariants of some splitting subgroups of the group $P(1,4)$ we have constructed the ansatzes, which reduce majority of the classes investigated to differential equations with fewer independent variables. The corresponding symmetry reduction is done.

Introduction

Among equations, important for theoretical and mathematical physics, mechanics, etc., there are also ones which have nontrivial symmetry groups. This type equations have been studied in many works (see, for example, [1–9]). In particular, the articles [7–9] are devoted to the construction and investigation of differential equations which are invariant under continuous subgroups of the generalized Poincaré group $P(1,4)$ and defined in the spaces $M(1,4) \times R(u)$ and $M(1,3) \times R(u)$. Here, and in what follows, $R(u)$ is the number axis of the dependent variable u. In [7], there are some first-order differential equations in the space $M(1,4) \times R(u)$ with nontrivial symmetry groups. Some differential equations of first- and second-order in the space $M(1,3) \times R(u)$ with given symmetry groups can be found in [8]. The article [9] contains some results concerning the symmetry reduction of classes of the first-order differential equations in the space $M(1,4) \times R(u)$ to ODEs.

In the present paper we concentrate our attention on the symmetry reduction of these classes to classes of differential equations with two-, three-, and four independent variables.

Our paper is organized as follows: in the first section we introduce some notations and results concerning the group $P(1,4)$ and its Lie algebra, which we use thereafter. Section 2 presents our main results.

1. The group $P(1,4)$ and its non-conjugate subgroups

The group $P(1,4)$ is a group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. Some applications of this group in theoretical and mathematical physics can be found in [2, 3, 6].

To study the subgroup structure of the group $P(1,4)$, we used the method proposed in [10]. Continuous subgroups of the group $P(1,4)$ have been found in [11–13]. Among these subgroups there are the Poincaré group $P(1,3)$ and the extended Galilei group $\widetilde{G}(1,3)$ (see also [14]).

The Lie algebra of the group $P(1,4)$ is given by 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ $(\mu,\nu = 0,1,2,3,4)$ and P'_μ $(\mu = 0,1,2,3,4)$, which satisfy the commutation relations:

$$\left[P'_\mu, P'_\nu\right] = 0, \qquad \left[M'_{\mu\nu}, P'_\sigma\right] = g_{\mu\sigma}P'_\nu - g_{\nu\sigma}P'_\mu,$$

$$\left[M'_{\mu\nu}, M'_{\rho\sigma}\right] = g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\sigma}M'_{\mu\rho} - g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$.

Here, we consider the following representation of the Lie algebra of the group $P(1,4)$:

$$P'_0 = \frac{\partial}{\partial x_0}, \quad P'_1 = -\frac{\partial}{\partial x_1}, \quad P'_2 = -\frac{\partial}{\partial x_2}, \quad P'_3 = -\frac{\partial}{\partial x_3},$$

$$P'_4 = -\frac{\partial}{\partial x_4}, \quad M'_{\mu\nu} = -\left(x_\mu P'_\nu - x_\nu P'_\mu\right).$$

Below, we will use the following basis elements:

$$G = M'_{40}, \quad L_1 = M'_{32}, \quad L_2 = -M'_{31}, \quad L_3 = M'_{21},$$

$$P_a = M'_{4a} - M'_{a0}, \quad C_a = M'_{4a} + M'_{a0}, \quad (a = 1,2,3),$$

$$X_0 = \frac{1}{2}\left(P'_0 - P'_4\right), \quad X_k = P'_k \quad (k = 1,2,3), \quad X_4 = \frac{1}{2}\left(P'_0 + P'_4\right).$$

The Lie algebra of the group $\widetilde{G}(1,3)$ is generated by the following basis elements:

$$L_1,\ L_2,\ L_3,\ P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4.$$

2. On some classes of the first-order differential equations in the space $M(1,4) \times R(u)$

In this section, we present some of the new results concerning the symmetry reduction of the classes of the first-order differential equations, which are defined in the space $M(1,4) \times R(u)$.

The classes of the first-order differential equations, which are invariant under splitting subgroups of the group $P(1,4)$ and defined in the space $M(1,4) \times R(u)$, can be written in the following form (see, for example, [1 ,4 ,5]):

$$F(J_1, J_2, ..., J_t) = 0,$$

where F is an arbitrary smooth function of its arguments, $\{J_1, J_2, ..., J_t\}$ is a functional basis of the first-order differential invariants of splitting subgroups of the group $P(1,4)$.

The classes of the first-order differential equations in the space $M(1,4) \times R(u)$, which are invariant under splitting subgroups of the group $P(1,4)$, have been constructed. Some of the results obtained can be found in [7].

We used only the symmetry properties of these classes in order to construct the ansatzes, which reduce them to classes of differential equations with fewer independent variables. The corresponding symmetry reduction is obtained. Among reduced equations, there are one-, two-, three-, and four-dimensional ones.

It is impossible to present here all the results obtained. Therefore, we will give a short rewiew of the results obtained.

First, we consider the classes of the first-order differential equations, which are constructed with the use of the functional bases of the first-oder differential invariants containing two usual invariants. One of them is always u, since it is one of the invariants of the group $P(1,4)$. Using these two invariants we have constructed the ansatzes, which reduce the classes considered to classes of ODEs. The corresponding symmetry reduction is done. We considered 90 of such classes of differential equations. From among them, 17 classes have been reduced to the classes of functional equations. More details about it can be found in [9].

Now, we consider the classes of the first-order differential equations, which are obtained with the use of the functional bases of the first-oder differential invariants containing three usual invariants. We investigated 52 of such classes. From among them, there are 30 ones, which are invariant under the splitting subgroups of the group $\widetilde{G}(1,3)$. The 20 classes are reduced to classes of two-dimensional PDEs. Below, we give some examples. In each example for some splitting subgroups of the group $P(1,4)$, we write the basis elements of their respective Lie algebras, the corresponding arguments $J_1, J_2, \ldots, J_t$ of the function F, the ansatzes obtained, as well as the reduced equation.

$$\begin{aligned}
&\langle P_1, P_2, X_3\rangle,\\
&J_1 = x_0 + x_4, \quad J_2 = \left(x_0^2 - x_1^2 - x_2^2 - x_4^2\right)^{1/2}, \quad J_3 = u,\\
&J_4 = u_1(x_0+x_4) + x_1(u_0 - u_4), \quad J_5 = u_2(x_0+x_4) + x_2(u_0-u_4), \quad J_6 = u_3,\\
&J_7 = u_0 - u_4, \quad J_8 = u_0^2 - u_1^2 - u_2^2 - u_4^2\ ; \qquad u = \varphi(\omega_1,\omega_2), \quad \omega_1 = x_0 + x_4,\\
&\omega_2 = \left(x_0^2 - x_1^2 - x_2^2 - x_4^2\right)^{1/2}\ ; \ F\left(\omega_1,\omega_2,\varphi,0,0,0,\tfrac{\omega_1}{\omega_2}\varphi_2, \varphi_2^2 + 2\tfrac{\omega_1}{\omega_2}\varphi_1\varphi_2\right) = 0\ ;\\
&\qquad u_\mu \equiv \tfrac{\partial u}{\partial x_\mu},\ \mu = 0,\ 1,\ 2,\ 3,\ 4; \qquad \varphi_i = \tfrac{\partial\varphi}{\partial\omega_i},\ i = 1,\ 2.
\end{aligned}$$

The 10 of these classes are reduced to classes of ODEs. Let us give an example:

$$\begin{aligned}
&\langle L_3, P_3, X_4\rangle,\\
&J_1 = x_0 + x_4, \quad J_2 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_3 = u, \quad J_4 = x_1u_2 - x_2u_1,\\
&J_5 = (x_0+x_4)u_3 + (u_0-u_4)x_3, \quad J_6 = u_0 - u_4, \quad J_7 = u_1^2 + u_2^2,\\
&J_8 = u_0^2 - u_3^2 - u_4^2\ ; \qquad u = \varphi(\omega_1,\omega_2), \quad \omega_1 = x_0 + x_4, \quad \omega_2 = \left(x_1^2 + x_2^2\right)^{1/2}\ ;\\
&F\left(\omega_1,\omega_2,\varphi,0,0,0,\varphi_2^2,0\right) = 0\ .
\end{aligned}$$

The 22 classes of differential equations are not invariant under the splitting subgroups of the group $\widetilde{G}(1,3)$. They are reduced to two-dimensional PDEs.

Let us consider the classes of the first-order differential equations, which are obtained with the use of the functional bases of the first-oder differential invariants containing four usual invariants. We investigated 23 of such classes. From among them, there are 16 ones, which are invariant under the splitting subgroups of the group $\widetilde{G}(1,3)$. The 13 classes are reduced to classes of three-dimensional PDEs. Below, we give an example:

$$\begin{aligned}
&\langle L_3, P_3\rangle,\\
&J_1 = x_0 + x_4, \quad J_2 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_3 = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}, \quad J_4 = u,\\
&J_5 = x_1u_2 - x_2u_1, \quad J_6 = (x_0+x_4)u_3 + (u_0-u_4)x_3, \quad J_7 = u_0 - u_4, \quad J_8 = u_1^2 + u_2^2,\\
&J_9 = u_0^2 - u_3^2 - u_4^2\ ; \qquad u = \varphi(\omega_1,\omega_2,\omega_3), \quad \omega_1 = x_0 + x_4, \quad \omega_2 = \left(x_1^2 + x_2^2\right)^{1/2},\\
&\omega_3 = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}\ ; \ F\left(\omega_1,\omega_2,\omega_3,\varphi,0,0,\omega_1\tfrac{\varphi_3}{\omega_3}, \varphi_2^2, \varphi_3^2 + 2\omega_1\varphi_1\tfrac{\varphi_3}{\omega_3}\right) = 0\ .
\end{aligned}$$

The 3 of these classes are reduced to classes of two-dimensional PDEs. Let us give an example:

$\langle L_3 - P_3, X_4\rangle$,
$J_1 = x_0 + x_4, \quad J_2 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_3 = \arctan\frac{x_1}{x_2} + \frac{x_3}{x_0+x_4}, \quad J_4 = u,$
$J_5 = x_2u_1 - x_1u_2, \quad J_6 = \frac{u_0-u_4}{x_0+x_4}x_3 + u_3, \quad J_7 = u_0 - u_4, \quad J_8 = u_1^2 + u_2^2,$
$J_9 = u_0^2 - u_3^2 - u_4^2\ ; \qquad u = \varphi(\omega_1,\omega_2,\omega_3), \quad \omega_1 = x_0 + x_4, \quad \omega_2 = \left(x_1^2 + x_2^2\right)^{1/2},$
$\omega_3 = \arctan\frac{x_1}{x_2} + \frac{x_3}{x_0+x_4}\ ; \qquad F\left(\omega_1,\omega_2,\omega_3,\varphi,\varphi_3,\frac{\varphi_3}{\omega_1},0,\frac{\varphi_3^2}{\omega_2^2}+\varphi_2^2,-\frac{\varphi_3^2}{\omega_1^2}\right) = 0\ .$

The 7 classes of differential equations are not invariant under the splitting subgroups of the group $\widetilde{G}(1,3)$. They are reduced to three-dimensional PDEs.

Finally, we consider the classes of the first-order differential equations, which are obtained with the use of the functional bases of the first-oder differential invariants containing five usual invariants. We investigated 8 of such classes. From among them, there are 6 ones, which are invariant under the splitting subgroups of the group $\widetilde{G}(1,3)$. All these classes are reduced to classes of four-dimensional PDEs. Below, we give an example:

$\langle P_3\rangle$,
$J_1 = x_1, \quad J_2 = x_2, \quad J_3 = x_0 + x_4, \quad J_4 = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}, \quad J_5 = u,$
$J_6 = (x_0 + x_4)u_3 + (u_0 - u_4)x_3, \quad J_7 = u_0 - u_4, \quad J_8 = u_0^2 - u_3^2 - u_4^2, \quad J_9 = u_1,$
$J_{10} = u_2\ ; \qquad u = \varphi(\omega_1,\omega_2,\omega_3,\omega_4), \quad \omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_0 + x_4,$
$\omega_4 = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}\ ;$
$F\left(\omega_1,\omega_2,\omega_3,\omega_4,\varphi,0,\omega_3\frac{\varphi_4}{\omega_4},\varphi_4^2 + 2\omega_3\varphi_3\frac{\varphi_4}{\omega_4},\varphi_1,\varphi_2\right) = 0\ .$

The 2 classes of differential equations are not invariant under the splitting subgroups of the group $\widetilde{G}(1,3)$. They are also reduced to four-dimensional PDEs. Let us give an example:

$\langle G\rangle$,
$J_1 = x_1, \quad J_2 = x_2, \quad J_3 = x_3, \quad J_4 = \left(x_0^2 - x_4^2\right)^{1/2}, \quad J_5 = u,$
$J_6 = (x_0 + x_4)(u_0 + u_4), \quad J_7 = u_1, \quad J_8 = u_2, \quad J_9 = u_3, \quad J_{10} = u_0^2 - u_4^2\ ;$
$u = \varphi(\omega_1,\omega_2,\omega_3,\omega_4), \quad \omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_3, \quad \omega_4 = \left(x_0^2 - x_4^2\right)^{1/2}\ ;$
$F\left(\omega_1,\omega_2,\omega_3,\omega_4,\varphi,\omega_4\varphi_4,\varphi_1,\varphi_2,\varphi_3,\varphi_4^2\right) = 0\ .$

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Weight factor of a set of traces in a stochastic graph

Ireneusz Krech

A method of calculating the weight factor of all traces leading to a fixed boundary node in a stochastic graph, is presented in this work. The results obtained here find application in calculating probabilities of some events in countable probabilistic spaces.

Introduction

Definition 1. *A directed graph* is a pair (S, Q_{0-1}), where $S = \{1, 2, \dots, n\}$ $(n > 1)$ and Q_{0-1} is a nonzero square matrix of a rank n and of elements from the set Q_{0-1}. The elements of the S set are called *the nodes of the graph.* The matrix $Q_{0-1} = [q_{jk}]$ is called *the transition matrix.* Let $j, k \in S$. If $q_{jk} = 1$, so the pair (j, k) is called *the arc* and we denote it by $j{\to}k$. The node j is called *the beginning*, and the node k is called *the end* of this arc. If $q_{jj} = 1$, so the arc $j{\to}j$ is called *the loop.* If a node j is a beginning just only of a loop, so we call it *a boundary node.* The set of all boundary nodes of a graph is denoted by $V(S, Q)$. If $\#V(S, Q) = l$, this graph is called *the graph of l boundary nodes.*

The sequence of arcs $(j_1{\to}j_2, j_2{\to}j_3, \dots, j_{m-1}{\to}j_m)$ is represented as a sequence of nodes $(j_1, j_2, j_3, \dots, j_{m-1}, j_m)$.

Definition 2. Any sequence of arcs $(j_1, j_2, j_3, \dots, j_{m-1}, j_m)$ of the directed graph (S, Q_{0-1}) complying with the condition:

$$q_{j_1 j_2} \cdot q_{j_2 j_3} \cdots q_{j_{m-1} j_m} = 1$$

is called *the transition from the node j_1 to the node j_m.* The number m i.e. the number of arcs in the transition is called *the length of transition.* Any node that is the beginning or the end of transition arc is called *a node connected with this transition.*

Definition 3. The directed graph (S, Q_{0-1}) complying with the below conditions is called *the proper directed graph*:

(1) $\forall j_1, j_2 \in S\ [(\forall k \in S\ q_{kj_1} = q_{kj_2} = 0) \Longrightarrow j_1 = j_2]$.

(2) $\forall j \in S\ \exists k \in S[k \neq j \Rightarrow (q_{jk} = 1 \vee q_{kj} = 1)]$,

(3) $\exists k \in S\ \ \forall j \in S[j \neq k \Rightarrow\ (\exists j_1, j_2, \dots, j_n \in S\ \ (q_{kj_1} \cdot q_{j_1 j_2} \cdot \ldots \cdot q_{j_n j} = 1))]$,

(4) $\forall j \in S\ \ \exists k \in V(S, Q)\ \exists j_1, j_2, \dots, j_n \in S[q_{jj_1} \cdot q_{j_1 j_2} \cdot \ldots \cdot q_{j_n k} = 1]$.

In the following the proper directed graph will be briefly called *graph.*

The node k of the graph (S, Q_{0-1}) complying with the condition:

$$\forall j \in S, j \neq k \ \ \exists j_1, j_2, \ldots, j_m \in S \ \ [q_{kj_1} = 1 \cdot q_{j_1 j_2} = 1 \cdot \ldots \cdot q_{j_n j} = 1],$$

and unambiguously determined by the context of the graph creation (in case if in the set S there are at least two nodes complying with the above condition) is called *the start node* and is denoted by s.

Definition 4. Any transition from the start node s in the graph (S, Q_{0-1}) to the boundary node j is called *the graph trace leading to the node* j, and the length of this transition is called *the length of the trace*. The length of the trace t is denoted by $|t|$.

The set of all traces of the graph (S, Q_{0-1}) is denoted by $B(S, Q_{0-1})$. The set of all traces leading to the boundary node j is denoted by B_j.

Definition 5. Any transition from the node j to the same node j is called *a cycle*. If a start node is connected with a cycle, so this cycle is called *the basic cycle*. In other case the cycle is called *the internal cycle*. A loop is a cycle of the length l. If a node of the trace t is simultaneously connected with the trace t, so we say that the cycle c is *connected with the trace* t. The set of all basic cycles of the graph (S, Q) is denoted by C.

A trace with which a basic cycle is connected, is called *a simple trace*.

A nonzero square matrix of the rank n of non-negative elements in which the sum of the elements in each row equals to 1 is called *the stochastic matrix of the rank* n.

Let Q be the stochastic matrix of the rank n. Let us consider the function f:
– defined on the set of all stochastic matrixes,
– of values in the set of all nonzero null-one matrixes,
that assigns null-one matrix $f(Q) = Q_{0-1} = [q_{jk}]$ to the stochastic matrix $Q = [p_{jk}]$ created from the matrix Q by replacement all its nonzero elements by the number 1. It follows that:

$$\forall j, k \in \{1, 2, \ldots, n\} \quad [q_{jk} = 1 \Longleftrightarrow p_{jk} > 0].$$

Definition 6. Let $S = \{1, 2, \ldots, n\}$ $(n \geq 1)$. Let us assume that Q is the stochastic matrix of a rank n. If a pair $(S, f(Q))$ is a graph, so the pair (S, Q) is called *the stochastic graph.*

The concepts already introduced for the graph are valid also for the stochastic graph.

Definition 7. Let (S, Q) be a stochastic graph, where $Q = [p_{jk}]$. The number p_{jk} is called *the weight of the arc* $j \rightarrow k$. If v is the transition on the graph (S, Q), so the product of weights of consecutive arcs in this transition is called *the weight of the transition* v and is denoted by $w(v)$.

If B is any set of traces in the stochastic graph (S, Q), so the number $w(B) = \sum_{t \in B} w(t)$ is called *the weight of the set* B.

For any stochastic graph the weight of all transitions of a common beginning of the first arc is equal to 1.

Let (S, Q) be a stochastic graph of n boundary nodes $(n \geq 2)$. The set of all simple traces of the graph (S, Q) is denoted by B^0. Let $B_j^0 = B_j \setminus B^0$.

In [8] *the cycle* was named as any multiple valued transition from the node j to the same node j. Assuming this definition the following theorem was proved:

Theorem 1. . *If (S, Q) is a stochastic graph of n boundary nodes, where $n \geq 2$, so*

$$\forall j \in V(S, Q) \quad [w(B_j) = \frac{w(B_j^0)}{1 - w(C)}], \tag{1}$$

where B_j^0 denotes a set of all traces leading to the boundary node j, that are not simple.

Below, assuming Definition 5, we will formulate and prove the theorem concerning the weight of set of all traces leading to the boundary node j.

Theorem 2. *. If (S,Q) is a stochastic graph of n boundary nodes, where $n \geq 2$, so*

$$\forall j \in V(S,Q) \quad [w(B_j) = w(C) \cdot w(B_j^0) + w(B_j^0)]. \tag{2}$$

Proof. Let j be any settled boundary node of the graph (S,Q). Here we have:

$$B_j = B_j^0 \cup B^0 \ \text{ and } \ B_j^0 \cap B^0 = \emptyset.$$

Let c_k be any cycle. Let us denote by A_{c_k} the set of all simple traces t leading to the boundary node j, which beginning of length $|c_k|$ creates a cycle c_k and ending of the length $|t| - |c_k|$ creates a trace from the set B_j^0. In this case we have:

$$B^0 = \bigcup_{c_k \in C} A_{c_k} \ \text{ and } \ \forall c_n, c_m \in C[c_n \neq c_m \Rightarrow \ A_{c_n} \cap A_{c_m} = \emptyset],$$

Therefore

$$w(c_k) \cdot w(B_j^0) = w(A_{c_k}).$$

Finally we get

$$w(B^0) = w(\bigcup_{c_k \in C} A_{c_k}) = \sum_{c_k \in C} w(A_{c_k}) = \sum_{c_k \in C} w(c_k) \cdot w(B_j^0) = w(C) \cdot w(B_j^0),$$

what is the end of the proof.

Theorem 3. *. If (S,Q) is a stochastic graph of n boundary nodes, where $n \geq 2$, so*

$$\forall j \in V(S,Q) \quad \left[w(B_j) = \frac{w(B_j^0)}{\sum_{k \in V(S,Q)} w(B_k^0)} \right].$$

Proof. >From the fact that $\sum\limits_{k \in V(S,Q)} w(B_k) = 1$ it results

$$w(B_j) = \frac{w(B_j)}{\sum_{k \in V(S,Q)} w(B_k)}.$$

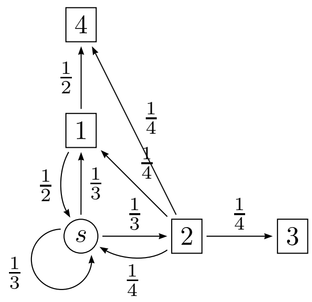
Using the Theorem 2 we receive:

$$w(B_j) = \frac{w(C) \cdot w(B_j^0) + w(B_j^0)}{\sum_{k \in V(S,Q)} w(C) \cdot w(B_k^0) + w(B_k^0)} = \frac{(w(C)+1) \cdot w(B_j^0)}{\sum_{k \in V(S,Q)} (w(C)+1) \cdot w(B_k^0)} =$$

$$= \frac{(w(C)+1) \cdot w(B_j^0)}{(w(C)+1) \cdot \sum_{k \in V(S,Q)} w(B_k^0)} = \frac{w(B_j^0)}{\sum_{k \in V(S,Q)} w(B_k^0)}.$$

- Let us consider the stochastic graph from the Figure 1. It is a pair of (S,Q), where:

$$S = \{s, 1, 2, 3, 4\} \quad \text{and} \quad Q = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Here we have
$B_4^0 = \{s{\to}1{\to}4, s \to 2{\to}1{\to}4, s \to 2{\to}4\}$ and $w(B_4^0) = \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{4} = \frac{7}{24}$,
$B_3^0 = \{s \to 2 \to 3\}$ and $w(A_3^0) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$.
Therefore according to the Theorem 3 we get:
$w(B_4) = \dfrac{\frac{7}{24}}{\frac{7}{24} + \frac{1}{12}} = \frac{7}{9}$.

The stochastic graph is an algebraic presentation of a homogenous Markow chain. The probabilistic space of Markow chain is equivalent (isomorphic) to the probabilistic space generated by the stochastic graph of this chain (see [6]). In the probabilistic space generated by the graph, the sets B_j are the events. Generally they are infinite sets in a countable probabilistic space, therefore their probabilities are the sums of certain series. The presented theorems provide us with some tools for calculating probabilities of events in countable probabilistic spaces.

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CTL Parser
(A tool for automated verification whether a formula belongs to the CTL language)

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The main aim of the paper is to present a new tool CTL Parser, which is designed for verification whether a formula belongs to the CTL language. CTL is the branching time temporal logic, which enables to express such "time" properties like: "eventually", "always", "next-time", "until", "release", "never", "inevitably", "optional", etc. CTL Parser is written in C and uses such well known parser generators like bison and flex. It takes a string and checks whether is it a CTL formula, and converts the formula either to RPN or PN.

Introduction

Main goals of our project is to create a separate tool for verification whether a formula belongs to the CTL language. It could be used with any external model checker such as Verics [1]. The main idea of our project was based on parser theory [3]. Typical parser takes input and transforms it into anexpected format.
Our main purpose is to check lexical analyses and syntax analyses. Finally, after checking that a formula is correct, we convert BNF format formula into such a format as RPN or PN. We can introduce formulae manually or as text files. The output may also be changed into a text file. The program is a text tool which is started in Unix environment.

Preliminaries

In this section we introduce formalisms and notations used in the rest of the paper. We start with introducing the syntax of the CTL language. Then, in Sections 2.2 and 2.3 we describe tools used to build the CTL parser.

CTL

Computation tree logic (CTL) is a propositional branching-time temporal logic that was introduced by Emerson and Clarke in [2] as a specification language for finite-state systems. The language contains two path quantifiers E and A with the intuitive meaning:

there is a path and for all paths, respectively. The CTL basic modalities are EX , EG, and EU (and their duals AX, AF, and AR).

Now, the formal syntax of CTL language is given.

Let $\mathcal{PV}$ be a set of propositional variables also containing the symbol $true$. The set of CTL formulae is defined inductively as follows:

- every member of $\mathcal{PV}$ is a formula,
- if α and β are formulae, then so are $\neg\alpha$, $\alpha \wedge \beta$ and $\alpha \vee \beta$,
- if α is formula, then so are $\mathrm{EX}\alpha$, $\mathrm{EG}\alpha$ and $\mathrm{E}\alpha\mathrm{U}\beta$.

The derived basic modalities are defined as follows: $\mathrm{EF}\alpha \overset{def}{=} \mathrm{E}(true\mathrm{U}\alpha)$, $\mathrm{AX}\alpha \overset{def}{=} \neg\mathrm{EX}\neg\alpha$, $\mathrm{AF}\alpha \overset{def}{=} \neg\mathrm{EG}\neg\alpha$, $\mathrm{A}(\alpha\mathrm{R}\beta) \overset{def}{=} \neg\mathrm{E}(\neg\alpha\mathrm{U}\neg\beta)$, $\mathrm{AG}\alpha \overset{def}{=} \neg\mathrm{EF}\neg\alpha$. Moreover, $\alpha \rightarrow \beta \overset{def}{=} \neg\alpha \vee \beta$.

Our tool is not designated for checking the validity of the CTL formulae, but it is designed for performing a syntax analysis. Therefore, we do not present any semantics of CTL in our paper.

Flex

Flex is a fast lexical analyser generator. It is a tool for generating scanners: programs which recognize lexical patterns in input text. Flex is a free implementation of the well known Lex program. The description is in the form of pairs of regular expressions and C code, called rules. The best choice is to link it with another great tool Bison.
Lexical analyser created by Lex co-operates with the syntax analyser in the following way. After starting by the syntax analyser, the lexical analyser begins to read in the next symbols from the input, till it finds the longest file-prefix of input suitable for one of the models. Afterwards it does an action which typically directs steering back to the syntax analyser. In the case of scanning white spaces or comments, the lexical analyser does not transfer any values to the syntax analyser, but returns to finding out the next lexem. Otherwise, the lexical analyser passes on to the syntax analyser a singular value, called a lexical symbol. To transferring values of an attribute consisting information of lexem, global variable YYLVAL is being used.

Bison

Bison is a tool parser generator. It is a tool to convert a grammar description for an LALR context-free grammar into a C program to parse that grammar. It can specify one or more syntactic groupings and give rules for constructing them from their parts. The input to Bison is essentially machine-readable BNF (Backus-Naur Form). Parsers generated by Bison use in their work the auxiliary stack. The state of calculation made by the parser is represented by a parser configuration. Such configuration is an ordered pair of sequences - stack and input. Stack element describes all symbols presented on the stack and input consists still not read input symbols. The most important actions of parser are shift and reduce with following meaning:
- shift - moves a current terminal symbol to the top of the stack of the parser
- reduce - changes an alfa production into A terminal

CTL Parser tool description

The tool is combined from 3 parts. The first task is to check the lexical correctness and return to the next part the recognized path qualifiers, temporal operators and propositional variables in the form of tokens and propositional variables. At the Flex level it is checked whether at the input are the adequate words from the set of CTL language. The second task rests with Bison where proper grammar rules were defined, which have to check the syntactical correctness of the specified language. We take ZM token and make non-terminal symbol. The next step is to create an expr non-terminal result from tokens AX AG ... , non-terminals of the type zm and term, which has been defined above and which is used recurrently, and an action in ANSI C code. By this method the syntax is checked.

With every rule an operation, which aim is to properly build a structure of a binary tree and to place it in a proper place of an output structure, is connected. With every rule an adequate action, which is suited by the longest match base, is related.

Using the default stack in bison the pointer is put in the yylval variable to the current branch of the tree. By this method a tree of expressions is created according to the polish notion by Lukasiewicz.

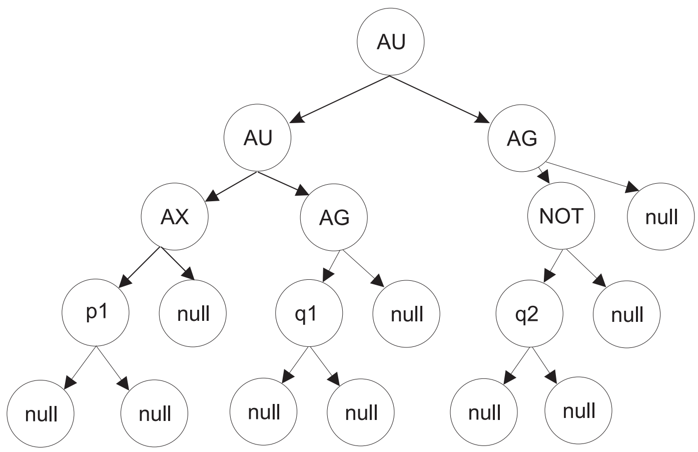

Figure 1: Tree structure filled with an example formula

The last step is the adequate writing out the structure in the given RPN or PN format.

(i) Input - standard input or text file (parameters given by user)

(ii) Flex - lexical analyser, generats tokens from input data

(iii) Bison - grammar parser, checks syntax analyses based on flex tokens and rule

(iv) C code - builds a tree using internal bison stack

(v) C code - generats output data in given format RPN or PN

(vi) Output - standard output or text file (parameters given by user)

Summary and Future works

This tool can take any formula in CTL language, which checking in standard method could take a very long time, and check syntax immediatly. In that state tool makes

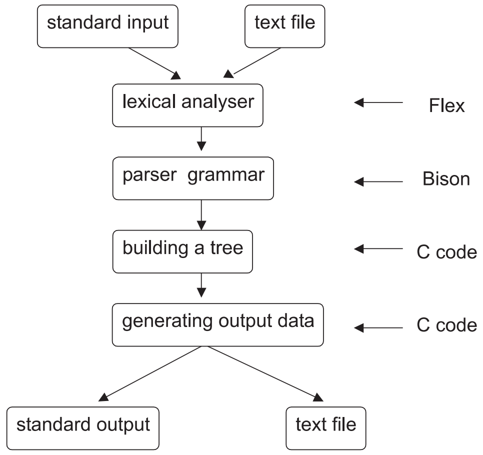

Figure 2: Workflow

lexical and syntax checking. Program operates interactive with command prompt or with files. In the future, we are going to parse other language like Star Computation Tree Logic (CTL*), Real Time Computation Tree Logic (TCTL), Computation Tree Logic of Knowledge(CTLK), and so on. Moreover, our tool will be one of the modules of symbolic model checker [1].

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The Methods of Realization of K-Valued Logical Function's Implicants

Sergiej Novikov

The efficient solution of the problem minimization of system $F = f_1, f_2, \ldots, f_s$ K-valued $(K > 2)$ of the partial logical functions n variables is reduced to generation of the (U, h) system-interval's set, where U is the word in alphabet $-, 0, 1, \ldots, K-1$, h is the set of functions from F, for which U is the interval. For construct of the analitical record for F each the interval must interpret as the formula which is the implicant of the functions from F. This paper presents methods of the implicant's realization as the formula for basis Rosser and Turquette, basis $\{Max,\ Min,\ x+1(modK)\}$ and basis $\{Max,\ Min,$ $x+B(modK),\ .\ \varphi_{K-1}(v)\ \}$, where $B \in \{0, 1, \ldots, K-1\}, \varphi_{K-1}(v) = K-1$ for $v = K-1$ and $\varphi_{K-1}(v) = 0$ for $v \neq i$. The method of realization of the system implicants as the circuit for K-valued Programmable Logic Array Chips and the estimation of the circuit's complication are also proposed.

The partial K-valued $(K > 2)$ logical function $f(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is defined as a set $\{\mathbf{M}^0, \mathbf{M}^1, \ldots, \mathbf{M}^{K-1}\}$, where $\mathbf{M}^i$, $i \in A = \{0, 1, \ldots, K-1\}$, is a set of words with length n in alphabet $\mathbf{A}$ such that $f(\alpha) = i$ for $\alpha = \alpha_1\alpha_2\ldots\alpha_n$, $a \in \mathbf{M}^i$ with $d = |\cup_{i=1}^{K-1} \mathbf{M}^i| \leq K^n$.

The problem of representation of a system $\mathbf{F} = \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_s\}$ of the partial K-valued logical functions as a set of formulas with minimum letters is very important from theoretical point of view (the partial function is a generalization of the function defined for the whole K^n of words – a range of definition the variables) and from applied science (elaboration of High Density Programmable Logic Array Chips for example K-PLA).

The problem of minimization K-valued logical functions has been frequently discussed. In the literature was described the results for case with alone $(s = 1)$ 3-valued function defined for the whole range of variables $(d = K^n)$. The difficulty of problem's solution for certain methods depends on the quantity of K and n [1, 2, 3].

The original method of K-valued logical function's minimization (case $K \geq 3,\ d \leq K^n$) was proposed in [4, 5].

The method is based on the new definitions of prime implicant, complete system of implicants, system-interval and special operations for K-space in order to generate system-intervals which cover range of definition of $\mathbf{F} = \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_s\}$. The difficulty of problem's solution for the method depends on d first of all.

The system-interval may be realized as prime implicant in the different bases.

The aim of this paper is to report the ways of realization of K-valued logical function's implicants.

First of all, each implicant may be written as formula for Rosser and Turquette's basis [1].

Let the interval $\mathbf{I}(\alpha)$ is described by $(K+1)$-valued vector $\alpha = \alpha_1\alpha_2\ldots\alpha_n$ with $\alpha_i \in \{-, 0, 1, \ldots, K-1\}$, where $\mathbf{f}(\alpha_1, \alpha_2, \ldots, \alpha_n) = j$.

The corresponding formula for Rosser and Turquette's basis has shape

$$\mathbf{P}j(\alpha) = \mathbf{C}j \wedge \Phi(\alpha), \tag{1}$$

where $\mathbf{P}j(\alpha) = \mathbf{j}$ for each $\beta \in \mathbf{J}(\alpha)$ and $\mathbf{P}j(\alpha) = 0$ for each $\gamma \notin \mathbf{J}(\alpha)$, a constant $\mathbf{C}j \in \{1, 2, \ldots, K-1\}$, $\Phi(\alpha)$ – product (conjunction) of functions characteristic $\varphi_i(\mathbf{v})$, where $\varphi_i(\mathbf{v}) = \mathbf{K-1}$ for $\mathbf{v} = \mathbf{i}$ and $\varphi_i(\mathbf{v}) = 0$ for $\mathbf{v} \neq \mathbf{i}$, $\wedge$ is the operation *Min*.

The first way for realization of implicants in basis

$$\{Max, Min, \varphi_0(\mathbf{v}_1), \ldots, \varphi_{K-1}(\mathbf{v}_n), 1, \ldots K-1\}$$

demands too many ($\leq K^*n$) functions $\varphi_j(v_i)$.

Let's consider the second method for realization of K-valued logical function's implicants in basis $\{Max, Min, x+1(ModK)\}$.

The characteristic function $\varphi_j(\mathbf{v}_i)$ may be written as formula $\mathbf{A}((\varphi_i(\mathbf{v}_j))$ in basis $\{Max, x+1(ModK), 1, 2, \ldots, K-1\}$.

We have for $0 < t < K-1$ and $j \in \{1, 2, ..., n\}$

$$\mathbf{A}(\varphi_t(\mathbf{v}_j)) = (\bigvee_{m=0}^{t-1} \mathbf{A}(\varphi_m(\mathbf{v}_j)) \quad \vee \bigvee_{s=0}^{(K-2)-t} (\mathbf{v}_j + \mathbf{s})) + 1, \tag{2}$$

where $\bigvee$ is meant through operation *Max*.

For the border ($\mathbf{t} = 0$, $\mathbf{t} = K-1$) we have

$$\mathbf{A}(\varphi_0(\mathbf{v}_j)) = (\bigvee_{s=0}^{K-2} (\mathbf{v}_j + s)) + 1 \tag{3}$$

$$\mathbf{A}(\varphi_{K-1}(\mathbf{v}_j)) = (\bigvee_{m=0}^{K-2} \mathbf{A}(\varphi_m(\mathbf{v}_j)) \ \vee (\mathbf{v}_j + (K-1)) + 1. \tag{4}$$

The constant $K-1$ can be written as formula

$$K - 1 = \mathbf{v}_j \vee (\mathbf{v}_j + 1) \vee ((\mathbf{v}_j + 1) + 1)(((\ldots((\mathbf{v}_j + 1) + 1) + \cdots + 1), \tag{5}$$

The editor replaced V-letter V by $\vee \ldots$ or. Is that right? where last item has $K-1$ round brackets inside.

The others constants are equal to

$$0 = (K-1) + 1,\ 1 = 0 + 1,\ 2 = 1 + 1, \ldots, K-2 = (K-3) + 1. \tag{6}$$

The conclusion may be stated according to the results (1) - (6).

The result of minimization of K-valued logical functions may be written as a formula in the basis $\{Max, Min, x+1(ModK)\}$.

It is a fine result from the mathematical point of view, because the basis includes three function only. However, this method does not permit to decrease the quantity of inside junctions in K-PLA. This problem may be decided with the help of third method.

Let us consider the function $\mathbf{v}_i^j = \varphi_{K-1}(\mathbf{v}i + Nj)$ for $i \in \{1, \ldots, n\}$, $j \in \{0, 1, \ldots, K-1\}$, where Nj is the Lukasiewicz's negation ($Nj = (K-1) - j$). That function is an efficient tool for writing the results of minimization of K-valued logical functions as a formula and also for realization of implicants by the logical circuit with K-PLA.

The function $\mathbf{v}_i^j$ has the following properties, where $\vee$ means the operation *Max* and $\wedge$ means the operation *Min*:

$$\mathbf{v}_i^j = \varphi_j(\mathbf{v}_i) \tag{7}$$

$$\bigvee_{j=0}^{K-1} = K-1 \qquad \text{and} \qquad \bigwedge_{j=0}^{K-1} \mathbf{v}_i^j = 0 \tag{8}$$

$$\mathbf{v}_i^j \wedge \mathbf{v}_t^s = \varphi_j(\mathbf{v}_i) \wedge \varphi_s(\mathbf{v}_t) = \varphi_{K-1}((\mathbf{v}_i + Nj) \wedge (\mathbf{v}_t + Ns)) \tag{9}$$

$$\mathbf{v}_i^j \vee \mathbf{v}_t^s = \varphi_j(\mathbf{v}_i) \vee \varphi_s(\mathbf{v}_t) = \varphi_{K-1}((\mathbf{v}_i + Nj) \vee (\mathbf{v}_t + Ns)) \tag{10}$$

$$(\mathbf{v}_i^j \wedge \mathbf{v}_t^s) \ \vee \mathbf{v}_i^j = \mathbf{v}_i^j \tag{11}$$

$$(\mathbf{v}_i^j \vee \mathbf{v}_t^s) \ \wedge \mathbf{v}_i^j = \mathbf{v}_i^j \tag{12}$$

$$(K-1) \ \wedge \mathbf{v}_i^j = \mathbf{v}_i^j \tag{13}$$

$$0 \wedge \mathbf{v}_i^j = \mathbf{v}_i^j \tag{14}$$

$$C \wedge \mathbf{v}_i^j = C \quad \text{for } \mathbf{v}_i^j = j \text{ and } C \wedge \mathbf{v}_i^j = 0 \text{ for } \mathbf{v}_i \neq j,\ C \in \{1, \ldots, K-2\} \tag{15}$$

$$\begin{aligned} f(\mathbf{v}_1\mathbf{v}_2, \ldots, \mathbf{v}_n) \ = \ & (\mathbf{v}_i^0 \wedge f(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, 0, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_k)) \vee \\ & \vee(\mathbf{v}_i^1 \wedge f(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, 1, \mathbf{v}_{i+1}, \ldots \mathbf{v}_n)) \vee \\ & \ldots \\ & \vee(\mathbf{v}_j^{K-1} \wedge f(\mathbf{v}_1, \ldots, \mathbf{v}_{i+1}, K-1, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_k)) \end{aligned} \tag{16}$$

$$\begin{aligned} f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) \ &= \ (\mathbf{v}_i^0 \wedge \mathbf{v}_2^0 \wedge \cdots \wedge \mathbf{v}_n^0 \wedge f(0, 0, \ldots, 0)) \vee \\ & \quad \ldots \\ & \quad \vee(\mathbf{v}_1^{K-1} \wedge \mathbf{v}_2^{K-1} \wedge \ldots \wedge \mathbf{v}_n^{K-1} \wedge f(K-1, \ldots, K-1)) = \\ &= \ \bigvee\nolimits_{\sigma 1 \sigma 2, \ldots, \sigma n} (\mathbf{v}_1^{\sigma 1}, \mathbf{v}_2^{\sigma 2}, \ldots, \mathbf{v}_n^{\sigma n} \wedge f(\sigma 1, \sigma 2, \ldots, \sigma n)) \end{aligned} \tag{17}$$

Theorem 1. *If we have the partial K-valued ($K > 2$) logical function $f(\mathbf{v}_1, \ldots, \mathbf{v}_n) = j > 0$ for $\beta \in I(\alpha)$, where $\alpha = \alpha_1\alpha_2 \ldots \alpha_n$ with $\alpha_i \in \{-, 0, 1, \ldots, K-1\}$, then the implicant covering $f(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ on the words from the set $\mathbf{I}(\alpha)$ is equal to*

$$\mathbf{P}j(\alpha) = Cj \wedge (\mathbf{v}_1^{\alpha_2} \wedge \mathbf{v}_2^{\alpha_2} \wedge \cdots \wedge \mathbf{v}_n^{\alpha_n}, \tag{18}$$

where for $\alpha_i = -$ a functor is absent, the constant $Cj \in \{1, 2, ..., K-1\}$, $\wedge$ is the operation Min.

Proof. If $\beta \in I(\alpha)$, then $f(\beta) = j > 0$. Let us consider the function $p'(\alpha) = (\mathbf{v}_1 + N\alpha_1) \wedge (\mathbf{v}_2 + N\alpha_2) \wedge \ldots \wedge (\mathbf{v}_n + N\alpha_n)$, where for $\alpha_i = -$ a functor is absent. Because $\beta \in I(\alpha)$, component $\beta_i = \alpha_i$ for $\alpha_i \neq -$. Hence we have $\beta_i + N\alpha_i = K - 1$ for $\beta_i = \alpha_i$ and

$$p'(\beta) = (\beta_1 + N\alpha_1) \wedge (\beta_2 + N\alpha_2) \wedge \ldots \wedge (\beta_n + N\alpha_n) = K - 1.$$

According to the definition of function $\phi_j(\mathbf{v}_i)$ we get that $p''(\beta) = \phi_{K-1}(p'(\beta))$ is equal to $K-1$ for each word from $I(\alpha)$. Hence we have $p''(\beta) = K - 1$ for each $\beta \in I(\alpha)$ and $p''(\gamma) = 0$ for each $\gamma \notin I(\alpha)$. We have from (18) for $Cj = j$ the following statement: $f(\beta) = j$ for each $\beta \in I(\alpha)$ and $f(\gamma) = 0$ on each word γ, where $f(\gamma) < j$. This is why the formula (18) is the implicant covering $f(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ on each word which belongs to $I(\alpha)$. □

Theorem 2. *If* $\{\mathbf{P}_{11}, \mathbf{P}_{12}, \ldots, \mathbf{P}_{1d_1}, \ldots, \mathbf{P}_{(K-1)1}, \mathbf{P}_{(K-1)2}, \ldots, \mathbf{P}_{(K-1)d(K-1)}\}$ *is the resulted system of implicants for* $f(\mathbf{v}_1, \ldots, \mathbf{v}_n)$*, where* Pij *is written as formula (18) and* j *is the number of implicants covering each of word* α *where* $f(\alpha) = i$ *with* $i > 0$ *and* $j \in \{1, \ldots, di\}$*, then*

$$f(\mathbf{v}_1, \ldots,\ \mathbf{v}_n) = \bigvee_{j=1}^{K-1} \ \bigvee_{j=1}^{di} Pij \tag{19}$$

Example 0.1. Let us consider the system of two functions from $K = 8$, $n = 6$, $d = 20$, which is represented as follows

$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{f}_1$	$\mathbf{f}_2$	
4	3	0	6	3	4	3	4	1
4	2	7	4	4	5	6	6	2
7	5	4	3	2	2	2	3	3
0	4	4	3	4	7	0	6	4
5	4	5	3	0	5	5	5	5
4	3	4	6	2	3	2	1	6
0	2	6	4	7	1	2	5	7
4	6	1	5	6	0	5	5	8
0	6	4	7	1	1	6	6	9
7	6	1	0	6	5	7	7	10
4	2	6	6	2	2	0	5	11
4	3	4	6	6	4	3	5	12
0	6	1	5	1	3	6	4	13
4	3	4	1	4	7	0	1	14
7	0	6	6	0	7	3	3	15
7	3	0	6	6	4	7	7	16
4	3	4	7	1	1	4	4	17
1	1	6	2	0	7	7	3	18
4	7	1	1	0	7	6	1	19
5	4	4	7	0	6	3	2	20

The results, where product of minimization (system-intervals) for stage i is described as $\mathbf{R}i$ and product of realization for stage i is described according to (18) as implicants $\mathbf{P}_{ij}$, are represented as follows:

$\mathbf{R}1 = \{(4 - - - - - \{2\})\}$

$\mathbf{P}_{11} = 1 \wedge \mathbf{v}_1^4$

$\mathbf{R}2 = \{(--4-2-\{1\}), (----7-\{1\}), (---7--\{2\})\}$

$\mathbf{P}_{21} = 2 \wedge (v_3^4 \wedge v_5^2),\ \mathbf{P}_{22} = 2 \wedge \mathbf{v}_5^7,\ \mathbf{P}_{23} = 2 \wedge \mathbf{v}_4^7$

$\mathbf{R}3 = \{(-----4\{1\}), (7-----\{2\}), (--6---\{2\}), (----0-\{1\})\}$

$\mathbf{P}_{31} = 3 \wedge \mathbf{v}_6^4,\ \mathbf{P}_{32} = 3 \wedge \mathbf{v}_1^7,\ \mathbf{P}_{33} = 3 \wedge \mathbf{v}_3^6,\ \mathbf{P}_{43} = 3 \wedge v_5^0$

$\mathbf{R}4 = \{(--0---\{2\}), (----1-\{1,2\})\}$

$\mathbf{P}_{41} = 4 \wedge \mathbf{v}_3^0,\ \mathbf{P}_{42} = 4 \wedge \mathbf{v}_5^1$

$\mathbf{R}5 = \{(-----5\{1,2\}), (-26---\{2\}), (----6-\{2\}), (-----0\{1\})\}$

$\mathbf{P}_{51} = 5 \wedge \mathbf{v}_6^5,\ \mathbf{P}_{52} = 5 \wedge (\mathbf{v}_2^2 \wedge \mathbf{v}_3^6),\ \mathbf{P}_{53} = 5 \wedge \mathbf{v}_5^6,\ \mathbf{P}_{54} = 5 \wedge \mathbf{v}_6^0$

$\mathbf{R}6 = \{(--7---\{1,2\}), (04----\{2\}), (06----\{1\}), (-7----\{1\}), (0-4---\{2\})\}$

$\mathbf{P}_{61} = 6 \wedge \mathbf{v}_3^7,\ \mathbf{P}_{62} = 6 \wedge (\mathbf{v}_1^0 \wedge \mathbf{v}_3^4),$

$\mathbf{P}_{63} = 6 \wedge (v_1^0 \wedge \mathbf{v}_2^6),\ \mathbf{P}_{64} = 6 \wedge \mathbf{v}_2^7,\ \mathbf{P}_{65} = 6 \wedge (\mathbf{v}_1^0 \wedge \mathbf{v}_3^4)$

$\mathbf{R}7 = \{(7---6-\{1,2\}), (1-----\{1\})\}$

$\mathbf{P}_{71} = 7 \wedge (\mathbf{v}_1^7 \wedge \mathbf{v}_5^6) = \mathbf{v}_1^7 \wedge \mathbf{v}_5^6,\ \mathbf{P}_{72} = 7 \wedge \mathbf{v}_1^1 = \mathbf{v}_1^1$

$\mathbf{R}7 = \{(7---6-\{1,2\}), (1-----\{1\})\}$

$\mathbf{P}_{71} = 7 \wedge (\mathbf{v}_1^7 \wedge \mathbf{v}_5^6) = \mathbf{v}_1^7 \wedge \mathbf{v}_5^6,\ \mathbf{P}_{72} = 7 \wedge \mathbf{v}_1^1 = \mathbf{v}_1^1$

According to (19) we have the representation of the system $\mathbf{F} = \{\mathbf{f}_1, \mathbf{f}_2\}$ of partial **8**-valued logical functions as the sum-of-product (SOP) system

$$\begin{aligned}
\mathbf{f}_1 &= \mathbf{P}_{21} \vee \mathbf{P}_{22} \vee \mathbf{P}_{31} \vee \mathbf{P}_{34} \vee \\
&\quad \mathbf{P}_{42} \vee \mathbf{P}_{51} \vee \mathbf{P}_{54} \vee \mathbf{P}_{61} \vee \mathbf{P}_{63} \vee \mathbf{P}_{64} \vee \mathbf{P}_{71} \vee \mathbf{P}_{72} = \\
&= (2 \wedge (v_3^4 \wedge v_5^2)) \vee (2 \wedge v_5^7) \vee (3 \wedge v_6^4) \vee (3 \wedge v_5^0) \vee (4 \wedge v_5^1) \vee \\
&\quad \vee (5 \wedge v_6^5) \vee (5 \wedge v_6^0) \vee (6 \wedge v_3^7) \vee \vee (6 \wedge (v_1^0 \wedge v_2^6)) \\
&\quad \vee (6 \wedge v_2^7) \vee (v_1^7 \wedge v_5^6) \vee v_1^1
\end{aligned}$$

$$\begin{aligned}
\mathbf{f}_2 &= \mathbf{P}_{11} \vee \mathbf{P}_{23} \vee \mathbf{P}_{32} \vee \mathbf{P}_{33} \\
&\quad \vee \mathbf{P}_{41} \vee \mathbf{P}_{42} \vee \mathbf{P}_{51} \vee \\
&\quad \mathbf{P}_{52} \vee \mathbf{P}_{53} \vee \mathbf{P}_{61} \vee \mathbf{P}_{62} \vee \mathbf{P}_{65} \vee \mathbf{P}_{71} = \\
&= (1 \wedge v_1^4) \vee (2 \wedge v_5^7) \vee (3 \wedge v_1^7) \vee (3 \wedge v_3^6) \vee (4 \wedge v_3^0) \vee (4 \wedge v_5^1) \vee \\
&\quad (5 \wedge v_6^5) \vee (5 \wedge (v_1^2 \wedge v_3^6)) \vee \\
&\quad (5 \wedge v_5^6) \vee (6 \wedge v_3^7) \vee (6 \wedge (v_1^0 \wedge v_2^4)) \vee (6 \wedge (v_1^0 \wedge v_2^6)) \vee (v_1^7 \wedge v_5^6)
\end{aligned}$$

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Balance equations for an interface between two phases and a contact line between three phases

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Interfacial region and three-phase line region are considered as two-dimensional and one-dimensional continua. Equations of the linear momentum balance generalize the Laplace equation for surfaces and the Young equation for lines.

Mathematical preliminaries

In this section we recall mathematical tools [1] which will be used in the paper.

The well-known three-dimensional Gauss divergence theorem reads

$$\iiint_V \boldsymbol{\nabla} \cdot \mathbf{T} \, \mathrm{d}V = \iint_A \mathbf{n} \cdot \mathbf{T} \, \mathrm{d}A, \tag{1}$$

where $\mathbf{T}$ is a tensor field, V is a volume with a boundary surface A having the outer unit normal vector $\mathbf{n}$, $\boldsymbol{\nabla}$ is the gradient operator.

In a case of surface Σ with a boundary line L the two-dimensional Gauss divergence theorem has the form

$$\iint_\Sigma \boldsymbol{\nabla}_\Sigma \cdot \mathbf{T} \, \mathrm{d}\Sigma = \int_L \mathbf{N} \cdot \mathbf{T} \, \mathrm{d}L - \iint_\Sigma 2H\mathbf{n} \cdot \mathbf{T} \, \mathrm{d}\Sigma, \tag{2}$$

where $\mathbf{N}$ is the outer unit normal to L lying in a tangent plane to Σ, H is the mean curvature of a surface, $\boldsymbol{\nabla}_\Sigma$ is the surface gradient operator.

The one-dimensional Gauss divergence theorem appears as

$$\int_L \boldsymbol{\nabla}_L \cdot \mathbf{T} \, \mathrm{d}L = (\boldsymbol{\lambda} \cdot \mathbf{T})_-^+ - \int_L \kappa \boldsymbol{\mu} \cdot \mathbf{T} \, \mathrm{d}L, \tag{3}$$

where $\boldsymbol{\nabla}_L$ is the line gradient operator, $\boldsymbol{\lambda}$ the unit tangential vector, $\boldsymbol{\mu}$ the principal normal vector, κ denotes the curvature of the curve L. The symbol f_-^+ means the difference between the values of a function f at the end and initial points of a curve.

The Lagrangian time derivative of the material volume element is expressed in terms of divergence of the velocity $\mathbf{v}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}V) = (\boldsymbol{\nabla} \cdot \mathbf{v})\mathrm{d}V. \tag{4}$$

Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}\Sigma) = (\boldsymbol{\nabla}_\Sigma \cdot \mathbf{v}_\Sigma)\mathrm{d}\Sigma \tag{5}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}L) = (\boldsymbol{\nabla}_L \cdot \mathbf{v}_L)\mathrm{d}L. \tag{6}$$

Balance equations for an interface between two phases

Consider a material volume

$$V = V_1 \cup V_2 \cup \Sigma \tag{7}$$

with a boundary $A_1 \cup A_2$ containing two-dimensional interface Σ with a boundary L separating surfaces A_1 and A_2. Any extensive quantity Ψ characterizing this volume can be written as the sum

$$\Psi = \iiint_{V_1} \rho_1 \psi_1 \,\mathrm{d}V + \iiint_{V_2} \rho_2 \psi_2 \,\mathrm{d}V + \iint_{\Sigma} \rho_\Sigma \psi_\Sigma \,\mathrm{d}\Sigma, \tag{8}$$

where ρ_1 and ρ_2 are mass densities, ψ_1 and ψ_2 are densities of Ψ per unit mass in V_1 and V_2; ρ_Σ is the surface mass density, ψ_Σ is treated either in accordance with Gibbs' approach as the surface excess density associated with the dividing surface [2] or presentation (8) can be substantiated by the measure theory [3–5].

The time change of the extensive quantity (8) is defined by the productions Δ_1, Δ_2, Δ_Σ and the fluxes $\mathbf{J}_1$, $\mathbf{J}_2$, $\mathbf{J}_\Sigma$:

$$\begin{gathered}\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \iiint_{V_1} \rho_1 \Delta_1 \,\mathrm{d}V + \iiint_{V_2} \rho_2 \Delta_2 \,\mathrm{d}V + \iint_{\Sigma} \rho_\Sigma \Delta_\Sigma \,\mathrm{d}\Sigma - \\ - \iint_{A_1} \mathbf{n}_1 \cdot \mathbf{J}_1 \,\mathrm{d}A - \iint_{A_2} \mathbf{n}_2 \cdot \mathbf{J}_2 \,\mathrm{d}A - \int_L \mathbf{N} \cdot \mathbf{J}_\Sigma \,\mathrm{d}L,\end{gathered} \tag{9}$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the outer unit normals to the boundaries A_1 and A_2.

Since the equation

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V_i} \rho_i \psi_i \,\mathrm{d}V = \iiint_{V_i} \rho_i \Delta_i \,\mathrm{d}V - \iint_{A_i \cup \Sigma} \mathbf{n}_i \cdot \mathbf{J}_i \,\mathrm{d}A - \\ - \iint_{A_{12}} \rho_i \psi_i \left(\mathbf{v}_i - \mathbf{v}_\Sigma\right) \cdot \mathbf{n}_i \,\mathrm{d}\Sigma, \qquad i = 1,\ 2\end{gathered} \tag{10}$$

is valid for every volume V_i, we obtain

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\Sigma} \rho_\Sigma \psi_\Sigma \,\mathrm{d}A = \iint_{\Sigma} \rho_\Sigma \Delta_\Sigma \,\mathrm{d}A - \int_L \mathbf{N} \cdot \mathbf{J}_\Sigma \,\mathrm{d}L + \\ + \iint_{\Sigma} \left(\mathbf{n}_1 \cdot \mathbf{J}_1 + \mathbf{n}_2 \cdot \mathbf{J}_2\right) \,\mathrm{d}\Sigma + \\ + \iint_{\Sigma} \left[\rho_1 \psi_1 \left(\mathbf{v}_1 - \mathbf{v}_\Sigma\right) \cdot \mathbf{n}_1 + \rho_2 \psi_2 \left(\mathbf{v}_2 - \mathbf{v}_\Sigma\right) \cdot \mathbf{n}_2\right] \,\mathrm{d}\Sigma.\end{gathered} \tag{11}$$

Differentiating the surface integral in the left-hand side of (11) with respect to time t and using the surface divergence theorem to reduce the line integral around L to the surface integral over Σ and utilizing the two-dimensional balance equation of the surface mass density

$$\frac{\mathrm{d}\rho_\Sigma}{\mathrm{d}t} + \rho_\Sigma \boldsymbol{\nabla}_\Sigma \cdot \mathbf{v}_\Sigma = \rho_1 \left(\mathbf{v}_1 - \mathbf{v}_\Sigma\right) \cdot \mathbf{n}_1 + \rho_2 \left(\mathbf{v}_2 - \mathbf{v}_\Sigma\right) \cdot \mathbf{n}_2 \tag{12}$$

we arrive at the balance equation for the quantity ψ_Σ at a propagating interface

$$\begin{gathered}\rho_\Sigma \frac{\mathrm{d}\psi_\Sigma}{\mathrm{d}t} = \rho_\Sigma \Delta_\Sigma - \boldsymbol{\nabla}_\Sigma \cdot \mathbf{J}_\Sigma + \mathbf{n}_1 \cdot \mathbf{J}_1 + \mathbf{n}_2 \cdot \mathbf{J}_2 + \\ + \rho_1 (\psi_1 - \psi_\Sigma) \left(\mathbf{v}_1 - \mathbf{v}_\Sigma\right) \cdot \mathbf{n}_1 + \rho_2 (\psi_2 - \psi_\Sigma) \left(\mathbf{v}_2 - \mathbf{v}_\Sigma\right) \cdot \mathbf{n}_2\end{gathered} \tag{13}$$

which can describe phase transition and wave propagation.

Balance equations for a contact line between three phases

Consider a material volume

$$V = V_1 \cup V_2 \cup V_3 \cup \Sigma_{12} \cup \Sigma_{13} \cup \Sigma_{23} \cup L \tag{14}$$

with a boundary $A_1 \cup A_2 \cup A_3$ containing three two-dimensional interfaces Σ_{12}, Σ_{13} and Σ_{23} with boundaris L_{12}, L_{13} and L_{23} separating surfaces A_1, A_1 and A_3. This volume also contains the three phase contact line L.

Any extensive quantity Ψ characterizing the considered material volume can be written as the sum

$$\Psi = \iiint_{V_1} \rho_1\psi_1 \, \mathrm{d}V + \iiint_{V_2} \rho_2\psi_2 \, \mathrm{d}V + \iiint_{V_3} \rho_3\psi_3 \, \mathrm{d}V + \iint_{\Sigma_{12}} \rho_{12}\psi_{12} \, \mathrm{d}\Sigma +$$
$$+ \iint_{\Sigma_{13}} \rho_{13}\psi_{13} \, \mathrm{d}\Sigma + \iint_{\Sigma_{23}} \rho_{23}\psi_{23} \, \mathrm{d}\Sigma + \int_L \rho_L\psi_L \, \mathrm{d}L. \tag{15}$$

For a volume V_1 and an interface Σ_{12} the following formulae

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V_1} \rho_1\psi_1 \, \mathrm{d}V = \iiint_{V_1} \rho_1\Delta_1 \, \mathrm{d}V - \iint_{A_1 \cup \Sigma_{12} \cup \Sigma_{13}} \mathbf{n}_1 \cdot \mathbf{J}_1 \, \mathrm{d}A -$$
$$- \iint_{\Sigma_{12}} \rho_1\psi_1 \left(\mathbf{v}_1 - \mathbf{v}_{12}\right) \cdot \mathbf{n}_1 \, \mathrm{d}\Sigma - \iint_{\Sigma_{13}} \rho_1\psi_1 \left(\mathbf{v}_1 - \mathbf{v}_{13}\right) \cdot \mathbf{n}_1 \, \mathrm{d}\Sigma \tag{16}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\Sigma_{12}} \rho_{12}\psi_{12} \, \mathrm{d}A = \iint_{\Sigma_{12}} \rho_{12}\Delta_{12} \, \mathrm{d}A + \iint_{\Sigma_{12}} \left(\mathbf{n}_1 \cdot \mathbf{J}_1 + \mathbf{n}_2 \cdot \mathbf{J}_2\right) \, \mathrm{d}\Sigma +$$
$$+ \iint_{\Sigma_{12}} \left[\rho_1\psi_1 \left(\mathbf{v}_1 - \mathbf{v}_{12}\right) \cdot \mathbf{n}_1 + \rho_2\psi_2 \left(\mathbf{v}_2 - \mathbf{v}_{12}\right) \cdot \mathbf{n}_2\right] \, \mathrm{d}\Sigma -$$
$$- \int_{L \cup L_{12}} \mathbf{N_{12}} \cdot \mathbf{J}_{12} \, \mathrm{d}L - \int_L \rho_{12}\psi_{12} \left(\mathbf{v}_{12} - \mathbf{v}_L\right) \cdot \mathbf{N_{12}} \, \mathrm{d}L \tag{17}$$

hold. Using (16), (17) and analogous relations for V_2, V_3, Σ_{13} and Σ_{13} we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_L \rho_L\psi_L \, \mathrm{d}L = \int_L \rho_L\Delta_L \, \mathrm{d}L - \left(\boldsymbol{\lambda} \cdot \mathbf{J}_L\right)_-^+ +$$
$$+ \int_L \rho_{12}\psi_{12} \left(\mathbf{v}_{12} - \mathbf{v}_L\right) \cdot \mathbf{N}_{12} \, \mathrm{d}L +$$
$$+ \int_L \rho_{13}\psi_{13} \left(\mathbf{v}_{13} - \mathbf{v}_L\right) \cdot \mathbf{N}_{13} \, \mathrm{d}L + \int_L \rho_{23}\psi_{23} \left(\mathbf{v}_{23} - \mathbf{v}_L\right) \cdot \mathbf{N}_{23} \, \mathrm{d}L +$$
$$+ \int_L \left[\mathbf{N}_{12} \cdot \mathbf{J}_{12} + \mathbf{N}_{13} \cdot \mathbf{J}_{13} + \mathbf{N}_{23} \cdot \mathbf{J}_{23}\right] \, \mathrm{d}L. \tag{18}$$

After differentiating the line integral on the left-hand side of (18) with respect to time and using the one-dimensional divergence theorem (the Newton–Leibniz formula) and one-dimensional balance equation for the line mass density

$$\frac{\mathrm{d}\rho_L}{\mathrm{d}t} + \rho_L \boldsymbol{\nabla}_L \cdot \mathbf{v}_L = \rho_{12} \left(\mathbf{v}_{12} - \mathbf{v}_L\right) \cdot \mathbf{N}_{12} +$$
$$+ \rho_{13} \left(\mathbf{v}_{13} - \mathbf{v}_L\right) \cdot \mathbf{N}_{13} + \rho_{23} \left(\mathbf{v}_{23} - \mathbf{v}_L\right) \cdot \mathbf{N}_{23} \tag{19}$$

we arrive at the following balance equation

$$\rho_L \frac{\mathrm{d}\psi_L}{\mathrm{d}t} = \rho_L\Delta_L - \boldsymbol{\nabla}_L \cdot \mathbf{J}_L + \mathbf{N}_{12} \cdot \mathbf{J}_{12} + \mathbf{N}_{13} \cdot \mathbf{J}_{13} + \mathbf{N}_{23} \cdot \mathbf{J}_{23} +$$
$$+ \rho_{12}(\psi_{12} - \psi_L) \left(\mathbf{v}_{12} - \mathbf{v}_L\right) \cdot \mathbf{N}_{12} + \rho_{13}(\psi_{13} - \psi_L) \left(\mathbf{v}_{13} - \mathbf{v}_L\right) \cdot \mathbf{N}_{13} +$$
$$+ \rho_{23}(\psi_{23} - \psi_L) \left(\mathbf{v}_{23} - \mathbf{v}_L\right) \cdot \mathbf{N}_{23}. \tag{20}$$

Applications of the results

Identifying the quantity ψ with the displacement velocity vector $\mathbf{v}$, the production Δ with the body force vector $\mathbf{f}$ and the flux $\mathbf{J}$ with the stress tensor $\boldsymbol{\sigma}$ (with the opposite sign) we obtain the generalization of the Laplace equation

$$\rho_\Sigma \frac{\mathrm{d}\mathbf{v}_\Sigma}{\mathrm{d}t} = \rho_\Sigma \mathbf{f}_\Sigma + \boldsymbol{\nabla}_\Sigma \cdot \boldsymbol{\sigma}_\Sigma - \mathbf{n}_1 \cdot \boldsymbol{\sigma}_1 + \mathbf{n}_2 \cdot \boldsymbol{\sigma}_2 +$$

$$+\rho_1 (\mathbf{v}_1 - \mathbf{v}_\Sigma)\,(\mathbf{v}_1 - \mathbf{v}_\Sigma) \cdot \mathbf{n}_1 + \rho_2 (\mathbf{v}_2 - \mathbf{v}_\Sigma)\,(\mathbf{v}_2 - \mathbf{v}_\Sigma) \cdot \mathbf{n}_2 \tag{21}$$

and the generalization of the Young equation

$$\rho_L \frac{\mathrm{d}\mathbf{v}_L}{\mathrm{d}t} = \rho_L \mathbf{f}_L + \boldsymbol{\nabla}_L \cdot \boldsymbol{\sigma}_L - \mathbf{N}_{12} \cdot \boldsymbol{\sigma}_{12} - \mathbf{N}_{13} \cdot \boldsymbol{\sigma}_{13} - \mathbf{N}_{23} \cdot \boldsymbol{\sigma}_{23} +$$

$$+\rho_{12} (\mathbf{v}_{12} - \mathbf{v}_L)\,(\mathbf{v}_{12} - \mathbf{v}_L) \cdot \mathbf{N}_{12} + \rho_{13} (\mathbf{v}_{13} - \mathbf{v}_L)\,(\mathbf{v}_{13} - \mathbf{v}_L) \cdot \mathbf{N}_{13} +$$

$$+\rho_{23} (\mathbf{v}_{23} - \mathbf{v}_L)\,(\mathbf{v}_{23} - \mathbf{v}_L) \cdot \mathbf{N}_{23}. \tag{22}$$

Indeed, in a case

$$\rho_\Sigma = 0, \quad (\mathbf{v}_1 - \mathbf{v}_\Sigma) \cdot \mathbf{n}_1 = 0, \quad (\mathbf{v}_2 - \mathbf{v}_\Sigma) \cdot \mathbf{n}_2 = 0, \quad \boldsymbol{\sigma}_\Sigma = \sigma_\Sigma \mathbf{a}_\Sigma,$$

where σ_Σ is the surface tension, $\mathbf{a}_\Sigma$ is the first fundamental form of a surface, we obtain from (21):

$$\boldsymbol{\nabla}_\Sigma \sigma_\Sigma + 2H\sigma_\Sigma \mathbf{n}_1 = \mathbf{n}_1 \cdot \boldsymbol{\sigma}_1 + \mathbf{n}_2 \cdot \boldsymbol{\sigma}_2. \tag{23}$$

Here H is the mean curvature of an interface.

Equation (23) is a basis of theoretical investigation of various physical phenomena caused by heterogeneous surface tension including wetting of heterogeneous surfaces and interaction of surface-active melts with metals [6].

Finally, if $\sigma_\Sigma =$const, then we arrive at the classical Laplace equation

$$2H\sigma_\Sigma = p_2 - p_1, \tag{24}$$

where p_1 and p_2 are the hydrostatic pressures in contacting bulk phases.

When

$$(\mathbf{v}_{12} - \mathbf{v}_L) \cdot \mathbf{N}_{12} = 0, \quad (\mathbf{v}_{13} - \mathbf{v}_L) \cdot \mathbf{N}_{13} = 0, \quad (\mathbf{v}_{23} - \mathbf{v}_L) \cdot \mathbf{N}_{23} = 0,$$

$$\boldsymbol{\sigma}_{12} = \sigma_{12} \mathbf{a}_{12}, \quad \boldsymbol{\sigma}_{13} = \sigma_{13} \mathbf{a}_{13}, \quad \boldsymbol{\sigma}_{23} = \sigma_{23} \mathbf{a}_{23},$$

$$\rho_L = 0, \quad \boldsymbol{\sigma}_L = \sigma_L \boldsymbol{\lambda} \otimes \boldsymbol{\lambda},$$

we have

$$\mathbf{N}_{12}\sigma_{12} + \mathbf{N}_{13}\sigma_{13} + \mathbf{N}_{23}\sigma_{23} - \frac{\partial \sigma_L}{\partial s} \boldsymbol{\lambda} - \kappa \sigma_L \boldsymbol{\mu} = 0. \tag{25}$$

For a plane interface and $\sigma_L = 0$ the classical Young equation for the contact angle θ follows from (25):

$$\sigma_{23} \cos\theta = \sigma_{13} - \sigma_{12}. \tag{26}$$

Further applications can be also found in the investigations [6-15].

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Automorphisms of Witt rings of elementary type

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Let $W = (R, G)$ be an abstract Witt ring in terminology of Marshall [3], where G denotes a subgroup of multiplicative group R^*. In this abstract case a Harrison automorphism of Witt ring $W = (R, G)$ is a ring automorphism σ of R, such that $\sigma(a) \in G$ for all $a \in G$. The purpose of the paper will be to describe all Harrison automorphisms of reduced Witt rings with the group G finite and to give a descriptions of the automorphisms of the other Witt rings of elementary type in some special cases.

1 Witt rings and associated quaternionic structures

There are three ways to look at automorphisms of Witt rings. The first way is to consider the usual ring automorphisms in the ring theory, the second one - the automorphisms of Witt ring $W(F)$ over the field F which arises from automorphisms of the field F. In this note we investigate the third type of automorphisms of Witt rings - that one which preserves dimension of forms (called here *Harrison automorphisms*).

In the paper we will consider abstract Witt rings and associated quaternionic structures introduced and described in details in [3]. Our goal is to describe all automorphisms of those Witt rings which are group rings (a special case of Witt rings of elementary type). According to Marshall the category of quaternionic structures is equivalent to the category of Witt rings. It follows that an isomorphism of two quaternionic structures determines the Harrison automorphism of corresponding Witt rings and conversely, every Harrison automorphism of Witt rings determines the isomorphism of associated quaternionic structures. These facts allows us to use quaternionic structures in order to prove facts about Harrison automorphisms of Witt rings when it is convenient. We introduce some special notations, but most of our notation and terminology will follow what is used in [3].

For the convenience of the reader we state fundamental notions and facts concerning Witt rings and quaternionic structures. A *Witt ring* is a pair $W = (R, G)$, where R is a commutative ring with unity 1 and G is a subgroup of the multiplicative group of units R^* which has exponent 2 and contains -1. In all this paper we assume that G is finite. Let I denotes the fundamental ideal of R generated by elements of the form $x + y$, where $x, y \in G$ and by I^k we denote the k-th power of I. Moreover the following three axioms hold:

$\mathbf{W_1}$: G generates R additively, that means every element of R is of the form $a = a_1 + \cdots + a_n$ with $a_1, a_2, \ldots, a_n \in G$ and $n \geq 1$.

$\mathbf{W_2}$: The following Arason-Pfister property holds for $k = 1$ and 2.
If $a = a_1 + \cdots + a_n \in I^k$ with $n < 2^k$, then $a = 0$.

$\mathbf{W_3}$: If $a_1+\cdots+a_n = b_1+\cdots+b_n$ and $n \geq 3$, then there exist $a, b, c_3, \ldots, c_n \in G$ such that $a_2+\cdots+a_n = a+c_3+\cdots+c_n$, $a_1+a = b_1+b$ (and, hence $b_2+\cdots+b_n = b+c_3+\cdots+c_n$).

Let $\sigma \in \mathrm{Aut}(R)$. If $\sigma(a) \in G$ for all $a \in G$, then σ will be called a *Harrison automorphism* of the Witt ring $W = (R, G)$.

Now let G be a group of exponent 2 (i.e., $a^2 = 1$ for all $a \in G$) with distinguished element -1, with $-a := -1 \cdot a$, Q be a pointed set with distinguished element θ, and let $q \colon G \times G \to \mathrm{Q}$ be a surjective mapping.

The triplet (G, Q, q) is said to be a quaternionic structure (Q-structure, for short), if q satisfies:

$\mathbf{Q}_1$: $q(a, b) = q(b, a)$

$\mathbf{Q}_2$: $q(a, -a) = \theta$

$\mathbf{Q}_3$: $q(a, b) = q(a, c) \Leftrightarrow q(a, bc) = \theta$

$\mathbf{Q}_4$: $q(a, b) = q(c, d) \Rightarrow$ there exists $x \in G$ such that $q(a, b) = q(a, x)$ and $q(c, d) = q(c, x)$, where $a, b, c, d \in G$.

The category of Witt rings and the category of Q-structures are naturally equivalent ([3], Th. 4.5). This means that for given an abstract Witt ring one can construct a quaternionic structure (G, Q, q) associated to it and conversely, for given a quaternionic structure (G, Q, q), there exists a Witt ring R with the quaternionic structure (G', Q', q') isomorphic to (G, Q, q).

For $W = (R, G)$ an arbitrary Witt ring we define Q to be the subset of R consisting of all elements $(1-a)(1-b)$, where $a, b \in G$ and the mapping $q \colon G \times G \to \mathrm{Q}$ by $q(a, b) = (1-a)(1-b)$. The triplet (G, Q, q) is the Q-structure associated to W (cf. [3], Prop. 4.2).

Conversely, let (G, Q, q) be a quaternionic structure. Define R to be the quotient of the integral group ring $\mathbb{Z}[G]$ obtained by factoring by the ideal J generated by $[1] + [-1]$ and all elements $([1] - [a])([1] - [b])$, where $a, b \in G$ satisfy $1 \in D\langle a, b\rangle$ and by $[a]$ we denote an element of $\mathbb{Z}[G]$ associated to $a \in G$. Then $W = (R, G)$ is a Witt ring associated to quaternionic structure (G, Q, q).

Let $\mathcal{G} = (G, \mathrm{Q}, q)$ and $\mathcal{G}' = (G', \mathrm{Q}', q')$ be quaternionic structures and let $\varphi \colon G \to G'$ be a group isomorphism with $\varphi(-1) = -1'$. If for every $a, b \in G$, $q(a, b) = \theta \Leftrightarrow q'(\varphi(a), \varphi(b)) = \theta'$ then φ is said to be a Q-*isomorphism.* The structures will be called *equivalent* and we shall write $(G, \mathrm{Q}, q) \sim (G', \mathrm{Q}', q')$. A Q-isomorphism σ of the structure (G, Q, q) onto itself, is said to be a Q-*automorphism* or an *automorphism of the quaternionic structure* (G, Q, q). The group of all Q-automorphisms of (G, Q, q) will be denoted by $\mathrm{Aut}(G, \mathrm{Q}, q)$.

If (G, Q, q) is the quaternionic structure of a Witt ring $W = (R, G)$ and $\sigma \in \mathrm{Aut}(G, \mathrm{Q}, q)$, then the mapping $a_1 + \cdots + a_n \to \sigma(a_1) + \cdots + \sigma(a_n)$ is well defined Harrison automorphism of W. This establishes a cannonical isomorphism of the group of all Q-automorphisms of (G, Q, q) with the group $\mathrm{Aut}_H(W)$ of all Harrison automorphisms of W.

2 Automorphisms of group Witt rings

Let (S, G_S) be a Witt ring and consider a group ring $S[\Delta]$. According to [3], Prop. 5.16 we get a Witt ring $R = (S[\Delta], G_S \times \Delta)$ (group Witt ring). Here we denote: $R := S[\Delta]$

and $G_R := G_S \times \Delta$. According to Marshall [3], we can assume that all basic elements of group Witt ring (R, G_R) are in the group G_S. Let us consider the associated Q-structure (G_R, Q_R, q_R). First notice, that $G_R = G_S \times \Delta$, next the surjective quaternionic mapping q_R can be defined by

$$q_R\colon (G_S \times \Delta) \times (G_S \times \Delta) \to G_S \oplus (G_S \otimes \Delta) \oplus (\Delta \wedge \Delta),$$

$$q_R((g_1, d_1), (g_2, d_2)) = (q_S(g_1, g_2), g_1 \otimes d_2 * g_2 \otimes d_1, d_1 \wedge d_2).$$

Above $*$ denotes operation in the group $G_S \otimes \Delta$ and $\wedge$ is an outer product.

Let (S, G_S) be a Witt ring such that it can not be expressed as a group ring and assume that G_S is finite. By [1], Theorem 2, the ring (S, G_S) is basic, that means every element $b \in G_S$ is basic ($D\langle 1, b\rangle \neq \{1, b\}$ or $D\langle 1, -b\rangle \neq \{1, -b\}$). Let Δ be any group of exponent 2, $\varphi \in \mathrm{Aut}(\Delta)$ and let $f \in \mathrm{Aut}_{\mathrm{H}}(S)$. Consider a group ring $S[\Delta]$. By universal property of group ring (Prop. 1, 2, pp. 111-112 of [2]) it follows that there exists an automorphism $h \in \mathrm{Aut}(R)$, such that the following diagram commutes:

$$\begin{tikzcd}
\Delta \arrow[r, "\varphi"] \arrow[d, "\varphi_0"'] & \Delta \arrow[d, "\varphi_0"] \\
S[\Delta] \arrow[r, "h"] & S[\Delta] \\
S \arrow[u, "f_0"] \arrow[r, "f"'] & S \arrow[u, "f_0"']
\end{tikzcd}$$

where: $f \in \mathrm{Aut}_H(S[\Delta])$, $\varphi \in \mathrm{Aut}(\Delta)$ and $f_0\colon S \hookrightarrow S[\Delta]$, $\varphi_0\colon \Delta \hookrightarrow S[\Delta]$ are canonical embeddings $f_0(b) = b \cdot 1$ (or $f_0(b) = \langle b, 1\rangle$) and $\varphi_0(x) = 1 \cdot x$ (or $\varphi(x) = \langle 1, x\rangle$).
In this case the automorphism h has the following form:

$$h\colon R \to R, \qquad h(\alpha) = \sum_{x \in \Delta} (f_0 \circ f)(b_x) \cdot (\varphi_0 \circ \varphi)(x),$$

for all $\alpha \in S[\Delta]$ of form $\alpha = b_1 x_1 + \ldots + b_n x_n$.

Proposition 3. *The automorphism h defined above is a Harrison automorphism of the Witt ring $(S[\Delta], G_S \times \Delta)$.*

Proof. Consider the restriction of $h\colon R \to R$ to $G_R = G_S \times \Delta$.
Let $g \in G_R$. Then g has a form: $bx \in G_S \times \Delta$ and assume that $x \neq 1$ ($bx \notin G_S \times \{1\}$). Then $h(bx) = (f_0 \circ f)(b) \cdot (\varphi_0 \circ \varphi)(x) = f(b \cdot 1) \cdot \varphi(1 \cdot x) = (f(b) \ \cdot f(1))(\varphi(1) \cdot \ \varphi(x)) = (f(b) \ \cdot \ 1)(1 \ \cdot \ \varphi(x)) \in G_S \times \Delta$ Consequently $h(bx) \in G_S \times \Delta = G_R$, that means $h \in \mathrm{Aut}_H(R)$ as claimed.■

Although every automorphism constructed in above method is a Harrison automorphism of the group Witt ring (R, G_R), these are not all Harrison automorphisms of (R, G_R).

Theorem 4. *Let (S, G_S) be a Witt ring such that it can not be expressed as a group ring and let Δ be any finite group of exponent 2. Let next σ be an automorphism of quaternionic structure associated to Witt ring (S, G_S), let $\varphi \in \mathrm{Aut}(\Delta)$ and $\delta \in \mathrm{Hom}(\Delta, G_S)$. Then every automorphism of the quaternionic structure associated to group Witt ring $(S[\Delta], G_S \times \Delta)$ has a form*

$$f\colon G_S \times \Delta \to G_S \times \Delta \qquad f(g, d) = (\sigma(g) \cdot \delta(d), \varphi(d)), \tag{1}$$

Proof. Let $g_1, g_2 \in G_S$ and $d_1, d_2 \in \Delta$. Then $f((g_1, d_1) \cdot (g_2, d_2)) = f(g_1 g_2, d_1 d_2) = (\sigma(g_1 g_2) \cdot \delta(d_1 d_2), \varphi(d_1 d_2))$. On the other hand we obtain $f(g_1, d_1) \cdot f(g_2, d_2) = (\sigma(g_1) \cdot \delta(d_1), \varphi(d_1)) \cdot (\sigma(g_2) \cdot \delta(d_2), \varphi(d_2)) = ((\sigma(g_1) \cdot \delta(d_1)) \cdot (\sigma(g_2) \cdot \delta(d_2)), \varphi(d_1) \cdot \varphi(d_2))$, what is equivalent to the previous expression, hence f is a group homomorphism.

Next it is obvious that $\ker f = 1$, since $\ker f = \{(g, d)\colon (\sigma(g) \cdot \delta(d), \varphi(d)) = (1, 1)\} = \{(1, 1)\}$, what means that f is an injection.

From the finitness of G_S, and hence $G_S \times \Delta$ we obtain that f is a surjection and consequently it is an automorphism of the group G_R.

In order to prove that f is an automorphism of quaternionic structure associated to group Witt ring we consider two cases.

<u>Case 1</u>. Let $(g_1, d_1), (g_2, d_2) \in G_R$ are basic elements of Witt ring $S[\Delta]$ That means $d_1, d_2 = 1$, $(g_1, d_1)(g_2, d_2) \in G_S \times \{1\}$. We shall show that:

$$q_R((g_1, 1), (g_2, 1)) = \theta_R \Leftrightarrow q_R(f(g_1, 1), f(g_2, 1)) = \theta_R.$$

If $q_R((g_1, 1), (g_2, 1)) = \theta_R$, then

$$(q_S(g_1, g_2),\ g_1 \otimes 1 * g_2 \otimes 1,\ 1 \wedge 1) = (\theta_S, \theta_{G_S \otimes \Delta}, \theta_\Delta) \tag{2}$$

On the other hand we get:

$q_R(f(g_1, 1), f(g_2, 1)) =$
$= q_R((\sigma(g_1) \cdot \delta(1), \varphi(1)), (\sigma(g_1) \cdot \delta(1), \varphi(1))) =$
$= q_R((\sigma(g_1), 1), \sigma(g_2), 1)) =$

$$= (q_S(\sigma(g_1), \sigma(g_2)),\ \sigma(g_2) \otimes 1 * \sigma(g_2) \otimes 1,\ 1 \wedge 1) = (\theta_S, \theta_{G_S \otimes \Delta}, \theta_\Delta). \tag{3}$$

Clearly conditions (2) and (3) are equivalent, hence f fulfils the condition of Q-automorphism in this case.

<u>Case 2</u>. Let$(g_1, d_1) \in G_R$ be birigid element. That means $d_1 \neq 1$. Then $f(g_1, d_1) = (\sigma(g_1) \cdot \delta(d_1), \varphi(d_1))$ and $\varphi(d_1) \neq 1$. Therefore $f(g_1, d_1) \notin G_S \times \{1\}$, so it is birigid too, then $\mathrm{D}\langle (1, 1), f(g_1, d_1)\rangle = \{(1, 1), f(g_1, d_1)\}$. On the other hand we get $f(\mathrm{D}\langle (1, 1), (g_1, d_1)\rangle) = f(\{(1, 1), (g_1, d_1)\}) = \{(1, 1), f(g_1, d_1)\}$. Finally we have got that $\mathrm{D}\langle 1, f(a)\rangle = f(\mathrm{D}\langle 1, a)$ for all $a \in G_R$, what is equivalent to the condition of Q-automorphism of quaternionic structure associated to the ring $(S[\Delta], G_S \times \Delta)$.

Moreover it is clear that $f(-1_{G_R}, 1_\Delta) = (\sigma(-1) \cdot \delta(1), \varphi(1)) = (-1 \cdot 1, 1) = (-1, 1)$, what finishes the proof that f is an automorphism of the quaternionic structure associated to the Witt ring $(S[\Delta], G_S \times \Delta)$.

Conversely, let h be an automorphism of the quaternionic structure associated to Witt ring $(S[\Delta], G_S \times \Delta)$. We are going to show that h has the form (1) and that this assertion is unique, that means we are going to prove that there are not another automorphisms in $(S[\Delta], G_S \times \Delta)$ but that of form (1).

Denote the projections of h to the group G_S and Δ by Π_{G_S} and Π_Δ respectively. Next consider the restriction $h_{G_S\times\{1\}}\colon G_S\times\{1\}\to G_S\times\{1\}$ of the mapping h to the group $G_S\times\{1\}$ and the restriction $h_{\{1\}\times\Delta}\colon \{1\}\times\Delta\to G_S\times\Delta$ to the group $\{1\}\times\Delta$. Finally we define the mappings $i_{G_S}\colon G_S\times\{1\}\to G_S,\ i_{G_S}(g,1)=g$ and $i_\Delta\colon\Delta\to\{1\}\times\Delta,\ i_\Delta(d)=(1,d)$. Now we can construct the mapping $f\colon G_S\times\Delta\to G_S\times\Delta$ by:

$$f(g,d)=(\sigma(g)\cdot\delta(d),\varphi(d)),$$

where $\sigma=i_{G_S}\circ h_{G_S\times\{1\}}\circ i_{G_S}^{-1},\quad \delta=\Pi_{G_S}\circ h_{\{1\}\times\Delta}\circ i_\Delta,\quad \varphi=\Pi_\Delta\circ h_{\{1\}\times\Delta}\circ i_\Delta.$

Assume that $h(g,d)=(g',d')=h(g,1)\cdot h(1,d)$ and let $h(g,1)=(g_1,1),\ \ h(1,d)=(g_2,d_2)$. Then $g_1g_2=g'$ and $d_2=d'$.

Now we calculate $f(g,d)=(\sigma(g)\cdot\delta(d),\varphi(d))=$
$=((i_{G_S}\circ h_{G_S\times\{1\}}\circ i_{G_S}^{-1})(g)\cdot(\Pi_{G_S}\circ h_{\{1\}\times\Delta}\circ i_\Delta)(d),(\Pi_\Delta\circ h_{\{1\}\times\Delta}\circ i_\Delta)(d))=$
$=(i_{G_S}(h_{G_S\times\{1\}}(i_{G_S}^{-1}(g)))\cdot\Pi_{G_S}(h_{\{1\}\times\Delta}(i_\Delta(d))),\Pi_\Delta(h_{\{1\}\times\Delta}(i_\Delta(d))))=$
$=(i_{G_S}(h_{G_S\times\{1\}}(g,1))\cdot\Pi_{G_S}(h_{\{1\}\times\Delta}(1,d)),\Pi_\Delta(h_{\{1\}\times\Delta}(1,d)))=$
$=(i_{G_S}(g_1,1)\cdot\Pi_{G_S}(g_2,d_2),\Pi_\Delta(g_2,d_2))=$
$=(g_1g_2,d_2)=(g',d')=h(g,d).$

It follows that any automorphism h of quaternionic structure $(G_S\times\Delta,\mathrm{Q}_{G_S\times\Delta},q_{G_S\times\Delta})$ associated to Witt ring $S[\Delta]$ can be expressed in the form (1).

Let $f_1,f_2\in(\mathrm{Aut}_H(G_S\times\Delta))$ such that $f_1(g,d)=(\sigma_1(g)\cdot\delta_1(d),varphi_1(d))$ and $f_2(g,d)=(\sigma_2(g)\cdot\delta_2(d),\varphi_2(d))$. We shall shaw that two mappings f_1 and f_2 defined with the triplet (σ,δ,φ) are different at least in one element. For this consider an element $(a,1)\in G_S\times\Delta$. Then $f_1(a,1)=(\sigma_1(a),1)\neq(\sigma_2(a),1)=f_2(a,1)$, so $f_1\neq f_2$ are different if $\sigma_1\neq\sigma_2$. If $delta_1\neq\delta_2$ or $\varphi_1\neq\varphi_2$ then it suffices to consider the element $(1,d)\in G_S\times\Delta$, and therefore we get $f_1(1,d)=(\delta_1(d),\varphi_1(d))\neq(\delta_2(d),\varphi_2(d))=f_2(1,d)$. This proves that the triplet (σ,δ,φ) defines in unique way an automorphism of the quaternionic structure associated to Witt ring $(S[\Delta],G_S\times\Delta)$.■

Remark 5. In the case S can be expressed as a group ring for instance $S_1[\Delta_1]$, then we can consider the ring $S_1[\Delta_1][\Delta]\cong S_1[\Delta_1\times\Delta]$. Repeating this procedure we will obtain some ring S_n, which is basic (and fulfilles the hypothesis of the above Theorem).

It is easy to create a group structure in the set of automorphisms of the form (1) of quaternionic structure of the group Witt ring $(S[\Delta],G_S\times\Delta)$. If f_1,f_2 are a such automorphisms defined by the triplets of mappings $(\sigma_1,\delta_1,\varphi_1)$ and $(\sigma_2,\delta_2,\varphi_2)$ then the product f_1*f_2 in $\mathrm{Aut}(G_S\times\Delta,\mathrm{Q}_{G_S\times\Delta},q_{G_S\times\Delta})$ will be described by the mappings: $(\sigma_1\circ\sigma_2,(\sigma_1\circ\delta_2)\cdot(\delta_1\circ\varphi_2),\varphi_1\circ\varphi_2)$. The order of the group equals to

$$|(\mathrm{Aut}_H(G_S\times\Delta))|=|(\mathrm{Aut}_H(G_S)|\cdot|\mathrm{Aut}(\Delta)|\cdot|\mathrm{Hom}(\Delta,G_S)|$$

what follows from the uniqueness of the mappings f_i defined as in (1).

Finally we wiil describe the groups of automorphisms of two special examples of Witt rings with the distinguished group G of cardinality 2^n.

Consider $W_0=(R_0,G_0)$ the Witt ring realized by the field $\mathbb{R}$ of real numbers and the associated Q-structure (G_0,Q_0,q_0). Hence $G_0=\{1,-1\}$ and the quaternionic mapping q_0 is defined by $q_0\colon\{1,-1\}\times\{1,-1\}\to\{0,1\},\ q_0(a,b)=1$ if $a=b=-1$ and $q_0(a,b)=0$ in all other cases. Then obviously $|\mathrm{Aut}_H(W_0)|=|\mathrm{Aut}(G_0,\mathrm{Q}_0,q_0)|=1$.

1) Let now $W=(R,G)$ be a product of n copies of W_0. It can be shown (compare the proof of Th. 2.4 in [4]) that the group $\mathrm{Aut}_H(W)$ consists of all automorphisms of the

form $f(a_1, \dots, a_n) = (b_1, \dots, b_n)$, where $a_i \in G_0$ and $(b_1, \dots, b_n)$ are all permutations of elements a_i.

2) Let now $W = (R, G)$ be a group Witt ring, where $R = R_0[\Delta]$, $G = G_0 \times \Delta$ with a group Δ such that $|\Delta| = 2^{n-1}$. Then by the Theorem 4 every Harrison automorphism of Witt ring W corresponds to automorphism f (of the associated quaternionic structure (G, Q, q)) of form $f(g, d) = (g \cdot \delta(d), \varphi(d))$, where $g \in G_0$, $d \in \Delta$ and $\varphi \in \mathrm{Aut}(\Delta)$.

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The method of subsidiary constant

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The contribution is focused on a method of proof, namely the method of subsidiary constant. This method is used in case of need to prove a property for all elements of given set or characteristic - it is the same, because there exists the set $A = \{x, \varphi(x)\}$ for every characteristic φ.

Basic concepts and terminology

We use common symbols $\neg, \vee, \wedge, \rightarrow, \leftrightarrow, \forall, \exists$ for the logical conjunctions. The formulas are marked by φ, ψ, if we want emphasize the fact, that formula φ depends on variables $x_1, \ldots, x_n$, we write $\varphi(x_1, \ldots, x_n)$. The variable is called local variable, if it immediately follows the quantifier, otherwise it is called global variable. The language of the mathematical theory is the set of symbols:

variables	$x, y, z, \ldots, x_1, x_2, \ldots$
logical conjunctions	$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
quantifier	$\forall, \exists$
subsidiary symbols	$(,)$
symbols of relations (predicates)	$\mathbf{P}, \mathbf{Q}, \mathbf{R}$
symbols of operations	$\mathbf{f}, \mathbf{g}, \mathbf{h}, \ldots$
arbitrary elements	$\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots, x, y, z, \ldots$

In contrast to variable, that presents an exactly undetermined element of the allowed objects set, the arbitrary element presents particular element of the allowed objects set.

The formula is called close, if it doesn't contain any global variables. Mathematical theory $\mathbf{T}$ is the set of close formulas in theory language. The proof in the theory $\mathbf{T}$ is finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ of the theory $\mathbf{T}$ language, where every formula φ_i is the axiom of predicate calculus, or the axiom of the theory $\mathbf{T}$, or it can be deduced from preceding terms of sequence by Modus Ponens or by the rule of generalisation. [1]

If there exists a proof in the theory $\mathbf{T}$ with last term φ, then φ is called provable in the theory $\mathbf{T}$ or the theorem of the theory $\mathbf{T}$, it is marked $\mathbf{T} \vdash \varphi$. If formula φ is provable in the theory $\mathbf{T}$ with added premise ψ (ψ is a new axiom of the theory $\mathbf{T}$), we write $\mathbf{T}, \psi \vdash \varphi$.

[1]Modus Ponens: from φ and $\varphi \rightarrow \psi$ can be deduced ψ. The rule of generalisation: from $\varphi(x)$ can be deduced $(\forall x)\varphi(x)$. More about the proof in the theory can be found in [1], eventually [3], [4].

The method of subsidiary constant

We begin by two well-known theorems of the mathematical logic. The first one deal with the proof of inclusion of close formulas.

Theorem 1. ***[The theorem on deduction]*** *Let* $\boldsymbol{T}$ *be mathematical theory,. Let* φ, ψ *be closed formulas of the theory* $\boldsymbol{T}$*. Then*

$$\mathbf{T}, \varphi \vdash \psi \qquad \Leftrightarrow \qquad \mathbf{T} \vdash \varphi \to \psi.$$

The following theorem deals with the relation of arbitrary elements and variables in connection with the provable formula, which depends on those variables.

Theorem 2. ***[The theorem on constants]*** *Let* $\boldsymbol{T}$*,* $\varphi(x_1, \dots, x_n)$ *be formula of the theory* $\boldsymbol{T}$*. Let* $c_1, \dots, c_n$ *be different arbitrary elements, which don't occur in any axiom of the theory* $\boldsymbol{T}$*. Then*

$$\mathbf{T} \vdash \varphi(c_1, \dots, c_n) \qquad \Leftrightarrow \qquad \mathbf{T} \vdash \varphi(x_1, \dots, x_n).$$

If we are interested in the proof of the inclusion $\varphi \to \psi$, following can be deduced from the theorem on constants:

$$\begin{aligned} & \mathbf{T} \vdash \varphi(x_1, \dots, x_n) \to \psi(x_1, \dots, x_n) \quad \Leftrightarrow \\ \Leftrightarrow \quad & \mathbf{T} \vdash \varphi(c_1, \dots, c_n) \to \psi(c_1, \dots, c_n). \end{aligned}$$

Hence $\varphi(c_1, \dots, c_n), \psi(c_1, \dots, c_n)$ are closed formulas, the theorem on deduction can be applied on the issued inclusion:

$$\begin{aligned} & \mathbf{T} \vdash \varphi(c_1, \dots, c_n) \to \psi(c_1, \dots, c_n) \quad \Leftrightarrow \\ \Leftrightarrow \quad & \mathbf{T}, \varphi(c_1, \dots, c_n) \vdash \psi(c_1, \dots, c_n). \end{aligned}$$

So we are getting a following consequence of the theorem on deduction and the theorem on constants.

Theorem 3. ***[The method of subsidiary constant]*** *Let* $\boldsymbol{T}$ *be the mathematical theory,* $\varphi(x_1, \dots, x_n)$ *and* $\psi(x_1, \dots, x_n)$ *be formulas of the theory* $\boldsymbol{T}$*. Let* $c_1, \dots, c_n$ *be different arbitrary elements, which don't occur in any axiom of the theory* $\boldsymbol{T}$*. Then*

$$\begin{aligned} & \mathbf{T}, \varphi(c_1, \dots, c_n) \vdash \psi(c_1, \dots, c_n) \quad \Leftrightarrow \\ \Leftrightarrow \quad & \mathbf{T} \vdash (\forall x_1) \dots (\forall x_n)(\varphi(x_1, \dots, x_n) \to \psi(x_1, \dots, x_n)). \end{aligned}$$

The method of subsidiary constant is often used by mathematics during proving (it is usually not explicit said). If we've got to prove the formula $(\forall x)\varphi(x) \to \psi(x)$, we begin: "Let x be arbitrary fixed such, that $\varphi(x)$." What does it mean? We made premise $\varphi(x)$, or we added new axiom to the theory $\mathbf{T}$. At the same time we made some limits to the variable x. Word "fixed" indicates, that we cann't use quantifier for variable x during proving, that is, x have to behave like arbitrary element. Word "arbitrary" expresses the fact, that there wasn't doing any premise about x (besides $\varphi(x)$ naturally), so x cann't occur in any axiom of the theory $\mathbf{T}$. See [1] or [2].

The proof is completed by proving $\psi(x)$ in the theory $\mathbf{T}$ with added premise $\varphi(x)$, what is the application of the method of subsidiary constant.

Typical situations for using the method of subsidiary constant are so-called ε proofs at mathematical analysis, where we have to prove $(\forall\varepsilon > 0)\psi(x)$, or $(\forall\varepsilon)(\varepsilon > 0 \to \psi(x))$ as the case may be. The proof is again begun saying: “Let ε be arbitrary fixed positive number.”

Example

There’s a need to prove:

$$\text{If}\quad \lim_{x\to a} f(x) = b \quad\text{and}\quad \lim_{x\to a} g(x) = c, \quad\text{then}\quad \lim_{x\to a}(f(x) + g(x)) = b + c.$$

Symbolic record:[2]

$$\mathbf{T} \vdash (\forall\varepsilon)[\varepsilon > 0 \to (\exists\delta > 0)\dots \mid f(x) + g(x) - (b + c) \mid < \varepsilon] \tag{1}$$

Our problem is proving the formula in the square brackets for all ε. Again we begin with magic words: “let ε be arbitrary fixed positive number” and we prove right side of the conclusion in the square brackets for such ε:

$$\mathbf{T}, \varepsilon > 0 \vdash (\exists\delta > 0)\dots \mid f(x) + g(x) - (b + c) \mid < \varepsilon \tag{2}$$

There is an important change of ε properties. While it is local variable in (1), it is arbitrary element, which doesn’t occur in any axiom of the theory $\mathbf{T}$, in (2).

Next proceeded proof is well known:

- From premise $\lim_{x\to a} f(x) = b$ conclude $\mid f(x) - b \mid < \varepsilon$, for all x such, that $\mid x - a \mid < \delta_b$.
- From premise $\lim_{x\to a} g(x) = c$ conclude $\mid g(x) - c \mid < \varepsilon$, for all x such, that $\mid x - a \mid < \delta_c$.

It already suffices to realize now, that for $\delta_a = \min(\delta_b, \delta_c)$ is $\mid f(x) + g(x) - (b + c) \mid < \varepsilon$ proved, for all x such as $\mid x - a \mid < \delta_a$. In this way, it is proved (2) and because of The method of subsidiary constant also (1).

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[2] Because of better arrange, the symbolic record is shorted by 3 dots and it is necessary to add premisses to the theory $\mathbf{T}$: $\lim_{x\to a} f(x) = b$ a $\lim_{x\to a} g(x) = c$.

Busy period characteristics for single-server queueing systems with non-homogeneous demands

Oleg Tikhonenko, Konstantin Klimovich

$BM/G/1/\infty$ queueing system with non-homogeneous demands is discussed. Each demand of the system is characterized by l-dimensional random non-negative indication vector $\boldsymbol{\zeta} = (\zeta_1, ..., \zeta_l)$. Service time of the demand arbitrary depends on its indications. Denote as Γ_i the sum of i-th components of indication vectors of demands that were served during a busy period. Let $\Gamma(\mathbf{x})$ be the distribution function of the random vector $\mathbf{\Gamma} = (\Gamma_1, ..., \Gamma_l)$. The Laplace-Stieltjes transform $\gamma(\mathbf{s})$ of $\mathbf{\Gamma}$ random vector is determined. All known "classical" characteristics of a busy period follow from the formula for $\gamma(\mathbf{s})$.

Introduction

Consider $BM/G/1/\infty$ queue, where the notation BM defines a stationary Poisson process with batch arrivals. Let the rate of the Poisson process be a and the probability that the batch size equals k be $g_k = \mathbf{P}\{\nu = k\}$, where ν is the number of demands in an arbitrary batch, $k \geq 1$. Denote as $G(z) = \sum_{k=1}^{\infty} g_k z^k$ the generation function of the batch size. Suppose that each demand is characterized by l-dimensional random non-negative indication vector $\boldsymbol{\zeta} = (\zeta_1, ..., \zeta_l)$. In the case of $l = 1$ the single indication ζ is named the demand volume or demand length [1, 2].

We assume, that the common distribution of an indication vector $\boldsymbol{\zeta}$ and service time ξ of the demand is done by the next joint distribution function:

$$F(\mathbf{x}, t) = \mathbf{P}\{\zeta_1 < x_1, ..., \zeta_l < x_l, \xi < t\} = \mathbf{P}\{\boldsymbol{\zeta} < \mathbf{x}, \xi < t\},$$

where $\mathbf{x} = (x_1, ..., x_l)$ and x_i, $i = 1, ..., l$, are non-negative real numbers.

Let $L(\mathbf{x}) = F(\mathbf{x}, \infty)$ be the distribution function of demand l-dimensional indication vector, $B(t) = F(\boldsymbol{\infty}, t)$ be the distribution function of its service time, where $\boldsymbol{\infty} = (\infty, ..., \infty)$ is an l-dimensional vector. Denote as

$$\alpha(\mathbf{s}, q) = \int_0^{\infty} ... \int_0^{\infty} \int_0^{\infty} e^{-(\mathbf{s},\mathbf{x})-qt} dF(\mathbf{x}, t)$$

the $(l+1)$-dimensional Laplace-Stieltjes transformation (LST) of $F(\mathbf{x}, t)$ function, where $\mathbf{s} = (s_1, ..., s_l)$, Re $s_i > 0$, $i = 1, ..., l$, Re $q > 0$. Now we can determine mixed moments (if they exist) of the random vector $(\boldsymbol{\zeta}, \xi)$ or its arbitrary partial vector:

$$\mathbf{E}({\zeta_1}^{i_1} ... {\zeta_l}^{i_l} \xi^j) = (-1)^{i_1+...+i_l+j} \frac{\partial^{i_1+...+i_l+j} \alpha(\mathbf{s}, q)}{\partial s_1^{i_1} ... \partial s_l^{i_l} \partial q^j} \Bigg|_{\mathbf{s}=\mathbf{0}, q=0},$$

where $i_k, j = 1, 2, ...; k = 1, ..., l$; $\mathbf{0} = (0, ..., 0)$ is an l-dimensional vector.

Denote as Γ_i the sum of i-th components of indication vectors of demands which were served during a busy period of the system under consideration, $i = 1, ..., l$, $\mathbf{\Gamma} = (\Gamma_1, ..., \Gamma_l)$. Let $\Gamma(\mathbf{x}) = \mathbf{P}\{\Gamma_1 < x_1, ..., \Gamma_l < x_l\}$ be the distribution function of the random vector $\mathbf{\Gamma}$, $\mathbf{x} = (x_1, ..., x_l)$, $\gamma(\mathbf{s})$ be the l-dimensional LST of $\Gamma(\mathbf{x})$ distribution function, i.e. $\gamma(\mathbf{s}) = \int_0^\infty ... \int_0^\infty e^{-(\mathbf{s},\mathbf{x})} d\Gamma(\mathbf{x})$.

The main purpose of the paper is to determine the function $\gamma(\mathbf{s})$.

The equation for $\gamma(\mathrm{s})$ function

Further we shall use the additional event method [3]. Suppose, that $s_i > 0$, $i = 1, ..., l$ and each demand characterized by $\mathbf{x}$ indication vector is red with probability $e^{-(\mathbf{s},\mathbf{x})}$ ore blue with probability $1 - e^{-(\mathbf{s},\mathbf{x})}$ independently of other demands. Then the LST $\varphi(\mathbf{s}) = \alpha(\mathbf{s}, 0)$ of $L(\mathbf{x})$ distribution function has the sense of probability, that an arbitrary demand is red. Suppose, that independently of the system behaviour some events take place. These events named catastrophes make up the stationary Poisson process with the parameter $q > 0$. Then $\alpha(\mathbf{s}, q)$ is the probability, that an arbitrary demand is red and there will be no catastrophes during its service time. In this case $\gamma(\mathbf{s})$ is the probability, that only red demands will be served during the busy period.

It's obvious, that $\mathbf{\Gamma}$ random vector characteristics not depend on the order of demands service inside the busy period. Assume, that 1) the order of demands service is arbitrary inside any batch; 2) batches entering to the system during demand from the batch opening the busy period service are served in the opposite to their entering order and have an advantage over demands from the opening batch; 3) if service of some demand from the batch entering inside the busy period begins, then all demands of this batch will served in the first place.

Now we can connect with each batch coming inside the busy period the time interval from the moment of the first demand from this batch service beginning to the finish of service of all demand from this batch and batches entering later. This interval we shall name the busy period corresponding to the batch. We shall name a batch (coming inside of the busy period) good, if only red demands are served during its corresponding busy period. It's obviously, that an arbitrary "inside" batch is good with probability $\gamma(\mathbf{s})$ for the discipline under consideration.

Let $P(\mathbf{x})$ be the conditional probability, that only good batches come to the system during demand from the "opening" batch service under condition, that its indication vector is equal to $\mathbf{x}$. Let us introduce the notation $B(t|\boldsymbol{\zeta} = \mathbf{x}) = \mathbf{P}\{\xi < t|\boldsymbol{\zeta} = \mathbf{x}\}$. Then we have

$$P(\mathbf{x}) = \int_0^\infty \sum_{k=0}^\infty \frac{(at\gamma(\mathbf{s}))^k}{k!} e^{-at} dB(t|\boldsymbol{\zeta} = \mathbf{x}) =$$

$$= \int_0^\infty e^{-(a - a\gamma(\mathbf{s}))t} dB(t|\boldsymbol{\zeta} = \mathbf{x}).$$

So, we obtain for probability $\kappa(\mathbf{s})$, that only good batches come to the system during servicing of a demand from the "opening" batch

$$\kappa(\mathbf{s}) = \int_0^\infty ... \int_0^\infty e^{-(\mathbf{s},\mathbf{x})} P(\mathbf{x}) dL(\mathbf{x}) =$$

$$= \int_0^\infty ... \int_0^\infty \int_0^\infty e^{-(\mathbf{s},\mathbf{x}) - (a - a\gamma(\mathbf{s}))t} dF(\mathbf{x}, t) =$$

$$= \alpha(\mathbf{s}, a - a\gamma(\mathbf{s})).$$

Then for $\gamma(\mathbf{s})$ probability we obtain the following functional equation:

$$\gamma(\mathbf{s}) = \sum_{k=1}^{\infty} g_k(\kappa(\mathbf{s}))^k = G(\alpha(\mathbf{s}, a - a\gamma(\mathbf{s}))), \tag{1}$$

which is true for such s_i, $i = 1, ..., l$, that Re $s_i > 0$, as it follows from the analytical continuation method. In particular, if each batch consists of a single demand ($G(z) = z$), we have from (1)

$$\gamma(\mathbf{s}) = \alpha(\mathbf{s}, a - a\gamma(\mathbf{s})).$$

Solution of (1) equation

Denote as $\beta(q) = \alpha(\mathbf{0}, q)$ the LST of $B(t)$ distribution function and as $\beta_1 = -\beta'(0)$ the first moment of service time.

Theorem. *a) The functional equation (1) determines a single analytical in the domain* Re $s_i > 0$, $i = 1, ..., l$, *function* $\gamma(\mathbf{s})$, *so that* $|\gamma(\mathbf{s})| < 1$ *in this domain;*

b) the function $\gamma(\mathbf{s})$ *may be represented in the following form:*

$$\gamma(\mathbf{s}) = \int_0^{\infty} ... \int_0^{\infty} e^{-(\mathbf{s},\mathbf{x})} d\Gamma(\mathbf{x}),$$

where $\Gamma(\mathbf{x})$ *is the proper distribution function, if* $\rho = a\beta_1 G'(1) \leq 1$, *and non-proper, if* $\rho > 1$ *(it means, that* $\Gamma(\boldsymbol{\infty}) = 1$, *if* $\rho \leq 1$, *and* $\Gamma(\boldsymbol{\infty}) < 1$, *if* $\rho > 1$*).*

Proof. Let us introduce the function $\psi(\mathbf{s}, z) = G(\alpha(\mathbf{s}, a - a\gamma(\mathbf{s}))) - z$. This function is convex with respect to z, if $\mathbf{s} = (s_1, ..., s_l)$ is fixed, where Re $s_i > 0$, $i = 1, ..., l$. Indeed, the function $G(\alpha(\mathbf{s}, q))$ is fully monotonous [3] with respect to each of its arguments, because it is an LST of some distribution function $R(\mathbf{x}, t) = \sum\limits_{k=1}^{\infty} g_k R_k(\mathbf{x}, t)$, where $R_k(\mathbf{x}, t)$ is the distribution function with LST being equal to $(\alpha(\mathbf{s}, a - a\gamma(\mathbf{s})))^k$. Then for real $s_i \geq 0$, $i = 1, ..., l$, $q \geq 0$ the inequality $\dfrac{\partial^2 G(\alpha(\mathbf{s}, q))}{\partial q^2} \geq 0$ takes place and we have, obviously,

$$\frac{\partial^2 \psi(\mathbf{s}, z)}{\partial z^2} = a^2 \frac{\partial^2 G(\alpha(\mathbf{s}, q))}{\partial q^2}\bigg|_{q=a-az} \geq 0.$$

Therefore the function $\psi(\mathbf{s}, z)$, being continuous on z (if $s_i \geq 0$, $i = 1, ..., l$, are fixed), has in the point of $z = 0$ and $z = 1$ the positive and negative value accordingly.

It follows from the convexity of $\psi(\mathbf{s}, z)$ function with respect to z, that a single solution $\gamma(\mathbf{s})$ of $G(\alpha(\mathbf{s}, a - a\gamma(\mathbf{s}))) - z = 0$ functional equation exists, when $s_i \geq 0$, $i = 1, ..., l$, $0 \leq z \leq 1$.

Let us introduce the next sequence of functions:

$$\gamma_0(\mathbf{s}) = 0, \gamma_{k+1}(\mathbf{s}) = G(\alpha(\mathbf{s}, a - a\gamma_k(\mathbf{s}))), k \geq 0, i = 1, ..., l. \tag{2}$$

By mathematical induction method we shall prove, that

$$0 \leq \gamma_k(\mathbf{s}) \leq \gamma_{k+1}(\mathbf{s}) \leq 1 \tag{3}$$

and the functions $\gamma_k(\mathbf{s})$ are fully monotonous with respect to each argument s_i, $i = 1, ..., l$, for each $k \geq 0$.

So far as the function $G(\alpha(\mathbf{s}, q))$ is fully monotonous on each argument, the truth of (3) inequality when $k = 0$ follows from (2), and the function $\gamma_0(\mathbf{s}) = 0$ is obviously fully monotonous. Suppose, that (3) inequality takes place for some k and the function $\gamma_k(\mathbf{s})$ is fully monotonous. Then it follows from the following inequalities

$$0 \le \gamma_k(\mathbf{s}) = G(\alpha(\mathbf{s}, a - a\gamma_{k-1}(\mathbf{s}))) \le G(\alpha(\mathbf{s}, a - a\gamma_k(\mathbf{s}))) = \gamma_{k+1}(\mathbf{s}) \le 1$$

and from the properties of fully monotonous functions [3], that inequality (3) takes place for all $k \ge 0$ and for all such k the function $\gamma_k(\mathbf{s})$ is fully monotonous.

Then the sequence of fully monotonous functions $\{\gamma_k(\mathbf{s})\}_{k=1}^{\infty}$ is monotonous and limited, therefore the limit $\gamma(\mathbf{s}) = \lim\limits_{k\to\infty} \gamma_k(\mathbf{s})$ exists. It's obviously, that $|\gamma(\mathbf{s})| < 1$ and the function $\gamma(\mathbf{s})$ is fully monotonous. Hence, it may be represented in the form

$$\gamma(\mathbf{s}) = \int_0^\infty ... \int_0^\infty e^{-(\mathbf{s},\mathbf{x})} d\Gamma(\mathbf{x}),$$

where $\Gamma(\mathbf{x})$ is some measure in the space $\mathbf{R}^n$.

It's clear, that $z = \gamma(\mathbf{0})$ is the minimal solution of $G(\alpha(\mathbf{0}, a - az)) - z = 0$ equation. But $z = 1$ is the solution of the equation too (the both solutions coincide in the case of $\gamma(\mathbf{0}) = 1$). It follows from the convexity on z of $G(\alpha(\mathbf{0}, a - az)) = G(\beta(a - az))$ function, that the second solution $z < 1$ exists, if and only if $-aG'(1)\beta'(0) = aG'(1)\beta_1 > 1$. Therefore only in this case the distribution, corresponding to $\gamma(\mathbf{s})$ function, will be non-proper (i.e. $\Gamma(\infty) < 1$). If $aG'(1)\beta_1 \le 1$, then $\gamma(\mathbf{0}) = 1$ and $\Gamma(\infty) = 1$, i.e. the corresponding distribution is proper.

The theorem is proved.

For example, let $l = 1$. Then from (1) we have

$$\gamma(s) = G(\alpha(s, a - a\gamma(s))). \tag{4}$$

Let α_{ij}, $i, j = 1, 2, ...$, be the $i + j$-th mixed moment of $F(x, t)$ distribution function, i.e.

$$\alpha_{ij} = \frac{\partial^{i+j}}{\partial s^i \partial q^j} \alpha(s, q) \bigg|_{s=0, q=0},$$

φ_i, β_i be the i-th moment of demand volume and service time accordingly. Then, as it follows from (4), we have the next formulae for the first (γ_1), and second (γ_2) moment of Γ random variable correspondingly:

$$\gamma_1 = \frac{G'(1)\varphi_1}{1-\rho};$$

$$\gamma_2 = \frac{G'(1)\varphi_2}{1-\rho} + \frac{2a(G'(1))^2\varphi_1\alpha_{11}}{(1-\rho)^2} + \frac{\varphi_1^2[G''(1) + a^2\beta_2(G'(1))^3]}{(1-\rho)^3},$$

when $\rho < 1$, or for the case of $G(z) = z$ we obtain

$$\gamma_1 = \frac{\varphi_1}{1-\rho};$$

$$\gamma_2 = \frac{\varphi_2}{1-\rho} + \frac{2a\varphi_1\alpha_{11}}{(1-\rho)^2} + \frac{a^2\beta_2\varphi_1^2}{(1-\rho)^3}.$$

We can calculate any mixed moment of $\mathbf{\Gamma}$ random vector by the same way.

Some special cases

1. Let $l = 1$. Suppose, that ζ and ξ random variables are independent, i.e. $\alpha(s, q) = \varphi(s)\beta(q)$, where $\beta(q) = \alpha(0, q)$ is the LST of $B(t)$ distribution function, Re $q > 0$. Then from (4) we have $\gamma(s) = G(\varphi(s)\beta(a - a\gamma(s)))$. In particular, if we suppose that all demands volumes are equal to one, then the total volume of demands being served during the busy period coincides with their number. In this case we have $\varphi(s) = e^{-s}$. Hence, denoting as $R(z)$ the generation function of number of demands being served during busy period, we have obviously, that $\gamma(s) = R(e^{-s})$, and obtain the well known equation

$$R(z) = G(z\beta(a - aR(z)))$$

for the generation function $R(z)$.

2. Let $l = 1$. Suppose, that demand service time is proportional to the demand volume, i.e. $\xi = c\zeta$, where $c = const > 0$. It's known [1], that in this case we have $\alpha(s, q) = \varphi(s + cq)$. So, the (4) equation looks as

$$\gamma(s) = G(\varphi(s + ca - ca\gamma(s))). \tag{5}$$

In particular, if demand volume equals numerically to its service time (in this case in (5) equation $\varphi(s)$ must be changed by $\beta(q)$), then the total volume of demands being served during the busy period coincides with with the duration of the period (i.e. $\gamma(s)$ in (5) must be changed by the LST $\pi(q)$ of the busy period). Then from (5) we obtain the well known equation for the function $\pi(q)$:

$$\pi(q) = G(\beta(q + a - a\pi(q))).$$

3. Let $l = 2$. Suppose, that for each demand we have $\zeta_1 = 1$, and ζ_2 is the demand volume. Let service time be proportional to the demand volume ($\xi = c\zeta_2$, $c > 0$). In this case we have

$$\alpha(\mathbf{s}, q) = \alpha(s_1, s_2, q) = e^{-s_1}\varphi(s_2 + cq)$$

and (1) equation looks as

$$\gamma(s_1, s_2) = G(e^{-s_1}\varphi(s_2 + ca - ca\gamma(s_1, s_2))). \tag{6}$$

Suppose, that ζ_2 equals numerically to service time of the demand and $c = 1$. Then the components Γ_1, Γ_2 of $\mathbf{\Gamma}$ random vector have the following sense: Γ_1 is the number η of demands being served during the busy period, Γ_2 is the duration of the period. Then (6) equation has the following form:

$$\gamma(s_1, q) = G(e^{-s_1}\beta(q + a - a\gamma(s_1, q))). \tag{7}$$

Let

$$g(z, q) = \mathbf{E}(z^{\eta}e^{-q\Pi}) = \sum_{n=1}^{\infty} z^n \mathbf{P}\{\eta = n\} \int_0^{\infty} e^{-qt} d\mathbf{P}\{\Pi < t | \eta = n\}.$$

Then we obviously have $\gamma(s_1, q) = g(e^{-s_1}, q)$, and for $g(z, q)$ function, describing the common distribution of the busy period duration and the number of demands being served during the period, we obtain from (7) the next well known equation:

$$g(z, q) = G(z\beta(q + a - ag(z, q))).$$

So, the last equation is the consequence of (1) equation too.

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A construction of d-tuples of orthogonal Latin hypercubes

Marián Trenkler

We give a construction of d-tuples of orthogonal Latin d-hypercubes. Our result generalizes well know result about orthogonal Latin squares published in 1960 by R.C.Bose, S.S.Shikhande and E.T.Parker.

A *Latin square* $\mathbf{R}^n = [\mathbf{r}(i_1, i_2); 1 \le i_1, i_2 \le n]$ of order n is an $n \times n$ matrix such that every row and every column is a permutation of the set of natural numbers $\{1, 2, \dots, n\}$. Two Latin squares $\mathbf{R}^n = [\mathbf{r}(i_1, i_2)]$ and $\mathbf{S}^n = [\mathbf{s}(i_1, i_2)]$ of order n are called *mutually orthogonal*, if whenever $i_1, i_1', i_2, i_2' \in \{1, 2, \dots, n\}$ are such that $\mathbf{r}(i_1, i_2) = \mathbf{r}(i_1', i_2')$ and $\mathbf{s}(i_1, i_2) = \mathbf{s}(i_1', i_2')$, then $i_1 = i_1'$ and $i_2 = i_2'$.

Two orthogonal Latin squares of order 4 are depicted in Fig. 1.

If n is odd then $\mathbf{R}^n = |\mathbf{r}(i_1, i_2) = (i_1 + i_2) \pmod n;\ 1 \le i_1, i_2 \le n|$ and $\mathbf{S}^n = |\mathbf{s}(i_1, i_2) = (i_1 - i_2) \pmod n)|$ are mutually orthogonal Latin squares. For $n \equiv 1 \pmod 2$ a couple of mutually orthogonal Latin squares is defined by relations $\mathbf{r}(i_1, i_2) = (i_1 + i_2) \pmod n$ and $\mathbf{s}(i_1, i_2) = (i_1 - i_2) \pmod n$.

The problem of existence of orthogonal Latin squares is very old. Already *L.Euler* knew the construction of a couple of mutually orthogonal Latin squares of any odd order. He conjectured that such a couple exists for every order n, $n \neq 2$. But only in 1960 did R.C.Bose, S.S.Shikhande and E.T.Parker [1] prove, that two mutually orthogonal Latin squares of order n exist if and only if $n \neq 2, 6$. For more information about these topics see [2].

A Latin squares may by thoutht of as a 2-dimensional analoque of permutation. A generalization of a Latin square is *Latin d-hypercube*. In this paper we generalize the well know result from [1] to a d-dimensional space for every natural number d.

Definition 1. A *Latin d-hypercube* of order n is a d-dimensional matrix

$$\mathbf{Q}^{d,n} = |\mathbf{q}(i_1, i_2, \dots, i_d);\ 1 \le i_1, i_2, \dots, i_d \le n|,$$

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

Figure 1: Mutually orthogonal Latin squares

such that every row is a permutation of the set of natural numbers $1, 2, \ldots, n$. By a *row* of $\mathbf{Q}^{d,n}$ we mean an n-tuple of elements $\mathbf{q}(i_1, i_2, \ldots, i_d)$ which have identical coordinates at $d-1$ places.

Definition 2. A d-tuple of Latin d-hypercubes

$$[\mathbf{Q}_k^{d,n} = |\mathbf{q}_k(i_1, i_2, \ldots, i_d);\ 1 \le i_1, i_2, \ldots, i_d \le n|, \quad k = 1, 2, \ldots, d]$$

of order n is called *orthogonal*, if whenever $i_1, i_2, \ldots, i_d, i'_1, \ldots, i'_d \in \{1, 2, \ldots, n\}$ are such that

$$\mathbf{q}_k(i_1, i_2, \ldots, i_d) = \mathbf{q}_k(i'_1, i'_2, \ldots, i'_d) \quad \text{for all} \quad k = 1, 2, \ldots, d,$$

then $i_k = i'_k$ for all $k = 1, 2, \ldots, d$.

The construction of a d-tuple of orthogonal Latin d-hypercubes is contained in the proof of the following theorem.

Theorem 3. *A d-tuple of orthogonal Latin d-hypercubes $\mathbf{Q}_k^{d,n}$ of order n exists for every natural number $n \neq 2, 6$ and every natural number $d \geq 2$.*

Proof. Let $\mathbf{R}^n = |\mathbf{r}(i_1, i_2);\ 1 \le i_1, i_2 \le n|$ and $\mathbf{S}^n = |\mathbf{s}(i_1, i_2)|$ are two orthogonal Latin squares of order n. They will play a crucial role in our construction of d orthogonal Latin d-hypercubes $\mathbf{Q}_k^{d,n}$, $k = 1, 2, \ldots, d$. The k-th hypercube arises using the square $\mathbf{R}^n$ $(k-1)$-times and the square $\mathbf{S}^n$ $(p-k)$-times.

We define the k-th Latin d-hypercube

$$\mathbf{Q}_k^{d,n} = |\mathbf{q}_k(i_1, i_2, \ldots, i_d)|$$

of order n by the following relation

$$\begin{aligned}
&\mathbf{q}_k(i_1, \ldots, i_d) = \\
&\mathbf{r}(i_1, \mathbf{r}(i_2, \mathbf{r}(i_3, \ldots, \mathbf{r}(i_{k-1}, \mathbf{s}(i_k, \mathbf{s}(i_{k+1}, \ldots, \mathbf{s}(i_{p-2}, \mathbf{s}(i_{d-1}, i_d)) \ldots))) \ldots)))
\end{aligned}$$

for every $1 \le i_1, i_2, \ldots, i_d \le n$.

1. Evidently, for every $k = 1, 2, \ldots, d$, the set

$$\{\mathbf{q}_k(i_1, i_2, \ldots, i_{j-1}, i_j, i_{j+1}, \ldots, i_d); \quad i_j = 1, 2, \ldots, n\}$$

is equal to the set $\{1, 2, \ldots, n\}$. From this it follows that $\mathbf{Q}_k^{d,n}$ is a Latin d-hypercube for all k.

2. We suppose that

$$\mathbf{q}_k(i_1, i_2, \ldots, i_d) = \mathbf{q}_k(i'_1, i'_2, \ldots, i'_d) \quad \text{for all} \quad k = 1, 2, \ldots, d \quad (E_k)$$

From (E_1) and (E_2) it follows

$$\begin{aligned}
\mathbf{s}(i_1, \mathbf{s}(i_2, \mathbf{s}(i_3, \ldots, \mathbf{s}(i_{d-1}, i_d) \ldots))) &= \mathbf{s}(i'_1, \mathbf{s}(i'_2, \mathbf{s}(i'_3, \ldots, \mathbf{s}(i'_{d-1}, i'_d) \ldots))) \\
\mathbf{r}(i_1, \mathbf{s}(i_2, \mathbf{s}(i_3, \ldots, \mathbf{s}(i_{d-1}, i_d) \ldots))) &= \mathbf{r}(i'_1, \mathbf{s}(i'_2, \mathbf{s}(i'_3, \ldots, \mathbf{s}(i'_{d-1}, i'_d) \ldots)))
\end{aligned}$$

Because $\mathbf{R}^n$ and $\mathbf{S}^n$ are mutually orthogonal Latin squares, we have

$$\begin{aligned}
&i_1 = i'_1 \qquad \text{and} \\
&\mathbf{s}(i_2, \mathbf{s}(i_3, \ldots, \mathbf{s}(i_{d-1}, i_d) \ldots)) = \mathbf{s}(i'_2, \mathbf{s}(i'_3, \ldots, \mathbf{s}(i'_{d-1}, i'_d) \ldots))
\end{aligned}$$

Replace i'_1 by i_1 in (E_k), $k = 1, 2, \ldots, d$. From (E_2) and (E_3) it follows

$$\begin{aligned}\mathbf{s}(i_2, \mathbf{s}(i_3, \ldots, \mathbf{s}(i_{d-1}, i_d) \ldots))) &= \mathbf{s}(i'_2, \mathbf{s}(i'_3, \ldots, \mathbf{s}(i'_{d-1}, i'_d) \ldots)))\\ \mathbf{r}(i_2, \mathbf{s}(i_3, \ldots, \mathbf{s}(i_{d-1}, i_d) \ldots))) &= \mathbf{r}(i'_2, \mathbf{s}(i'_3, \ldots, \mathbf{s}(i'_{d-1}, i'_d) \ldots)))\end{aligned}$$

and so

$$\begin{aligned}&i_2 = i'_2 \qquad \text{and}\\ &\mathbf{s}(i_3, \mathbf{s}(i_4, \ldots, \mathbf{s}(i_{d-1}, i_d) \ldots)) = \mathbf{s}(i'_3, \mathbf{s}(i'_4, \ldots, \mathbf{s}(i'_{d-1}, i'_d) \ldots))\end{aligned}$$

Continuing in this manner, after $(d-1)$ steps from (E_{p-1}) and (E_p) we get

$$\begin{aligned}\mathbf{s}(i_{d-1}, i_d) &= \mathbf{s}(i'_{d-1}, i'_d)\\ \mathbf{r}(i_{d-1}, i_d) &= \mathbf{r}(i'_{d-1}, i'_d)\end{aligned}$$

From the assumption that $\mathbf{R}^n$ and $\mathbf{S}^n$ are orthogonal we get

$$i'_{p-2} = i_{p-2} \qquad \text{and} \qquad i'_{d-1} = i_{d-1}$$

which completes the proof. □

Remark 4. Already L.Euler knew how to use orthogonal Latin squares in the construction of magic squares. In a similar way we can use a d-tuple of orthogonal Latin cubes to construct a d-dimensional magic cube. It was proved in [3] that a magic d-dimensional cube of odd order n exists in every dimension d. Because the construction of Latin hypercubes of even order is complicated, magic d-dimensional cubes of ever order (see [4]) were made using different methods.

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On expressing a continuous function of two variables as a sum of continuous functions of one variable

Eva Trenklerová

A compactum $K \subset \mathbb{R}^2$ is said to be basically embedded in $\mathbb{R}^2$ if for each continuous function $f : K \to \mathbb{R}$ there exist continuous functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in K$. Let an array be a sequence of points in $\mathbb{R}^2$ such that each consecutive pair bounds a vertical or a horizontal segment, and any two consecutive segments are orthogonal. The following simple characterization is due to Y.Sternfeld *(1989): a compactum $K \subset \mathbb{R}^2$ is basically embedded in $\mathbb{R}^2$ if and only if it does not contain any array of n points for some n. The original proof uses advanced arguments of functional analysis. I shall talk about a new topological proof, which uses only elementary arguments.*

The notion of basic embeddings is closely related to the 13th of the famous 23 problems of D.Hilbert *(1900) and the theorem of* V.I.Kolmogorov *(1956) about representation of continuous function of more variables by means of continuous functions of one variable.*

Introduction

Let us J be a compact subset of the plane $\mathbb{R}^2$, as on Figure 1.

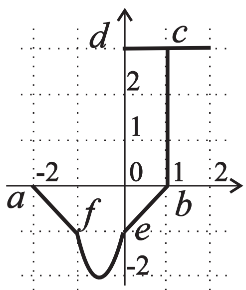

Figure 1: Set J

Let us have a real valued function f continuous on this set, $f \in C(J)$. We would like to find two real valued continuous functions g and h of one real variable, $g, h \in C(\mathbb{R})$, such that for all points $(x, y) \in J$ the following holds:

$$f(x, y) = g(x) + h(y). \tag{1}$$

We can do it in the following way. We will denote by h_0, g_1, h_2, g_3 restrictions of functions g and h to certain intervals – the indices will denote the order in which parts of the resulting functions will be defined. We start by putting $h_0(3) = 0$ (it can be equal to any constant) and defining $g_1(x) = f(x, 3) - h_0(3) = f(x, 3)$ for $x \in [0, 2]$. Thus g_1 and h_0 are both defined on the segment joining the points $d = (0, 3)$ and $(2, 3)$. On this segment obviously $f = g_1 + h_0 = g + h$.

The function f should be equal to $g + h$ on J. In the point $b = (1, 0)$ the value of g is already given: $g(1) = g_1(1) = f(1, 3)$. Therefore $h(0) = f(1, 0) - f(1, 3) = f(1, 0) - g_1(1)$. Next, in the point $a = (-2, 0)$ we get $g(-2) = f(-2, 0) - h(0)$. Similarly the values of g and h in the point $d = (0, 3)$ determine their values in the points $e = (0, -1)$ and $f = (-1, -1)$.

In this sense the values of g and h on segment joining the points $d = (0, 3)$ and $(2, 3)$ determine their values on the part of J in the rectangle given by the points $(-2, -1)$, $(2, -1)$, $(2, 3)$, $(-2, 3)$.

Thus we have to define $h_2(y) = f(x, y) - g_1(x)$ for $y \in [-1, 3)$, with $x \in [0, 2]$. We check the continuity of h in the point 3. We have $\lim_{y \to 3^-} h_2(y) = f(1, 3) - g_1(1) = f(1, 3) - f(1, 3) = 0 = h_0(3)$.

Next we have certian freedom; we extend h_2 arbitrarily but continuously over $[-1, -2]$. And again we have to put $g_3(x) = f(x, y) - h(y)$ for $x \in [-2, 0)$. As previously, the resulting function g is continuous in the point 0.

If a compact subset K of the plane $\mathbb{R}^2$ is such that every function $f \in C(K)$ is expressible in the form (1), then we say that K is a basic subset of the plane. This notion was introduced by Sternfeld in 1985 (naturally we have a similar definition for $K \subseteq \mathrm{E}m$):

Definition 1. [7] We say that $K \subseteq$ E2 is a *basic subset of* E2 if for each function $f \in C(K)$ there exist functions $g, h \in C(\mathbb{R})$ such that for all points $(x, y) \in K$:

$$f(x, y) = g(x) + h(y).$$

We say that a set K is *basically embeddable* into the plane if it can be embedded as a basic subset of the plane. (By an *embedding* we mean a continuous one-to-one map $f : X \to Y$ such that $f^{-1} : f(X) \to X$ is also a continuous one-to-one map. By a well-known result, if the space X is compact and f is continuous and one-to-one, then f is an embedding.)

We have seen that important for this expressing are sequences of points like a, b, c, d, e, f (Figure 1). We will call such sequences arrays:

Definition 2. A sequence of points $\{a_i\}_{i \in I} \subseteq \mathbb{R}^2$, where $I = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$ or $I = \mathbb{N}$, is said to be an *array* if

- each two consecutive points a_i, a_{i+1} are different and bound a segment parallel to one of the coordinate axes and
- each two consecutive segments $[a_{i-1}, a_i]$, $[a_i, a_{i+1}]$ are orthogonal to each other.

If I is a finite set of cardinality n then we say that the array $\{a_i\}_{i \in I} = \{a_i\}_{i=1}^{n}$ has *length* $n - 1$. If an array of length m is such that $a_1 = a_{m+1}$ then we say that the array is *closed*.

As we have seen, an array has the following property: the values of the functions g and h in one of its points determine their values in the rest of its points. It is possible to characterize compact basic subsets of the plane by means of arrays. Before stating the theorem we will give three more examples.

Example 3. If $K \subseteq \mathbb{R}^2$ is the graph of a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, then it is a basic subset of the plane.

Indeed, for every point $(x, y) \in K$ we have $y = \varphi(x)$. Thus for any $f \in C(K)$ we can set $g(x) = f(x, \varphi(x)) = f(x, y)$ and $h(y) = 0$. In this set the maximal length of an array is one. Moreover every array of length one forms a segment parallel to the vertical coordinate axis.

Example 4. The circle $C \subseteq \mathbb{R}^2$ from Figure 2 is not a basic subset of the plane, because it contains a closed array $\{a_i\}_{i=1}^5$, $a_1 = a_5$, of length 4.

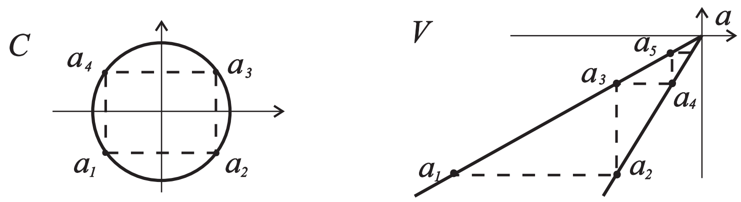

Figure 2: Embeddings of C and V

Let us denote the coordinates of the points of the array $a_1 = a_5 = (x_1, y_1)$, $a_2 = (x_2, y_1)$, $a_3 = (x_2, y_2)$, $a_4 = (x_1, y_2)$. Let us assume that C is a basic subset of the plane. Then any function $f \in C(C)$ is expressible in the form (1), i.e. as $f = g + h$, for some $g, h \in C(\mathbb{R})$. This implies

$$\begin{aligned} f(x_1, y_1) &= g(x_1) + h(y_1) \\ f(x_2, y_1) &= g(x_2) + h(y_1) \\ f(x_2, y_2) &= g(x_2) + h(y_2) \\ f(x_1, y_2) &= g(x_1) + h(y_2). \end{aligned}$$

When we add up these equations with changing signs we get

$$f(x_1, y_1) - f(x_2, y_1) + f(x_2, y_2) - f(x_1, y_2) = 0.$$

Thus, if the function f is such that this equality does not hold, then we get a contradiction. In general, no set containing a closed array is a basic subset of the plane. The proof is a direct generalization of this argument, see [4]. In [4] it is also proved that no embedding of the circle in the plane is basic (see also [6]).

Example 5. The set $V \subseteq$ E2 from Figure 2, which is an embedding of the letter V, contains an infinite convergent array $\{a_i\}_{i \in \mathbb{N}}$ and is not a basic subset of the plane.

Let us denote the coordinates of the points of the array $a_1 = (x_1, y_1)$, $a_2 = (x_2, y_1)$, $a_3 = (x_2, y_2)$, $a_4 = (x_3, y_2)$, ..., $a_{2k} = (x_{k+1}, y_k)$, ... and the coordinates of its limit point $a = (x_a, y_a)$.

We will define a function $f \in C(V)$ which can not be expressed as a sum $f = g + h$ (1) for any $g, h \in C(\mathbb{R})$. We put

$$f(a_i) = \frac{(-1)^i}{i} \quad \text{and} \quad f(a) = 0.$$

Such an f is continuous on the closure of $\bigcup_{i \in \mathbb{N}} a_i$, i.e. on $\bigcup_{i \in \mathbb{N}} a_i \cup a$, and we can extend it continuously to V. Let us assume that $f = g + h$ for some $g, h \in C(\mathbb{R})$. Then for every

$k \in \mathbb{N}$ we have

$$\begin{aligned} f(a_1) &= g(x_1) + h(y_1) \\ f(a_2) &= g(x_2) + h(y_1) \\ &\dots \\ f(a_{2k-1}) &= g(x_k) + h(y_k) \\ f(a_{2k}) &= g(x_{k+1}) + h(y_k). \end{aligned}$$

By summing up with changing signs we get

$$\sum_{i=1}^{2k} (-1)^i f(a_i) = \sum_{i=1}^{2k} \frac{1}{i} = -g(x_1) + g(x_{k+1}).$$

By letting k go to infinity we obtain

$$\sum_{i=1}^{\infty} \frac{1}{i} = -g(x_1) + \lim_{k\to\infty} g(x_{k+1}).$$

Since the function g is continuous, $\lim_{k\to\infty} g(x_{k-1}) = g(x_a)$. But the series on the left side is divergent, and we have a contradiction.

Sternfeld in 1980 proved the following theorem:

Theorem 6. [8] *For a compactum $K \subseteq \mathbb{R}^2$ the following conditions are equivalent:*
(A) K does not contain any array of n points for some $n \in \mathbb{N}$
(B) K is a basic subset of the plane.

In Example 3 the set K does not contain any array of length 2. The circle C from Example 4 contains a closed array, and thus it contains an array of any length: $\{a_1, a_2\}$, $\{a_1, a_2, a_3\}$,..., $\{a_1, a_2, a_3, a_4, a_1, a_2\}$,.... The set V from Example 5 contains an infinite array and thus an array of every length.

Using this theorem, Skopenkov in 1995 gave a characterization of graphs which are basically embeddable into the plane. His statement, which uses certain forbidden subsets, resembles Kuratowski's characterization of planar graphs. We give the formulation for finite graphs:

Theorem 7. [6] *A finite graph is basically embeddable into* E2 *if and only if it does not contain any subset homeomorphic to one of the following graphs: a circle, a pentod and a cross with branched ends (see Figure 3).*

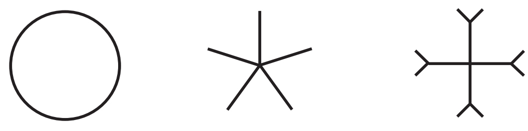

Figure 3: Skopenkov's forbidden subsets

Let us return to Sternfeld's statement 6. Its motivation dates to the year 1900. Back then, at the International Congress of Mathematics in Paris Hilbert formulated 23 problems for the mathematicians of the 20th century. In the thirteenth of these problems he conjectured that not all continuous functions of three variables were representable as compositions of continuous functions of two variables:

> "Prove that the equation of seventh degree $x^7 + ax^3 + bx^2 + cx + 1 = 0$ is not solvable by means of any continuous functions of only two variables."

This conjecture was refuted in 1950-ies by Kolmogorov and Arnold. However the functions they used are 'highly non-smooth', they have 'fractal graphs', and such functions were considered pathological at Hilbert's time and were not considered. And indeed, if a certain degree of smoothness is required, then Hilbert's conjecture is true.

Kolmogorov in 1957 proved a much stronger result:

Theorem 8. [3] *For each $n \geq 2$, there exist $2n+1$ functions $\{\varphi_i\}_{i=1}^{2n+1} \subseteq C(\mathbb{I}^n)$ of the form*

$$\varphi_i(x) = \textstyle\sum_{j=1}^{n} \varphi_{i,j}(x_j), \;\; \varphi_{i,j} \in C(\mathbb{I}), \;\; j = 1, \ldots, n$$

such that for each $f \in C(\mathbb{I}^n)$ there exist functions $\{g_i\}_{i=1}^{2n+1} \subseteq C(\mathbb{R})$ such that

$$f(x) = \sum_{i=1}^{2n+1} g_i(\varphi_i(x)), \;\; x \in \mathbb{I}^n.$$

Functions $\{\varphi_i\}_{i=1}^{2n+1} \subseteq C(\mathbb{I}^n)$ determine a continuous function

$$\varphi = (\varphi_1, \ldots, \varphi_{2n+1}) \;:\; \mathbb{I}^n \to \mathrm{E}2n+1.$$

Obviously the function φ has to be one-to-one. In the opposite case, if there existed two points $x, y \in \mathbb{I}^n$ such that $\varphi(x) = \varphi(y)$, then a function $f \in C(\mathbb{I}^n)$ such that $f(x) \neq f(y)$ would not be expressible as in the theorem. Thus, since $\mathbb{I}^n$ is compact, *φ is an embedding of $\mathbb{I}^n$ into* $\mathrm{E}2n+1$. And this embedding is *basic*. Thus Kolmogorov proved that the n-dimensional unit cube is basically embeddable into the $(2n+2)$-dimensional Euclidean space (it can be emdedded as a basic subset of $\mathrm{E}2n+1$). Ostrand generalized this result (the original formulation is not in terms of basic embeddings):

Theorem 9. [5] *Every n-dimensional compact metric space, $n \geq 0$, is basically embeddable into* $\mathrm{E}2n+1$.

It is of interest to compare this theorem to the famous result of dimension theory from 1930-ies (Menger, Urysohn, Nöbeling, Pontrjagin, Tolstova and others). For compact sets it says:

Theorem 10. [1] *Every n-dimensional compact metric space, $n \geq 0$, is embeddable into* $\mathrm{E}2n+1$.

The results of these two theorems seem to be similar. Is the first one only a stronger version of the second one? It turns out that the difference is more substantial. To see why, we have to consider statements in the opposite direction. Namely, we know that the embedding theorem is optimal in the following sense (this result from 1930-ies is due to van Kampen and Flores):

Theorem 11. [2, 10] *The n-skeleton of the $2n+2$-dimensional simplex, $n \geq 1$, is not embeddable into* $\mathrm{E}2n$.

Note that for $n = 1$, the skeleton of the 4-dimensional simplex is the complete graph on 5 vertices. This graph is not planar. In other words it is not embeddable into the 2-dimensional Euclidean space E2.

But surely some n-dimensional spaces are embeddable into $\mathrm{E}2n$ and thus the opposite direction "the dimension of a compact space embeddable into $\mathrm{E}2n+1$ is at most n" does not hold. This is where basic embeddings are different. Sternfeld in 1980 proved:

Theorem 12. [7] *If a compactum X is basically embeddable into* Em *and $m \geq 3$, then* $\dim X < m/2$.

Thus basic embeddings also give a characterization of dimension.

Let us return to basic subsets of the plane and Sternfeld's Theorem 6. Although the statement gives a simple topological characterization, its proof uses the machinery developed for the proof of Sternfeld's Theorem 12. It comprises a reduction to linear operators and non-trivial results of functional analysis, see [9]. Therefore it would be desirable to find a topological proof. In the paper [4] of this author and Neža Marmor-Kosta of University of Ljubljana there is such a proof of the implication $(B) \Rightarrow (A)$ from Theorem 6 (see also [9]). Presently the author is working on a topological proof of the other (and more difficult) implication $(A) \Rightarrow (B)$.

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PART II

MATHEMATICS TEACHING

Regular tessellations

Jiří Cihlář

The article dissert on the discussion about various ways of enriching of the teaching process by means of issues connected to the plane, especially how to cover the plane by regular polygons.

One of interesting topics that might be useful in teaching mathematics for both elementary and high school students is problems of tessellations.

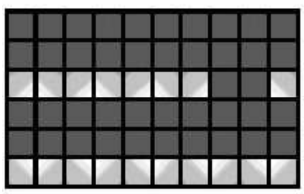
By the definition **tessellation** means such arrangement of regular many-sided polygons in a plane that fulfils the following requirements:

- all of the many-sided polygons are equilateral (the length of its sides is identical)
- there are no overlapping polygons there,
- adjoining polygons share their sides (they border one to another by their sides),
- identical configuration of many-sided polygons is repeated at each vertex.

Elementary-school pupils usually find a limited number of solutions only, senior high school students might want to discuss the problem in greater detail – for example by using algebraic equation – to find out the solutions.

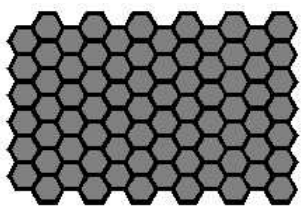
Tessellations of the following kind, where only one type of regular polygon is used, are fairly simple:

Tessellation 4-4-4-4



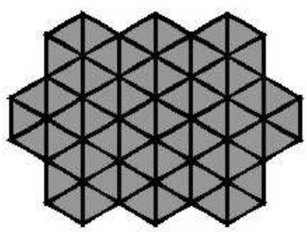
At each vertex four squares adjoin.

Tessellation 6-6-6



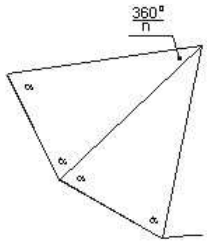
At each vertex three regular hexagons adjoin.

Tessellation 3-3-3-3-3-3



At each vertex six triangles adjoin.

The strategy for finding out further solutions is to calculate the interior angle of the regular many-sided polygon.



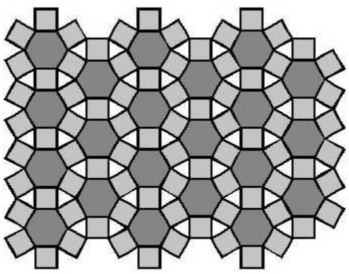
Based on the figure it clearly follows that the interior angle α_n is

$$\alpha_n = 2\alpha = 180° - \frac{360°}{n} = \left(\frac{1}{2} - \frac{1}{n}\right) \cdot 360°.$$

Elementary-school pupils then compute the values contained in the following table and try to determine such a combination of interior angles that has got 360 degrees as its sum in total.

n	α_n	n	α_n	n	α_n
3	60°	7	128,57°	11	147,27°
4	90°	8	135°	12	150°
5	108°	9	140°	13	152,31°
6	120°	10	144°	etc.	

For example, it is valid that $120° + 90° + 90° + 60° = 360°$, then at one vertex one regular hexagon, two squares and an equilateral triangle adjoin. Such tessellation can be found as the following figure testifies. By analogous methods further tessellations can be found as well.



The following chart enumerates all variants of determining the full angle based on the interior angles of regular n-shaped polygons. Some configurations have more variants; others, on the other hand, have no corresponding tessellation.

Number of n-shaped polygons adjoining at each vertex	Types of these n-shaped polygons	Tessellation possible
6	3-3-3-3-3-3	yes
5	3-3-3-3-6	yes
5	3-3-3-4-4	two variants possible
4	3-3-4-12	no
4	3-3-6-6	two variants possible
4	3-4-4-6	three variants possible
4	4-4-4-4	yes
3	3-7-42	no
3	3-8-24	no
3	3-9-18	no
3	3-10-15	no
3	3-12-12	yes
3	4-5-20	no
3	4-6-12	yes
3	4-8-8	yes
3	5-5-10	no
3	6-6-6	yes

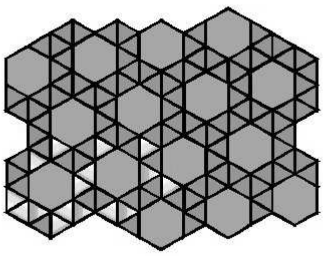
Using the above-stated formula, senior high school students will find out that the condition for the existence of tessellation is the possibility of solution $(n_1, n_2, n_3, \dots)$ of the following equation

$$\left(\frac{1}{2} - \frac{1}{n_1}\right) + \left(\frac{1}{2} - \frac{1}{n_2}\right) + \left(\frac{1}{2} - \frac{1}{n_3}\right) + \dots = 1.$$

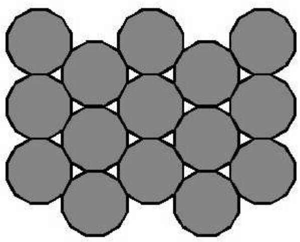
It is convenient to start with the assumption that $n_1 \leq n_2 \leq n_3 \leq \dots$ and to divide the solution into separate cases in relation to the number of n-shaped polygons that adjoin at one vertex. Of similar interest might also be the elucidation why some solutions of the equation have no corresponding tessellation.

The following figures exemplify all other possible tessellations.

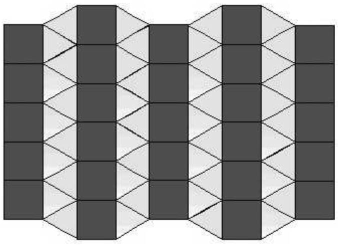
Tessellation 3-3-3-3-6

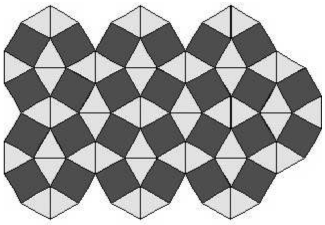


Tessellation 3-12-12

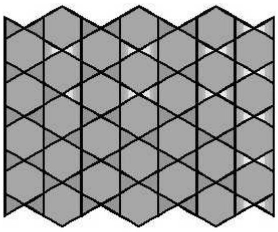


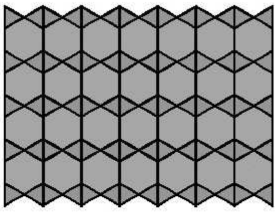
Tessellations 3-3-3-4-4 (two variants)



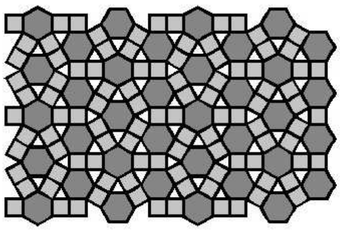


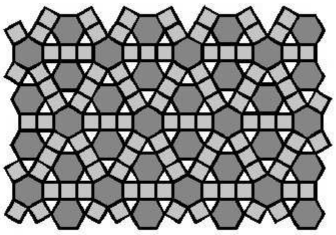
Tessellations 3-3-6-6 (two variants)



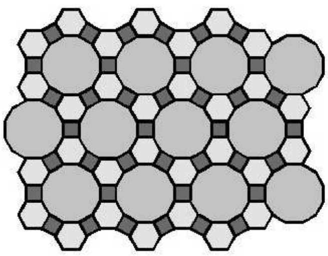


Tessellations 3-4-4-6 (remaining two variants)

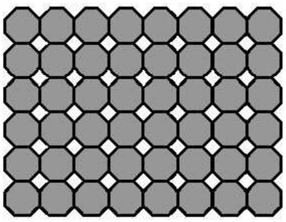




Tessellation 4-6-12



Tessellation 4-8-8



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How to write mathematical print materials for distance learning

Jindřiška Eberová, Anna Stopenová

This article deals with motivation of creation and creation of textbooks for distance learning of teacher training in mathematical instruction and characteristic features of the textbooks.

Introduction

Learning opens way to creating a substantially independent, satisfying and productive human life. Nowadays education becomes more important than ever before in influencing an individual's chance to find his or her place in the society, to do well and to improve their situation. An individual's motivation to learning and variety of learning opportunities are the key conditions to a successful realisation of lifelong learning. That is why it is necessary to extend the opportunities of lifelong learning. The educational systems should be adjusted to individual needs and to the demands of society.

Now it appears that people begin to realise the importance of their education and each individual will plan his or her learning activities during their lives only on condition that following suppositions are fulfilled :

(i) Learning experience in childhood was positive.

(ii) Learning opportunities are adequate and realistic (e.g. timetable, tempo, place, financial conditions, etc.).

(iii) Students are motivated to such a learning whose content and methods take into account their social and cultural prospects and life experience.

(iv) Students want to invest time, effort and money in further learning.

(v) Their capacities, abilities and professional skills were substantially appreciated for either personal reasons or for a job promotion.

Prospective elementary school teachers education

Up to now our experience in training prospective elementary school teachers are in accord with standpoints stressing the necessity to join theoretical subjects and teacher training together. This is the reason for both optimising the structure and content of individual subjects and for further usage of activating forms of education. This fact has led us to working out a project which was included in "Programme for support and development of public universities' educational activities" in its sub-programme "Support and development of new approaches in educational activities". We decided to select some parts of mathematics and didactics of mathematics and in order to make them more accessible to students, supported by grant resources, we worked them up in the form of educational texts differing in character from common educational texts.

Why write educational texts in the form of distance learning materials

One year ago when we decided to create distance learning materials we did not know how difficult task we were going to face. We were willing to sacrifice a lot of our free time and to dedicate substantial effort to create a text written in a new form. We found out that the most difficult thing for us is to acquire a specific style of elaborating the text. Preparation for this work was completely new for us since up to that moment we had worked only on texts destined for full-time studies. It is more difficult in all respects to write a learning material for directed self-education than to write a common educational text, as they are usually written in a stringent form consisting of a definition, a theorem, a proof and an example.

The 111/1998Coll. law states three forms of university studies :

Full-time study is performed by direct teaching at a university. Student's physical presence at the lectures is expected. Teacher and student are in direct contact at one time and place. The term " presence study" is also used to describe this type of study.

Distance study is performed solely by self-learning and it does not require student's presence at lectures. Student keeps contact with the educational institution. The educational institution controls and supports the student. Contact between the two of them is not expected to be at one place and it does not have to be at one time. The teacher/tutor does not lecture, he or she only helps methodologically and evaluates given assignments. The student, in order to be successful, needs some technical devices such as texts, video and audio records and CD-ROMs which enable him to cope with a subject matter as difficult as that for full-time students with minimal tutor's help. The contact between the distance study teacher and student is limited to a necessary minimum. This form of education is completely new, even though it continues the correspondence education tradition. Nevertheless the essence of this form is completely different, as it is based on *directed independent study.*

Combined study is a combination of full-time and distance study, i.e. it combines direct teaching at school and self-learning. It is in fact a condensed full-time study but the study results are only theoretically comparable to those of full-time study. Learning materials should thus mainly focus on that part of study which is performed by distance learning. The teacher's task is to help and support students who should perform most of their studies by self-learning. Active learning assumes motivation to learning, ability of critical reasoning and knowledge of self-education ways.

At our faculty elementary school teaching is accredited both in full-time and combined form. The number of students in combined study rises and it is comparable to the number of students in full-time study. We have prepared texts for the distance study part of combined study which contain only selected part of some mathematical disciplines. As Novák [3] highlights, the way of working up a text must provide the user with basis of concepts, symbols and methods necessary for a systematic study of mathematics. We want to help prospective teachers view the world of mathematics as a science with its specific system, structure, order and precise and given language. We must also keep in mind our target group of students of the education programme *teaching for elementary schools* and that is why we need to create motivating atmosphere for a successful self-education.

To fulfil this task as well as possible we attended a specialised course for authors of distance learning materials where we got information about the character of such a text. We must admit that we have never seen mathematical materials worked up in such a form and that this task was new for us. The lectures and experience from the seminars helped us to create the text.

General characteristics of each chapter of distance learning materials

- In the introduction to each chapter target skills and abilities are formulated. Students can check if they meet **educational aims**. Educational aim must be set clearly and precisely in each chapter and it must be worded in a student-friendly way so as to motivate the student to reach it. The laid-out cognitive aims are hierarchically ordered according to Bloom's taxonomy, beginning with the least demanding and ending with the most difficult ones.

- After the introduction to the topic follows the **explanatory part** which must be written in short paragraphs to be well-arranged. The formulations must be understandable, unambiguous and must be a pleasant read. The author of the text must stress the important things as at this moment the student is not able to learn it by himself. The author should also highlight problematic parts in which students often make mistakes and explain why they make them. In the text students are invited to revise often while studying the chapter.

- In the text there are also **activating elements** represented by various types of question tasks:

  insight questions which give the students time to think independently, but later the author explains it.

  questions checking comprehension corresponding with target skills in the introduction to each chapter. Students find the answer in the text.

  self-corrective tasks concerning key problems. The students check his or her answer in the clue.

  test which enables the students to check his or her understanding and knowledge. Each test must be accompanied by the clue and point evaluation. Students must learn the number of correct answers.

- In the distance learning material students are given **assignments** and **tasks** to exercise their skills. Correct answer is given in the clue.

- **Correspondence tasks** are registered and evaluated by the tutor.

- Explanatory part of the text is accompanied and supported by **solved examples**.

- **Study guide** is a specific part of a distance learning material. The author enters the text, makes contact with the students, give them advice how to proceed, encourages and motivates them. The author makes reference to the reader's experience and opinions. The text can be written in an easy way. This part stands as a substitute for the direct contact between the student and the teacher in the full-time study.
- The text contains **references to other learning materials**, such as additional materials and other media.
- **Concepts to remember** form an important part.
- At the end of every chapter there is a **conclusion**.

In respect to taxonomy of aims when giving tasks it is necessary to choose adequate active verbs, such as follows:

Concerning knowledge: define, reproduce, state, select, name, complete, write, revise, describe, order, explain

Concerning comprehension: explain, demonstrate, explain in own words, explain in another way

Concerning application: demonstrate, interpret, use, apply, try out

Concerning analysis: compare, analyse, decide, explain why, discern, find a principle

Concerning synthesis: sum up, clarify, propose, modify, solve, set, predict

Concerning evaluation: defend, disprove, judge, examine, explain, evaluate, decide, find difference, make a standpoint, discuss

General requirements on a good distance learning text

- The text must substitute the lacking direct contact between the student and the teacher.
- The subject matter of the material must be limited to basic theoretical knowledge.
- It is necessary to write as clearly, precisely and understandably as possible.
- The text must include elements stimulating students' activity, such as questions, different types of tasks, examples, resume, key terms, etc.
- The source of motivation for further studies is the information about student's progress, i.e. feedback.
- The text contains parts in which the student is encouraged and invited to exercise stamina.
- The learning material must respect students' needs.
- The material must always motivate students to study.
- Students' own experience is encouraged and stressed.

Advantages and disadvantages of distance learning material

One of the great advantages of printed materials is their mobility. One can use them at any time and place. They are also convenient for students and they belong to the most used ones. They are also financially affordable. One can use older texts or texts written by someone else. There are no problems with publishing further editions. On the other hand we have not had so far any students' feedback on working with the new text. This is still a matter to explore. As far as the disadvantages are concerned, printed materials are not interactive. They must be accompanied by additional technologies, such as e-mail, audio records, etc. And they lack sound and visual elements, they require reader's ability to comprehend and the distribution is sometimes slow.

We felt the text was too long-winded and the way of writing was very unusual for us. We had to adapt the text to the methodological requirements on a distance learning text which stress permanent students' motivation, activation, encouragement and appraisal. Apart from a mathematician's report, publishing a distance learning material requires a further report which evaluates how the author is aware of specific diversities of target students group and if he enables directed self-education.

When writing this type of text we tried to avoid pseudo-activities, to include things students are not able to cope with and unimportant things. We tried to follow core subject matter. We took advantage of our teaching experience to draw students' attention to difficult things in the text and to the things easily confused. All our effort was aimed at achieving maximal effectiveness and productivity of students' learning.

Areas we have worked on

We have written three distance learning materials: Basics of mathematics I, by Anna Stopenová, 60 pages. It deals with introduction to set theory and predicate calculus. This part of subject matter is taught at secondary schools and the author tried to revise and deepen the knowledge.

The second text continues the first one and it deals with subject matter concerning relations. It is called Basics of mathematics II by Jindřiška Eberová, 63 pages.

The third text deals with didactics of mathematics: Selected problems of didactics of mathematics I, by Bohumil Novák, 67 pages.

All above-mentioned texts are written according to principles for writing distance learning materials. The most difficult for us was to adopt a new style of writing. To establish contact with the student, as it was done mostly in the Study Guide, was particularly difficult. The Study Guide must be often included in the text, particularly at places where the author assumes the students will be helpless or tired. Helped by own experience with teaching given areas we encouraged the students, motivated them and encouraged their stamina. We tried to work up the text emphatically. We hope we have managed to predict and to draw students' attention to crucial points, difficult parts and practical needs. We provided them with our observations, experience and recommendation.

What is the benefit of self-education?

Students will gain the ability of working independently and of self-educating. If they have enough adequate distance learning materials, they will learn how to read and understand them. Their ability to read with comprehension will improve. They will develop the sense for detail and precision. Apart from professional skills and abilities their personality develops as well, hand in hand with abstract thinking, logical reasoning and precision in naming things. The students' aim should not only be to get the knowledge but also to feel satisfaction having mastered the subject matter.

Conclusion

We hope to have done the first step to making the prospective elementary teacher training in combined study effective. We have tried to bear in mind all the principles concerning directed self-education. The effectiveness of our distance learning materials will be continuously monitored after the publishing. The character and content of the material enables them to be used even for full-time students. We leave this issue of whether this will happen open.

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Products and Projections

Roman Frič

Let A_1, A_2 be sets and let $A_1 \times A_2$ be their Cartesian product. Usually $A_1 \times A_2$ is defined as the set of all ordered pairs $[a_1, a_2]$ such that a_i belongs to $A_i, i = 1, 2$, and, even though the definition of a general (categorical) product is based on projections, the role of projections pr_1 and pr_2 (pr_i maps $[a_1, a_2]$ to $a_i, i = 1, 2$) is not explained. We present some examples and remarks related to teaching products and some other abstract notions.

Introduction

Let me start with a well-known mathematical puzzle about a solid which looks like a circle from above, like a triangle from one side, and like a rectangle from the other side. It has a simple solution. Just take a cylinder and cut off two equal pieces, each of them looking like a half of the circle from below, like a half of the triangle from the first side, and like a half of elipse (determined by the rectangle) from the other side.

The puzzle came to my mind when explaining the notion of a product to my students. The students are familiar with the Cartesian product of (finitely many) sets, with the product of groups and, perhaps with the product of (two) vector spaces or topological spaces. If you look into commonly used textbooks, you will find the standard definitions not mentioning the projections of the product onto its factors. Indeed, on the more advanced level of mathematical structures (concrete categories), the notion of a product of two (or of an indexed set) objects is defined exactly via the projections, and exactly the role of projections in the definition comes to students as a surprise. It is my experience that the understanding of the role of projections leads to a much better understanding of particular cases of products, and to the better insight into mathematics.

The other thing which came to my mind was the beautiful and seminal paper [3] by M. Hejný. It deals with “mathematical structures for kindergarten”. It is about notions like mother, father, ..., family, and their relationships. When comparing two familes, we discuss similarities and differences. Hence, in turn, we talk about the “structure of a family” and about the the “isomorphism of two families”. Clearly, kids and their parents do not use the language of mathematical structures (sets, relations, arrows, ...) explicitly, but what they use are pretty much mathematical constructions and mathematical reasoning (cf. [1], [2]). So, why not to try “projections and products for kindergarten” in some natural everyday situations.

In the pesent paper we recall some facts about products, mention some examples, and point out some simple aspects of other abstract constructions. Our goal is to call your attention to related problems, to motivate teachers, and to encourage students to write a thesis along these lines (cf. [4]).

More about the puzzle

First, in fact, in the puzzle we ask about the existence of an object (a solid), the three projections of which are known objects (a circle, a triangle, a rectangle) - the answer is YES. Second, in fact, we should ask about the uniqueness of solution. The answer would be that there are (infinitely) many solutions, but there exists "the best one".

Indeed, from the solid we have described in the introductory section (a solution of the puzzle) we can cut off (infinitely) many additional pieces without changing the projections in question. Observe that the initial solution is maximal in the sense that any additional point would change (enlarge) at least one projection. Further, other solutions are smaller, i.e., they can be considered as subsets of (uniquely embedded into) the maximal solid. This way we have turned a puzzle into a mathematical problem and our observations lead to the notion of a product in mathematical structures. The good point about the puzzle and its discussion is that it can be easily modelled (e.g. on a PC) and understood by pupils "almost on the kindergaten level".

More about the problem

On a more advanced level of mathematical structures, the previous discussion can be summarized as follows. Within a given family of objects and a given family of admissible maps, we try to "legalize" the existence of a unique new object via specifying its "contours". Here by a "contour" we undestand the target object of an admissible map, the domain of which is the new object. The families of objects and admissible maps constitute a given universum in which we carry out all our particular mathematical activities. Objects and maps of such universum constitute a mathematical structure. In fact, it is a concrete category and we also shall use this more standard terminology. We avoid formal definitions of a mathematical structure and related notions. It suffices to be familiar with the naive set theory and basics of groups, linear spaces and metric spaces, order and similar relations. Such well-known mathematical structures provide natural illustrative examples and the properties of a product (coproduct) can be understood via familiar situations.

Now, assume that $\mathcal{S}$ a mathematical structure. Informally, we have the objects of $\mathcal{S}$ (each object is a set equipped with a structure, e.g., a partial order, a metric, a group structure, ...) and the morphisms of $\mathcal{S}$ (maps "preserving" the structure in question, i.e., preserving order, distance, group homomorphisms, ... , we assume that for each object of $\mathcal{S}$ the identity map is a morphism and the morphisms are closed with respect to compositions of maps).

Definition 1. *Let o_1, o_2, and o be objects of $\mathcal{S}$ and let $pr_i : o \longrightarrow o_i$, $i = 1, 2$, be morphisms of $\mathcal{S}$ such that*

(PR) For each object o' of $\mathcal{S}$ and for each pair of morphisms $f_i : o' \longrightarrow o_i$, $i = 1, 2$, of $\mathcal{S}$ there exists a unique morphism $f : o' \longrightarrow o$ of $\mathcal{S}$ such that $f_i = pr_i \circ f$, $i = 1, 2$, i.e., f_i is the composition of f and pr_i.

Then o, together with the morphisms pr_i, $i = 1, 2$, is said to be a **product** *of o_1 and o_2 in $\mathcal{S}$, o_1 and o_2 are called* **factors**, *and pr_i, $i = 1, 2$ are called* **projections**.

Observe that not in all mathematical structures products do exist. On the other hand, if a product exists, then it is unique up to isomorphisms in $\mathcal{S}$ (recall that two objects o and o' are isomorphic in $\mathcal{S}$ whenever there exist two morphisms $f : o \longrightarrow o'$ and $f' : o' \longrightarrow o$ in $\mathcal{S}$ such that the composition $f' \circ f$ of f and f' is the identity map on o, and the composition $f \circ f'$ of f' and f is the identity map on o'; see further comments at the end of article). The product of o_1 and o_2 is usually abbreviated to $o_1 \times o_2$. The product of an indexed set $\{o_t, t \in T\}$ of objects in $\mathcal{S}$ is defined analogously and it is usually abbreviated to $\prod_{t\in T} o_t$.

There are at least two good reasons for which products are fundamental in mathematics. First, it is a construction of new objects from old ones. Second, undoubtedly, relations play in mathematics and its applications a crucial role. By definition, relations are subsets of products.

In the next section, as an illustration, we discuss products in a number categories.

Examples

Example 2. *The category SET consists of the sets as objects and the maps as morphisms. The usual Cartesian product $A_1 \times A_2$ of two sets A_1 and A_2, together with the projections $pr_i : A_1 \times A_2 \longrightarrow A_i$, $i = 1, 2$, is the product of A_1 and A_2 in SET. Indeed, let A be a set and let $f_i : A \longrightarrow A_i$, $i = 1, 2$, be maps. For $a \in A$ define $f(a) = [f_1(a), f_2(a)] \in A_1 \times A_2$. Clearly, this yields a unique map f of A into $A_1 \times A_2$ such that $f_i(a) = (pr_i \circ f)(a)$ for all $a \in A$.*

Example 3. *By a partial order we understand a reflexive, transitive, and antisymmetric relation. Let $(A_1, \leq_1)$ and $(A_2, \leq_2)$ be partially ordered sets. A map $f : A_1 \longrightarrow A_2$ is order preserving if $a{\leq_1}b$ implies $f(a){\leq_2}f(b)$. The category POSET consists of partially ordered sets as objects and order preserving maps as morphisms.*

It follows from the previous example that $A_1 \times A_2$ is the obvious candidate for the underlying set of the product of $(A_1, \leq_1)$ and $(A_2, \leq_2)$ in POSET. There are many ways how to define a partial order on $A_1 \times A_2$. Consider the lexicographic order $\leq_l$ on $A_1 \times A_2$ defined as follows: $[a, b]{\leq_l}[c, d]$ whenever either $a{\leq_1}c$ or $a = c$ and $b{\leq_2}d$. The coordinatewise order $\leq_c$ on $A_1 \times A_2$ is defined by $[a, b]{\leq_c}[c, d]$ whenever both $a{\leq_1}c$ and $b{\leq_2}d$. The later one is the product order. Indeed, let $(A, \leq)$ be a partially ordered set and let $f_i : A \longrightarrow A_i$, $i = 1, 2$, be order preserving maps. Then $f(a) = [f_1(a), f_2(a)], a \in A$, defines the unique map $f : A \longrightarrow A_1 \times A_2$ such that $f_i(a) = (pr_i \circ f)(a)$ for all $a \in A$.

Observe that if $A_1 \times A_2$ carries $\leq_l$, then pr_2 does not preserve the order. Hence (barred trivial cases) $\leq_l$ fails to be the product order (it suffices to consider $A_1 = A_2 = \{0, 1\}$ carrying the natural order, $(A, \leq) = (A_1 \times A_2, \leq_c)$, and $f_i = pr_i, i = 1, 2$).

Example 4. *Consider the category of groups and group homomorphisms. Let X_1 and X_2 be groups. Their product $X_1 \times X_2$ is the group of all pairs $[a, b]$, $a \in X_1$ and $b \in X_2$, together with the usual projections, where the group operations are defined coordinatewise, e.g. $[a, b][c, d]$ is defined as $[ac, bd]$. It is a routine to verify that that coordinatewise structure is the "best one" with respect to the projections. Similarly, the product of two linear spaces over a scalar field is the set of all pairs equipped with the coordinatewise linear operations.*

Since in mathematical structures the objects are sets equipped with a structure and each morphism is a map of one set into another ("preserving the structure"), the previous examples indicate that the underlying set of a product (if it does exist at all) will be the Cartesian product of the underlying sets. Usually, the harder problem is to find the product structure. The next example shows that the product in some cases does not exist.

Example 5. *Let X be a set and let d be a map of $X \times X$ into the real numbers assigning each pair $[x, y] \in X \times X$ a nonnegative distance $d(x, y)$; moreover, we assume that the distance is symmetric, $d(x, x) = 0$ for all $x \in X$, for $x \neq y$ the distance $d(x, y)$ is positive, and d satisfies the so-called triangle inequality $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$. We call (X, d) a metric space. Denote $ISOMET$ the category having the metric spaces as objects and the maps preserving distance, or isometries (if (X_i, d_i), $i = 1, 2$, are metric spaces and $f : X_1 \longrightarrow X_2$ is a map, then f is an isometry whenever $d_2(f(x), f(y)) = d_1(x, y)$ for all $x, y \in X$), as morhisms.*

We claim that $ISOMET$ does not have products. For example, let $X = \{x, y\}$ and let $d(x, y) = d(y, x) = 1$, $d(x, x) = d(y, y) = 0$. Then (X, d) is a metric space. Contrariwise, assume that (Y, d_Y) is the product of (X, d) and (X, d) in $ISOMET$. Then the set Y contains exactly two points. For, in $ISOMET$, two distinct points are always mapped onto two distinct points and since each projection maps (Y, d_Y) onto (X, d), Y canot have more than two points. On the other hand, the identity map $id_X : X \longrightarrow X$ maps (X, d) onto (X, d) and hence (according to the definition of a product) there exists an isometry of (X, d) into (Y, d_Y). Without loss of generality, let $a, b \in Y, a \neq b$, and let $pr_1(a) = x$, $pr_1(b) = y$. Now, there are two possibilities.

1. $pr_2(a) = x$. Define $f_2 : X \longrightarrow X$ as follows: $f_2(x) = y$, $f_2(y) = x$. Then f_2 is an isometry of (X, d) onto (X, d). Let $f_1 = id_X$. Then (according to the definition of a product) there exits a unique isometry f of (X, d) onto (Y, d_Y) such that $f_i = pr_i \circ f$, $i = 1, 2$. Then $(pr_1 \circ f)(x) = id_X(x) = x$ implies $f(x) = a$. But $(pr_2 \circ f)(x) = f_2(x) = y$, cotradicting $pr_2(a) = x$.

2. $pr_2(a) = y$. We arrive to a contradiction analogously as in case 1.

The argumentation looks a bit complicated, but if we draw arrows and pictures (diagrams), all partial arguments are "easy to see".

Observe that isometries are very special morphisms. If we admit less restrictive "preservation of the structure", e.g. contractive maps, then the situation with the product in various categories of metric spaces can be different. More information can be found in [1], [2].

Coproducts

In the category theory there are dual notions and dual constructions ("coconstructions"). Formally, it suffices to reverse arrows. For example, a coproduct of two objects o_1 and o_2 in a given category is defined as an object o together with coprojections (reversed arrows) $copr_i : o_i \longrightarrow o$ such that for each object o' and each pair of morphisms $f_i : o_i \longrightarrow o'$, $i = 1, 2$, there exists a unique morphism $f : o \longrightarrow o'$ such that $f_i = f \circ copr_i$, $i = 1, 2$. If a coproduct exists, then it is unique up to isomorphisms. Usually it is denoted by $o_1 \coprod o_2$.

Example 6. *As expected, the coproduct of two (more) sets is their disjoint union and the coprojections are the embeddings. Let X_1 and X_2 be sets.*

1. If $X_1 \cap X_2 = \emptyset$, then $X_1 \cup X_2$ is the disjoint union of X_1 and X_2 and $copr_i : X_i \longrightarrow X_1 \cup X_2$, $i = 1, 2$, are the usual embeddings sending $x \in X_i$ to $x \in X_1 \cup X_2$. Let Y be a set and let $f_i : X_i \longrightarrow Y$, $i = 1, 2$ be maps. Then each $x \in X_1 \cup X_2$ belongs exactly to one of the sets X_1, X_2. Hence $f : X_1 \cup X_2 \longrightarrow Y$, defined by $f(x) = f_i(x), x \in X_i$, is the unique map such that $f_i = f \circ copr_i$, $i = 1, 2$.

2. If $X_1 \cap X_2 \neq \emptyset$, then the disjoint union of X_1 and X_2 can be visualized as the union of two disjoint sets $X'_1 = \{[x,1];\ x \in X_1\}$ and $X'_2 = \{[x,2];\ x \in X_2\}$ (indeed, if $a \in X_1$ and $b \in X_2$, then $[a,1] \neq [b,2]$) and $copr_i : X_i \longrightarrow X'_1 \cup X'_2$, $i = 1,2$, are the usual embeddings defined by $copr_i(x) = [x,i]$. Let Y be a set and $f_i : X_i \longrightarrow Y$, $i = 1,2$, be maps. Define $f : X'_1 \cup X'_2 \longrightarrow Y$ by $f([x,i]) = f_i(x)$, $i = 1,2$. Then f is the unique map of $X'_1 \cup X'_2$ into Y such that $f_i = f \circ copr_i$, $i = 1,2$.

Note that for $x \in X_1 \cap X_2$ the symbol $[x,i]$ represents $x \in X_i$ considered in the i-th context $i = 1,2$.

Example 7. *It can be easily verified that in POSET the coproducts exist. On the other hand, this is not true in the catgory of linearly ordered sets. Indeed, each singleton (a set consisting of exactly one point) is linearly ordered, but the disjoint union of the two singletons cannot be linearly ordered in "the best way".*

Final remarks

It might be interesting to describe situations in the real life leading to products and coproducts. For example, by n-tuples we model a database in such a way, that on the k-th place we store a particular k-th information independently of the others and putting together particular information about, e.g., a person we get a structured information in the form of n-tuples; the k-th projection yields the k-th information. In fact, we deal with products.

On the other hand, Yellow Pages (in a Telephone Directory) can be considered as a coproduct: the same person (or a company) can be listed (as coprojections) in several "disjoint contexts", each representing a different service. Similarly, in a heritage dictionary to the same entry (word) the coprojections assigne separately listed meanings.

When defining products, coproducts, isomorphisms, and other basic notions of mathematical structures, morphisms play a fundamental role. This of course means to teach students "see and understand" morphisms in the everyday life. I believe that "structures in the everyday life" provide suitable topics for diploma theses.

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MS Excel and interactive exercise books

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The idea of really interactive exercise computer-books is as follows: The user enters arbitrary input values and then checks the results step by step. It can simply be prepared by using MS-Excel. Note, that the user does not have to be familiar with MS-Excel, he or she does not have to use any commands. The aim is to introduce the books here.

Introduction

Many university students of different study programmes have to study mathematics and statistics. Not only knowledge of theory, but also practical skills are usually required during examinations. As concerns classical exercise books, they always contain only a limited number of solved exercises. Students are thus encouraged to use more or less specialized software to check their results, when trying other similar exercises, but not all the students can access it or deal with it and, beyond this fact, such software does not always show all the steps made by non-computer calculation, so readers can not find the cause of the wrong result. The idea of really interactive exercise computer-books is thus as follows: The user enters arbitrary input values and then checks the results step by step. It can simply be prepared by using MS-Excel. Note, that the user does not have to be familiar with MS-Excel, he or she does not have to use any commands. Such an exercise book does exist for statistics (in Czech only) and a similar book for matrix computations is under construction ([3], [4]).

Interactive exercise book for statistics

The realized interactive exercise text book is organized as follows. The main page [2] serves as the contents page with links to the rtf document for each chapter. There are also links to the www pages of statistical offices, statistical SW and some on-line statistical text books ([1], [5]-[8]), at its bottom.

The chapters cover the common syllabus of statistics, taught at non- mathematical faculties, i.e. descriptive statistics (frequencies, quantiles, moments), probabilistic distributions (binomial, hypergeometric, Poisson, exponential, normal), confidence intervals (for probability, mean, variance), tests (one- and two-sample, ANOVA, chi-square) and other methods (simple linear regression, time series description).

Each chapter contains a brief theoretical summary (definitions, formulas, properties) and at least one solved exercise. The graphical link (in the form of a pie-chart) leads immediately to the appropriate Excel file (sheet "ZADÁNÍ" – setting), where the only non-locked cells are those for entering the data or parameters. The choice of another sheet (e.g. "Momenty", "Testy" and so on) provides the concrete statistical problem solved step by step.

For example, the graphical link from the chapter Normal distribution leads to the file STA-norm.xls. Only two values have to be entered at the sheet "ZADÁNÍ", mean and variance namely. The variance has to be positive, which is automatically checked. The sheet "Hustota a distribuční funkce" provides values and graphs of the density and distribution function. The next sheet ("Pravděpodobnosti") provides the calculation of given probabilities or quantiles, vice versa.

Similarly, the graphical link from the chapter Chi-square test of independence leads to the file STA-chi2.xls. The frequencies in the contingency table have to be entered at the sheet "ZADÁNÍ". They all have to be naturals, this is automatically checked. The sheet "Test" provides calculation of expected frequencies and of the test statistics, defines the critical region and determines the statistical decision.

Interactive exercise book for matrix algebra

The same approach can be used in the exercise book for matrix algebra. The first topic is under construction at present: The calculation of determinants for the matrix of the type 2x2, 3x3 and 4x4 respectively. The theory and some exemplary exercises shall be in a rtf file again. Graphical links from this file will lead to the MS-Excel files Det2x2.xls, Det3x3.xls and Det4x4.xls.

For example, there are 3x3 entry cells at the sheet "ZADÁNÍ" in the file Det3x3.xls. The calculation of appropriate determinant using permutations can be found at the sheet "Determinant podle definice", step by step. The calculation according to the Sarrus' rule is provided at the next sheet "Determinant křížovým pravidlem". And the calculation using Laplacian expansion by minors is performed at the last sheet "Determinant rozvojem" (where all the six possibilities are described).

Another similar topic under construction is the calculation of a product matrix.

Remarks and acknowledgement

Students of three universities (VŠEM and UJEP in Ústí nad Labem and ČVUT in Děčín), mainly in the combined form of the study, have already used the statistical exercise text book [2] and their first reactions are very positive. This approach seems to be very useful to support self-study. The advantage is, that no special software (except for the commonly used) is required for the preparation and working.

The interactive exercise book for statistics was prepared at the J.E.Purkyně University in Ústí nad Labem thanks to the FRVŠ grant n. 2659 of the Czech Ministry of Education, Youth and Sports.

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Exhibits from a book in preparation: *The Investigative Approach in School Mathematics*

Jan Kopka, George Feissner

Mathematical theories have an experimental and inductive character at their beginning, and they gain a deductive character only after they have been investigated. If we really want to practice mathematics with our students on their level, then we should respect and explain how mathematical theories come into existence, how they develop, and how they finally gain their deductive character. Investigation is one of the methods of teaching that can contribute to this.

The book in preparation - "The Investigative Approach in School Mathematics" will show teachers what a powerful educational instrument problem-solving is and will also show the value of mathematical experimentation in new situations. In addition, it will supply sufficient resources to enable them to use these techniques in their classrooms. The book will contain a large number of solved problems and many situations with investigations carried out. These problems and situations can be used at many different levels of instruction.

Now we give an example of an investigation of a situation. This problem is not only interesting in its own right, but can be used to improve a student's understanding of the properties of negative numbers.

Problem 1: The square $ABCD$ is divided into nine cells, as shown below. In each of the cells, we write either the number $+1$ or the number -1. We construct a new square $A_1B_1C_1D_1$ from $ABCD$ as follows: the number in each cell of $ABCD$ is replaced by the product of the numbers in the neighboring cells. (Two cells are called neighboring if they have an edge in common).

What happens when this process is repeated?

Experiment: We start with the square $ABCD$ shown in Figure 1. Using the procedure described above, we obtain $A_1B_1C_1D_1$. Be sure to notice that some of the cells in the square have two neighbors, some have three, and one has four! Upon repeating the procedure, we obtain the square $A_2B_2C_2D_2$. Carry out at least two more steps.

Do several experiments; that is, start with different arrangements of the numbers $+1$ and -1 in the cells of the square $ABCD$. You will almost certainly find that the squares generated have certain properties. Before reading further, write down in your own words what you have found.

Here are some possible findings, which we call **conjectures**:

$$
\begin{array}{ccc}
D \qquad\qquad\quad C & & D_1 \qquad\qquad\quad C_1 & & D_2 \qquad\qquad\quad C_2 \\
\begin{array}{|c|c|c|}\hline -1 & 1 & 1 \\ \hline -1 & -1 & 1 \\ \hline 1 & 1 & -1 \\ \hline \end{array} & \Rightarrow & \begin{array}{|c|c|c|}\hline -1 & 1 & 1 \\ \hline 1 & -1 & 1 \\ \hline -1 & 1 & 1 \\ \hline \end{array} & \Rightarrow & \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline -1 & 1 & -1 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \\
A \qquad\qquad\quad B & & A_1 \qquad\qquad\quad B_1 & & A_2 \qquad\qquad\quad B_2
\end{array}
$$

Figure 1: Problem 1 - Experiment

(i) After at most two steps, the number in the middle cell becomes $+1$ and never changes, regardless of the number of steps carried out (see Figure 2a).

(ii) After at most two steps, the initial square $ABCD$ is transformed into a square which has the symmetry pattern shown in Figure 2b.

(iii) After at most three steps the initial square $ABCD$ is transformed into a square which is symmetric about all four axes of symmetry of a square (see Figure 2c).

(iv) After at most four steps, the initial square $ABCD$ is transformed into a square in which each cell contains the number $+1$ (see Figure 2d).

$$
\begin{array}{cccc}
\begin{array}{|c|c|c|}\hline & & \\ \hline & 1 & \\ \hline & & \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & 1 & d \\ \hline c & b & a \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline a & b & a \\ \hline b & 1 & b \\ \hline a & b & a \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \\
a & b & c & d
\end{array}
$$

Figure 2: Problem 1 - Conjectures

It is important to note that at this point, these conclusions are only conjectures. Unless we had investigated all possible arrangements (there are 512 of them), we couldn't be sure without more thought that the conjectures are correct. However, we can prove that our conjectures are always true.

Proof: We fill the cells of the square $ABCD$ with the variables $a_1, a_2, \ldots, a_9$, each of which is either $+1$ or -1. We now perform four iterations of the operation to generate new squares. Along the way, we can simplify our work by taking advantage of the fact that $(a_i)^2 = 1$. This is true since $1^2 = 1$ and $(-1)^2 = 1$. We show this process in Figure 3.

Since we have proven our conjectures, we can rename them as **theorems**.

We can easily generalize our fourth conclusion; for instance if we allow each cell of the square $ABCD$ to contain any nonzero real number and not just $+1$ or -1, then after at most four steps we will get a square containing positive numbers only. Also, if we place either integers or the square roots of integers in the cells of the square $ABCD$, then after at most four steps all cells will cointains only integers.

Problem 2: Place a 0 in one of the cells of the square $ABCD$. In all others, place either the number $+1$ or the number -1. Investigate what squares we can obtain by repetitions of the above procedure.

Note that we can conduct many experiments: the single 0 could be in the center cell, in one of the four corner cells, or in one of the remaining four cells. Also, for each location of the 0, there are 256 different arrangments for the remaining eight cells.

Before going on, experiment and make a conjecture.

$$
\begin{array}{|c|c|c|}\hline a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ \hline a_7 & a_8 & a_9 \\ \hline \end{array}
\Rightarrow
\begin{array}{|c|c|c|}\hline a_2a_4 & a_1a_3a_5 & a_2a_6 \\ \hline a_1a_5a_7 & a_2a_4a_6a_8 & a_3a_5a_9 \\ \hline a_4a_8 & a_5a_7a_9 & a_6a_8 \\ \hline \end{array}
\Rightarrow
$$

$$
\Rightarrow
\begin{array}{|c|c|c|}\hline a_1^2a_3a_5^2a_7 & a_2^3a_4^2a_6^2a_8 & a_1a_3^2a_5^2a_9 \\ \hline a_2^2a_4^3a_6a_8^2 & a_1^2a_3^2a_5^4a_7^2a_9^2 & a_2^2a_4a_6^3a_8^2 \\ \hline a_1a_5^2a_7^2a_9 & a_2a_4^2a_6^2a_8^3 & a_3a_5^2a_7a_9^2 \\ \hline \end{array}
=
\begin{array}{|c|c|c|}\hline a_3a_7 & a_2a_8 & a_1a_9 \\ \hline a_4a_6 & 1 & a_4a_6 \\ \hline a_1a_9 & a_2a_8 & a_3a_7 \\ \hline \end{array}
\Rightarrow
$$

$$
\Rightarrow
\begin{array}{|c|c|c|}\hline a_2a_4a_6a_8 & a_1a_3a_7a_9 & a_2a_4a_6a_8 \\ \hline a_1a_3a_7a_9 & 1 & a_1a_3a_7a_9 \\ \hline a_2a_4a_6a_8 & a_1a_3a_7a_9 & a_2a_4a_6a_8 \\ \hline \end{array}
\Rightarrow
\begin{array}{|c|c|c|}\hline a_1^2a_3^2a_7^2a_9^2 & a_2^2a_4^2a_6^2a_8^2 & a_1^2a_3^2a_7^2a_9^2 \\ \hline a_2^2a_4^2a_6^2a_8^2 & 1 & a_2^2a_4^2a_6^2a_8^2 \\ \hline a_1^2a_3^2a_7^2a_9^2 & a_2^2a_4^2a_6^2a_8^2 & a_1^2a_3^2a_7^2a_9^2 \\ \hline \end{array}
=
$$

$$
=
\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}
$$

Figure 3: Problem 1 - Proof

Conjecture: No matter where we insert the 0, after at most four steps the situation shown below will occur (see Figure 4).

$$
\begin{array}{|c|c|c|}\hline 1 & 0 & 1 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array}
\;\underset{\longleftarrow}{\overset{\longrightarrow}{}}\;
\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array}
$$

Figure 4: Problem 2 - Conjecture

We could attempt to prove that our conjecture is correct by experimentation. The amount of work we would have to do could be lessened by taking advantage of symmetry. However, it may be preferable to use the algebraic technique illustrated above. Can you show that putting the 0 in the middle cell allows us to reach the situation in Figure 4 in the fewest number of steps and putting it in one of the corner cells requires the greatest number? If we have proven our conjecture, either by much experimentation (all cases must be checked) or by an algebraic argument, we can then rename conjecture as a theorem.

Problem 3: We wish to obtain a 0 in every one of the cells of the square after some number of repetitions of the operation.

(i) What is the smallest number of zeros which must be placed in the original square $ABCD$ to guarantee that this will happen?

(ii) Where must these zeros be placed?

Answer: We must start with two zeros - one must be in one of the positions labeled a in Figure 5 and the other must be in one of the positions labeled b. We will always obtain a square filled with only zeros after the second step.

a	b	a
b	a	b
a	b	a

Figure 5: Problem 3 - Answer

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About the schoolbooks of probability for the Secondary grammar schools

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At the present time we use the schoolbooks of T. Hecht ([2]) and B. Riečan ([4]) for teaching the probability at the Slovak secondary grammar schools. The schoolbook of E. Calda and V. Dupač ([1]) is used for teaching in the Czech Republic. Each of them has a special structure and level of difficulty. The aim of this paper is to inform teachers about the contents and the structure of these three schoolbooks.

E. Calda - V. Dupač: Mathematics for the Secondary grammar schools - Combinatorics, probability and statistics

First of all we will pay attention to less known Czech schoolbook in our country. Probability is discussed there in the second chapter. Right in the introduction we find the examples of random experiments and their usage with the explanation which experiments are called random. We can find the division of standard and other experiments with lot of practical examples, too.

Section *The random experiments* is followed by *The set of all possible results of experiment.* Authors proceed from known to unknown there. They show the construction of the set Ω on example of tossing four coins. At the same time they define which set we call the set of all possible results of experiment and they establish its denotation. They point out that it is important how fine we want to distinguish elements of the set Ω. Once again this fact is demonstrated at particular example. We can find good tasks with application in the book on page 82. They are concretely number *2.2, 2.3* and *2.4*.

The introductory example about the casting two dice and about the pulling the lottery Športka creates a nice entry to the definition of a event in section *Events* on page 83. Authors introduce terminology and the operations with events. They also explain it schematically on two solved examples (*Example 1, Example 2*).

In section *Probabilities* we can see the nice passage from relative frequency to probability without using strict definition. The exact value of probability is considered as unknown constant that is lying somewhere near known value of relative frequency. Analogies of defining the notions from physics, casting a die, entries from statistical annuals, large-scaling records from manufacturing processes of components and many more are used as examples. So students get gradually to the following information: if we have the random experiment with the set Ω of all possible results, then the probabilities $p(\omega)$ of these results are non-negative numbers whose sum equals one.

In section *Probabilities of events* there is an example of a family with three children that presents the interpretation of the following information. The probability of an event A is the sum of probabilities of the results belonging to the event A. We can find there a suitable illustration of the case, where the results do not have same value of probability. It is the solved *Example 4* about the casting two dice and the note listed below this example on page 94. Authors encourage the student to decide which of the given events is more probable in two solved examples (*Example 7, Example 8*). As a random experiment they use casting a die. The tasks numbered *2.15, 2.16, 2.20* are suitable for this exercise. Their topics are: pulling questions on an exam, pulling in lottery, accepting the supply delivery by consumer in accidental check out.

The section *The addition of probabilities* deals with the properties of probability as probability of a sum of events, probability of a complementary event and inequalities. All of these sections end up with a short summary of the notions and formulas.

A lot of interesting tasks for exercising the notions explained in previous sections are on page 104 in section *The revision tasks.* We should mention e.g. a nice task *2.31* for thinking, which defines the question: Are given values of probabilities for a horse victory possible? Similar to this task is the task *2.33* about probability of victory of football team Sparta or Slávia. Task *2.32* about the assumed dice is also suitable. The tasks number *2.37, 2.39, 2.40, 2.41, 2.42, 2.44, 2.45* belong to the tasks from ordinary life. Their topics are school (class) and students (teacher), shop and its customer, football cup competition, family and children.

Next sections are dedicated to the topic of independence and topics related to it. In an introduction of the first section *The independent events (Multiplication of probabilities)* authors use the notion of independence in context "does not have influence". The variety examples lead gradually to the definition of the independence of two events. In this way is also introduced the independence of multiple events. Chosen approach using the exploring spirit of students, their natural curiosity and the will to discover something new turned out to be convenient. It can be seen on page 108, where students are asked the question: "Let A and B be the independent events. Are then events A' and B' independent? Afterwards the representation of events by set diagrams and use of gained knowledge follow. Similarly they solve this problem for events A', B and A', B'. The nice examples from real life are tasks number *2.50, 2.51, 2.52* with topics: technical university and the percentage of failing students, device and blocks that function, product and its visual and functional defect. The solved *Example 2* on page 109 about the product and three kinds of errors can be also used.

Section *The independent experiments* is very good introduction to the section *Binomial distribution (Bernoulli scheme).* On examples of independent and dependent experiments authors explain when the individual experiments are independent. Solved *Example 1* with interesting assignment can be found on page 112. It deals with a pair of tosses of non-ideal coin, whose heads falls with probability $p = \frac{1}{2} + \varepsilon$ and tails with $p = \frac{1}{2} - \varepsilon$. Task is to compute the probability of event A "both tosses get the same result". Nice tasks from practice are tasks number *2.54, 2.59* about two doctors and one diagnosis of patient and about two controllers of account book.

Previously mentioned section called *Binomial distribution* begins with natural reasoning about n independent experiments, such that each of the experiments has only two possible results (success Z, failure N). Special case, where all experiments are finished with the same results success (or failure) is illustrated on the particular example (*Example 1*) about phone line and ten of its users. Questions How can we proceed in the case, when k experiments are successful and n are failures? What is the probability of such result? refer to the above mentioned *Example 1.* The formula for probability of events "precisely k of n experiments are successful" is the outcome of the example. In this section we can find the explanation of the meaning of $P(B_k)$ and $P(C_k)$, where B_k is "at least k experiments of n are successful" and C_k means "at most k experiments from n are successful". The solved *Example 3* about family with three children and *Example 4* about a student and test with ten questions are here for exercising. To practical tasks we can add tasks number *2.61, 2.62, 2.63, 2.64, 2.66* with topics: familial disease, basketball, success of a medicament, tests, illness diagnostic tests, automats and its failures. The task *2.67* about a device and its parts that can fail is good thinking task.

An interesting way of explaining the notion of conditional probability can be found in section *The conditional probabilities.* It begins with following reasoning: Let us have an experiment with possible results ω with probabilities $p(\omega)$. Let us imagine that in the process of the experiment we get to know that event B occurred. It means that we have to replace the "old" probabilities $p(\omega)$ with the "new ones" which we denote $p(\omega|B)$. We name them the conditional probabilities by condition B. We put $p(\omega|B) = 0$ for $\omega \notin B$ because these results can not occur. We have to raise the probabilities $p(\omega|B)$ for $\omega \in B$ proportionally in order to obtain their sum equal to one. So $p(\omega|B) = \frac{p(\omega)}{P(B)}$ for $\omega \in B$. The conditional probability of each event A by condition B is defined in a common way as a sum of probabilities of results belonging to the event A. So $p(A|B) = \sum\limits_{\omega\in A} p(\omega|B)$. Substituting $p(\omega|B)$ with $p(\omega|B) = \frac{p(\omega)}{P(B)}$, for $\omega \in B$ we obtain $p(A|B) = \sum\limits_{\omega\in A\cap B} \frac{p(\omega)}{P(B)} = \frac{P(A\cap B)}{P(B)}$. The nice *Example 2* about golden and silver coins and drawers is on page 122. The tasks number *2.70, 2.74, 2.75* are suitable for exercising and training the logical reasoning. They tasks are of the following types "What is the value of $P(B|A)$, if $B \subset A$?", "What is the value of $P(A|A)$?", etc. The practical applications concerning horse race, production of goods by two lines and population and disease, respectively are topics of the tasks *2.71, 2.72, 2.75.* This part is again concluded with *Revision II* of basic notions and formulas.

The last section *The revision tasks* is composed by a set of tasks ranged from the independence of events to the conditional probability. Tasks number *2.78, 2.84, 2.85, 2.87, 2.88, 2.91, 2.92, 2.93, 2.94, 2,96* with topics device and its blocks, secretaries and their arrival to work, riflemen and target, company and defective products, diabetics and smokers, race, a workshop and its workers, criminals and warden are again very nice tasks.

T. Hecht et al.: Mathematics for 4^{th} year of Secondary grammar schools. Probability. Statistics

This schoolbook belongs to the series of thin fascicle, which process the content of mathematic education at Secondary grammar schools. These schoolbooks present new concept of education. It is based on practical and theoretical experimentation. Aforementioned schoolbook is the third fascicle of mathematics for the fourth year. We can find the probability in the fourth chapter there although it is the first subject in this fascicle. The second one is statistics.

Topic of probability begins with the nice incentive introduction. The author explains what is the object of probability theory interest on the particular examples with applications. At the same time he gives to the student the overview of the topics concerned in next pages.

The tasks oriented on experimentation and sorting the events into five groups (most likely, likely, random, seldom, unlikely) are listed in section *The probability and the relative frequency.* The proper events for experimentation are the events from number 1 to 9. They are related to the random experiments like tossing a coin, casting a die, birthday, pulling balls from a bag, etc. So students acquire the practise from this experimentation, which is the key for developing the estimate, the sense for probability. In this section we can find the short explanation of the notions like the random experiment, the event, the relative frequency and the probability, too. The natural connection between the notions is missing there. On contrary there is the nice categorization of the probability estimate. The author introduces four ways of this estimate: the real experimentation, the simulation of experimentation on computer, the thought experiment and the calculation by formulas. The thought experiment in Hecht's sense is very similar to the way the "probability abacus" works.

Previous section is followed by section *The experimentation, the simulation.* It is dedicated to two kinds of simulations: the simulation by computer and the simulation by thought experiment. It would be good to choose other task as a sample for the simulation on computer, because it is quite difficult for understanding. As the convenient replacement we can use *Task 1* from *Exercise* on page 7. In *Example 1* it is necessary to explain the constriction of the tree used in the thought experiment, which can be treated as the scheme of probability abacus. The suitable task from exercise is *Task 2* about a figure and a hole.

The section *The probability tasks solved with the use of combinatorics* begins with the theoretical introduction to the problems of probability calculation by Laplace scheme. The nice task from ordinary life is the solved *Example 3* about a lottery for sightseeing tour. The author uses the combinatorics for solving this example. *Example 4* about dividing the coins with incorrect setting is a good task for exercising the logical reasoning.

The next section is *Laplace scheme, the properties of probability.* It again begins with the abstract explanation of general Laplace scheme. The natural derivation why the sum of probability of all elementary events equals to one is missing there. Here we can find the formulas for computing probabilities of complementary event and union of disjoint events A and B. All acquired knowledge is exercised on solved examples. We can mention e.g. the classic example about assumed dice. The interesting task in this section is *Example 4* about three riflemen shooting at target solved only by thought experiment. It is the task oriented on calculation of the conditional probability. The nice task is *Task 3* about journeyer and four ways on page 13, too. It is again the task on the conditional probability. The tasks from practice are tasks number 1, 2 with topics: horse race, lottery Športka in the exercises on page 13.

The section *The conditional probability* begins with solved *Example 1* with nice introductory explanation of the change of the probability of events, if the further information is known. This task is followed by *Example 2* about cancer and medical tests. It is solved by two methods: the thought experimentation and the use of formulas. In the thought experiment, the tree, which would illustrate the relations between the events, is missing. I would propose the following modification. I would skip the solution using the formulas because it uses the results, e.g. the formula for complete probability that were not explained before. Similar problem occurs in *Examples 3* about students and their talents for writing poems. This example is solved only by formulas. The task number 2 with topic of product quality and its production in two plants belongs to the applied tasks.

The section *The independence of events* also begins with very nice example about tossing the coin. It would be good to add the tree by the thought experiment and to explain better why $P(B|A) = P(B)$. The suitable tasks for logical reasoning are solved *Example 2, Example 3* with incorrect setting. The interesting tasks are the tasks number 1, 2 and 5 about the quality of consingnment, the solving a task by a group of students, and the shooting on target.

The author begins the section *The sequence of independent experiments (Bernoulli scheme)* with motivation about an experiment, where the event A occurs with the probability p. Naturally he gets to the thought that we can look at the r iterations of this experiment in the same conditions like at compound experiment. Then we are so near to derive the Bernoulli scheme while using acquired knowledge. The solved *Example 1, Example 2* with topic about tests and random answers of students on questions and tasks number 1,2 and 4 with topics: control of series of products, the sex of kids in family are nice practical tasks.

The last section is *Bazaar* with tasks on geometrical probability (*Example 2, Example 3*). In conclusion the part called *Blockbuster or what is important*, where is the short revision of notions could be found.

B. Riečan: Mathematics for 3^{rd} year of the Secondary grammar schools. Probability and statistics

Many students of Secondary grammar schools know the schoolbook *Probability and statistics* as a offprint of schoolbook *Mathematics for* 3^{rd} *year of Secondary grammar schools*. The second chapter called *Elementary notions of probability theory* is devoted to the probability. The applications of probability and statistics are discussed in the third chapter.

The second chapter begins with the section *The notion of probability*. The author introduces the classic definition of probability (Laplace scheme). However, the explanation of basic notions as random experiment, the set of all results of random experiment Ω is missing there. He explains the choice of the set Ω and the calculation of probability by using Laplace scheme on classic examples of random experiments as casting a die and tossing a coin. The nice tasks from ordinary life are the solved *Example 4* about the selection of components from a box and *Example 5* about ticket automats. The first one specializes on Laplace scheme and the second one on general Laplace scheme. On page 36 we can find the suitable practical tasks with combinatorial approach. They are tasks number *2.1, 2.2, 2.4, 2.5, 2.6* with the topics: a selection of components and the control of defective components, a draw of numbers in lottery Mates, a shop and TV sets of first quality, furniture workshop and the quality of chairs, mineral waters and unstuck tags. Task number *2.11* about non-ideal die is the nice task for reasoning.

In the second section *The properties of probability* the properties of probability and their proofs with the use of the Laplace scheme are discussed. On the example the author explains the formula for probability of complementary event. We can find there the nice reflection on the probability of an event as an area and a suitable graphic representation by using the set diagram. Solved *Example 2* about group of students and their date of birth and tasks number *2.18, 2.19, 2.21, 2.24, 2.25* with the topics: control of consingnment of shoes, school canteen and oversweet compotes, control of homework, cloakroom and hats, group of people and birthday can be included to tasks from practice. The task *2.26* is the task for logical reasoning.

The section *Independences* begins with the example that explains the notion of independence of experiments. There is no remark about meaning of used denotation $P(B|A)$. In solved *Example 2*, the representation of events B_1, B_2 by set diagram is missing in the list of elements of the set Ω. Author formulates the condition about independence of two events in the conclusion. The nice task *Example 4* is about riflemen which is solved by calculation and by logical reasoning, too. *Example 5* about the reliability of a system of two indicator lamps and the tasks number *2.29, 2.30, 2.31, 2.32, 2.33* with the topics: a measuring apparatus and vacuum valve tubes, a price of publishing houses, an inspection and the approval of flats, a shop and goods after warranty, a system of three lights in the corridor are the suitable tasks from ordinary life.

The last section in this chapter is *Bernoulli scheme*. It begins with nice historical introduction and with explanation what is the main point in Bernoulli scheme. The problem of calculation the probability of an event and its generalization to n experiments with k successes in given experiment is analyzed there. We can see it in *Example 1* on seven casts a die. We can include the solved *Example 3* about automatic lathe and defective components and the tasks number *2.39, 2.40* about family with four kids, student and test with ten questions among applied tasks. The tasks for exercising are listed in the conclusion under name *Let examine yourselves*. From these are nice the tasks 1A, 2A, 3A, 3C on pages 52, 53.

Third chapter called *Others applications of probability and statistics* begins with section *The estimate of probability by using the relative frequency*. Right in the introduction we can find good motivation on particular examples from life (*Example 1, Example 2*). This motivation says: Even though we know the value of relative frequency (ascertaining percentage), but we do not know the real percentage, the real probability. The following cases are discussed on particular example of twelve casts a die as follows: ***(i)*** what is the change of reliability changed, if we specify the estimate, ***(ii)*** if we want to increase the reliability, then we must increase the number of iterations of experiment, ***(iii)*** what is the interpretation of 90-percentage certainty that the probability p lies in some interval. The nice tasks from ordinary life are tasks number *3.1, 3.2, 3.3, 3.5, 3.6* with the topics: visitors of Slovak national theatre and suitable hour of performance, a preparation against the potato beetle, a birth of boy, a plant for manufacturing ball bearings and rejects and effectiveness of medicament.

The second and last section is *Testing of statistical hypothesis*. Author gets to student basic idea about the testing hypothesis right in the introduction. Next follows the sequence of solved examples. He explains how we compose the hypothesis, the critical area for given α, the criterion of flat refusal of hypothesis, etc. This sequence of examples is not very suitable, because students get the information how to solve these tasks only at the end. It would be more suitable to explain the following: ***(i)*** how we compose the critical area W for concrete α (solved *Example 5, Example 3*), ***(ii)*** how we calculate the rejection or not rejection of the hypothesis for given α. First explanation of these notions from definition on page 60 could be before this sequence of examples. *Example 4* about testing the medicament on reducing blood pressure and tasks number *3.10, 3.11* about student and test with three or two alternative answers, respectively, are tasks from practice. The set of tasks for exercise in part *Let examine yourselves* are in the conclusion of the chapter. The suitable tasks from real life are tasks *1A, 1B, 1C, 2A, 2B, 2C, 3A, 3B, 3C* with topics: the Slovak philharmonic orchestra and the music of twentieth century, an audience of TV program, a test of the medicament, a control of the rejects and an examination of the student.

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Some remarks on students' knowledge of the absolute value

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In the paper the results of research with applying the knowledge and mathematical abilities in solving problems related to absolute value (by students) are presented.

Every teacher of mathematics, at every level of education, teaches certain facts (included in the curriculum), certain methods and rules, as well as the language of the subject, which, with new facts covered, develops and becomes gradually more extensive. There is a danger, however, that too heavy an emphasis laid on certain methods of solving problems or continuous demonstration of the same patterns may result in an incomplete, one-sided development of cognitive structures in a pupil's mind.

One of the important mathematical notions with which pupils come into contact in gymnasium and which is developed at subsequent levels of mathematical education is the absolute value. Because of its properties of being non-negative, subadditive and positively homogeneous, it has wide applications and generalisations in nearly all branches of mathematics. At the same time, it has a relatively lucid definition and a suggestive geometrical interpretation (as the distance of the point associated with a given number from the point 0 on the real axis).

The analysis of problems and examples presented in textbooks and workbooks currently used at various levels of mathematical education in which the notion of the absolute value appears is a source of the reflection that in the majority of them the stress is laid on algorithmic approach to solving equations and inequalities with absolute value, namely on solving equations and inequalities by the method of equivalent equations (or inequalities) combined with considering suitable cases with respect to the range of applicability of particular formulae. Hardly ever does a pupil encounter exercises which could be solved by applying either properties of the notion or the denotative approach. Simultaneously, various materials designed for a pupil present such solutions only sporadically. Teaching the notion of the absolute value is frequently reduced to developing computational techniques related to algorithmic methods of solving, while devoid of any reflection upon possibilities of solving them in some other way. This opinion is confirmed by the remark of B. Bugajska-Jaszczołt and G. Trelinski: *The analysis of textbooks' presentations shows that in numerous cases the authors do not focus their efforts on developing the concept but, instead, on examples which enable to compose a „smooth" formulation (solely) of the text of its definition. As classic examples (at the level of gymnasium) one may quote the no-*

tions of the function or of the absolute value of a number. (Bugajska-Jaszczołt, Treliński, 2002).

According to W. Wierzbicki, *The absolute value is an important, though underestimated (in school education),mathematical notion, without which familiarity with numbers is incomplete. By no means is this notion simple for a pupil and it may be understood only if illustrated with carefully selected examples and after solving suitable problems.* (see Wierzbicki, 1967).

In view of the above remarks it seems crucial that in the case of students of mathematics the notion of the absolute value is properly developed, that these students are aware of its various applications and, what is most important, that they are able to make an operational use of it.

The objective of this paper is to present the results of research concerning students work on series of equations and inequalities with absolute value. Methods of approach towards these problems and characteristic difficulties that appeared in the course of the students' work are also discussed.

As research instruments the following two series of problems were employed. Problem. Solve the equations and inequalities:

Series 1	Series 2
a) $\lvert x^2\rvert + \lvert x^2+1\rvert = 2x^2+1$;	a') $\lvert 4^x\rvert + \lvert x^2\rvert = 4^x + x^2$;
b) $\lvert 2x-4\rvert \geq 0$;	b') $\lvert \cos x - \ln x + \sin x\rvert \geq 0$;
c) $\lvert 2x+3\rvert + \lvert x-7\rvert = 0$;	c') $\lvert \log x\rvert + \lvert -\cos x\rvert = 0$;
d) $\lvert x+3\rvert\lvert x-4\rvert \geq -2$;	d') $\lvert\sqrt{\log(2^x+1)}+1\rvert\lvert x-4\rvert \geq -3$;
e) $\lvert 2x-8\rvert + \lvert x-4\rvert \leq 0$;	e') $\lvert \sin x\rvert + \lvert 2^x-4\rvert \leq 0$;
f) $\lvert x^3+2x-3\rvert \leq \lvert x^3\rvert + \lvert 2x-3\rvert$;	f') $\lvert x^3+2^x-4\rvert \leq \lvert x^3\rvert + \lvert 2^x-4\rvert$;
g) $\lvert 2x-x^2\rvert \geq \lvert 2x\rvert - \lvert x^2\rvert$;	g') $\lvert\sqrt{x+1}-12^x\rvert \geq \lvert\sqrt{x+1}\rvert - \lvert 12^x\rvert$;
h) $\lvert\frac{x+3}{x+5}\rvert\lvert x+5\rvert = 2$;	h') $\lvert\frac{2^x-8}{\sin x+\frac{1}{2}}\rvert\lvert \sin x + \frac{1}{2}\rvert = 8$;
i) $\lvert x-6\rvert + \lvert x^3-3\rvert = -3$.	i') $\lvert 2^x-5^x\rvert + \lvert\log(x+1)^2\rvert = -3$.

The problems of the first series are formulated in such a way that it is possible to solve them by applying the method of equivalent equations (inequalities), in an algorithmic way, that is to say, by considering the cases with respect to the range of applicability of particular formulae. As for the problems of the second series, solving them by the same method is far from easy. At the same time, both series of problems may be solved by applying suitable properties of the absolute value. Problems a and a' (similarly: b and b' etc.) may be solved by employing the same property of the absolute value.

The study was conducted among a group of intramural students of the first, third and fourth year of mathematics of Pedagogical University of Cracow. The total of 103 student participated in the study. They were divided into groups. Part of them were working (for 45 min) on one of the series of problems. The problems of the first series were being solved by 5 students of the first year and 5 students of the third year, the second series was presented to 21 first year students, 5 third year students and 24 students of the third year. The remaining persons — 26 first year students and 17 students of the third year of mathematics — were solving (for 90 min) both series of problems.

The analysis of the students' solutions enabled distinguishing three methods of solving the problems of series 1 and 2:

α) algorithmic approach employing the method of equivalent equations (inequalities) accompanied by applying the definition of the absolute value and considering the cases with respect to the range of applicability of particular formulae,

β) algebraic approach to solving equations and inequalities employing properties of the absolute value,

γ) graphic methods of solving equations and inequalities.

Table 1. shows the quantitative classification of choices of solving method for the problems of series 1 and 2 made by particular groups of students.

Group	Solving	Series 1									Series 2								
	method	a	b	c	d	e	f	g	h	i	a'	b'	c'	d'	e'	f'	g'	h'	i'
I year	α	16	22	23	17	20	9	14	13	7	4	3	4	2	6	3	1	3	4
26 per-	β	13	5	3	9	6	9	6	13	9	20	14	8	9	9	14	14	16	11
sons	γ	0	0	0	0	1	0	0	0	0	0	1	5	0	3	0	0	0	0
I year	α										3	9	12	6	4	4	3	9	1
21 per-	β										18	7	4	7	11	12	14	12	12
sons	γ										0	0	2	0	1	0	0	0	0
III year	α	3	11	11	10	9	6	7	4	4	0	1	0	2	1	1	0	1	1
17 per-	β	13	7	8	6	7	9	4	13	8	14	12	13	11	10	13	9	13	11
sons	γ	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0
III year	α	2	1	2	1	2	2	0	1	0									
5 per-	β	3	4	3	3	3	3	3	3	3									
sons	γ	0	0	0	0	0	0	0	0	0									
III year	α										2	2	2	3	2	1	0	2	1
5 per-	β										3	3	2	3	2	3	3	3	3
sons	γ										0	1	1	0	0	0	0	0	0
IV year	α	0	1	2	2	1	0	0	2	0									
5 per-	β	5	3	2	3	3	4	2	3	4									
sons	γ	0	0	0	0	0	0	0	0	0									
IV years	α										1	11	13	9	6	5	2	7	2
24 per-	β										21	9	10	11	12	12	12	17	11
sons	γ										0	1	2	0	3	0	0	0	0

Table 1.

It is necessary to point out that the number of persons solving a problem by a particular method is in some cases greater than the number of persons taking part in the study. The reason lies in the fact that sometimes students gave two solutions of the same problem or, like in the case of problems h and h'), used properties of the absolute value and employed method α described above further on in their solutions.

The observation of the students' work and the subsequent analysis of their papers suggests that at the beginning nearly all the participating students used the definition of the absolute value and considered the cases with respect to the range of applicability of particular formulae. In the course of their work certain participants of the study gradually changed their attitude towards the problems. Occasionally they were coming back to the problems they had already solved and offered a different justification of the answer.

The data in the table show that the majority of the participants of the study used the first of the methods of approach characterised above (method α). In the case of both series1 and series 2 this method was used for solving problems b and c (b' and c', respectively)

by the majority of students. In both series of equations and inequalities most participants chose method β to solve problems a and h (a' and h', respectively).
Only in a small number of students' papers did sketches of graphs appear. The problems solved by the graphic method were problems e, b', c', e'. This method of solving was employed most frequently in the case of problem c' (11 persons chose this approach).

There were certain persons among the participants of the study (approximately 10%) who attempted to solve all the problems by the method of equivalent equations (inequalities) and, when it failed to bring results, undertook no further attempts to solve a problem by some other method.

It is worth stressing that over a half of the whole group of the students using the first of the described methods of solving made mistakes in their solutions and, as a consequence, failed to obtain the correct answers. In the cases when one of the remaining methods was chosen the number of persons who obtained correct answers was considerably higher.

A limited number of the participants chose the same approach in both series, using properties of the absolute value from the beginning. Among students' papers there are some in which no attempt was made to solve certain problems, and the number of such persons was notably greater among those who worked on series 2. This happened predominantly in the case of problems g' and i'.

The results of the study confirm the opinion that students have difficulties with understanding the notion of the absolute value. One may therefore quote Z. Krygowska's words that they have mastered skills in applying procedures but without being fully conscious of facts (see Krygowska, 1989). Such a students' attitude was undoubtedly influenced by activities confined to the plane of superficial structures, favoured during the process of education (see Z. Semadeni, 2002).

The examples from the students' papers presented below demonstrate a lack of knowledge and skills related to the absolute value, as well as a lack of competence in solving equations and inequalities in which this notion appears.

Example 1. (an extract from a solution of problem b')

$|\cos x - \ln x + \sin x| \geq 0.$

I consider two cases:

$$\begin{cases} \cos x - \ln x + \sin x \geq 0 \\ \cos x - \ln x + \sin x < 0 \end{cases} \mid \Rightarrow x \in \emptyset.$$

The student applied the definition of the absolute value, namely considered two cases with respect to the range of applicability of formulae; however, further on a mistake was made. The cases were treated as the conjunction of conditions and the student draws the conclusion that the initial inequality has no solutions.

Example 2a. (an extract from a solution of problem d)

$|x + 3||x - 4| \geq -2.$

The inequality is always satisfied since the left-hand side is always greater than 0 or equal to 0 as the product of two ***positive*** *numbers.*

Example 2b. (an extract from a solution of problem i')

$|2^x - 5^x| + |\log(x + 1)^2| = -3.$

The sum of ***positive*** *numbers is positive, so there is no x that satisfies this equation.*

Example 2c. (an extract from a solution of problem e')

$|\sin x| + |2^x - 4| \leq 0$ *It is not true!!!*

The sum of two ***positive*** *numbers is positive!*

In examples 2a, 2b and 2c we present extracts from students' papers which contain mistakes revealing the fact that the persons solving the problems identify the absolute value of every real number with a positive number. Example 2a may also indicate difficulties at the level of language, namely that the student assigns the meaning of "nonnegative" to the word "positive".

Example 3. (an extract from a solution of problem i)

$|x-6|+|x^3-3|=-3$

$|x-6|+|x^3-3| \geq |x-6+x^3-3|$ *by the triangle inequality*

$|x-6+x^3-3|=-3$

The source of this mistake lies in an incorrect use made of the correctly formulated theorem. As a result, the student obtained the equation which is not equivalent to the initial one.

Example 4. (an extract from a solution of problem g)

$|2x-x^2| \geq |2x|-|x^2|.$

By the triangle inequality, this inequality is always false.

The error committed here reveals inability to apply or ignorance of mathematical theorems. The author of the solution quotes the triangle inequality in the situation where it cannot be applied (the assumptions of the theorem are not satisfied).

Example 5. (an extract from a solution of problem a')

$|4^x|+|x^2|=4^x+x^2$

The equation is equivalent to the disjunction of systems:

$\begin{cases} x \geq 0, \\ 4^x+x^2=4^x+x^2; \end{cases} \quad \begin{cases} x<0, \\ -4^x-x^2=4^x+x^2. \end{cases}$

Example 5 indicates the student's difficulties connected with specification of the definition of the absolute value. Similar mistakes were made by a considerable number of students in solutions of other problems. These were the most "popular" mistakes, that is to say, were committed by the greatest number of the participants.

In summary, it may be stated that that the students working on the problems came across various difficulties. The occurrence of theses difficulties was independent of the method chosen for solving particular problems. They were related to specification of the definition of the absolute value, with transformations of graphs (creating the graphs of functions $|f(x)|$, $f(|x|)$ when the sketch of a function $f(x)$ is given). What gives rise to certain concern is the fact of applying "false theorems" during the work on the problems and incorrect application of mathematical theorems, which may be connected with misunderstanding them.

It seems that as basic causes of these mistakes the following circumstances may be pointed out:

— instrumental familiarity with the definition and theorems concerning the absolute value, with absence of thorough comprehension,

— inappropriate application of the definition and theorems concerning the absolute value (including widening the range of the definition and theorems concerning the absolute value),

— applying the "intuitive definition" of the absolute value,

— applying imprecise expressions of everyday language in a way which is inconsistent with the language of mathematics.

It is worth pointing out that no notable difference in approach was noticed between the groups of participants. The students at all levels of mathematical education were making similar mistakes and encountered the same difficulties. One may put forward the hypothesis that mathematical facts (including applications and generalisations of the absolute value) conveyed to students during their academic education have no significant influence on the way they approach elementary (school) examples of equations and inequalities in which the absolute value appears. What also seems to find its confirmation is the opinion that working on methods of solving equations and inequalities with the absolute value at school results in a meagre, one-sided development of cognitive patterns in pupils' minds. In our opinion, a content and form of problems in which the absolute value appears should be enriched. Diverse problems and their various solutions should be therefore offered to pupils and students at every level of mathematical education.

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Intuitions associated with conditional probability

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In the paper intuitional to conditional probability graduates and students (before their starting learn theory of probability) are presented.

The present paper deals with intuitive understanding of conditional probability at various levels of mathematical education, manifested by entrants and students. Our observations suggest that various students' difficulties are connected with this notion. The "minimum curriculum" of basic mathematical education for secondary schools does not include conditional probability, and for this reason it is during their studies, on probability classes, when the majority of students come into contact with this notion for the first time. The paper presents the results of research investigating the intuitive understanding of conditional probability on the basis of one particular problem. Our intention was to find out in what way secondary school graduates and students of mathematics would choose to solve a problem related to this notion. We wanted to find answers to the following questions:

— what intuitions associated with this notion do the participants of the study have?

— in what way do they approach the problem?

— how do they understand the mathematical text of the problem?

— how do they choose their argumentation?

The research was conducted among secondary school graduates and students of the first and the third year of mathematics who had not yet done their course in probability theory. We analysed papers containing solutions of the following problem:

Problem 1. Three dice were rolled and the greatest score obtained was 5. What is the probability that the sum of all the scores is 13?

It is a problem on conditional probability. Let us cite its definition.

> Definition: If A and B are events in the sample space (Ω, Z, P) and $P(B) > 0$, then the *conditional probability* of A given that B has occurred is defined as the quotient $\frac{P(A\cap B)}{P(B)}$ and is denoted as $P(A|B)$.

Let us notice that, according to the definition, conditional probability is a new probability of the same event but with different data. The data change, therefore the probability changes, whereas the event remains the same. The symbol $P(A|B)$ may suggest that what appears here is the probability of a new event $A|B$. This denotation may be a source of mistakes and difficulties in understanding the notion of conditional probability.

Solution of problem 1.

The experiment described in the problem is rolling three dice. If the dice are treated as distinguishable objects, the outcome of this experiment may be coded as a triple of numbers (x, y, z), where x is the score obtained on the first dice, y is the score obtained on the second one and z is the score on the third. Such a triple may be represented as a three-digit number $100x + 10y + z$. If Ω is the set of all outcomes of this experiment, then

$$\Omega = \{100x + 10y + z : x, y, z \in \{1, 2, 3, 4, 5, 6\}\}.$$

There is 6^3 (i.e. 216) possible outcomes and they are all equiprobable. As a model of a roll of three dice a classic sample space (Ω, Z, P) may be chosen.

With this experiment two events will be associated:

$A = \{$*the sum of the three scores is 13*$\}$,

$B = \{$*the greatest of the scores is 5*$\}$.

Then

$$\overline{\overline{B}} = 5^3 - 4^3 = 125 - 64 = 61, \quad \text{and therefore} \quad P(B) = \frac{61}{216},$$

$$A \cap B = \{553,\ 544,\ 535,\ 355,\ 445,\ 454\}, \quad \text{so} \quad P(A \cap B) = \frac{6}{216},$$

and consequently

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{6}{216}}{\frac{61}{216}} = \frac{6}{61}.$$

The probability of the total score of 13 when it is known that the greatest score obtained is 5 equals $\frac{6}{61}$.

Problem 1 may be solved without applying the definition of conditional probability, namely by employing the notion of probability of an event within a proper sample space. A solution of problem 1 which is not based on the definition of conditional probability is presented below. In this solution the classic sample space constructed previously is employed.

Second solution of problem 1.

Let us assume that the event B has occurred, that is to say, that the greatest of the separate scores is 5. Since the event B is known to have occurred, the only outcomes that count are those which are favourable to the event B.

Therefore we may restrict our considerations to those outcomes in which the greatest score is 5. Let us denote this new set by Ω_B. Therefore $\Omega_B = B$ and $\overline{\overline{\Omega_B}} = 61$. Now consider those of the outcomes from the set Ω_B which are favourable to the event A. Let us notice that the event A is treated now as a subset of the set Ω_B. In order to stress this fact we denote it by A_B. We want to find the probability of A_B is in this situation. We have $A_B = \{553,\ 544,\ 535,\ 355,\ 445,\ 454\}$. There are only 6 outcomes favourable to A_B among 61 possible ones, so $P(A_B) = \frac{6}{61}$. The probability of obtaining the total score of 13 given that the greatest score is 5 equals $\frac{6}{61}$.

What is actually calculated in the solution based on the predefinitional approach to conditional probability, when the set Ω is restricted to the set Ω_B, is the probability of a different event $A_B \subset \Omega_B$ (although verbal description of this event does not change), which, in fact, is the event $A \cap B$, unlike in the first solution, where a new (different) probability of the same event was calculated. It is a different, intuitive approach to conditional probability.

Problem 1. was being solved by 157 entrants sitting the entrance exam (applied informatics) to Wyższa Szkoła Zawodowa in Nowy Sącz in July, 2002.

Out of this number 126 persons (80% of the examinees) undertook an attempt to solve this problem; the remaining 31 persons (20% of the examinees) returned blank papers. None of the examinees succeeded in solving the problem correctly.

Only one third of the participants of the research understood the issue described in the problem correctly.

For the purpose of presentation of the results the papers were divided into groups (numbered by Roman numerals) from the point of view of the method of solving the problem. In the analysis of the solutions the denotations introduced in the solution of problem1. were preserved.

I. Three persons applied the definition of conditional probability in their solutions, but mistakes made in calculation of the probability of the event $B = \{$*the greatest of the scores is 5*$\}$ made it impossible for them to obtain the correct result.

II. In the solutions of 51 persons (30% of the participants) the idea presented in the second solution of the problem was employed. However, all of them made mistakes in determining the potency of the set B (as many as 34 persons stated that it is equal to 53, i.e. 125). In numerous cases mistakes were made in finding the number of the outcomes favourable to the event B which are also favourable to the event A (quite often just two outcomes were indicated: 553, 445).

III. Approximately 9% of the participants (14 persons) calculated the "regular" probability of the event A instead of its conditional probability. Only 4 of them did it correctly, whereas others made mistakes in describing the set of all possible outcomes Ω (2 persons) or in calculating the probability of the event A (10 persons).

IV. Nearly 20% of the participants (31 persons) instead of calculating conditional probability attempted to find the "regular" probability of the event $A \cap B$. Half of them (15 persons) succeeded in doing it correctly. The remaining examinees made mistakes in describing the set of all possible outcomes Ω (4 persons), in determining which of the outcomes are favourable to the intersection of events $A \cap B$ (6 persons) or in describing both the set of all possible outcomes Ω and the set $A \cap B$ (6 persons).

V. Approximately 17% of the participants (27 persons) started solving the problem. Their papers contain, at the most part, either mere attempts (often incorrect) to construct a probabilistic model or nothing more but attempts to determine the number of all the outcomes of the experiment. In numerous cases the participants tried to find the potencies of various sets, but these attempts were made incorrectly, by mechanical "manipulation" with symbols.

The analysis of the solutions of the problem from the entrance exam is a source of certain reflections on intuitions and probabilistic skills of secondary school graduates. Hardly anyone of the examinees understands the mathematical text and, in the situation when their conceptual instruments are insufficient, seeks new ways of solving the problem, restricting their activities to trying to make use of familiar formulae, algorithms or patterns without even attempting to adapt them to the new situation.

The obtained results were an inspiration to continuing the research, this time among students of mathematics of Pedagogical University of Kraków. The participants were students of the first and the third year of mathematics who had not yet done their course in probability theory.

We expected that because of their wider mathematical experience students of mathematics would cope with solving an analogous problem more successfully.

What was the source of the entrants' considerable difficulties with problem 1. was calculations connected with finding the set of outcomes favourable to particular events. In order to focus the participants' attention on probabilistic notions rather than on combinatorial calculations the text of the problem was slightly modified and, as the result, the students were presented with problem 2.

Problem 2. The following events are associated with two subsequent rolls of a dice:

$A = \{$*the total score is equals 7*$\}$,

$B = \{$*at least one of the scores is 2*$\}$.

Let us assume that the event B has occurred. Find the probability of the event A under these circumstances.

Problem 2. also deals with conditional probability but its formulation differs from the one of problem 1. In problem 2. a throw of three dice (from problem 1.) was replaced with a double throw of a dice. The reason behind this modification was to simplify both the description of the sample space and calculation of the potencies of the events so that it does not require knowing combinatorial formulae. Moreover, the events were defined explicitly in order to focus one's attention on them. Additionally, the expression "in this situation" was inserted, whose role was to emphasise the fact that the information that the event B has occurred is available.

Let us now present two methods of solving problem 2., similarly to the case of problem 1.

First solution of problem 2.

The experiment described in this problem consists in a double throw of a dice. By coding the outcomes as pairs (similarly to the solution of problem 1.) and identifying them with two-digit numbers we obtain

$$\Omega = \{10x + y : x, y \in \{1, 2, 3, 4, 5, 6\}\}.$$

There are 6^2, i.e. 36, possible outcomes and they are equiprobable.

Consider the following events.

$A = \{$*the total score is equals 7*$\}$,

$B = \{$*at least one of the scores is 2*$\}$.

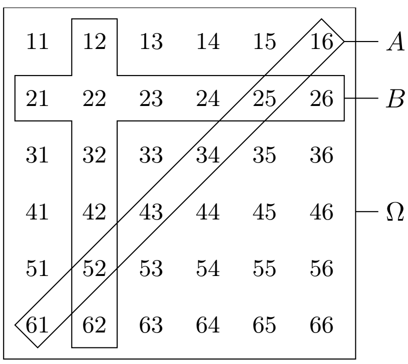

Figure 1:

Figure 1. shows the set Ω and the events A and B as its subsets. We have

$$B = \{26, 25, 24, 23, 22, 21, 12, 32, 42, 52, 62\}, \quad \text{so} \quad P(B) = \frac{11}{36},$$

$$A \cap B = \{52, 25\}, \quad \text{therefore} \quad P(A \cap B) = \frac{2}{36},$$

and consequently

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{36}}{\frac{11}{36}} = \frac{2}{11}.$$

The probability of obtaining the total score of 7 given that at least one of the scores is 2 equals $\frac{2}{11}$.

Second solution of problem 2.

Let us assume that the event B has occurred, namely that at least once a 2 is rolled. We want to know the probability of the event A under these circumstances. Since the event B has occurred, only 11 out of 36 equiprobable outcomes are taken into account (see fig. 1.). Of these 11 outcomes only two (25 and 52) are favourable to the event A, therefore the probability we were looking for is $\frac{2}{11}$.

Problem 2. was being solved by 124 persons (63 students of the first year and 61 students of the third year of mathematics). The approach of the students reveals similarities in "methods" of solving the problem in comparison with the case of the entrants. In spite of a lower degree of difficulties at the level of calculations, the errors made by them are similar to those made by the entrants.

An attempt to solve this problem was made by 105 persons (84% of the participants); the remaining 19 persons returned blank papers. Among the latter there were 11 persons who commented on the absence of solution as resulting from an insufficient knowledge of the notions or formulae necessary for solving the problem.

I. Twelve persons (10% of the students) applied the definition of conditional probability in their solutions and obtained the correct result.

II. Fifteen persons (15% of the participants) employed the idea shown in the second solution of problem 2., but some of them (8 persons) obtained an incorrect value of potency of either the set B (giving its value as 12) or the set of these outcomes belonging to B which are favourable to the event A, for example, indicating only one outcome: 25.

III. Approximately 8% of the participants (10 persons) calculate the "regular" probability of the event A instead of its conditional probability. One of these persons does it incorrectly.

IV. Nearly 25% of the participants (30 persons) instead of calculating conditional probability attempt to find the "regular" probability of the intersection $A \cap B$. Almost all of them (26 persons) do it correctly. The mistakes of the remaining 4 students lie in an incorrect description of the outcomes favourable to the intersection of events $A \cap B$.

V. Approximately 16% of the participants (21 persons) merely started solving the problem. Their papers, similarly to the ones of the entrants, contain, for the most part, mere attempts (often incorrect) to construct a probabilistic model or nothing more but attempts to determine the number of all the outcomes of the experiment.

VI. In 17 papers the probability calculated by their authors is the one of the event A given that the first score was 2. The condition assumed here was different from the one described in of the problem.

The following extract from the paper of a student of the third year illustrates the nature of this error.

Since at least one of the scores is 2 and the total score is 7, the other one must be 5, and that a 5 is rolled in a single throw this probability equals $\frac{1}{6}$.

The table shows the breakdown of the results of the research.

Method of calculating the probability			Group of participants: Entrants	First year students	Third year students
I.	I. Applying the definition of conditional probability $P(A) = \frac{P(A \cap B)}{P(B)}$	correct	0	11	1
		incorrect	3	0	0
II.	Solution based on correct intuitions associated with conditional probability	correct	0	1	6
		incorrect	51[*]	3	5
III.	Calculation of the "regular" probability of the event A	correct	4	4	5
		incorrect	10	0	1
IV.	Calculation of the "regular" probability of the intersection of events $A \cap B$	correct	15	22	4
		incorrect	16	2	2
V.	Attempts of constructing a model	correct	-	-	3
		incorrect	27	5	13
VI.	Calculating the conditional probability of the event A under a different condition	correct		2	15
		incorrect	-	-	-
VII.	Blank papers		31	4	4
VIII.	Absence of solution with the comment "I don't remember"		-	9	2[**]
	Total number of participants		157	63	61

Table 1:

The analysis of papers of both the entrants and the students of mathematics shows that conditional probability is a notion which is difficult for the participants of the study. Those participants who knew the definition of conditional probability had no problems in applying the formula. From the point of view of our study such solutions are not interesting, since what they amount to is basically mechanical application of the familiar formula.

What was interesting was the papers of those of the participants who, despite the lack of knowledge of definitions and patterns to follow, undertook an attempt to solve the problem (see point II. of the table). It is worth pointing out that as many as one third of the entrants participating in the study would have obtained correct solution of the problem had they not made combinatorial mistakes in calculating the potencies of events. In this context, it is quite astonishing that only approximately 13% of the students (15 persons) presented solutions based on a predefinitional, intuitive understanding of the notion of conditional probability. Students should not have difficulties with finding the potency of events since the problem they were solving required no particular information on combinatorics, and in spite of this fact only seven students who were solving it in this way obtained the correct result.

A large number of the participants calculated probabilities of events other than the one mentioned in the problem. The events whose probabilities were calculated were, for example, the event A or the event $A \cap B$. This fact may suggest difficulties in understanding or interpreting a mathematical text. This supposition is confirmed by the papers assigned to groups III, IV and VI. In a number of papers instead of the event: at least one roll gives a 2, the authors considered the event {*all separate scores are at least 2*}, or the event {*the*

[*] Within this group there were 25 persons whose only mistake lied in an incorrect calculation of the potency of the set B as the product $5 \cdot 5 \cdot 5$.

[*] Having declared that they did not remember the necessary formulae, these persons did nonetheless undertake an attempt to construct a sample space.

second score is 5} given that the event {*the first score is 2*} has occurred.

Certain differences in the numbers of the participants solving the problem with the aid of the conditional probability formula may be noticed within the group of students of the first and the third year. There were 11 such persons among the first year students and just one among the third year students. One may suppose that the first year students remember the formula from secondary school, whereas the third year students have already forgotten it. Moreover, as many as 9 students of the first year wrote that they did not remember the formula and made no other attempt to solve the problem. Among the students of the third year there were two who admitted that they did not remember the formula and who attempted to construct a sample space and to calculate the probability of the event. It is noticeable that the third year students, when lacking necessary tools, are more inclined to look for a solution. In all probability, it is the result of their wider mathematical experience.

In the opening we have mentioned the difficulties arising from the notion of conditional probability (from its symbolical notation, methods of calculating it, from understanding it). The results of our study seem to confirm this supposition. Only a relatively small number of the participants (approximately 29%) understand the instruction contained in the problem and undertake an attempt to carry it out (see table 1., rows I and II). It is the group of 81 persons from the total of 281 of the participants.

A considerable number of the participants do not understand the notion of conditional probability and identify it incorrectly with the "regular" probability. It is the case of 24 participants (8,5% of the whole group participating in the study). For the most part, they calculate the probability of the event A as though no information on the occurrence of the event B were given, although the information on the occurrence of the event B is relevant here (it has an influence on the probability of the event A). The participants intuitively feel that they are supposed to calculate the probability of the same event and they actually do it, although forgetting about the additional piece of information that the event B has occurred. In this particular situation their intuitions are erroneous.

The wording of a problem concerning conditional probability may sometimes suggest that it refers to the probability of the intersection of events. This is how the problem was understood by those of the participants who were calculating the probability of the event $A \cap B$ (see table 1., row IV). There were as many as 61 such persons (22% of the whole group). It may be argued that they also have certain intuitions associated with conditional probability. Indeed, let us observe that by definition conditional probability is calculated as the quotient $\frac{P(A\cap B)}{P(B)}$. The participants are therefore right in calculating $P(A \cap B)$. It is nonetheless regrettable that they fail to notice that it does not complete their task.

What is more, what plays a vital role when conditional probability is considered is time: we know that the event B has occurred (a completed incident), and under these circumstances we ask about the probability of the event A (a future possibility). What appears here is the sequence of various moments in time, which may be easily misunderstood. Such a misunderstanding of the sequence of moments in time is manifested in the papers of the students (see table 1., row VI) who in the case of the information that: *at least one of the scores equals* 2 and when asked, under these particular circumstances, about the probability of the event that the *total score is* 7 reason in the following way: *since at least one of the scores is* 2 *and the total score is* 7, *the other one must be* 5, *and that a* 5 *is rolled in a single throw this probability equals* $\frac{1}{6}$.

From this statement it may be inferred that in the students' reception the event: a 2 is rolled is the former of the two. The condition »must« precede the other event and is, for this reason, associated with the first throw of a dice (the former one), and the event $\{$*the total score is* $7\}$ is a consequence and, as such, »must« come next, that is to say, on the second (later) roll. This reasoning is not devoid of logic. It is possible to reflect here on intuitions which may be quite useful in certain situations; in this particular case, however, they result in an incorrect solution of the problem.

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ICT in elementary mathematics education: two critical questions

Tibor Marcinek

The article poses two questions an elementary school teacher should ask himself/herself before introducing pupils to the computer-assisted teaching in a computer lab. The first one questions the real motivation behind a youngster's interest in working on the computer. The second analyzes sensory experience during computer-based activities and its appropriateness for a child of primary school age.

The role and impact of ICT in elementary education has been present in studies and research for quite a while. With sound arguments on both sides, the issue has not been resolved and there is probably no simple, "short statement" solution whatsoever. At the current state of knowledge we suggest to provide elementary school teachers with as many well-founded views as possible and let them to take the responsibility for when, how, or how much of ICT to implement in their mathematics instruction.

In the paper, we discuss two critical questions that an elementary school teacher should ask before introducing children to a computer lab.

Question of Motivation

"Natural curiosity" of a child sitting behind a technical instrument is very often cited as a strong motivational impulse. The impulse can be employed to engage a pupil in a number of valuable activities. At the same time, however, it is necessary to provide a teacher with sound methodical background (Partova, Marcinek 2000). On the other hand, J. Healy, based on her research (Healy, 1999), believes that child's real motivation is a mere superficial "incentive" to work with the computer. She expresses her concerns that too early exposure to technical means may hinder seeking *inner motivation* to solve problems and achieve success. Outer stimuli, such as multimedia effects, reduce or even subdue valuable inner ones. Thus, it may seem relevant to observe more closely how a child behaves while it works with ICT. The first question comes about - a question important for an elementary school teacher to answer:

1. *What is the child's inner motivation to work with the computer?* Is the child driven by a genuine effort to solve the problem or by a mere superficial desire to work with the technology and/or win a reward (collect points, play games, receive multimedia effects, etc.)?

The relevance of the question is (at least partially) justified by some observations from the research of E. Partova (research in progress). For example, in one situation a group of second graders was to practice basic multiplication facts using a selected software. Manipulatives, (cubes) were also available whenever a pupil needed to model the situation. Interestingly, if a pupil was not sure as to the right answer AND he or she felt he/she had not been observed by a teacher, he/she simply "clicked off" an assumed answer without trying to reason further and went on to the next problem. Pupils that felt they had been observed were more careful and did not simply enter assumed answers. They seemed to be trying to reflect on them. However, they used manipulatives mostly only after being encouraged by a teacher.

Question of lacking sensory experience

Working on the computer tends to narrow down sensory perceptions and puts them out of their regular context (comp. Healy, 2000). Manipulation of objects displayed on 2D monitor using a very limited space of a mouse pad undoubtedly provides a different sensory experience than manipulation of real object in the real world. As the integration of sensory perceptions is critical in forming child's academic skills, too frequent exposure to ICT - at the current state of technology development at least - may deprive a child of inevitable "real world objects" manipulations. Thus, the second critical question for a teacher to answer should be:

2. *What kind of "real world" sensory experience is being missed while a pupil is working on the computer?* Is the ICT - in the particular case of my classroom - a necessary substitution for the real, 3-D modeling and experience?

Undoubtedly, different sensory experience provided by the computer and its irreplaceable capability of dynamic modeling can be very well employed to gain a deeper insight into a wide variety of concepts in higher grades. However, as to the elementary school grades, another question pops up rather distinctly: *When is it appropriate to introduce pupils to the ICT* ... so that the technology may work as an extension (fostering that deeper insight for example) rather than a tool that takes away from pupils' opportunities to build fundamental concepts in a natural way using real objects and manipulation? Healy answers the last question surprisingly precisely. To put it simply: Absolutely no contact with the computer until 7 years of age and no significant time spent on the computer until 10-11 years of age. Inasmuch as Healy's opinion is based on her extensive research and long-time experience, we might take her age limits sufficiently valid. Anyway, we rather adopt a disinterested stance leaving the abovementioned limits open for further discussion.

However, we will provide one argument in favor of Healy's views. "Typical" personal computer technology can be dated back to the late 80's or early 90's, i.e. when today's top ICT experts were in their teen or college age. Thus, it seems that there is nothing like a "head start" or "falling behind" in mastering computer technology and one may easily catch up even in his/her teen age or later to become the ICT expert. This probably cannot be said about building fundamental math concepts or acquiring social skills and these should be given preference in elementary education.

By no means we advocate putting off the ICT education until a teen age. We would like to increase teachers' sensitivity and awareness to these important issues. We fully rely on teacher's experience and expertise; we believe that a teacher who takes his/her pupils to the computer lab after frankly answering the two critical questions, have made the best choice in the context of the class or topics.

In place of conclusion, a look ahead. The thoughts and opinions presented in the paper are probably valid only at the current stage of development of computers' interfaces. Since, for example, writing with a pen is now nothing exceptional in communication with the computer, it is a question of time when, say, a "real" compass and ruler will form a computer's interface or modeling with real, 3-D objects becomes available as a way to communicate with the computer. This may adumbrate a new time of ICT in education when the two question will no longer be *critical*.

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E-learning and Blended Learning as a Support of Education of Mathematics

Jan Melichar

The Department of Mathematics of the Pedagogical Faculty UJEP in Ústí nad Labem have prepared several articles about utilization of E-learning and blended learning as a support of education of mathematics, arithmetic, methodology, geometry and statistics in the subject Development of Mathematics Thought taught during the studies of education for basic school and kindergarten. The article explains terms E-learning and blended learning and shows possibilities of their utilization.

Currant professional literature characterised E-learning as an electronic education. PC, browser and software are essential items which enables users to work in network (internet and intranet). Multimedia platform based on CD-ROM and DVD is other component of the complex. Usage of PC, network and visual and interactive environment is primary. The main item is to focus on educational aims. With respect to the fact that e-learning is not all-powerful and together with the new pedagogical approaches, a new term "blended learning" occurred. Blended learning is considered to be combination of e-learning and presentation forms of education (see elearningeuropa.info).

At the Institute of Mathematics, Department of the Primary School Education at the Pedagogical Faculty of the University of Jan Evangelista Pukyně in Ústí nad Labem we are giving mathematical training not only to students of Primary School Education but also to students of Pedagogy (Master Degree) and students of Kindergarten Education (Bachelor Degree). In all those courses we have students from combined studies and the low amount of lessons can be filled up by electronic education. Publishing of schoolbooks is time consuming in addition to that mathematics is very static in schoolbooks. E-learning and blended learning enables us to show mathematics in a dynamic way and explain it better. The process of schoolbook preparation and publishing is minimised. E-learning enables to place new text directly to web sites. Beside that E-learning gives the user opportunity of feedback, user s can control that they understood the text and it gives the feedback to the author that users are learning. Users are solving problems and their results are passed to the author. It is also important that users are writing seminar works with use of the E-learning and those works are controlled by the author. In the seminar works the author can identify how the user is able to use his/her knowledge in practise. As in the E-learning as well as in the blended learning authors are using pictograms to facilitate user's work during studying.

Following texts have been prepared for combined studies at the Department of Primary School Education: Doc. PaedDr. Miroslav Bělík, CSc. Úvod do studia matematiky v kombinovaném studiu učitelství pro 1. stupeň základní školy (Introduction into the Studies of Mathematics for Combined Studies of Primary School Education) (http://pf1.ujep.cz/_KMA/Belik_Mat_TS.html) and Geometrie s didaktikou v kombinovaném studiu pro učitelství pro 1. stupeň základní školy (Geometry with Methodology in Combined Studies of Primary School Education) (http://pf1.ujep.cz/_KMA/Belik_GEO/kosgts.html), Prof. RNDr. Jan Kopka, CSc. Aritmetika - Přirozená čísla pro studium učitelství pro 1. stupeň základní školy (Arithmetic - Natural Numbers for Primary School Education Studies) (http://nella.ujep.cz/pf/kmat/aritmetika_pro_NS) and Aritmetika - Celá čísla pro studium učitelství pro 1. stupeň základní školy (Arithmetic - Integral Numbers for Primary School Education Studies) (http://nella.ujep.cz/pf/kmat/aritmetika_pro_NS/nove/) Prof. RNDr. Jan Melichar, CSc. a PaedDr. Josef Svoboda: Statistika pro studium učitelství pro 1. stupeň základní školy (Statistics for Primary School Education Studies) (http://pf1.ujep.cz/data/KMA_statistika.pdf) and Rozvoj matematického myšlení I pro obor učitelství pro mateřské školy (Development of Mathematic Thought I for Kindergarten Studies) (http://pf1.ujep.cz/KMA_melichar01.asp.)

It is clear that by means of web sites students are informed about detailed syllabuses of individual subjects, studying sources, requirements for credits, exams and final exams. It provides quick transfer of information and possibility of quick adjustment of information to general changes. It is not pure E-learning or blended learning but this quick exchange of information supports the process of education Literature:

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Relations in preschool mathematics

Edita Partová

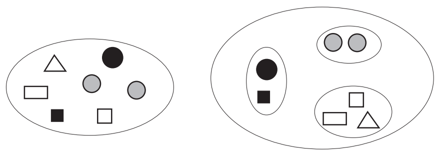
The article describes using of relations in preschool education and their misuse.

For the psychical development of children is very important to create logical-mathematical structure. Children from the very early age are discovering relations and dependency on the base working with subjects. In preschool education children are connected with relations and functions for example: recognizing the subset of set, sorting the subjects, making comparison between them and classifying them by certain signs, construction of geometrical shape according to a picture etc. The most dangerous methodical defect in the early age of children is when they create false image of basic mathematical notions. For teachers in preschool education it's important to recognize the existence of the relations and functions in their everyday work, and also understand their basic character and consciously use it.

1.2. Relations in preschool mathematics

The most common understanding of relations is relations as relationships. In everyday life we can see a lot of relations for example: mother-daughter, number-symbol, person-photo, merchandise-price, chair-student… Some of the relations definite on one set have interesting and important properties. (reflexive, symmetric, transitive, asymmetric, trichotomic act). We can demonstrate these properties to children.

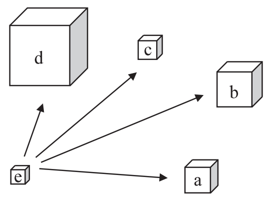
Example: Let's choose relation R on the set of colored shapes: R = { **object X has the same color as object Y.** }



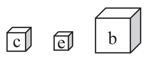
Relations with all three properties (reflexive, symmetric, transitive) what we also call as **equivalence relation**, define a partition of set of colored shapes. In preschool education classification based on partition of set is a frequent activity, so it is important for the teacher to understand, which type of the relations are appropriate for classification. Now let see the relation: S = { **ball A is smaller than ball B** }. At first sight it is obvious that it isn't symmetric, it means we cannot classify the balls for "smaller than" and "larger than". This relation does not fulfill the requirements for classification of the objects, but is good for ordering.

Ordering

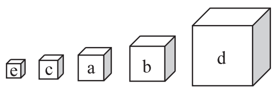
Relation with asymmetric, transitive and trichotomic properties is called a relation of ordering. It is important to realize that in ordering is necessary to make comparison for each pair of elements, as is apparent from the picture.



If cube c smaller than cube b, and e is smaller than b it does not necessary mean that c is smaller than cube e. We must compare cube c and e too.



By the relation: T={ **cube X is smaller than cube Y** } we put them in order as follows:



In this connection we must understand, that relations, even if at the first sight seems to be relations of ordering are not necessary the case.

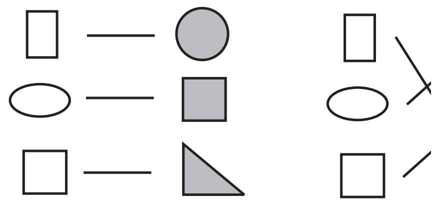
Let's make a comparison of the relation **before** and **immediately before**.

Relation x is before y is asymmetric, transitive and trichotomic, so it is a relation of ordering. Relation **x is immediately before y** is not transitive, that is why it cannot be relation of ordering. It is possible to find the exact mathematical formulating for the above-mentioned properties of relations, but for the needs of preschool preparing is more adequate to use illustration, what later the teacher (after small changing) can use in their own practice.

2.2.Functions in preschool education

Curriculum of preschool education involves lots of activity, where is very important to apply functions. Function is a relation that matches each element of a first set to an element of the second set in such a way that no element in the first set is assigned to two different elements in the second set. According to our case this is the proper definition.

The most frequent application, children try to make is the one to one correspondence. They use it to find out if two set have the same number of elements. Often occurs a methodic mistake, when the one to one correspondence is considered only by what is seen on the picture a). It is obvious, that picture b) is a one to one correspondence too and also suits for finding out equality of number of elements of sets.



Picture a) Picture b)

Conclusion The basic elements of abstract thinking are creating and using relations with different quality. Forming relations is very simple and clear in preschool activities. Despite the facts teacher can form fault image, with misinterpret even the most simple task if they do not understand the above-mentioned mathematical regularity.

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On false convictions in mathematics students' reasoning

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In my paper I present a subject of my several-year-long observations — false convictions — assumptive foundations of mistaken reasoning existing in students' solutions of problems.

I show some examples of based on false convictions solutions of problems concerning geometric transformations of the plane. I also present short characterization of false convictions and selected conclusions drown from my research.

From 1994 to 2003 I was running courses in elementary geometry for the first year students of mathematics as prospective teachers. Topics of those courses mostly concern geometric transformations of the plane. I was conducting the research parallel to my teaching activities.

The main inspiration for undertaking the research was one solution of an examination question. The solution released the unexpected and non-standard (other than discussed within the framework of classes) and, most importantly, erroneous approach of the student to the problem presented.

> Sometimes a single fact, single example, apparently isolated, and even apparently strange, reflects with a sudden flash in our pedagogical intuition and may direct us towards considerably important problems ([8]).

Is the reasoning observed a single case or perhaps a symptom of difficulties met by a number of students? The question became for me a research challenge.

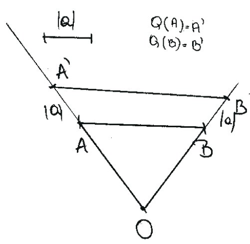
In the course of my several-year-long observations of various groups of students, as a result of in-depth analyzing of the erroneous solutions of problems given by students I noticed numerous incorrect modes of reasoning. I analyzed these types of reasoning from the point of view of logic and mathematics in order to disclose the essence of an error and its hypothetical sources. Assumptive foundations (misconceptions, erroneous inference rules or mental schemes) of mistaken reasoning I called *false convictions* which is analogous to observed by Bell ([2]) students' false views like: "multiplication increases and division decreases".

I will present some examples of students' *false convictions* to illustrate the problem.

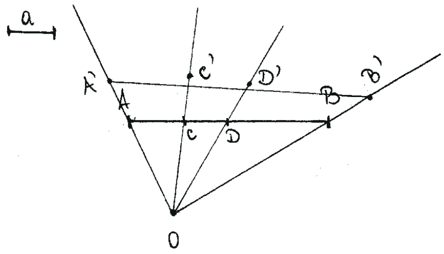
Question 1. Point O and the segment a are given. Transformation Q is defined as follows: for each $X \neq O$, $Q(X) = X'$, such that $X' \in OX^{\rightarrow} \wedge |OX'| = |OX| + |a|$ ($OX^{\rightarrow}$ —

half-line beginning in point O in direction X). Find the image of a segment AB under transformation Q.

Answer 1



Answer 2



Both answers are obviously wrong. The author of answer 1 did not search for images of points in the segment but of its endpoints. She identified images of points A and B and joined them. See that the author of answer 2, in spite of undertaking some attempts of testing the Q transformation, ignored the obtained images of points C and D, and drew line segment $A'B'$. Authors of presented drawings acted as if they were applying the following mental scheme:

FC "segment"

The image of a segment XY under any transformation P is the segment $X'Y'$ such that $P(X) = X'$, $P(Y) = Y'$
(regardless of the definition of transformation P)

In years 1999-2003 sixty three out of the one hundred forty two observed students gave at least one incorrect solution based on *FC "segment"*. The numbers indicates the rank of the difficulty we are talking about*.

Possibly this scheme results from earlier school experience of students. In the elementary school they learned about transformations that preserve collinearity and order of points. There, images of segments were constructed exactly in this way in all cases. It seems that, repeated many times, this procedure became universal for many students. Supposedly, in many of them the term "line segment image" produces the association with only one procedure of finding the image, which they learned about at school — *FC "segment"*. The reason for that action is the conviction that for testing the transformation it is sufficient to test some "boundary conditions".

Difficulty of this sort was observed at a lower school level already in other research ([3]). In her paper in the following words Ciosek commented one of pupil's incorrect answers:

> (...) each of geometrical transformations, discussed in details at the primary school, preserves the center of symmetry of a figure (as an affine transformation). This may cause the consolidation of the pupil's view that the center of symmetry of a figure is an invariant to **each** geometrical transformation. The pupil (of class 7) does not realize that it is not true.

Question 2. (see [5])

Given are two lines a and b, point A and a non-zero vector $\vec{u}$ such that $a \perp b \wedge A \in a \wedge A \notin b \wedge \vec{u} \parallel b$. Transformation P of plane π is defined as follows:

$P = S_a$ for points of the plane belonging to half-plane $bA^{\rightarrow}$,

$P = T_{\vec{u}}$ for each other point of the plane*.

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*Another example of application of this scheme is presented in my paper ([9]).

*S_a — reflection in line a, $T_{\vec{u}}$ — translation by vector $\vec{u}$, $bA^{\rightarrow}$ — half-plane based on line b and point A.

Examine if: a) P is an isometry, b) P is a similarity, c) P preserves collinearity.

Answer 3. (referring to question 2a)

Let $X \in bA^{\rightarrow}$ $S_a(X) = X' \longrightarrow$ *reflection in line is an isometry, so for* $X \in bA^{\rightarrow}$ $P = S_a$ *is an isometry.*

Let $Y \notin bA^{\rightarrow}$ $T_{\vec{u}}(Y) = Y' \Longleftrightarrow \overrightarrow{YY'} = \vec{u} \longrightarrow$ *translation is an isometry, so for* $Y \notin bA^{\rightarrow}$ $P = T_{\vec{u}}$ *then* P *is an isometry.*

Transformation P *is an isometry.*

Answer 4. (referring to question 2c)

As this transformation is not similarity, we can conclude that it does not preserve collinearity of points, as by theorem, similarity preserves collinearity of points.

Answer 3 is obviously erroneous — transformation P is not an isometry. It shouldn't be left unnoticed that the student, in her solution, did not take advantage of the definition of an isometry. She was reasoning as if she were convinced about the existence of the following false relation:

> If both transformations P_1, P_2 defined on sets Z_1, Z_2 respectively, have property W, then the transformation P defined on the set $Z = Z_1 \cup Z_2$ as sticking together
> $$P = \begin{cases} P_1 & \text{for points of the plane belonging to } Z_1, \\ P_2 & \text{for points of the plane belonging to } Z_2, \end{cases}$$
> has property W as well.

It is likely that the author of answer 3, regarding the transformation, which is two isometries sticked together, was reasoning as is being reasoned in the test of composition of two isometries. She knows that the composition of two isometries is an isometry. In the assembly referred to, also two isometries occur. Although she did not find the sameness between these types of transformations (assembly, composition), her solution of the task shows that she may have considered these two "combinations" of transformations to have similar properties (see [6]).

Conclusively, her reasoning might have taken place in accordance with the following, obviously incorrect, scheme (erroneous inference rule):

FC "EIR 1"

If A is C and B is C then any combination of A and B is C

Examples of theorems exist, whose structure reminds of the above-mentioned scheme. For example, these may be theorems regarding adding even numbers or composing of various kinds of transformations — an isometry, a bijection. It is not ruled out that the occurrence of the connection or relation between the objects under consideration may be perceived by some students as "some combination". Perhaps the association of "a new combination", which, for a number of beginner students the assembly is, with "combinations to date", matching the incorrect rule, *FC "EIR 1"* constitutes the reason for the errors discussed.

Answer 4 is an example of an erroneous reasoning, ended with the correct conclusion. Let us pay attention to the fact that the student, while solving the question, did not test anything. In the first part of her written answer, she referred to the previous section of the question, based upon which she had already known that the transformation in question was not a similarity. In further part of the solution she was inferring, based upon the erroneous rule, whose implied form is as follows:

FC "EIR 2"
If A is not B and B is C then A is not C

The aforesaid rule is related to the erroneous understanding of the implication as the equivalence. This type of reasoning with regard to the relation "$p \Longrightarrow q$", observed in 10 and 11-year-old children, is mentioned in Ciosek's paper ([4]).

Question 3. Point A and a non-zero vector $\vec{u}$ are given. Does the composition $T_{\vec{u}} \circ S_A$ have a fixed point? Justify your answer.

Answer 5.

This transformation has no fixed points because the unique fixed point of reflection in a point is the center of symmetry, which undergoes this translation.

In the academic year of 2002/2003, I obtained similar answers from 35% of students questioned. It is not difficult to see that the composition $T_{\vec{u}} \circ S_A$ is a central symmetry, therefore it must have a fixed point. The student, whose solution I presented, reasoned otherwise. By his erroneous answer, he argued as if he were certain about the existence of the following false relation:

FC "translation"
The composition of two transformations, of which one has fixed points and the other one is a translation by a non-zero vector, has no fixed points

$\downarrow$ more general than *FC "translation"*

FC "reduction of fixed points"
The composition of two transformations, of which one has fixed points and the other one has no fixed points, has no fixed points

$\downarrow$

FC "EIR 3"
If A is C and B is not C then combination of A and B is not C (B spoils C)

False convictions

A common feature to the solutions based upon *false convictions* is that the solver produces the solution of the question as if he already knew the answer. He/she simply does not check anything, does not search for anything and does not refer to any definitions or theorems learned before. Seemingly, he simply forms the answer through materializing the first thought or idea that hit his mind. Vinner ([11]) calls this reasoning pseudo-analytic. In addition, we cannot find in students' notes any traces of verifying the solution provided, which implies the absence of any doubts of the author as to the correctness of the solution.

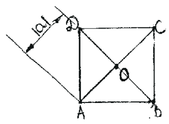
The experience to date, observations and gained knowledge — all have considerable impact on thoughts and associations of a human. He cannot influence the thoughts and associations hitting his mind in a given problem situation. However, he can influence further handling of this thought or association, for example, whether or not this thought or association needs verification.

The attitude of authors of the solutions presented is not a mathematical attitude. In the context of *false convictions*, the problem formulated by Freudenthal (see [1]) "How to develop the mathematical attitude in students?" seems to be current and timely. This problem is in the close connection with my research.

Examples of didactic exertions leading to elimination of *false convictions*

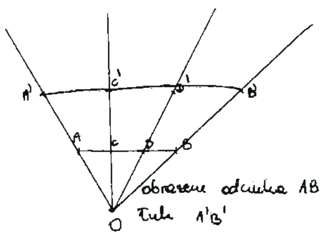
I made a catalogue of revealed *false convictions*. But it is not making such an inventory that is most important. My main goal is to remove rooted *false convictions*. Elimination of *false convictions* is difficult, as students have no doubts if their procedures are correct. The following example may serve confirming the thesis on the durability of *false convictions*.

At the starting lecture in the course of academic year of 2002/2003 the students met examples of numerous transformations, including Q mapping (defined in question 1). They learned that the image of a straight-line in the Q transformation is a conchoid. They also saw that the picture of a line segment AB in this transformation is the $A'B'$ arc of the determined curve line. After solving question 1, the students received the following question.



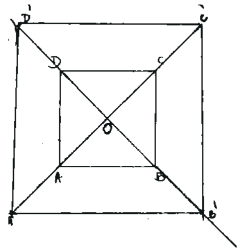
Question 4. Square $ABCD$ is given.
Find the image of the border of this square under transformation Q.

Answer 6 (referring to question 1)



The image of the segment AB is $A'B'$ arc

Answer 7 (referring to question 4)



Result: 24 students, out of 48 students solving the question 4 made the drawing similar to one presented in answer 7. The presented combination of two contradictory solutions, proposed by the same person, is astonishing. This apparently stupid mistake (which students should not have committed following previous experience with the Q transformation), shows how difficult it is for them to abandon what they previously found as an unquestionable fact. This is not surprising, as highly likely many of them did not think about the scheme applied. Until that time "it had gone well", so they did not think that it might be erroneous in some situation.

The idea that **a procedure leading to creating the object required, when once mastered, always functions** may be the subsequent *false conviction* anchored in the minds of a number of students, constituting the reason for errors that appear in their reasoning.

It is likely that the more general form of *FC "segment"* — *FC "figure"* may be the reason of the error in the solution presented in answer 7.

FC "figure"
The image of a figure under transformation P is a figure of the same kind

It has turned out that the single didactic exertion, in particular the monologue of a teacher, involving conveying to the students the information essential to them, may be insufficient if its objective is to eliminate the *false conviction*. Although the information described here influenced the change of the answer of the students in the analogical situation as previously considered (a straight-line and line segment), it failed to have any influence on his strategy in the changed situation (a square edge). This is why searching for and verifying didactic exertions, which in the course of working with students enable revealing and elimination of *false convictions* existing in their minds, became an integral part of may didactical and research practice. Krygowska ([8]) wrote:

> What method should be taken to reveal a misunderstanding in good time? Should one provoke an error purposefully setting traps? It does not sound proper from the pedagogical point of view. But such steps may be taken in a fair way during the process of solving problems.

In order for the student to change his erroneous reasoning, at first he must perceive an error in this reasoning and to understand what it means (to understand its essence). The didactic activities towards this change should persuade the student for reflections about his reasoning. I prepare for students problems that help to create situations, in which *error occurs, error is being discovered, and finally the truth is found* ([10]). The objective of the most of questions formulated by me is error-provoking. A number of them is virtually based on authentic, erroneous reasoning of the students. The adequate combination of questions is to present the students with relation to the conflict provoking situation — the solutions obtained by them are inter-contradictory.

Here are one example of combinations of questions, solved one after another.

Question (1)	**Question (2)**
Point O and a non-zero vector $\vec{u}$ are given. Do you think, that the composition $T_{\vec{u}} \circ S_O$ has any fixed points? Justify your answer.	Point A and a non-zero vector $\vec{w}$ are given. Describe, what you will do to find an answer to the following question: Does the composition $T_{\vec{w}} \circ J_A^{-2}$ have any fixed points?

One of the students, while solving question (1) stated that the composition $T_{\vec{u}} \circ S_O$ had no fixed points. However, in question (2) she empirically determined the fixed point of the composition given. After this experience, she returned to question (1) and wrote: *I changed my mind after solving question* (2). *The composition will have a fixed point.*

The reflection of the student, provoked by the appropriate exercise, changed her reasoning. It is essential that the student changed her view, and more importantly, under the influence of her own reasoning. I recorded such change in seven out of ten persons, who revealed the *false conviction* in the question (1).

The sentences presented played an essential role in the example discussed. In question (1), in the question regarding the opinion, also the suggestion is contained that the composition provided may fail to have fixed points. Question (2) causes that the student as if assumes that the fixed points of the composition provided may exist and starts thinking of how to find them.

Selected conclusions drown from my research

The answers based upon *false convictions* seem to be obvious for their authors. According to Fischbein, the obviousness is characteristic for the intuitional knowledge. The intuitional statement is directly accepted as true and, as such, obvious without the necessity of any proof. What is intuitional seems to be the absolute and unchangeable truth. Fischbein refers to psychology research, which has revealed in people strong inclination for excessive confidence in one's own answers, solutions and decisions. This inclination is not found in the direct relation with competence; however, statistically bigger extent of confidence has been observed in the case of erroneous answers (i.e. based upon lower extent of competence). By contrast, the information from among any new one, which supports the intuitional answer, solution or decision, assumed at the beginning, strengthens this confidence. The facts contradictory to that are ignored or rejected. We also have an inclination to feel confidence to the answer intuitionally accepted, however erroneous rather than correct, but requiring some support from outside (e.g. was learned during school classes). The knowledge internally credible is remembered and used better than the knowledge based upon the formal or empirical argumentation. The erroneous axioms, rules, postulates or *apriori* ideas, accepted intuitionally, constitute the basic didactical dilemma. **In order to change them, the pupil must admit that he had considered something an absolute truth, which actually is erroneous** (see [7]).

There is some evidence that although *false convictions* preserve with considerable force, they may be eliminated. However, this is not easy. The elimination of the *false conviction* v is not the question of a single exertion. It either does not involve achieving the correct answer from the student. This change is only an apparent elimination of the problem, constituting an extemporaneous success. A elimination of *false convictions* can only result of long-lasting process related to the change of the reasoning and creating the mathematical attitude. It is important that the students search for, test rather than confine themselves on the first thought or association. Not less essential is the fact that the students remember about the verification and self-control and that they can recognize the situations where they happened to commit an error before and that they understand its essence.

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Net of a Solid and Space Imagination

Jaroslav Perný

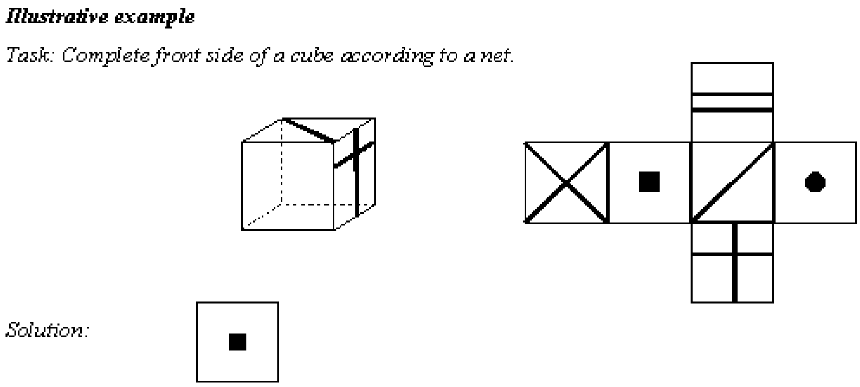
The contribution presents some results of the space imagination research on pupils of the basic school on the solution tasks in spontaneous stereometry range. The experiment is based on a task situation called "Imagining a solid given by its net". Differences of success between boys and girls, older and younger students and so on were investigated. article also contains a concrete example of this calculation.

Introduction

Space imagination is an important ability of a man, which takes place in both the occupation and everyday life. It can be developed among others in the school, e.g. in the art lessons, manual work and in mathematics in the stereometry lessons. In order to support desired development of a space imagination, the options of application of so called spontaneous stereometry (creation of basic knowledge about solids and their properties) has been investigated. It includes for example distinguishing and determining of solids, their representing with various methods, joining of solids, movement of solid on solid and work with nets of solids. This contribution deals with the last mentioned theme. It includes task situations, where pupils in their mind manipulate given nets of solids and use them to create models of solids. They are requested for example to fill the correct symbol from the net on the face of a solid or to find and equivalent solid to a given net of solid. These tasks can be practiced by suitable methods with pupils of various ages. In this contribution I mention some of experiment sets used in creation of "Imagining a solid from the net".

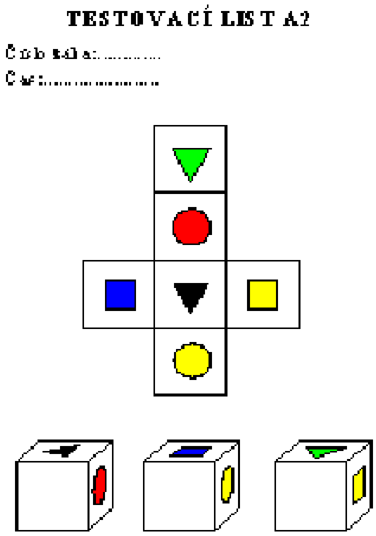
First Experiments

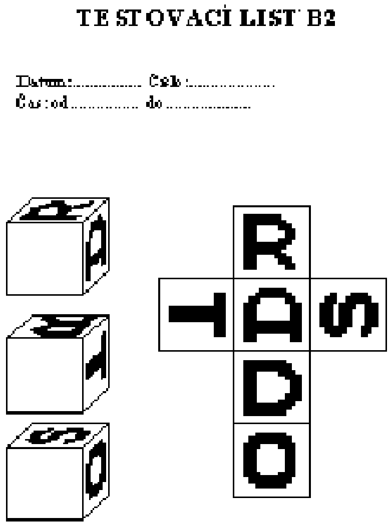
First set of experiments is based on a task situation "imagining a cube" in which a student creates the cube from its given net with differently marked faces in his/her mind. The method of the research is an experiment with a group of students completed with observing. Students are acquainted with the task and then the practice task follows.



Pupils then solve set of various level tasks in two variants: color and next day black and white.

Tasks: *(for example)*





These tasks investigate, how the pupil manages "in mind" assembly of a net of cube, how fast and how correctly fills the front face of cube.

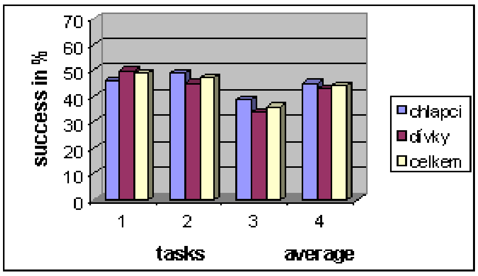
Some findings: (investigations were carried out with 5th and 7th grade pupils) The success of solving of tasks 1 to 3 and the average for boys, girls and grade is displayed on graphs. (Gr. 1-4)

" Older pupils do not posses significantly higher level of space imagination then younger ones.

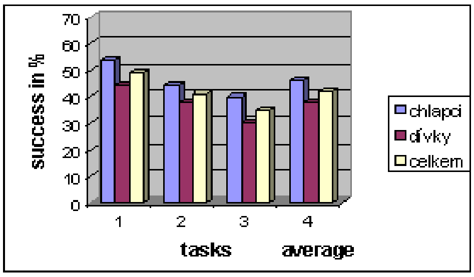
" The success in variant B engaged one day later increased significantly only in 7th grade pupils.

" There is only a small difference in success between boys and girls in 5th grade, but it is noticeable in 7th grade. Boys are improving with age; girls are stagnating, sometimes decreasing."

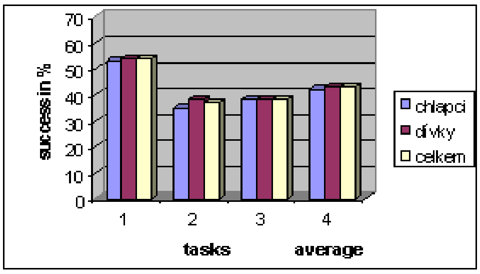
Success of tasks A1-3 and ø 5.g.(color)



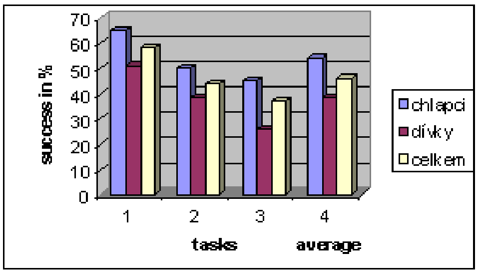
Gr.2: Success of tasks A1-3 and ø 7.g.(color)



Gr. 3: Success of tasks B1-3 and ø 5.g.(BW)



Gr. 4: Success of tasks B1-3 and ø 7.g.(BW)



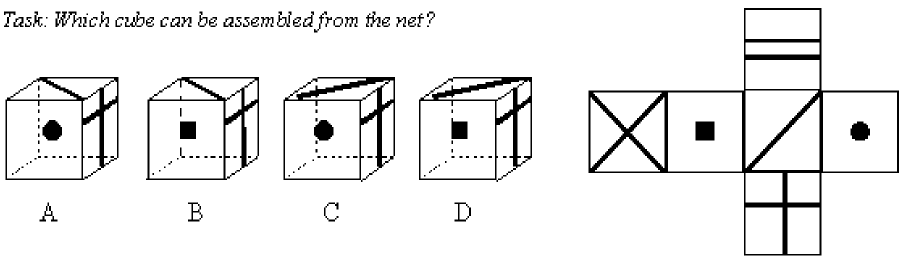
Pupils encounter significant problems in case, when not only the position, but also the orientation of the symbol is important.

Second Set of Experiments

Second set of experimentsis based on task situation "Imagining a solid", in which the student in his mind creates model of a solid from given net having different symbols on its faces and compares it to a set of pictured solids. As this task was given also to the youngest pupils, also nets and models of houses were used for motivation instead of geometrical solids. The method of the research is an experiment with a group of students completed with observing. Students are acquainted with the task and then the practice task follows.

Illustrative example

Task: Which cube can be assembled from the net?



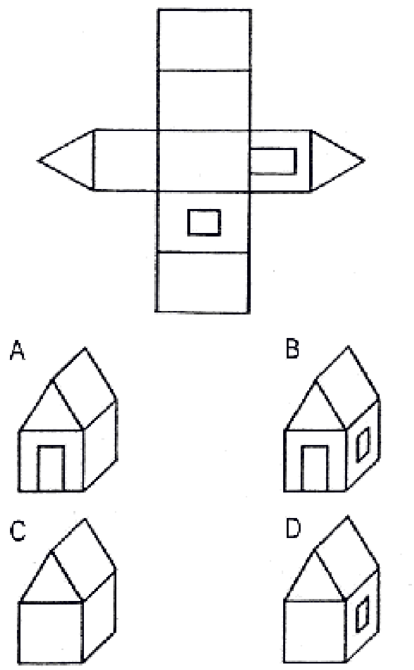
Solution: B

Pupils then solve set of different level tasks in two variants: houses and after a month the cubes.

Tasks: *(for example)*

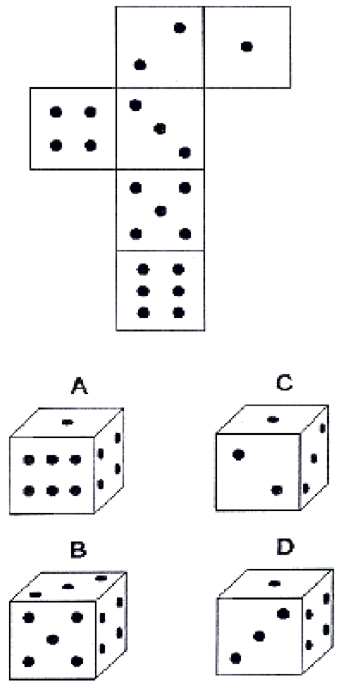
Task C2:

Which of the houses does not originate from the net?



Task D2:

Which of the cubes can be assembled from the net?



These tasks investigate, how the pupil manages "in mind" assembly of a net of solid, how fast and how correctly chooses correct solid.

<u>*Solution:*</u> Task C2: A a D Task D2: C

Some findings: (investigations were carried out with 3rd and 5th grade pupils) The success of solving of tasks 1 to 3 and the average for boys, girls and grade is displayed on graphs. (Gr. 5-8)
" Older pupils are more successful.
" The success of variant D engaged one month later increased mainly in 5th grade.
" Small differences among boys and girls were observed. In the second variant in 5th g. girls have been more successful.

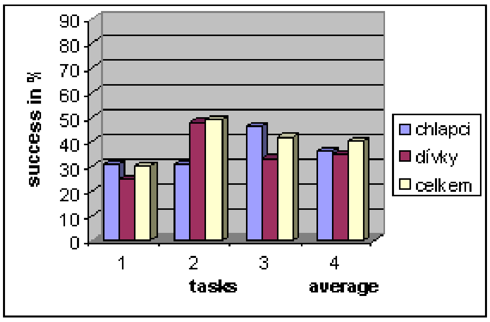
Conclusion

Investigations proved, that space imagination development can be started by suitable methods in the basic school and that these tasks are suitable for mathematics lessons, even when the geometry is not on schedule.

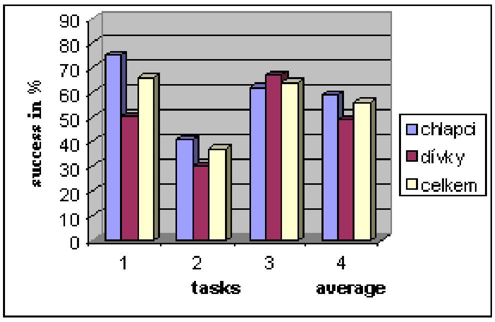
Pupils admire these tasks; they are desired diversification and suitable motivation.

It is sought-after, that future mathematics teachers, even for basic school, at the biggest possible rate meet these tasks and topics, which can support development of space imagination.

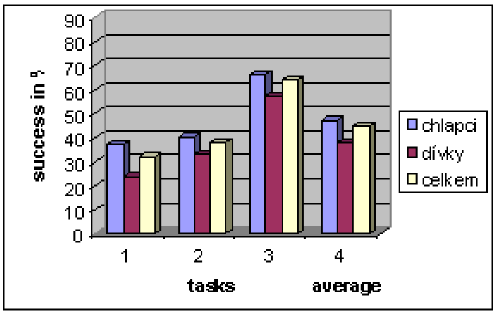
Gr.5: Success of tasks C1-3 and ø 3.g.(houses)



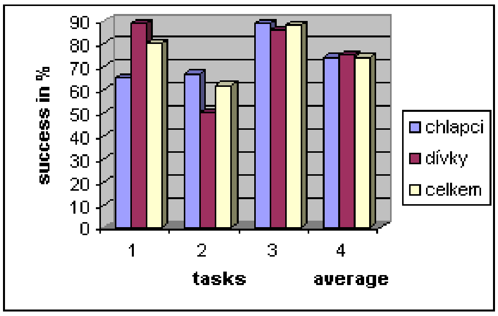
Gr.6: Success of tasks C1-3 and ø 5.g.(houses)



Gr.7: Success of tasks D1-3 and ø 3.g.(cubes)



Gr.8: Success of tasks D1-3 and ø 5.g.(cubes)



" Elimination of the wrong house was someday more difficult for 5. g. then for 3. g.

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"Lotto" type game coupons and icon presentation of the combinatorics notions

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In the following work we propose the iconic representations of classes or series as probabilistic notions as well as objects constructed by a student through getting to know the motivations for counting them. The main issue of this work is a chance game as a source of mathematical problems and exercises with the solutions being discovering the probabilistic notions and formulas.

The notions of combinatorics are tools for constructing and exploring probabilistic spaces as models of probabilistic experiments (see [2], p. 37). The outcomes of those experiments are usually permutations, variations or combinations of certain classes.

In this work we consider chance experiments, for which the class Ω of possible results is finite and for every result it is possible to state *a priori* or estimate *a posteriori* the probability of getting it. If p is a function, which connects every outcome with its probability, then a pair (Ω, p) is a particular discrete probabilistic space. We call it a *stochastic model* of the experiment (see [2], p. 37). By $U_{1\to s}$ we mark the urn, in which there are s balls numbered 1 to s.

Chance games of "Lotto" type

We will consider two types of chance games here. During the game a chance experiment δ is conducted with a probabilistic model of (Ω, p), which can be stated *a priori.* Before the experiment is held, a player bets on the result and wins a point (or other prize), if his guess is correct. In this case, the player's guessing may be of two kinds. The player may bet either

1° on the result of the experiment δ, or

2° on what an event will take place.

In the case of 1° it is a game of g_w type, where the player bets on the outcome of the experiment δ. In case of 2° it is a game of g_z type, when the player bets on the event connected with the experiment δ. The estimation mentioned above inspires two questions to be asked:

1. What can the player bet on?
2. What is worth betting on?

In case of a g_w type game question 1. concerns the set Ω of possible outcomes of the experiment δ. Let d_ω mean a decision: *I bet on the outcome* ω, where $\omega \in \Omega$.

Let $\mathcal{Z} = 2^\Omega$. Defined in this way, $\mathcal{Z}$ is the class of all the events in the probabilistic space of (Ω, p), and so the class of events connected to the experiment δ. In the case of g_z type game question 1. is about the $\mathcal{Z}^*$ class of events which may happen during the game and can be bet on. It is obvious, that $\mathcal{Z}^* \subset \mathcal{Z}$ and $\mathcal{Z}^* \neq \mathcal{Z}$, because:
– we can exclude betting on an event that is certain (because it wouldn't be a chance game), so $\Omega \notin \mathcal{Z}^*$,
– betting on the event with only one favorable outcome is in fact betting on this outcome (in this case it is really a game of g_w type), so $\{\omega\} \notin \mathcal{Z}^*$ for $\omega \in \Omega$.

If $A \in \mathcal{Z}^*$, then by d_A we mark the decision: *I bet on the event* A.

Let (Ω, p) be a probabilistic model of the experiment δ. In case of a g_w type game question 2. is about the p function. In the other case the same question is about the P function, that is about the probability in the spcae (Ω, p) (see [2], p. 98).

Let $C = \{1, 2, 3, \ldots, 49\}$. The Lotto chance experiment is taking 6 numbers out of the set C and the number taken is also the number of the ball taken from the urn $U_{1\to 49}$ (experiment δ_1). It is either
– simultaneous drawing six balls out of the urn, the result is a six-element combinatio of the set C, or
– a six-time drawing a ball without returning it to the urn (experiment δ_2), its result is a six-element no repetition variation of the C class.

Before drawing the player fills in a coupon by crossing out 6 numbers in the C class. The coupon is a representation of a six-element combination of the C class.

In case of the drawing δ_1 the player, who has filled in the form, has bet on the experiment result. In this version of drawing the Lotto game is a game of g_w type.

Let's assume that drawing the Lotto numbers is a experiment δ_2. The player who has filled in the coupon bets on the numbers drawn without considering their order. The coupon is now a representation of an event connected to the experiment δ_2. The results favoring this event are permutations of the class of numbers crossed out by the player. In this case the Lotto game is a g_z type game.

In both versions of the Lotto game the prize for betting correctly is money. In games g_z and g_w the prize is a point. Because of the similarities stated above both g_z and g_w games are called the Lotto type games. The question what is worth betting on is a matter of a strategy of the player.

If one wants to take part in the game of g_w type rationally he/she has to estimate the chance (probability) of getting every possible result of the δ experiment. If all the results are equally possible, it doesn't really matter which result one decides to bet on. If the results are not equally possible it is worth betting on the more probable result. If in the set Ω the most probable result exists, betting on it is the optimal decision. To settle the matter if there is an optimal decision of betting in the game is actually a problem of stating the probabilistic space (Ω, p), which is a model of the δ experiment.

A Lotto type game is an illustration of a process of making a decision in risky conditions (see [3], p. 182). It also shows the importance of time as a parameter (see pic.1). The order of procedures taken is important: filling in the coupon (Saturday), drawing of numbers (Sunday) and paying the prize out (Monday).

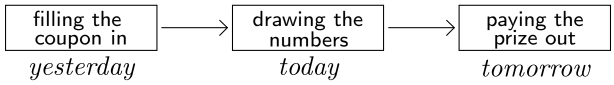

Figure 1. Time as an important parameter in a Lotto type game

A Lotto coupon vs. different schemes of number drawing

In pic. 2 we see three Lotto coupons, already filled in. Each of them shows a six-element combination of the set C stated by the crossed out numbers. The coupon in pic. 2a shows the $\{7, 16, 13, 33, 34, 45\}$ combination of the set C.

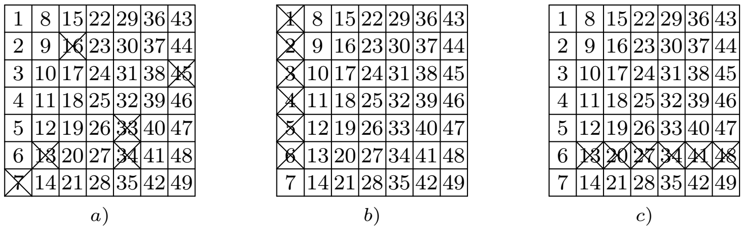

Figure 2. Filled in Lotto coupons as a representation of a six-element combination of the $\{1, 2, 3, \ldots, 49\}$ class

Let's assume that in a Lotto game we draw 6 balls from the urn $U_{1\to 49}$ (experiment δ_1). Now, some mathematical problems show up:

Ex. 1. How many coupons do you have to fill in and how, to make sure that in the nearest drawing you will be the winner of first prize, which is given only to those, who have crossed all the 6 numbers out correctly? How much would it cost? Is it worth trying?

Ex. 2 Let's assume that you spend 10 seconds filling in the coupon. How much time would you spend on filling in that much coupons?

To be sure to get the first prize the player has to fill in the number of coupons equal to the number of possible outcomes of the game, and each coupon must differ from the others with at least one number. The result of drawing 6 balls from the urn $U_{1\to 49}$ is a six-element combination of the set C. There are $\binom{49}{6}$ such outcome, which gives 13 983 816 possibilities. The player scores the six when drawing six balls out of 49 ends with exactly the outcome he bets on. The probability of getting six correct choices, not considering which specific numbers the player crossed out, equals $\frac{1}{13\ 983\ 816}$.

Ex. 3. Suggest a Lotto coupon, treating Lotto as a g_w type game for a drawing numbers experiment δ_2.

The outcome of the experiment δ_2 is a six-element variation with no returns of the set C. A filled-in coupon has to include not only what numbers are to be drawn, but also in what order. A suggestion of such a coupon we can see in pic. 3.

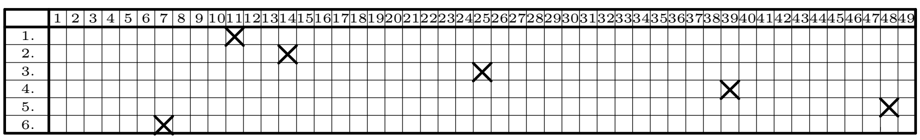

Figure 3. A coupon for betting in a six-time drawing with no returns from the $U_{1\to 49}$ urn

A filled-in coupon in pic. 3 is a representation of a $(11, 14, 25, 39, 48, 7)$ series, that is a six-element variation with no returns of the C class as an outcome of δ_2 drawing, which the player has et on.

Ex. 4. How many coupons shown in pic. 3 the player has to fill in to make sure he/she wins the g_w type game? How much time does the player need for filling them in, if he/she fills in one coupon in 10 seconds?

Examples of Lotto type games and coupons as representations of probabilistic notions

A Mini-Lotto game and a two-element combination of a finite class and its representations

The outcome of a simultaneous drawing two balls out of the urn $U_{1\to 5}$ is a two-element subset of a set of five numbered balls. This outcome we can show as a two-element combination of the set $\{1,2,3,4,5\}$. In picture 4 this combination is shown as a line connecting two drawn balls. Therefore, picture 4 shows:

a) the set Ω of all possible outcomes of drawing two balls out of the urn $U_{1\to 5}$;

b) the set of all two-element combinations of the set $\{1,2,3,4,5\}$.

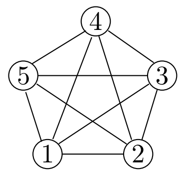

Figure 4. Segment as outcomes of drawing two balls out of the urn $U_{1\to 5}$

1	2	3	4	5

A miniature Lotto game coupon

A g_w type game, in which a player bets on an outcome of drawing two balls out of the urn $U_{1\to 5}$ is a miniature of a real Lotto game. Picture 5 shows a coupon of such a game. To bet on an outcome a player has to cross out two numbers. But picture 6 may also be a coupon for this game. To fill in such a coupon means to connect two balls with a line. Picture 7 shows such a coupon already filled in. The outcome bet on is in this case a $\{2,4\}$ combination.

Picture 4 suggests some idea of counting all the two-element combinations of any s-element set Q.

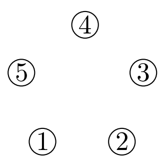

Figure 6. A shape of a Lotto type game coupon

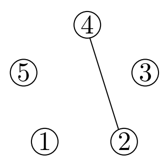

Figure 7. A filled-in coupon. The player bets on the outcome $\{2,4\}$

Let's place the elements of the s-element class on the tops of a convex s-gon. Every segment connecting its two tops represents a two-element sub-class of the set Q. There are as many subsets as lines connecting tops of the convex s-gon. From each top there are $(s-1)$ segments drawn. There are s tops, so the ratio $s\cdot(s-1)$ would be the number of segments, but each of them would be counted twice. The number of segments, and so the number of two-element combinations of the Q class equals then $\frac{s\cdot(s-1)}{2}$. Hence $\binom{s}{2}=\frac{s\cdot(s-1)}{2}$.

A g_w type game of a Lotto kind as a strategic-chance game

In the game, two balls are drawn simultaneously from the urn U in picture 8. The outcome is stated by the sum of the numbers on the balls. So there are 3 outcomes of such a game:
0: `the number on the both drawn balls is 0`,
1: `the number on one of the balls is 0, and on the other 1`,
2: `the number on the both drawn balls is 1`.
We can talk about drawing from the set $\{0, 1, 2\}$. Before the drawing the player bets on one of the outcomes and scores a point, if his/her guess is correct. Betting on the outcome is equivalent to crossing out a number on a coupon in pic. 8.

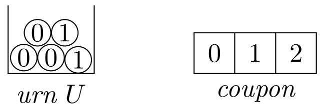

Figure 8. The urn U and a coupon for a Lotto type game

Let d_j mean the decision: *I bet on the j number for* $j = 0, 1, 2$. The optimal decision is the one which gives the biggest chance of scoring the point. To state if there is an optimal decision in the set $\{d_0, d_1, d_2\}$ we need to make a model of this experiment. If this model is a pair (Ω, p), then $\Omega = \{0, 1, 2\}$. From picture 9 we see that:

$$p(0) = \tfrac{3}{10},\ \ p(1) = \tfrac{6}{10}\ \text{ i }\ p(2) = \tfrac{1}{10}.$$

The (Ω, p) probabilistic space is stated with the method of classifying equally possible cases (see [2], p. 39). The drawing outcomes are not equally possible. Betting on the most possible outcome is a rational decision. The decision d_1 is therefore the best. This g_w type game is therefore a strategic-chance game.

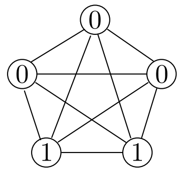

Figure 9. Equally possible cases in drawing two balls from the urn U

Betting on the outcome of n-time throwing a dice in a g_w type game

In a g_w type game the player bets on a outcome of throwing a dice n times. Picture 10 shows a filled-in coupon as a representation of an outcome which is bet on, when $n = 5$. To fill in the coupon means to put exactly one cross in each line of the coupon. The filled-in coupon in picture 10 shows the 53652 variation as a outcome of throwing the dice five times.

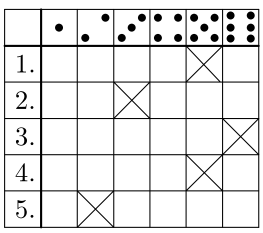

Figure 10. A coupon representing a outcome of a quintuple toss of a dice

Throwing six dices in a Lotto type game

The following are the exercises created by a Lotto type game in which six dices are thrown.
Ex. 5. Suggest a coupon for a g_w type game in which a player bets on a outcome of throwing six dices.
Ex. 6. The rules of a g_z type game allow betting on one of the three events:
$A = \{$*all the numbers show up*$\}$,
$B = \{$*one of the numbers shows up on at least two dices*$\}$,
$C = \{$*the same number shows up on all the dices*$\}$.
We may take three decisions here: d_A, d_B and d_C. Is there an optimal decision here?

A suggestion of a coupon and a way of filling it in is shown in pic. 11. The outcome shown in this picture favors the B event. It does not favor the A event.

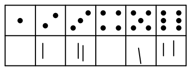

Figure 11. A filled-in coupon for a game, in which a player bets on a outcome of throwing six dices

In a model of throwing six dices the A and B events are opposite, and $P(A) = \frac{6!}{6^6} \approx 0,015432$, $P(B) = 1 - P(A) = 1 - \frac{6!}{6^6} \approx 0,984568$. The event A is therefore practically impossible, while the event B is practically sure (see [2], p. 118). This probabilistic feature of those events we can discover during the lesson using statistic methods. The basis of estimating the probability of the events may be the statistic data gathered in class (see [3], p. 93).

Betting on the outcome of a n-time coin toss in a Lotto type game

Let's assume that a n-time coin toss is a chance experiment δ in a g_w type game. A coupon for this game is suggested in the next exercise.

The outcome of a n-time coin toss is a n-element variation of the set $\{o, r\}$. Picture 12 shows a filled-in coupon for a game in which a player bets on a outcome of a 5-time coin toss before throwing the coin. Here a question appears: which outcome is shown in the picture?

	1.	2.	3.	4.	5.
o			X	X	
r	X	X			X

Figure 12. A filled-in coupon for a game in which one bets on the outcome of a 5-time coin toss

Making such coupons for different versions of Lotto type games is a mathematical exercise connected to coding some probabilistic objects. Deciphering of the filled-in coupon is decoding of information written in an iconic-symbolical form. Coding and decoding information from different forms of notation and to different notations is one of the forms of mathematical activity stated by didactics ([1]).

A tower as a outcome of a chance experiment – the certain, impossible and probable events

From a box containing four blocks identical in touch, two of which are white and two red, we will draw one three times without returning. The drawn blocks will form the ground, first and second floors of a tower. This tower is a outcome of an experiment δ_W. Picture 13 shows the set of towers as the set Ω of the experiment δ_W outcomes.

c	b	c	b	c	b	
b	c	c	b	b	c	Ω
b	b	b	c	c	c	

Figure 13. A class of towers as a class of the experiment outcomes

Before the tower is created a player estimates what colors will the floors be of and if he/she guesses the outcome of a experiment δ_W correctly, he/she scores a point. It is a g_w type game. Betting on the tower is the same as filling in the coupon (pic. 14) by coloring one or two squares red.

	second floor
	first floor
	ground floor

Figure 14. A coupon for a game in which one bets on a tower

bbc	bcb	bcc	cbb	cbc	ccb

Figure 15. A coupon for a game in which one bets on an event

A code of a tower as a outcome of the experiment δ_W is a three-element variation of the set $\{\mathsf{b}, \mathsf{c}\}$, where b stands for a white block and c stands for a red block. The first element is the color of the ground floor, the second – the color of the first floor and the third – the color of the second floor. In this situation the outcomes of the experiment δ_W make the set of $\Omega = \{\mathsf{bbc}, \mathsf{bcb}, \mathsf{bcc}, \mathsf{cbb}, \mathsf{cbc}, \mathsf{ccb}\}$.

The outcome bet on we note as a series of three letters. It is about translating certain information into the language of mathematics, and at the same time about passing over unimportant facts, simplifying, getting a mathematical point of view on the situation.

Let's alter the rules of the game. Instead of betting on what color the floors of the tower are we will predict what the whole tower may be. The following are some of such predictions:

$A = \{$*the tower will be in one color*$\}$,
$B = \{$*the tower will be in two colors*$\}$,
$C = \{$*the ground floor will be red*$\}$,
$D = \{$*the ground floor will be white*$\}$,
$E = \{$*the tower will have more red floors than white ones*$\}$,
$F = \{$*the tower will have more white floors than red ones*$\}$.

Now the issue is to bet on different events connected to the experiment δ_W. If a player bets on an event which actually happens – he/she scores a point. It is now a g_z type game. The A event is impossible. The B event is certain. The rest of events mentioned are possible. The events C and D are opposite. Are the events E and F opposite too? Betting on an event is the same as filling in the chart in picture 15. The player writes 1 under the outcome he/she bets on, and 0 under the outcome he/she does not bet on. Now betting on the event means defining it with its indicator ([3], p. 62).

Lotto type card games

A tool in this game is a set of 13 spade cards. After shuffling the cards will be laid out in a row. Before it happens a player bets on a specific cards' positions in the row. When the outcome is shown the player scores as many points (zlotys) as he guessed correctly. This chance game inspires exercises:

Ex. 7. Suggest a coupon for betting on an experiment outcome in this game.

Ex. 8. State the set Ω of all the outcomes of this chance experiment. How many elements has this class got?

Ex. 9. How many outcomes of „seven" on the third position and „three" on the seventh are there?

The outcome of laying the 13 cards out in a row is a permutation of the set $\{\mathsf{A}, 2, 3, 4, 5, 6, 7, 8, 9, 10, \mathsf{J}, \mathsf{Q}, \mathsf{K}\}$. This permutation is an order set in the set of 13 cards. The „author" of this order is fortune.

	A	2	3	4	5	6	7	8	9	10	J	Q	K
1								X					
2		X											
3				X									
4	X												
5					X								
6									X				
7							X						
8												X	
9										X			
10			X										
11						X							
12											X		
13													X

a)

1	2	3	4	5	6	7	8	9	10	11	12	13
8	2	4	A	5	9	7	Q	10	3	6	J	K

b) a simplified version of a coupon

the filled-in coupon shows the
(*permutation* 8,2,4,A,5,9,7,Q,10,3,6,J,K)

Figure 16. Two versions of a coupon for betting on the outcome of laying 13 cards out in a row

The two suggestions of a coupon (as two representations of a permutation of a 13 card class) are shown in pic. 16. Filling in the coupon in pic. 16a) means putting crosses in the spaces, considering that in each row and each column only one cross should appear. Filling in the coupon in pic. 16b) means putting the card symbols in the lower row (under the number of the card's position).

Let's assume that a player pays 1 \$ for taking part in the game, and the prize for 13 correct guesses is as much as 5 000 \$. Before laying the cards out a player may fill in any number of coupons shown in pic. 16. Here other problems occur:

Ex. 10. How many coupons should a player fill in and how to make sure he/she gets the first prize?

Ex. 11. Assuming that filling in one coupon lasts 30 seconds, how much time does a player need for filling in all the coupons from ex. 10? Is it worth to fill in that many coupons to be sure of winning 5 000 \$?

There are 13! outcomes of the game, which counts up to 6 227 020 800 coupons. In this game a player may win 5 000 \$ only when it ends in exactly the outcome he/she has bet on. The question was about the time required for filling in 6 227 020 800 coupons.

After shuffling 13 spades cards 5 of them are laid out in a row. Before it happens a player bets on what card gets to the first, second etc. position. For five correct guesses he/she wins the first prize.

Ex. 12. Pic. 17 shows a coupon for this game. Describe the way of filling it in. How many coupons does a player need to fill in to be sure to win the first prize in such a Lotto type game?

	A	2	3	4	5	6	7	8	9	10	J	Q	K
1.													
2.													
3.													
4.													
5.													

Figure 17. A coupon for betting on an outcome of laying out in a row 5 out of 13 cards

1	2	3
4	5	6
7	8	9
$*$	0	#

Figure 18. A door-board at the McDonald's back door

Ex. 13. In the McDonald's restaurants the back door (a staff entry) have usually got a code lock. The lock has got 10 buttons numbered 0 – 10 and two other marked $*$ and # (pic. 18). To open the door one has to press four buttons in the correct order. How many ways are there to state such a code? What is this code from the probabilistic point of view?

Roulette as a Lotto type game

Another Lotto type game is the well known Roulette. In the game a number is drawn from the set $\{0, 1, 2, 3, \ldots, 36\}$. A model of this experiment δ_R is the pair (Ω, p), where $\Omega = \{0, 1, 2, 3, \ldots, 36\}$ and $p(\omega) = \frac{1}{37}$ for $\omega \in \Omega$. Before the drawing δ_R takes place a player places a counter he/she has bought before on the board with betting fields (pic. 19, see also [2], p. 94-97). Let's assume that a counter costs \$ 1. On the board there are two kinds of betting fields:

a) *numbers*, that is squares with numbers 0 – 36 on them and

b) *chances*, that is squares representing at least two-element sub-classes of the set $\{0, 1, 2, 3, \ldots, 36\}$.

– Placing a counter on a *number* type square a player bets on a outcome of drawing a number. If Roulette draws out the number a player has put his/her counter on, he/she wins \$ 30. In case of betting on the numbers Roulette is a g_w type game.

– Placing a counter on a *chance* type square a player bets on some event. If it actually happens, he/she wins an amount of money dependant on the cardinality of bets on this event. In this case Roulette is a g_z type game.

A square with a black rhombus on it represents black-colored numbers:

2, 4, 6, 8, 10, 11, 13, 15, 17, 20, 22, 24, 26, 28, 29, 31, 33, 35.

The Manque square represents numbers from 1 to 18, and the 12^P – numbers from 1 to 12 (those form the first dozen).

The Pair square represents even numbers. If a player places his/her counter on this square, he/she bets on the event

$C = \{\textit{the number drawn is even}\}$.

This position of the counter determines the set

$C = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36\}$, and so it is a code of this set. It is also some form of a filled-in coupon (a player could cross out numbers in the middle of the board; here – even numbers). Placing the counter on the Pair square is a specific rationalization of filling in the coupons.

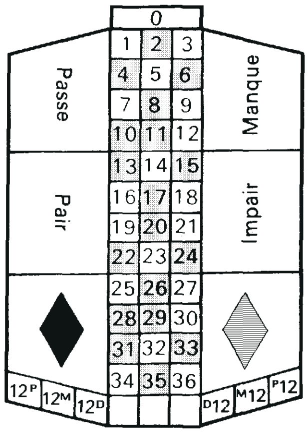

Figure 19. A board for betting in Roulette

A counter may also be placed on the side or top of the numbered squares (see [2], p. 95). Every such placing of the counter states a subset of the set $\{1, 2, 3, \dots, 36\}$, and so a kind of betting. If a player places a counter on the edge of squares numbered 8 and 11, he/she bets on the event:

$A = \{$*the Roulette draws number 8 or number 11*$\}$.

If the event A happens, the player wins 18 counters. It is a prize for guessing one of two numbers correctly.

Another Lotto type game is *A roundabout with horses* described in [4]. It is a g_w type game. To fill in a coupon means in this case to place a coin on an adequate square of the game board.

The Pools Tote and a Lotto type game

Each of the football meetings ends up in the host's victory (1), their loss (2) or a tie ($\times$). In 1956 – 1994 in Poland there was a game known as the Pools Tote, in which the players bet on the outcomes of football matches. A coupon consisted of n rows, each of which represented a pair of football teams. Filling in the coupon a player guessed the outcomes of n matches for the following week. At first a set consisted of $n = 14$ matches, then it changed into $n = 13$.

In the 1950s a player had to put a cross in one of the three squares in each row of the coupon. Picture 20 shows three filled-in coupons for the 10th June 1956 (here $n = 14$). The cross in the first column of the first bet means that a player bets on the hosts' victory, in the second column – on a tie, in the third – on the guests' victory. Each row of the coupon represents one match. A filled-in coupon for $n = 13$ is shown in pic. 21.

A filled-in coupon (the outcomes of the Sunday's matches included in the Tote) is a n-element variation of the set $\{1, \times, 2\}$. Pic. 21 shows a outcome coded in a variation: $(1, \times, 2, 2, 1, \times, 1, \times, 2, 2, 1, \times, 2)$.

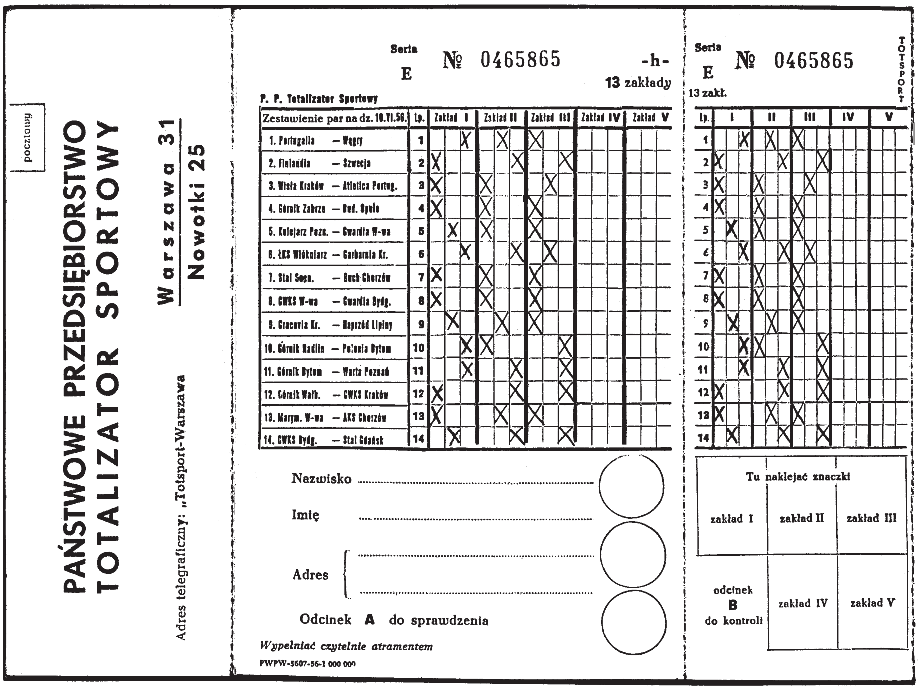

Figure 20. A filled-in coupon of the Pools Tote

In the Pools Tote game a player bets on the outcomes of 13 football matches.

Ex. 14. How many coupons should a player fill in to be sure of winning the first prize, which one could get for guessing the outcomes of all the matches of the week correctly?

Ex. 15. How much time would a player spend on filling the coupons in, assuming that filling in one coupon lasts 30 seconds?

Note: In this game the bet is on the outcome of an experiment, which is rather hard to be considered a chance one, because the outcome of a game depends on the ability of players, not on the fortune. The reciprocal of the number from ex. 14 is not the probability of winning the first prize in the Pools Tote. The game considered here is not a Lotto type game, because it is not a chance game.

	1	×	2
1.	1		
2.		×	
3.			2
4.			2
5.	1		
6.		×	
7.	1		
8.		×	
9.			2
10.			2
11.	1		
12.		×	
13.			2

Figure 21. A filled in coupon of the Pools Tote

Exercise 15 is a combinatorial, not a probabilistic one.

In this work we have shown how to motivate discovering of the probabilistic notions in school in context of specific analysis of the chance games.

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Epistemological Obstacles and the Phenomenon of Infinity

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The contribution will engage in aspects of the theory of epistemological obstacle according to the phenomenon of infinity in geometry. We can define that an obstacle demonstrates a set of mistakes related to previous knowledge. These mistakes are not unstable, but on the contrary, they are recurrent and permanent. The fundamental epistemological obstacle related to the phenomenon of infinity is knowledge about the finite world, which is so familiar to us. We can see our tendency, in history and in contemporary students as well, to attribute infinite mathematical objects with the same properties that finite objects have. My attention will focus on a possible knowledge, which can play the role of an epistemological obstacle within students' understanding of infinity, especially in context of one-to-one correspondence.

Introduction

Many educators and researchers in education attempted to answer the question how can the familiarity with the historical development of a particular mathematical knowledge help to better understand the cognitive process of contemporary students. There began to spread the Biogenetic, or Recapitulation Law formulated by E. H. Haeckel around the turn of the 19th and 20th century. He transformed the theory of recapitulation of main developmental stages from biology toward psychology. The concept of genetic development was elaborated in the work of J. Piaget and R. Garcia, as a reaction to this simplistic psychological version of recapitulationism. They assumed that cognition is not an accumulative process. Development is realized via particular stages and from one stage to the next one is necessary to reorganize the previous structure. G. Brousseau formulated an idea of epistemological obstacles, where the construction of knowledge is not determined by the stages but also by the negative impact of various norms and beliefs and ways of thinking that constitute obstacles. These stages and/or obstacles respectively seem to be analogous in history as well in contemporary students' thinking. On the other hand, there could be an argument, that today students develop in a different socio-cultural clime than mathematical concepts arose in the history, as social psychologists (for example Vygotsky) opposed.

Theoretical background

The theory of epistemological obstacles is based on ideas of G. Bachelard whose philosophy of science was developed between 1927 and 1953. He used the term epistemological obstacle for the first time in his work The Development of scientic thought (*La formation de l'esprit scientifique*) published in 1938. In his theory, he formulated the idea of a parallel between a psychological theory of individual mental development and maturation and a genetic theory of the development of concepts in the history of science. (Rogers 2000)

G. Brousseau transposed Bachelard's ideas into mathematics education in the 1970's. We can define that *an obstacle* demonstrates a set of mistakes related to previous knowledge. These mistakes are not unstable, but on the contrary, they are recurrent and permanent. The obstacle can be

- Knowledge (there is a domain where particular knowledge is used fruitfully, this domain is usually investigated, the knowledge is verified by many experiences).
- But there is a domain where the knowledge fails and produces wrong results, the knowledge is not able to transfer to another context (because there is a different viewpoint of thought or/and it is thought in a more general context).
- Knowledge, which resists contradictions and discrepancies with which it is confronted and hence, does not lead to a creation of 'better' knowledge. (It is the difference between an obstacle and a difficulty.) Knowledge presents in the same way whenever there is a repetition of such situation.
- Knowledge establishes after its integration into a system of cognition. Because there are other notions connecting to an original knowledge – obstacle.

The guiding idea of the theory is that an obstacle does not represent an absence of knowledge, the knowledge is there but it does not succeed in a particular situation. Knowledge, as an obstacle, is resistant to rejection. It has a tendency to adapt locally, to modify itself with an change as small as possible, and to optimise within a narrower domain. The reason is that an obstacle is knowledge related to a concept, that means a mathematical notion with a set of situations - problems which give to the notion a sense, a set of meanings which an individual is able to connect with the notion, and a set of tools, theorems and algorithms which an individual is able to use in working with the notion. We can classify the three following sources of obstacles: an ontogenetic source; a didactic source; an epistemological source. Each of these sources is connected with a different system, which enters into a pedagogical interaction. The most important for us, an epistemological source relates to the process of gaining the knowledge itself. These are obstacles, which we cannot and we should not divest ourselves of, because they are fundamental and essential for a formation of target knowledge. We can meet these obstacles within a history of notions themselves. (Radford 1997; Spagnolo and Čižmár, 2003)

The interpretation of the student's understanding of mathematics is framed here by the idea that the development of knowledge is a sequence of conceptions and obstacles to overcome. Consequently, the pedagogical action is focused on the elaboration and organisation of teaching situations built on carefully chosen problems that will challenge the student's previous conceptions and make it possible to overcome the epistemological obstacles, opening new avenues for richer conceptualisations (Radford, Boero and Vasco 2000, pp.162-167).

Epistemological obstacles related to the phenomenon of infinity in one-to-one correspondence context

When we realize that an epistemological obstacle should be knowledge, so the fundamental epistemological obstacle related to the phenomenon of infinity must be knowledge about the finite world, which is so familiar to us. We can see our tendency, in history and in contemporary students as well, to attribute infinite mathematical objects with the same properties that finite objects have. For example, it is obvious that a proper subset of a finite set must have fewer elements than this set. This is knowledge, which is reinforced in many situations and we have much experience from our life. But this knowledge fails when we work with an infinite set.

G. Galileo (1564-1642) in the *Discorsi* in his work *Dialogue Concerning Two New Sciences*, published in 1638, noted that by putting the natural numbers into one-to-one correspondence with their squares we arrive at an apparent contradiction in which infinity of numbers is equal in size to an infinite part of itself. In the same work, Galileo also noted that the points of two line segments of unequal length can be put into one-to-one correspondence, and that therefore the unequal segments must each contain the same number of points. These difficulties led to Galileo rejecting infinite collection of objects because 'they are not amenable to reason'.

C. F. Gauss (1777-1855), like Galileo, rejected the actually infinite in mathematics. In a letter, written in 1831, he says:

> I protest against the use of an infinite quantity as an actual entity; this is never allowed in mathematics. The infinite is only a manner of speaking, in which one properly speaks of limits to which certain ratios can come as near as desired, while others are permitted to increase without bound.

A. L. Cauchy (1789-1857) was unable to accept the idea of an infinite collection because of the apparent contradiction Galileo had noted - that an infinite set can be put into one-to-one correspondence with a part of itself.

Probably the first mathematician who accepted the consequences following from one-to-one correspondence on infinite sets was Bernhard Bolzano (1781-1848). In a work *Paradoxes of the Infinite*, published in 1851, after his death, he defended the existence of the actually infinite, and stressed that a concept of the equivalence of two sets applied equally to infinite and finite sets. He accepted that it was a commonplace for infinite sets to be 'equivalent' to a part of themselves. (Flegg 1975, pp. 9-11)

Bolzano was relatively isolated in Prague and his work had little influence in his time. Georg Cantor* (1845–1918) made the fundamental step toward the theory of sets. He accepted infinite sets with all their consequences. So we can say that it took at least 250 years until mathematicians coped with this property of an infinite set. Thus it cannot be surprising when a contemporary student has so many difficulties with understanding infinite sets. They often reject the results which are given by one-to-one correspondence similar to Galileo or Cauchy, and they do not accept the actually infinite as a 'real object' like Gauss.

I interviewed 13 prospective teachers and beginning teachers. The question was: *'What about infinity was for you unexpectedly surprising and/or what did you reject to accept?'*

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*Nevertheless Cantor's ideas were not accepted without comments. His most fundamental and history-making paper was at first denied publication by the journal to which he submitted it. And although his papers on the infinite were finally published, the reception accorded them was by no means cordial. (Wilder 1978)

10 from those 13 participants mentioned some aspect of the comparison of infinite sets. Many other experiments (Jirotková 2003, Prokopová 2003a,b, Tsamir 1999) show how difficult part of mathematics it is. Students frequently reject the principle of one-to-one correspondence, including those who followed a Cantorian set theory course. Students intuitively use other criteria, when compare infinite sets, but they generally neglect to use one-to-one correspondence. (Tsamir 1999). The most common criteria are:

- visual criterion – incorrect transfer of measurement
- criterion of inclusion – a set that is included in another set has fewer elements than that set
- criterion of single infinity – all infinite sets are equal
- incomparability – it is not possible to compare infinite sets

All of these intuitive criteria are connected with knowledge about finite sets. The visual criterion is usual case of intuitive law, which says 'a longer segment – more points'. This statement repeats even by students who got more familiarity with Cantorian theory. For example, when they have to compare a straight line and a plane, they use a criterion 'higher dimension – more points'. The similar situation is when student uses the criterion of inclusion. This property of finite sets is so familiar to us that we use it again and again. These two criteria are linked to principle of adding: 'if something is added the product is bigger'. For example, a student accepts demonstrated one-to-one correspondence in the case of two open intervals but they fail when they have to compare one open and other close one.

The last two criteria could be used when acquired results about infinite sets are in contradiction with previous knowledge (about finite sets). For example, the young teacher answered that *'number of points on two concentric circles are incomparable because they are both infinite'* even she created one-to-one mapping between the point sets. The result was surprising for her because she expected that the 'bigger' circle has more points than other one, on the other hand, she knew that number of points on a circle is infinite and more that *'it is not possible that one infinite is more than other one'*.

Conclusions

The one-to-one correspondence is so easy to use it but so difficult to understand it. How we can see in the history it took more than 450 years until mathematicians accepted infinitive sets with their strange properties as a real mathematical objects. There had to overcome many obstacles. The main of them was knowledge about finite word, because the results, which were given by theory of infinite sets were in contradiction with other conceptions.

Our students have the same obstacle, which they have to overcome. They use intuitive criteria – as visual, inclusion, single infinity and incomparability – for comparison of infinity sets. They reject to accept the one-to-one correspondence. For example because *'there could be created other mapping which shows that one infinite set has more elements than other one'*.

There is one important phenomenon. It is that the intuitive criteria repeat in different situations again even the student got familiar with the one-to-one correspondence. It is characteristic to epistemological obstacles. I believe that the investigation of the phenomenon of infinity greatly helped to develop human knowledge in a history and hence reflection on infinity by students could significantly contribute to their cognitive development. Development of students' understanding can be realized by help overcoming obstacles, which we can identify in history.

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Cabri Geometry in the Poincaré 's Model of a Lobachevskian Geometry

Petr Rys, Tomáš Zdráhal

Non-Euclidean geometries run counter to common sense. In order to understand them, students need to draw pictures. First the teacher must give them models, however. Once students have models, they can draw pictures. Why do not they use here the well known program Cabri Geometry? It was found just this program is both attractive and comfortable for them and enable them to solve many problems concerning e.g. a Lobachevskian geometry.

The ten axioms of Euclid describe the world around us with extraordinary accuracy. Therefore the main reason for the initial resistance to non-Euclidean geometry was practical experience. In our perception of the "flatness" of the immediate world in which we leave we accept the Euclid's fifth postulate ("Parallel Postulate") as true. That is, the axioms of Euclid describe the geometry of plane surfaces. By changing the fifth postulate, we can describe the geometry of other surfaces. So, other geometric systems exist as much as Euclidean geometry exists, and they even can be demonstrated in our world. They are just not as familiar. A system of geometry in which fifth postulate is changed is called a non-Euclidean geometry. In this paper we deal with a Lobachevskian geometry. The model used to express this geometry is the interior of a half-plane and was describe by the Frenchman Henri Poincaré. The Poincaré's model defines a plane to be all points "inside" a half-plane; the points on the line are excluded. Poincaré's model of a Lobachevskian geometry defines a line to be either of the following:

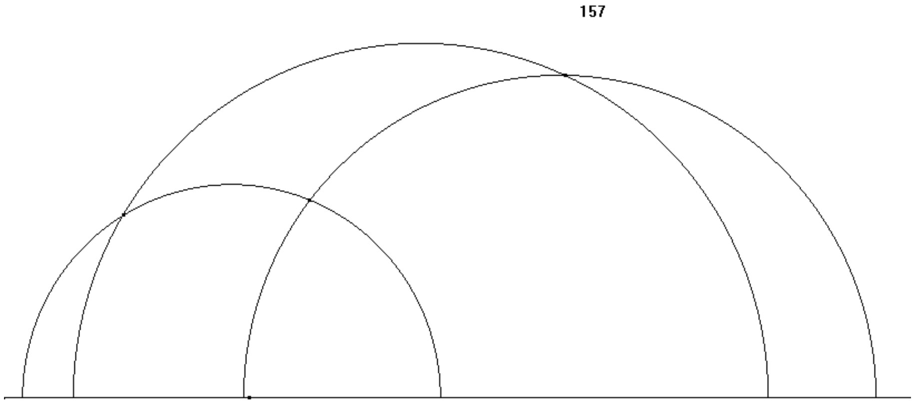
(i) a half-line orthogonal on the line

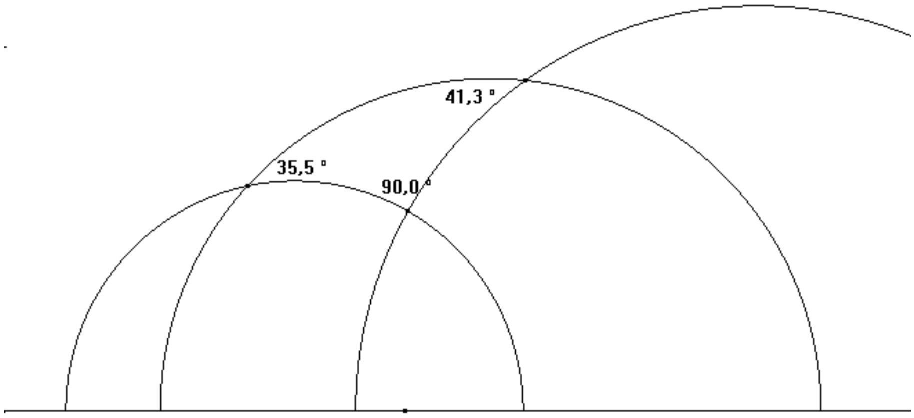
(ii) a circular arc with midpoint on the line

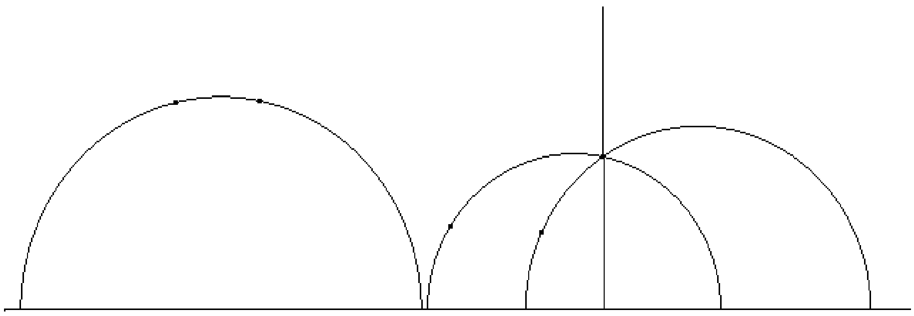
Because the boundary of the disk is excluded, a line has no endpoints. Using Poincaré's disk model, we can construct more than one parallel line through a given point. It seems very useful for students to use the program Cabri Geometry to draw some pictures in Poincaré's disk model and solve problems as follows:

- What can be said about the sum of the angles of a triangle?
- How many right angles can a triangle have?
- In how many points can a pair of distinct lines intersect?

Further are enclosed some pictures that can be done by means of the program Cabri Geometry by the students themselves:







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Optimal stopping rule in stochastic game

Pavel Tlustý

The article deals with two stochastic games (with an unbiased dice and coin). Some situations and problems connected with the search in the optimal stopping rules are investigated.

When dealing with practical problems, we can get into a situation when it is necessary to make a certain decision. It is clear that we try to make, in a certain sense, the best decision. However, it is not always possible to take into consideration all circumstances, as some of them are of a random nature. The following two examples illustrate the way to approach the problem of decision-making and the way of choosing "optimality criterion".

First game

This game is meant for an arbitrary number of players taking their turns regularly. The players cast a dice in a beforehand given order. If the players score 2, 3, 4, 5 or 6, they obtain the corresponding number of points. The players can anytime decide whether to go on or to stop casting the dice. In the case they decide to stop, all the points in that particular series of casting (within their turn) are added together and credited to their account. Then it is time for another player to cast the dice. If they go on with the casting and score 1, all the points obtained in that series are lost. The winner is the player who scores first 1,000 points.

Before we start searching for the optimal solution, let us imagine the following situation. In a particular series, one of the players has scored 25 points so far and s/he is about to make a decision whether to go on with the casting or to stop. For a better illustration, let us assume that this situation happens 6 times in a row. The player has two possible strategies:

- 1st strategy - *"stop casting the dice".*

  The casting is stopped and 25 points are credited to his/her account. In total after 6 turns, 6 times $25 = 150$ points have been reached.

- 2nd strategy - *"go on with casting the dice".*

  What can we expect in this situation?

  In this case in average, the player scores once 1 (no points are credited), the player scores once 2 (27 points are credited), ... and the player scores 6 (31 points are credited). In total,

$$27 + 28 + 29 + 30 + 31 = 145$$

points are credited.

We can see that the first strategy can guarantee a higher point credit and it is therefore better.

Regarding the theory of probability, we have calculated the mean of the random value and because the expected profit is lower then the expected lost, it is not profitable to go on with casting the dice. The rational decision is therefore to stop casting the dice. It is easy to prove that "the optimal stopping rule" in this game is as follows:

- *if the sum is less than 20 – go on with casting.*
- *if the sum is equal to 20 – it is equally sensible to go on or to stop casting.*
- *if the sum is more than 20 – stop casting.*

The following simulative study shows that the above-mentioned strategy is really the best. Using the Maple 8 software, the simulative program has been made and particular strategies have been compared. In each situation, 10,000 games were modelled and "the average number of series" necessary for reaching 1,000 points was determined. The results are summed up in the following table:

stopping rule	10	18	19	20	21	22	30
number of series	146,50	125,71	124,34	123,96	124,17	124,90	134,01

The table shows that the theoretical solution agrees with the solution found when using the Monte Carlo method.

We showed how to utilise "the increase and decrease" of the mean value to determine the optimal stopping rule. Unfortunately, it is not possible to apply the algorithm from above in all cases without prior thinking, as the second game illustrates.

Second game

This game is played with an unbiased coin. If we stop tossing the coin after the nth toss, we get

$$\frac{n}{n+1} \cdot 2^n$$

points, if we toss only tails, or we get nothing if the last toss is a head. When is it optimal to stop tossing?

Let us denote the random value with S_n - the number of obtained points after n tosses. If we use the same reasoning as in the previous example, it means that we calculate ES_n – the mean of the random value S_n for $n = 1, 2, 3, 4, \dots$ and until the mean value increases, we go on with tossing.

$$\begin{aligned}
n = 1, \qquad ES_1 &= \frac{1}{2} \cdot \frac{1}{1+1} \cdot 2^1 + \frac{1}{2} \cdot 0 = \frac{1}{2}, \\
n = 2, \qquad ES_2 &= \frac{1}{4} \cdot \frac{2}{2+1} \cdot 2^2 + \frac{3}{4} \cdot 0 = \frac{2}{3}, \\
n = 3, \qquad ES_3 &= \frac{1}{8} \cdot \frac{3}{3+1} \cdot 2^3 + \frac{7}{8} \cdot 0 = \frac{3}{4},
\end{aligned}$$

$$n = 4, \quad ES_4 = \frac{1}{16} \cdot \frac{4}{4+1} \cdot 2^4 + \frac{15}{16} \cdot 0 = \frac{4}{5},$$

$$\vdots$$

We can see that the mean of the random value S_n is monotonously increasing. From that it is obvious that it is optimal keep tossing (endlessly, in theory). Regarding the theory of probability again, we know with probability of 1 that we toss at least one head in the endless series of tosses. If we toss a head, we lose everything. There is a clear paradox:

It is optimal to keep tossing until everything is lost.

It is clear from the presented example that it is necessary to be careful when solving this type of problems.

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Reasoning facilitating generalising and applying a letter symbol by pupils aged 13-14

Lidia Zaręba

The paper presents an extract from the wider project which concerned the types of generalisations employed by children aged 13-14, these children's capability of applying a letter symbol during the process of generalising, their methods of proceeding and difficulties they come across in the course of the process of solving problems designed to lead to generalising. The study on which I focus my attention below was aimed at classifying typical methods of proceeding which represent the reasoning of pupils participating in the study and at determining what influence a particular method of proceeding may have on the final outcome of these pupils' work. After a very short introduction into the research issues, the organisation of the study and the method and instruments applied are presented, followed by the summary of the results obtained by analysing the solutions of the problems of a certain test. The results of the research suggest that the type of reasoning conducted by a pupil influences the final result of this pupil's work. Presumably, the geometrical type of reasoning produces a positive effect on the process of generalising and on ability to describe a regularity noticed by a pupil with the aid of a letter symbol.

1. Research issues

Epistemological basis of forming mathematical notions and, in particular, beliefs concerning their abstract character are principal sources of revising school curricula and teaching methods of this subject. Currently, a certain shift of the main stress in teaching mathematics may be observed, namely from commanding knowledge onto commanding a method. The focus, therefore, is on a pupil's creative activity, on his or her active and conscious participation in discovering notions, formulae, theorems, proofs, in schematising situations, in mathematising and **generalising** these situations ([1], p. 13-14).

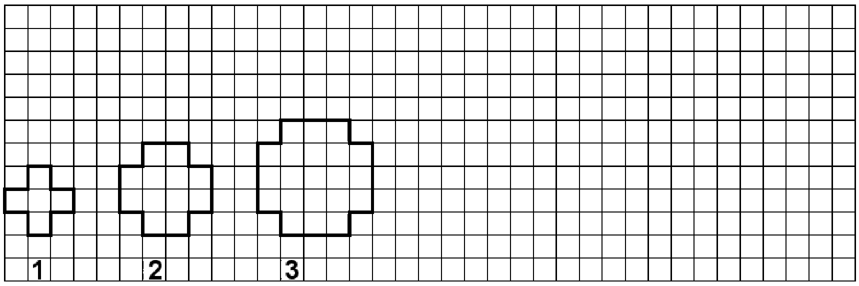
Generalising in mathematics is understood as an activity as well as an instrument for creating notions and theorems, whereas a **generalisation** is a result of the process of generalising, that is to say, a notion, opinion, theorem, regularity noticed and formulated, hypothesis.

Mathematical-didactical literature mentions **generalising notions** and **generalising theorems**. In the course of the study presented here I concentrated on a particular type of **generalising theorems**, namely **inductive generalising**. I examined the process during which after conducting a sequence of mathematical trials a pupil is supposed to notice a certain regularity in the results of these trials and formulate a mathematical hypothesis.

2. Description of the study

The paper presents an extract from the wider project one of the objectives of which was distinguishing typical modes of proceeding displaying the reasoning of pupils aged 13-14 conducted during the process of solving certain problems aimed at triggering the process of generalising, and establishing the influence of the mode of proceeding taken by a pupil on the ultimate outcome of their work.

1) Draw the fourth and the fifth cross in the series.
Find the permiter D and the surface area P of each of these crosses.



D= …… D= …... D=……. ……………. …………..
P= …… P= …... P=……. ……………. …………..

Note the results in the table.

Number of the cross	1	2	3	4	5	
Perimeter - D						
Surface area - P						

2) Without drawing the sixth cross guess what its perimeter and surface area is.
Describe how you guessed (expressing it in words).
…………………………………………………………………………………………
…………………………………………………………………………………………
…………………………………………………………………………………………

3) Fill in the table.

Number of the cross	7	8	9	10	20	...	n
Perimeter - D							
Surface area - P							

How did you calculate the perimeter of the cross number n?
…………………………………………………………………………………………
…………………………………………………………………………………………
…………………………………………………………………………………………

How did you calculate the surface area of the cross number n?
…………………………………………………………………………………………
…………………………………………………………………………………………
…………………………………………………………………………………………

TEST „CROSSES” part 1.

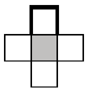
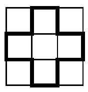
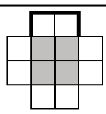
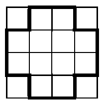
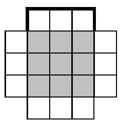
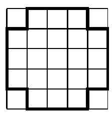
The study was based on the analysis of solutions of sequences of problems (tests) designed for this particular study. As a research instrument the test "Crosses", consisting of two parts, was applied; the test required finding perimeters and surface areas of certain cross-shaped figures. Part 1 of the test, by the principle assumed, contained no suggestion of a method of solving; part 2 offered such a suggestion in the form of two "geometrical" methods of solving this problem: Magda's method and Przemek's method.

The test was being solved during the school years 1996/97 and 1997/98 by pupils aged 13-14 from Polish and Czech schools. The project involved 90 Polish and 114 Czech pupils. I excluded from the analysis those of the pupils' papers which in my opinion either displayed the lack of understanding of the notion of perimeter or surface area or whose authors did not notice analogies in the shapes of consecutive crosses, did not follow the way in which the sketch of the fourth and the fifth cross was created. After these exclusions the analysis covered 87 Polish and 78 Czech papers.

Magda and Przemek used different methods of calculating
the perimeter D and the surface area P of the crosses.

This is what Magda wrote:	**This is what Przemek wrote:**
<u>Calculation of the perimeter:</u> I multiply the length of the indicated sides of the crosses by 4. <u>Calculation of the surface area:</u> I add the surface areas of the four white arms of the cross the to surface area of the shaded square.	<u>The perimeter of the cross</u> is calculated just like the perimeter of the square containing it, since the length of its one arm is the same as the length of the side of this square. <u>Surface area:</u> If I take 4 small squares in the vertices I get the cross. That is why I calculate the surface area of the square and then subtract 4.

Complete Magda's and Przemek's notes:

(1)  D = 4 · 3 = P = 1 + 4 =	(1)  D = 4 · 3 = P = 3 · 3 – 4 =
(2)  D = 4 · ... = P = 2 · 2 + 4 · ... =	(2)  D = 4 · ... = P = 4 · ... – 4 =
(3)  D = 4 · ... = P = ... · ... + 4 · ... =	(3)  D = ... · 5 = P = ... · ... - ... =

Using these methods calculate the perimeter and the surface area
of the cross number 4, 5, 10, ..., n.

(4)	(4)
(5)	(5)
(10)	(10)
(n)	(n)

Justify the answer to the last example (n).

TEST „CROSSES" part 2.

3. Conclusions of the study

In the course of the study my attention was focused on the form taken by the generalisation made by a pupil and on the fact whether it would be expressed with the use of a letter symbol.

On the basis of the solutions of problems which required performing calculations in a general situation of n three types of generalising employed by pupils of the sample group were distinguished: generalisation based on an example, when a problem is solved in the case of a specific number chosen by a pupil, verbal generalising, describing operations performed by a pupil, and symbolic generalising, expressing relationships in the form of a formula containing a variable (the author's terminology assumed for the purpose of the study). Presumably, for a pupil of the sample group generalising based on an example and verbal generalising were easier than symbolic generalising, which means that the generalisation performed by such a pupil manifested, in the great measure, features of a paradigmatic example or a verbal description of relations.

As a result of the analysis of the solutions of the first part of the "Crosses" test I distinguished four characteristic modes of proceeding displaying reasoning of the pupils of the sample group which, at the same time, facilitate undertaking attempts to express the regularity implanted in the problem symbolically:

1. **Numerical reasoning** based on the recursive rule - a pupil notices that the perimeter of each successive figure is greater by 4 units.
2. **Geometrical reasoning**, consisting in noticing a regularity in the course of the process of observing geometrical figures; for example, a pupil identifies the perimeter of a cross with the perimeter of the minimal square containing this cross.
3. **Numerical and geometrical reasoning** (appearing in a pupil's reasoning parallelly).
4. **Guessing**, **counting** or **no specified mode of reasoning** performed by a pupil.

From the point of view of the mode of proceeding, the first part of the "Crosses" test confronted a pupil with an open situation. In this situation pupils felt more comfor with numerical reasoning. For example, in the case of the problems dealing with the perimeter 69 out of 87 Polish participants noticed only the recurrence relation, 4 persons noticed only the geometric way, and 6 persons noticed both.

The analysis of the solutions of part 1. of the test suggests the influence of the type of reasoning conducted by a pupil on the final result of this pupil's work. In the case of the sample group the following regularity may be distinguished: the greatest capability to perform symbolic or verbal generalising was manifested by those who noticed both the numerical and the geometrical way of reasoning; sticking solely to geometrical reasoning turned out to be less productive; noticing only the method employing recurrence relation brought even lower results.

What affords an additional opportunity to observe the influence of the reasoning on the ultimate result is the analysis of the second part of the "Crosses" test. In this part the geometrical way of approach was suggested. For both parts of the test I made the breakdown of the numbers of pupils who employed particular types of reasoning. In the case of the second part of the test I separated the records for the problems solved by Magda's method and for those solved by Przemek's method. The example of such a breakdown for the problems concerning the surface area of the n-th cross is shown in table 1.

Test	Type of generalisation			
	based on an example	verbal	symbolic	absence of generalisation
part 1	2	5	14	66
part 2 Magda's method	0	2	44	41
part 2 Przemek's method	1	1	27	58

Table 1: Quantitative distribution of answers in the case of the surface area of the figure number n (based on the sample group of Polish pupils)

As a result of directing pupils' proceeding towards geometrical reasoning (which was put into effect in part 2. of the test) the number of the pupils who achieved symbolic generalisation increased. The analysis of the problems on perimeter revealed similar dependence.

The analysis of the written version of solutions of the test problems suggests that the type of reasoning conducted by a pupil influences the final result of this pupil's work. Presumably, the way to symbolic or verbal generalisation through geometrical type of reasoning is quicker that the one restricted solely to numerical reasoning. At the present stage of the project this conclusion is only hypothetical in its character. For the research focused on generalising as a process, basing solely on the analysis of the solutions of problems of a test is not sufficient. Such an approach enables formulating purely preliminary hypotheses. What seems necessary is to complete and verify it by individual observations.

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Capabilities of pupils aged 10-14 to apply a letter symbol in the process of generalising - report from research

Lidia Zaręba

The paper presents an extract from the project which concerned the capabilities of pupils aged 10-14 to apply a letter symbol in the process of generalising, in other words, formulating an answer to the question whether the correct application of such a symbol lies within the zone of discussed group development or within the zone of proximate development or maybe it lies beyond the capabilities of an average pupil of the discussed group.

Generalising in mathematics is understood as an activity as well as an instrument for creating notions and theorems, whereas a generalisation is a result of the process of generalising, that is to say a notion, opinion, theorem, regularity noticed and formulated, hypothesis. A particular form of generalising theorems is the **inductive type of generalising**. This specific type of activity lies within the sphere of my research interests. What draws my interest is the process during which after conducting a sequence of mathematical trials a pupil is supposed to notice a certain regularity in the results of these trials and formulate a mathematical hypothesis. My particular attention is focused on whether a pupil spontaneously uses a letter symbol in the formulated generalisation. The present paper is a report from research aimed at finding an answer to the question whether correct application of such a symbol lies within the zone of development of an average pupil aged 10-14, within the zone of their proximate development or beyond the capability of such a pupil.

The project was based on the analysis of the solutions of sets of problems (tests) designed for this particular study and aimed at triggering the process of generalising. The tests were solved by pupils aged 10-14 from Polish and Czech schools (the total of 757 participants). The research instruments employed in this study were in the form of tests: 'Straight lines', 'Frames' (both appended at the end of the paper) and 'Crosses'- parts 1 and 2 (see [2] or http://fedu.ku.sk/konferencia/Zareba.pdf).

In the process of analysing the solutions of the test problems I applied H. Siwek's interpretation of the notion of Vygotski's zones of development ([1], p. 65). In this case the zones refer to a group of pupils participating in the study rather than to an individual child. According to this interpretation, a given activity lies:

a) within **the zone of development** of an average child from a class group when a

Denotation	Description of the test
PR	"Straight lines"
R	"Frames"
K-1D	"Crosses" - part 1; section concerning the perimeters of figures
K-1P	"Crosses" - section concerning the surface areas of figures
K-2D-M	"Crosses" - part 2; section concerning the perimeters of figures calculated by Magda's method
K-2D-P	"Crosses" - part 2; section concerning the perimeters of figures calculated by Przemek's method
K-2P-M	"Crosses" - part 2; section concerning the surface areas of figures calculated by Magda's method
K-2P-P	"Crosses" - part 2; section concerning the surface areas of figures calculated by Przemek's method

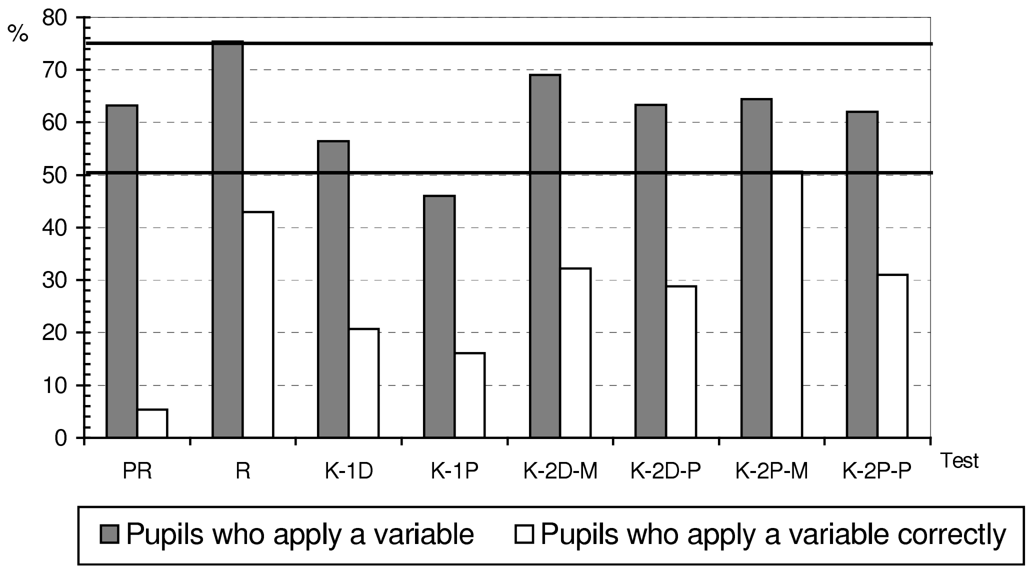
Table 1: The list of short denotations of the tests

problem requiring this activity was solved by 75%-100% of the whole group,

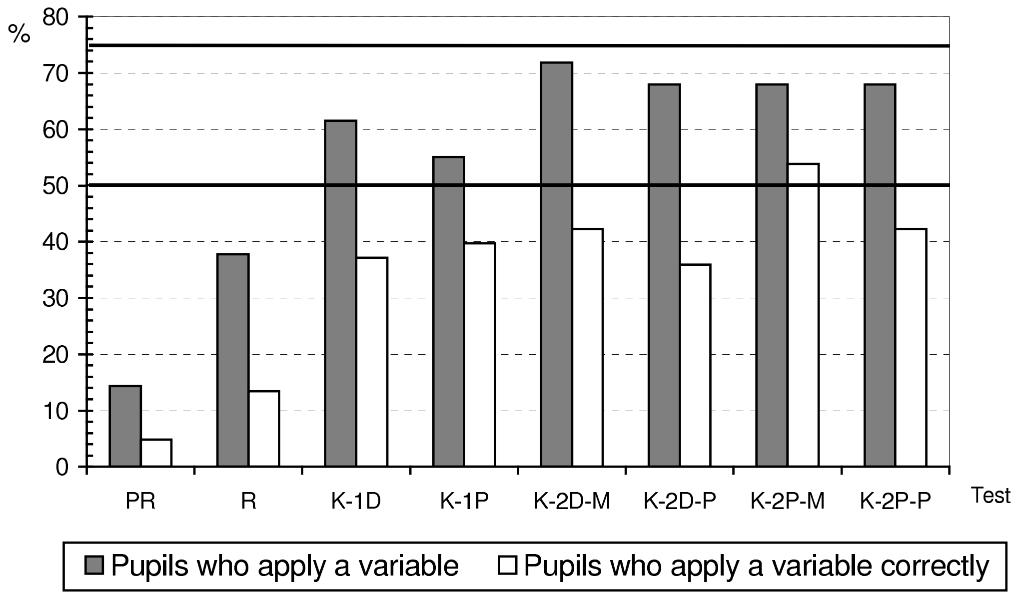
b) within **the zone of proximate development** if the percentage of the correct answers is between 50 and 75,

c) **beyond their capabilities** if the percentage does not reach the level of 50.

For each type of the test the percentage of the answers whose authors made use of a variable to express a relationship was calculated, as well as the percentage of the answers containing a correctly applied letter symbol. The results are presented in graphs I and II. Because of the large number of the tests they are described symbolically (according to the denotation of table 1).



GRAPH I. Application of a variable to expressing relationships - the percentage record for the sample group of Polish pupils



GRAPH II. Application of a variable to expressing relationships - the percentage record for the sample group of Czech pupils

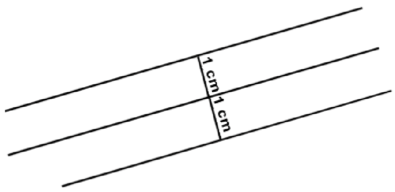
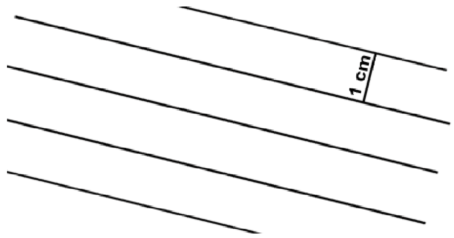
The results indicate that the correct symbolic notation of a logical relationship surpasses the capabilities of an average pupil of the tested group of Polish pupils, whereas applying a variable to expressing numerical relationships (the correctness of this notation aside) lies within the zone of proximate development of the pupils from this group.

The corresponding record made for the Czech group (graph II) suggests that, similarly, for an average Czech child the ability of creating a correct symbolic notation of a logical relationship lies beyond their zone of development. Presumably, applying a symbol lies within the zone of development of an average pupil aged 13-14. Commanding this skill is beyond the capabilities of an average pupil aged 10-12.

The comparison of the results of the analysis made for the sample groups of Polish and Czech pupils suggests that their age is not the decisive factor situating a pupil, in Piaget's terminology, at the formal operational stage. A pupil's status is strongly influenced by the extensiveness and quality of his o her experience with letter symbols. In Polish curriculum and workbooks formal mathematics appears earlier than in the Czech ones.

At the present stage of the research all the conclusions are only of hypothetical character. The verification of these hypotheses is the objective of individual observations, which are being analysed at present.

Two sets of parallel straight lines are given. The distance between every two neighbouring lines equals 1cm. In each case find the distance between the first and the last straight line.

a) Three straight lines are given.  Answer: The distance equals	b) Five straight lines are given.  Answer: The distance equals ......
c) Seven straight lines are given. Answer: The distance equals	d) 25 straight lines are given. Answer: The distance equals ......
e) n straight lines are given. Answer: The distance equals	f) Justify the last answer.

TEST „STRAIGHT LINES”

1) Complete the frame up to 100.

	2	3	4	5	6	7	8	9	10
1	12	13	14	15	16	17	18	19	20
2	22	23	24	25	26	27	28	29	30
3	32	33	34	35	36	37	38	39	40
4	42	43	44	45	46	47	48	49	50
5	52	53							

2) Four numbers may be framed together. Two frames are visible in the table:

the frame of the number 8

8	9
18	19

the frame of the number 33

33	34
43	44

In the table draw the frame of the number 57.

3) Complete the frames of the following numbers:

a) 84

84	
94	

b) 102

102	
112	

c) 155

155	
	166

d) 71

71	72

e) 147

147	
	158

f) 289

289	290

g) 43

43	

h) 176

i) 382

4) Complete the frame of a number n.
Describe (step by step) what you did and why.

....

....

TEST „FRAMES”

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Generalising reasoning of pupils aged 13-14

Lidia Zaręba

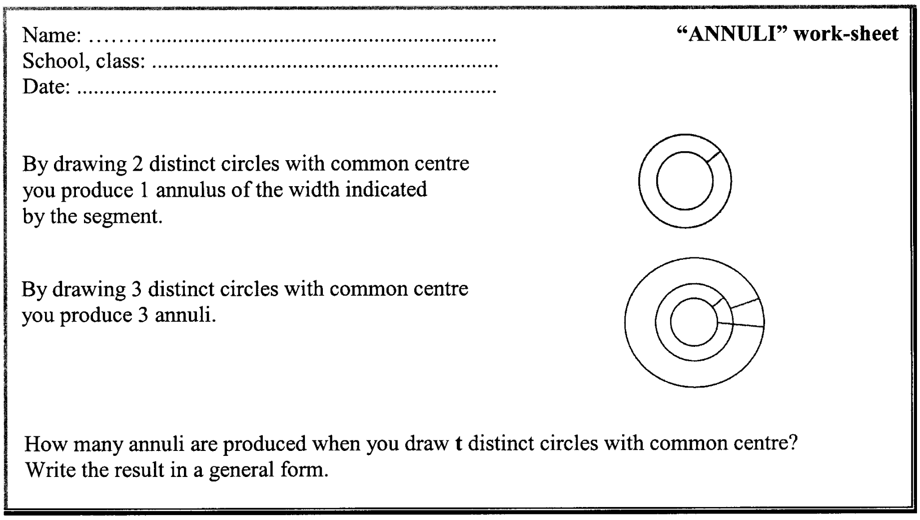
Inductive generalising is a specific type of generalising and is understood as the process of discovering rules on the basis of observation and comparison of particular cases. This specific class of activities lies within the domain of my scientific interest. In my presentation I discuss the results of individual observations conducted with a sample of pupils aged 13-14. These results indicate a general pattern of a route followed by a pupil on his or her way to generalising of the inductive type.

The problem, objective and method of the study

The presented study is an extract from the wider research concerning the process of generalisation and the problem of applying a letter symbol during this process by children aged 13-14. I restrict my attention here to problems which primarily trigger **generalising of the inductive type**. In didactics of mathematics it is understood as the process of discovering general rules on the basis of observation and comparison of particular cases ([1], [2], [3], [4]). The fact that during the research process my attention was focused on this particular type of generalising does not rule out the possibility of other types of generalising appearing in these problems (for example, generalising a reasoning by variating parameters, by noticing a recursive rule, generalising a result or the method of generalising by unification).

The objective of the study is to describe the route followed by pupils aged 13-14 in order to obtain an inductive generalisation and expressing it with the aid of a letter symbol. It is therefore aimed at distinguishing and characterising typical ways of proceeding displaying the reasoning conducted by the pupils participating in the study during the process of generalising and establishing the relation between the mode of proceeding taken by a pupil and his or her achieving a symbolical generalisation, i.e. the notation of the result of the generalising process in the symbolical language of mathematics.

The method applied during the study was individual observation combined with the researchers conversation with a participant of the study. The observer was following and registering the participant's proceeding while the pupil was solving certain problems, designed for the purpose of the study, presented on A-4 work-sheets. Two of the work-sheets, which were of key importance to the study, are displayed on figures 1 and 2. The problem on annuli, called the leading problem, was addressed to every pupil. "W-P Hint" was of an auxiliary character and was intended for those pupils who in their attempts to find the number of annuli for the case of t circles did not examine particular cases of the problem or did it in a chaotic way, which made it impossible for them to notice the general solution. The main goal of this work-sheet was to indicate the inductive way leading to the solution of the problem.


Name: .... "ANNULI" work-sheet
School, class: ....
Date: ....

By drawing 2 distinct circles with common centre you produce 1 annulus of the width indicated by the segment.

By drawing 3 distinct circles with common centre you produce 3 annuli.

How many annuli are produced when you draw **t** distinct circles with common centre? Write the result in a general form.

fig. 1 Annuli (Leading problem)

Organisation of the study

Individual observations were performed during the school year 2002/03. Thirty four pupils aged 13-14 [*] participated in the study. They were chosen from 9 classes by 9 teachers. From the group of 34 performances of only 33 pupils were included in the comprehensive analysis. It was the result of eliminating one of them because of the fact that the pupil failed to understand the problem's content.

Qualitative analysis

In order to solve a particular case of the problem the participants create certain schemes, models of the situation. Solving the problem within the framework of a particular model is connected with an interpretation based on a produced sketch or arithmetical scheme. In the work-sheets of the participants I have distinguished six mathematical models [*]:

1. **Geometrical (G)** — a pupil indicates annuli at the drawing and arrives at the solution by counting the previously indicated annuli.
2. **Geometrical-arithmetical (G-A)** — in order to find the number of annuli in a given particular case a pupil indicates them in a way which guaranties that no annulus has been omitted. Later on he or she makes use of this way of indicating in writing the sought number in the form of an arithmetic expression.
3. **Geometrical-recursive (G-R)** — a participant notices a recursive relationship between the number of annuli obtained for a few particular cases; consecutive differences by which the numbers of annuli increase are noticed with the help of a previously produced sketch.

[*]The group of participants consisted of 11 pupils "poor at mathematics", 12 pupils of average abilities and 11 "very good " ones. The choice of pupils was made by their teachers of mathematics.

[*]My classification of the models is based on the problem on circles and annuli. These models function in the problem on polygons and straight lines as well.

4. **Arithmetical (A)** — a participant identifies the number of annuli with a respective arithmetic expression without appealing to geometric interpretation of the solution of this problem; for example, the number of the annuli defined by 28 circles is the sum $1 + 2 + 3 + ... + 27$.

5. **Recursive (R)** — while analysing (in isolation from the sketch) the number of annuli for a few particular cases, a pupil notices a recursive relationship between them, notices the consecutive increase by 2, 3, 4, ... of the number of annuli with respect to the number of annuli for the preceding number of circles.

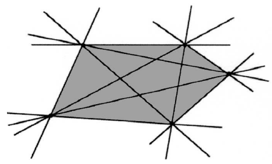
6. **Recursive-arithmetical (R-A)** — a pupil notices a recursive relationship, the consecutive increase by 2, 3, 4, ... of the number of annuli with respect to the number of annuli for the preceding number of circles, and on the basis of this relationship builds an arithmetic expression which enables finding the number of annuli for a given number of circles, this time without appealing to the preceding case.

Having a model at their disposal, the participants direct their activities at transforming it so that it enables them to narrow their actions to the ones which are really necessary and preserve only those of the model's elements which are relevant from the point of view of the ultimate solution of the problem. This kind of activities I describe as "reducing" a model. It is frequently equivalent to changing a model for some other one. The analysis of the actions of a pupil who is directed at generalising suggest a significant role which such actions conducted within the framework of mathematical models plays in the process of generalising. In particular, before the final emergence of a definite form of a generalisation in a pupil's reasoning one may distinguish a characteristic order of models applied by him or her.

The activities of all the pupils who understood the content of the problem arise form either model **G** or a model related to it (**G-A** or **G-R**). Undoubtedly, this situation results in great measure from the character of the problems: the formulations of both problems, the one on annuli and the one on polygons, include drawings. What is interesting, however, is the fact that after such a geometrical opening the actions of the most part of the participants become gradually detached from geometry and become for the most part directed at model **A**. This model provides, in my opinion, the key to creating the ultimate form of the generalisation. A pupil who tries to express a generalisation attempts, in all probability, to create a background of the same character as the sought generalisation. Although at the very beginning he or she does not know what the form the final solution will take on, the pupil is aware of looking for a certain "formula", that is to say, an object of an arithmetical rather than geometrical character. This is a possible explanation of the direction the actions most children enter upon; geometric point of view is advantageous at the stage of noticing an arithmetical relationship, which, in turn, may be "transformed" into an algebraic notation.

Name:.. "W-P Hint" work-sheet
School, class:..
Date:...

1. A convex pentagon is given. Every two vertices of this polygon define one straight line.
   Were all the possible straight lines drawn? ....
   What is the number of possible straight lines? ....



2. Draw a triangle and a convex quadrilateral.
   In both cases draw all the straight lines defined by each pair of vertices of the figure.
   How many such lines are defined by the vertices of a triangle?..................
   Of a quadrilateral?................
3. Fill in the table.

Number of the angles of a polygon	Number of straight lines defined by each pair of the polygon's vertices
3	
4	
5	
6	
7	
10	
24	
345	

4. Let n denote a number of vertices of a polygon. How can the number of the straight lines defined by this n-gon's vertices be described?

fig. 2 W-P Hint (Auxiliary problem)

Conclusions

The outcome of the study suggest that it is easier for a pupil to notice regularities while acting within the model with geometrical touch. Whereas in order to obtain a symbolical notation of this regularity he or she would rather resort to models of more arithmetical character. It seems therefore that this is the route which organising the didactical process directed at developing the activity of generalising crowned with a symbolical notation with the aid of a variable should point out: the route from geometrical viewpoint on a given problem to arithmetical phrasing. This the reason why it would seem advantageous if initial problems as tools for modelling this type of activities included a visual representation, and if in his or her first steps towards this aim a pupil were accompanied by models of geometrical character recommended by the teacher.

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Functional Equations and Some Elementary Functions

Petr Rys, Tomáš Zdráhal

Functional equations provide on exciting teaching opportunity and encourage students to discover and further investigate mathematical concepts of functions and their graphs. This paper deals with the process of defining some elementary functions by means of the functional equations. Especially from the pedagogical and didactics point of view. Developing student's thinking is one of the main goals of teaching mathematics.

The aim of this article is to show how we can define quadratic function, cubic function and so on, i.e. a polynomial of the n-th degree by means of functional equations. Appropriate functional equation we obtain by means of generalization of functional equation which we can define linear function, i. e. the equation $l(x+y) - l(x) = l(y) - l(0)$.

Let us start from the following lemma

Lemma 1. *Functional equations*

$$f(x+h) - f(x) = f(h) - f(0), \tag{1}$$

and

$$f(x+2h) - 2f(x+h) + f(x) = 0, \tag{2}$$

$x, h \in \mathbb{R}$ *are equivalent.*

Proof

Suppose that f is the solution of the equation (1) and prove that one is a solution of the equation (2) as well. Let $x := x + h$. We get

$$f(x+2h) - f(x+h) = f(h) - f(0).$$

According to (1)

$$f(h) - f(0) = f(x+h) - f(x)$$

and together we have

$$f(x+2h) - 2f(x+h) + f(x) = 0,$$

i.e. the equation (2).

Conversely, let $f(x)$ is the solution of the equation (2). Set here $x := 0$ then

$$f(2h) + f(0) = 2f(h).$$

Now let us set $h := x + h$, we get

$$f(2x+2h) + f(0) = 2f(x+h).$$

By (2) is

$$2f(x+h) = f(x+2h) + f(x)$$

and therefore

$$f(2x+2h) + f(0) = f(x+2h) + f(x).$$

For $x + 2h := h$ we get

$$f(x+h) - f(x) = f(h) - f(0),$$

i.e. equation (1).

Let us define for arbitrary function $f : \mathbb{R} \to \mathbb{R}$ and arbitrary real number h

$$\Delta_h f(x) := f(x+h) - f(x)$$

for such all x from the domain of definition of the function f that also $x + h$ belongs to this domain. Further let us define iterations Δ_h^p of the operator Δ_h for $p = 0, 1, 2, \dots$ in this way

$$\Delta_h^0 f = f, \quad \Delta_h^1 f = \Delta_h f, \quad \Delta_h^{p+1} f = \Delta_h \left(\Delta_h^p f\right).$$

The expression Δ_h^{p+1} will have the sense only for such x for which $x+h, x+2h, \dots, x+ph$ will belong to the domain of definition of the function f; futher we'll suppose this fact.

We express the functional formula for $\Delta_h^p f(x)$ which can be proved by induction on p.

Lemma 2. *For integer p is valid*

$$\Delta_h^p f(x) = \sum_{k=0}^{p} (-1)^{p-k} \binom{p}{k} f(x+kh) \tag{3}$$

and

$$\Delta_{-h}^p f(x) = (-1)^p \Delta_h^p f(x - ph).$$

Now, for $p = 1, 2, 3, \dots$ we get step by step

$$\Delta_h^1 f(x) = f(x+h) - f(x)$$

$$\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$$

$$\Delta_h^3 f(x) = f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x).$$

Putting in the previous equations $x := 0$, we get

$$\Delta_h^1 f(0) = f(h) - f(0)$$

$$\Delta_h^2 f(0) = f(2h) - 2f(h) + f(0)$$

$$\Delta_h^3 f(0) = f(3h) - 3f(2h) + 3f(h) - f(0)$$

From this is obvious that e.g. equation (1) we could write in the form

$$\Delta_h^1 f(x) = \Delta_h^1 f(0)$$

and equation (2) in the form

$$\Delta_h^2 f(x) = 0.$$

As we have proved both equations are equivalent and define linear function (rather: their continuous solution is linear function), it seems make sense next generalization

Definition 1. Continuous function $f : \mathbb{R} \to \mathbb{R}$ is called the polynomial of the p-th degree if it satisfies to the functional equation

$$\Delta_h^{p+1} f(x) = 0 \tag{4}$$

or (which is the same) to the functional equation

$$\Delta_h^p f(x) = \Delta_h^p f(0). \tag{5}$$

It's indeed surely necessary to show justification of our definition, i. e. to prove, that from secondary school known polynominal

$$a_0 x^p + a_1 x^{p-1} + \cdots + a_p, \tag{6}$$

$a_i \in \mathbb{R}$, $i = 0, 1, \ldots, p$ is the solution of the equation (4) or (5). The idea of this proof is simple, the proof is elaborate, however.

Term with the largest power in the polynomial (6) is dominant in the sense, that designates behaviour of whole polynominal for x in absolute value going to the infinity. Indeed, if we extract in (6) x^p, we get

$$f(x) = x^p \left(a_0 + \frac{a_1}{x} + \cdots + \frac{a_p}{x^p} \right) \qquad (a_0 \neq 0,\ x \neq 0).$$

Every term inside parentheses except the first one is going to zero for growing $|x|$. Thus

$$f(x) \doteq a_0 x^p \qquad \text{for enough large value of } |x|.$$

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On Some Errors in Projection of the Circle on Oval Solids

Oliver Židek

The article points out at some errors in the projection of the circle on oval solids. Since the errors often appear in publications intended for mathematics instruction, it is rather common for a teacher to come across such an incorrect projection.

It is the intuitive approach to learning geometrical shapes that is emphasized in the present-day school geometry. Attention is given to plane shapes (a square, rectangular, triangle, circle) as well as to some 3-dimensional solids. It is necessary to use models in instruction so that sense of touch can be employed in explorations. Later, at higher level of abstraction, a student should be able to identify a solid from its plane projection (a picture). This implies a teacher being able to sketch a projection of a given solid on the plane of chalkboard or paper. Although projecting some specific prisms and pyramids is usually relatively simple with the use of – for example – the Pohlke's theorem, an attempt to sketch oval solids containing circles may cause troubles in some cases. As to the school practice, troubles refer mostly to projections of cylinders, cones and specific circles on the sphere. Parallel and orthogonal projections are the most frequent projections used in pedagogical practice. Here, the problem refers to a projection of the circle and its specific tangents. It is known that the orthogonal projection of the circle can be a circle, ellipse or line segment; depending on the position of the plane of the circle and the projection plane. The most illustrative projections of oval solids are usually those, in which the ellipse represents the respective circle. Besides other characteristics of ellipse, one is worth mentioning here: the ellipse is an oval curve, i.e. with no cusp-like vertex or apex. This fact is very often disregarded in the projections, mainly if the eccentricity of the ellipse is considerable. Another type of errors often occurs in projections of cones and cylinders (outlines). There is a principle to observe: the outline of the curved surface projects onto tangents to the respective ellipse, which is the outline of the respective circular base. As to the cylinder, the error might not be obvious since the outline of the curved surface coincide with main vertices of both ellipses – projections of the bases. However, it is obvious in case of the projections of the cone or slanted cylinders or their parts. We will document the errors on a few illustrations taken from French and American textbooks. Similar errors can be found in other home or foreign resources.

Illustration 1.. The cone with indicated section shows one of the abovementioned errors. Author of the picture takes the line segment SA for the outline of the curved surface; it is shown on the detailed picture. Correct projection of the outline should be a tangent coinciding with the point S to the ellipse, which is the outline of the projection of the base. The error is less noticeable if the ellipse is more eccentric and if the height is relatively long. To make it more graphical, we may show a picture where the respective ellipse is less eccentric (getting closer to a circle) and the height of the cone is relatively short.

Illustration 2. This is to illustrate an error referring to cusp-like vertices of the projected circle. The error can be observed in attempts to depict (at least a part of) a cartographic grid on the reference sphere, which corresponds at the respective scale to the planet Earth. The picture should have shown 3 circles in azimuthal orthographic projection: the outlining circle, equatorial circle and a meridional circle. While the outline is in accordance with principles of projection, this cannot be said about the equatorial and meridional circles.

Illustration 3 shows a position of a sphere and a plane, while the projection of the point of tangency (the pole) does not correspond to the projecting principles.

Conclusion. Many of the abovementioned errors come from the fact that the illustrator is usually not the same person as the writer. With regard to the pace at which didactical material are produced, it is not exceptional that incorrect pictures like these escape authors' observations. The paper was intended to point out that there should be more attention given to the projecting principles in school teaching, especially in the teacher education.

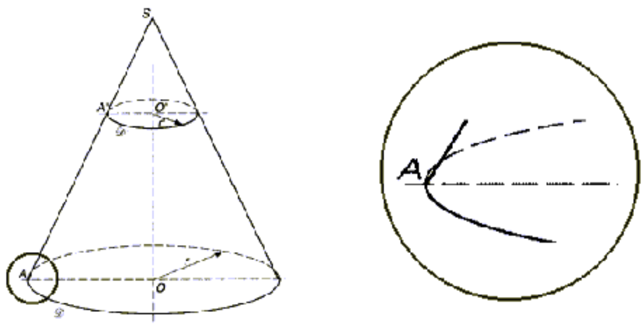

Figure 1

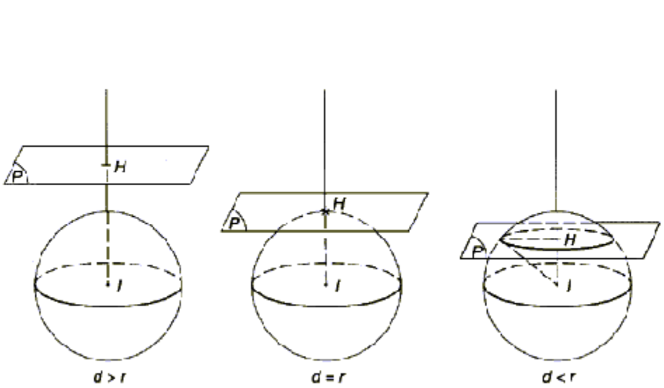

Figure 2

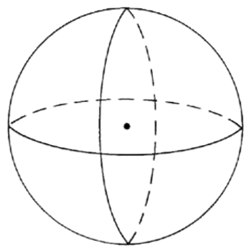

Figure 3

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