Further sequenced problems for functional equations

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Abstract

In this paper we will create a new sequence of problems for functional equations of type

$$H(f(x+y), f(x-y), f(x), f(y), x, y) = 0,$$

where H is known function and f is the unknown function to be determined. Some possible ways of generalization are also given in this note.

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1 Introduction

In [2], motivated some results of J. Aczél (see eg. [1][2][3]), we created a sequence of problems for the fostering of talented students on various levels of mathematical education. On the other hand, we found a possible way of generalization (see [5]).

Here we intend to present another sequence of problems for functional equations of the form

$$H(f(x+y), f(x-y), f(x), f(y), x, y) = 0 \tag{1}$$

where H is known function and f is the unknown function to be determined. On the other hand, some possible ways of generalization are also given in this paper.

2 A new sequenced problems for equations of type (1)

Problem 1. *Determine all solutions $f : \mathbb{R} \to \mathbb{R}$ of functional equation*

$$f(x) + f(x+y) = f(y) + f(x-y) \qquad (x, y \in \mathbb{R}). \tag{2}$$

Solution. Substitute $y = 0$ in (2). Then we get that

$$2f(x) = f(0) + f(x) \qquad (x \in \mathbb{R}).$$

This implies that f satisfies (2) if, and only if,

$$f(x) = c \qquad (x \in \mathbb{R}),$$

where $c = f(0)$ is an arbitrary constant.
Our solution implies the following result.

Theorem 1. *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies functional equation* (2) *if, and only if,*

$$f(x) = c \qquad (x \in \mathbb{R}),$$

where $c \in \mathbb{R}$ is an arbitrary constant.

Problem 2. *Find all solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation*

$$f(x) + f(x+y) = 2f(y) + 2f(x-y) \qquad (x, y \in \mathbb{R}). \tag{3}$$

Solution. Substitute $x = y = 0$ in (3) to obtain the equality $2f(0) = 4f(0)$. This implies that $f(0) = 0$. Put $x = 0$ into (3) to get

$$f(0) + f(y) = 2f(y) + 2f(-y) \qquad (y \in \mathbb{R})$$

then $f(0) = 0$ implies

$$f(y) + 2f(-y) = 0 \qquad (y \in \mathbb{R}). \tag{4}$$

Substituting $y = -y$ here we get

$$f(-y) + 2f(y) = 0 \qquad (y \in \mathbb{R}). \tag{5}$$

It is easy to see that the system of equations (4) and (5) can be solved for $f(y)$ for any $y \in \mathbb{R}$ and we get

$$f(y) = 0 \qquad (y \in \mathbb{R}). \tag{6}$$

This implies that $f(x) = 0 \quad (x \in \mathbb{R})$ is the only possible solution of equation (3). On the other hand, (6) indeed satisfies (3).

Thus we have proved the following result.

Theorem 2. *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies functional equation* (3) *if, and only if,*

$$f(x) = 0 \qquad (x \in \mathbb{R}).$$

Problem 3. *Determine all solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation*

$$f(x) + f(x+y) = 2f(y) + 2f(x-y) + ay + b \qquad (x, y \in \mathbb{R}), \tag{7}$$

where $a, b \in \mathbb{R}$ are arbitrary constants.

Solution 1. Set $x = y = 0$ in (7) to get the equality

$$2f(0) = 4f(0) + b.$$

This shows that $f(0) = -\frac{b}{2}$. Using the substitutions $x = 0$ and $x = 0, y = -y$ in (7), respectively we get, for all $y \in \mathbb{R}$, the system of equation

$$\begin{aligned} f(y) + 2f(-y) &= -ay - \frac{3}{2}b \\ f(-y) + 2f(y) &= ay - \frac{3}{2}b \end{aligned} \tag{8}$$

for $f(y)$ and $f(-y)$. After eliminating $f(-y)$, we get from (8) that

$$f(y) = ay - \frac{b}{2} \qquad (y \in \mathbb{R}), \tag{9}$$

which is the only possible solution of equation (7). (9) indeed satisfies (7).

Solution 2. One can easy verify the identity

$$-(ax - \frac{b}{2}) - [a(x+y) - \frac{b}{2}] = -2(ay - \frac{b}{2}) - 2[a(x-y) - \frac{b}{2}] - (ay + b)$$

Adding this identity and equation (7) side by side, we get the functional equation

$$f(x) - (ax - \frac{b}{2}) + f(x+y) - (a(x+y) - \frac{b}{2}) = 2[f(y) - (ay - \frac{b}{2})] + 2[f(x-y) - (a(x-y) - \frac{b}{2})]$$

for all $x, y \in \mathbb{R}$, which shows that the function

$$F(x) = f(x) - (ax - \frac{b}{2}) \qquad (x \in \mathbb{R}) \tag{10}$$

satisfies (3). Therefore, Theorem 2 implies that $F(x) = 0$ for all $x \in \mathbb{R}$. Thus, it follows from (10) that

$$f(x) = ax - \frac{b}{2} \qquad (x \in \mathbb{R})$$

again.

Summarizing, we have proved the following

Theorem 3. *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies functional equation* (7) *if, and only if,*

$$f(x) = ax - \frac{b}{2} \qquad (x \in \mathbb{R}).$$

Problem 4. *Find all solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation*

$$f(x) + f(x+y) = 2f(y) + 2f(x-y) + cy^2 + ay + b \qquad (x, y \in \mathbb{R}), \tag{11}$$

where $a, b, c \in \mathbb{R}$ are arbitrary constants.

Solution. Using the same argument as at the Solution 1 of Problem 3 we get that the function

$$f(x) = -\frac{c}{3}x^2 + ax - \frac{b}{2} \qquad (x \in \mathbb{R}) \tag{12}$$

the only possible solution of equation (11). But an easy calculation shows that (12) satisfies (11) only if $c = 0$.

This implies the following result for equation (11).

Theorem 4. *The functional equation* (11) *has solution if, and only if, c=0 and then*

$$f(x) = ax - \frac{b}{2} \qquad (x \in \mathbb{R}).$$

Remark 1. One can easy verify that the functional equation

$$f(x) + f(x+y) = 2f(y) + 2f(x-y) + a \cdot \sin y + b \qquad (x, y \in \mathbb{R}) \tag{13}$$

has solution only if $a = 0$ and then

$$f(x) = -\frac{b}{2} \qquad (x \in \mathbb{R}).$$

Problem 5. *Find all solutions* $f : \mathbb{R} \to \mathbb{R}$ *of the functional equation*

$$f(x) + f(x+y) = [f(y)]^2 + [f(x-y)]^2 \qquad (x, y \in \mathbb{R}). \tag{14}$$

Solution 1. Substitute $x = y = 0$ in (14) to obtain the equality $f(0) = [f(0)]^2$. This implies that either $f(0) = 0$ or $f(0) = 1$. Using the substitutions $x = 0$ and $x = 0, y = -y$ in (14) , respectively, we get for all $y \in \mathbb{R}$ the system

$$\begin{aligned} f(0) + f(y) &= [f(y)]^2 + [f(-y)]^2 \\ f(0) + f(-y) &= [f(-y)]^2 + [f(y)]^2 \end{aligned} \tag{15}$$

This shows that $f(-y) = f(y) \;\; (y \in \mathbb{R})$. Thus, from the first equation of (15), we get

$$2[f(y)]^2 - f(y) - f(0) = 0 \qquad (y \in \mathbb{R}). \tag{16}$$

In case $f(0) = 0$, we get from (16) the equation

$$2[f(y)]^2 - f(y) = 0 \qquad (y \in \mathbb{R}),$$

which implies that $\forall y \in \mathbb{R} : f(y) \in \{0, \frac{1}{2}\}$. Let us assume that $\exists x_0 \in \mathbb{R} : f(x_0) = \frac{1}{2}$. Put $x = y = x_0$ in (14) to get $f(x_0) + f(2x_0) = [f(x_0)]^2$. This implies that $f(2x_0) = -\frac{1}{4}$, which is a contradiction. Thus, in case $f(0) = 0$, we get $f(x) = 0 \;\; (x \in \mathbb{R})$.
In case $f(0) = 1$, we get from (16) the equation

$$2[f(y)]^2 - f(y) - 1 = 0 \qquad (y \in \mathbb{R}),$$

which implies $f(y) \in \{-\frac{1}{2}, 1\}$ for all $y \in \mathbb{R}$. Assuming the existence of $x_0 \in \mathbb{R}$ such that $f(x_0) = -\frac{1}{2}$, then by the substitutions $x = y = x_0$ in (14) , we get

$$f(x_0) + f(2x_0) = [f(x_0)]^2 + 1,$$

that is $f(2x_0) = \frac{7}{4}$, which is also a contradiction. This implies, in case $f(0) = 1$, that $f(x) = 1 \;\; (x \in \mathbb{R})$.
It is easy to see that functions $f(x) = 0 \;\; (x \in \mathbb{R})$ and $f(x) = 1 \;\; (x \in \mathbb{R})$ indeed satisfy (14).

Solution 2. Substitutions $x = y = 0$ in (14) imply again that either $f(0) = 0$ or $f(0) = 1$. Using now the substitution $y = 0$ in (14), we get

$$[f(x)]^2 = 2f(x) - f(0) \qquad (x \in \mathbb{R}). \tag{17}$$

Then by (15) and (17) , we obtain functional equation

$$f(x) + f(x + y) = 2f(y) + 2f(x - y) - 2f(0) \qquad (x, y \in \mathbb{R}),$$

which implies that the function F, defined by

$$F(x) = f(x) - f(0) \qquad (x \in \mathbb{R}), \tag{18}$$

satisfies the equation (3) of Problem 2. Thus, Theorem 2 implies that $F(x) = 0$ $(x \in \mathbb{R})$. It follows from (18) that $f(x) = f(0)$ $(x \in \mathbb{R})$ and so the functions $f(x) = 0$ $(x \in \mathbb{R})$ and $f(x) = 1$ $(x \in \mathbb{R})$ are the only possible solutions of equations (14).

We have proved the following result for equation (14).

Theorem 5. *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation* (14) *if, and only if, either $f(x) = 0$ $(x \in \mathbb{R})$ or $f(x) = 1$ $(x \in \mathbb{R})$.*

Problem 6. *Determine all solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation*

$$f(x) + f(x + y) = [f(y)]^3 + [f(x - y)]^3 \qquad (x, y \in \mathbb{R}) \tag{19}$$

Solution 1. Set $x = y = 0$ in (19). We get the equality $f(0) = [f(0)]^3$, which implies that $f(0) \in \{-1, 0, 1\}$. the substitutions $x = 0$ and $x = 0, y = -y$, respectively, imply the following system of equations

$$\begin{aligned} f(0) + f(y) &= [f(y)]^3 + [f(-y)]^3 \\ f(0) + f(-y) &= [f(-y)]^3 + [f(y)]^3 \end{aligned} \tag{20}$$

This shows that $f(-y) = f(y)$ $(y \in \mathbb{R})$. Therefore, the first equation of (20) implies

$$2[f(y)]^3 - f(y) - f(0) = 0 \qquad (y \in \mathbb{R}). \tag{21}$$

In case $f(0) = 0$, we get from (21) the equation

$$f(y)(2[f(y)]^2 - 1) = 0 \qquad (y \in \mathbb{R}),$$

which implies that $\forall y \in \mathbb{R} : f(y) \in \{0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\}$. Assuming that $\exists x_0 \in \mathbb{R} : f(x_0) = \frac{1}{\sqrt{2}}$ or $f(x_0) = -\frac{1}{\sqrt{2}}$, similarly then before, we get contradictions. Thus, in case $f(0) = 0$, it follows $f(x) = 0$ $(x \in \mathbb{R})$, which satisfies (19).
In case $f(0) = 1$, we obtain from (21) the equation

$$2[f(y)]^3 - f(y) - 1 = 0 \qquad (y \in \mathbb{R}).$$

This is equivalent to the equation

$$(f(y)-1)\cdot([f(y)]^2+[f(y)+1]^2)=0 \qquad (y\in\mathbb{R}),$$

which has only the solution $f(y)=1$ for all $y\in\mathbb{R}$ and this function satisfies (19).
In case $f(0)=1$, (21) implies that

$$[f(y)]^3-f(y)+1=0 \qquad (y\in\mathbb{R}).$$

This equation can be written in the form

$$(f(y)+1)\cdot([f(y)]^2+[f(y)-1]^2)=0 \qquad (y\in\mathbb{R}),$$

which satisfies if, and only if, $f(y)=-1$ for all $y\in\mathbb{R}$ and this function satisfies the equation (19).
Solution 2. Substituting $x=y=0$ in (19), we get again that either $f(0)=0$ or $f(0)=1$ or $f(0)=-1$. Set $y=0$ in (19) to obtain

$$[f(x)]^3=2f(x)-f(0) \qquad (x\in\mathbb{R}). \tag{22}$$

(19) and (22) imply the equation

$$f(x)+f(x+y)=2f(y)+2f(x-y)-2f(0) \qquad (x,y\in\mathbb{R}),$$

as at the Solution 2 of Problem 5. Thus, we have again that $f(x)=f(0)$ $(x\in\mathbb{R})$, which implies that either $f(x)=0$ $(x\in\mathbb{R})$ or $f(x)=1$ $(x\in\mathbb{R})$ or $f(x)=-1$ $(x\in\mathbb{R})$ are the only possible solutions of (19). These functions satisfy the functional equation (19).

Summarizing, we have proved

Theorem 6. *Function $f:\mathbb{R}\to\mathbb{R}$ satisfies the functional equation* (19) *if, and only if, either $f(x)=0$ $(x\in\mathbb{R})$ or $f(x)=1$ $(x\in\mathbb{R})$ or $f(x)=-1$ $(x\in\mathbb{R})$.*

Problem 7. *Find all solutions $f:\mathbb{R}\to\mathbb{R}$ of the functional equation*

$$f(x)+f(x+y)=[f(y)]^n+[f(x-y)]^n \qquad (x,y\in\mathbb{R}), \tag{23}$$

where n is a given natural number.

Solution. If $n=1,2,3$ in (23) then we get equations (3), (14) and (19), respectively.
Therefore, let us assume that $n>3$ and $n\in\mathbb{N}$. Set $x=y=0$ in (23). Then we get that $f(0)=[f(0)]^n$, which can be written in the form

$$f(0)\cdot([f(0)]^{n-1}-1)=0.$$

This implies that $f(0)\in\{0,1\}$ if n is even and $f(0)\in\{0,1,-1\}$ if n is odd.
Put $y=0$ in (23) to get

$$2f(x)=[f(0)]^n+[f(x)]^n \qquad (x\in\mathbb{R}),$$

which, together with $[f(0)]^n = f(0)$ and (23), implies the functional equation

$$f(x) + f(x+y) = 2f(y) + 2f(x-y) - 2f(0) \qquad (x, y \in \mathbb{R}).$$

This shows that the function

$$F(x) = f(x) - f(0) \qquad (x \in \mathbb{R}) \tag{24}$$

satisfies (3), that is,

$$F(x) + F(x+y) = 2F(y) + 2F(x-y) \qquad (x, y \in \mathbb{R}) \tag{25}$$

So, by Theorem 2, F satisfies (25) if, and only if $F(x) = 0$ for $x \in \mathbb{R}$.
Then it follows from (24) that $f(x) = f(0)$ $(x \in \mathbb{R})$. This equality together with the first part of our solution, implies that

- either $f(x) = 0$ $(x \in \mathbb{R})$ or $f(x) = 1$ $(x \in \mathbb{R})$ if n is even
- either $f(x) = 0$ $(x \in \mathbb{R})$ or $f(x) = 1$ $(x \in \mathbb{R})$ or $f(x) = -1$ $(x \in \mathbb{R})$ if n is odd

are the only possible solutions of equation (23).
It is easy to see that these functions satisfy (23). Thus, we have proved the following

Theorem 7. *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation* (23)*, where $n > 3$ is a given natural number, if, and only if*

- *either $f(x) = 0$ $(x \in \mathbb{R})$ or $f(x) = 1$ $(x \in \mathbb{R})$ if n is even*
- *either $f(x) = 0$ $(x \in \mathbb{R})$ or $f(x) = 1$ $(x \in \mathbb{R})$ or $f(x) = -1$ $(x \in \mathbb{R})$ if n is odd.*

3 A common generalization of Problem 2,3,4

A natural generalization of problems 2,3,4 is the following

Problem 8. *Find all solutions $f, g : \mathbb{R} \to \mathbb{R}$ of the functional equation*

$$f(x) + f(x+y) = 2f(y) + 2f(x-y) + g(y) \qquad (x, y \in \mathbb{R}). \tag{26}$$

Solution. Let us write (26) in the form

$$g(y) = f(x) + f(x+y) - 2f(y) - 2f(x-y) \qquad (x, y \in \mathbb{R}). \tag{27}$$

Write here $-y$ instead of y to get

$$g(-y) = f(x) + f(x-y) - 2f(-y) - 2f(x+y) \qquad (x, y \in \mathbb{R}). \tag{28}$$

With the substitution $x = 0$ we get from (27) and (28) that

$$g(y) = f(0) - f(y) - 2f(-y) \qquad (y \in \mathbb{R}) \tag{29}$$

and

$$g(-y) = f(0) - f(-y) - 2f(y) \qquad (y \in \mathbb{R}), \tag{30}$$

respectively.
By the help of equations (27), (28) and (29), (30) we get equations

$$g(y) + 2g(-y) = 3f(x) - 2f(y) - 4f(-y) - 3f(x+y) \qquad (x, y \in \mathbb{R}) \tag{31}$$

and

$$g(y) + 2g(-y) = 3f(0) - 5f(y) - 4f(-y) \qquad (x, y \in \mathbb{R}), \tag{32}$$

respectively.

Comparing equations (31) and (32) we find that

$$f(x+y) + f(0) = f(x) + f(y) \qquad (x, y \in \mathbb{R}),$$

which shows that the function $A(x) = f(x) - f(0)$ $(x \in \mathbb{R})$ is additive, i.e. satisfies the Cauchy functional equation

$$A(x+y) = A(x) + A(y) \qquad (x, y \in \mathbb{R}).$$

Thus we have

$$f(x) = A(x) + f(0) \qquad (x \in \mathbb{R}). \tag{33}$$

On the other hand, it follows from (29) and (32) that

$$g(x) = A(x) - 2f(0) \qquad (x \in \mathbb{R}). \tag{34}$$

It is easy to see that functions (32) and (33) satisfy the functional equation (26) for any choice of $f(0)$.
Our solution implies

Theorem 8. *Functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation* (26) *if, and only if,*

$$f(x) = A(x) + c \qquad (x \in \mathbb{R})$$
$$g(x) = A(x) - 2c \qquad (x \in \mathbb{R})$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and $c \in \mathbb{R}$ is an arbitrary constant.

Remark 2. If $A(y) = 0$ $(y \in \mathbb{R})$ and $b = 0$ then Theorem 8 gives the solution of Problem 2. By (33) the continuity of g implies the continuity of A and so $A(y) = ay$ $(y \in \mathbb{R})$, where $a \in \mathbb{R}$ is an arbitrary constant and then $g(y) = ay - 2c$ $(y \in \mathbb{R})$. In this case, with $c = -\frac{b}{2}$, we get the solution of Problem 3.

4 Further generalizations

The pexiderization of equation (2) leads to some other possible generalizations of our previous problems. We will investigate here the following two problems.

Problem 9. *Find all solutions* $f, g : \mathbb{R} \to \mathbb{R}$ *of the functional equation*

(35) $$f(x) + f(x+y) = g(y) + g(x-y) \qquad (x, y \in \mathbb{R}).$$

Problem 10. *Find all solutions* $f, g, h : \mathbb{R} \to \mathbb{R}$ *of the functional equation*

(36) $$f(x) + f(x+y) = g(y) + h(x-y) \qquad (x, y \in \mathbb{R}).$$

Solution of Problem 9. Put $x = y = 0$ in (35) to get

(37) $$f(0) = g(0)$$

Now substituting $y = 0$ in (35), we get

(38) $$2f(x) = g(0) + g(x) \qquad (x \in \mathbb{R}),$$

which, together with (35), implies the equation

$$f(x) + f(x+y) = 2f(y) + 2f(x-y) - 2f(0) \ \ (x, y \in \mathbb{R}).$$

This equation shows that the function $F(x) = f(x) - f(0)$ $(x \in \mathbb{R})$ staisfies the functional equation (3) of Problem 2 and so $F(x) = 0$ $(x \in \mathbb{R})$. Hence,

(39) $$f(x) = f(0) \qquad (x \in \mathbb{R})$$

follows. Using (37), (38) and (39), we get for g that

(40) $$g(x) = f(0) \qquad (x \in \mathbb{R}).$$

A simple calculation shows that functions (39) and (40) satisfy (35) for any choice of $f(0)$. Thus, we have proved the following result.

Theorem 9. *Functions* $f, g : \mathbb{R} \to \mathbb{R}$ *satisfy the functional equation* (35) *if, and only if,*

(41) $$f(x) = g(x) = c \qquad (x \in \mathbb{R}),$$

where $c \in \mathbb{R}$ *is an arbitrary constant.*

Remark 3. Using Theorem 9, one can easily get the solution of Problems 5,6 and 7. Namely, with notations

(42) $$[f(x)]^2 = g(x) \text{ or } [f(x)]^3 = g(x) \text{ or } [f(x)]^n = g(x) \qquad (x \in \mathbb{R}),$$

Problems 5,6 and 7 go over into the Problem 9, respectively. Theorem 9 implies that $f(x) = c$ $(x \in \mathbb{R})$.
On the other hand, it follows from (42) that

$$c^2 = c \text{ or } c^3 = c \text{ or } c^n = c,$$

respectively, which together with $f(x) = c$ $(x \in \mathbb{R})$, immediately imply the solution of Problems 5,6 and 7.

Now, let us investigate our last Problem.

Solution of Problem 10. Putting $y = 0$ in (36) we get

$$2f(x) = g(0) + h(x) \qquad (x \in \mathbb{R}). \tag{43}$$

Now (36) and (43) imply the equation

$$g(y) = f(x) + f(x+y) - 2f(x-y) + g(0) \qquad (x \in \mathbb{R}), \tag{44}$$

hence, replacing y by $-y$,

$$g(-y) = f(x) + f(x-y) - 2f(x+y) + g(0) \qquad (x \in \mathbb{R}) \tag{45}$$

follows. With the substitution $x = 0$ we infer from (44) and (45) the equations

$$g(y) = f(0) + f(y) - 2f(-y) + g(0) \qquad (y \in \mathbb{R}), \tag{46}$$

and

$$g(-y) = f(0) + f(-y) - 2f(y) + g(0) \qquad (y \in \mathbb{R}), \tag{47}$$

respectively. Calculating $g(y) + 2g(-y)$ from equations (44), (45) and (46), (47) and comparing the resulting equations, we get that f satisfies the functional equation

$$f(x+y) + f(0) = f(x) + f(y) \qquad (x, y \in \mathbb{R}).$$

This shows that the function $A(x) = f(x) - f(0)$ $(x \in \mathbb{R})$ is additive on $\mathbb{R}^2$. Thus we have

$$f(x) = A(x) + f(0) \qquad (x \in \mathbb{R}). \tag{48}$$

Further, from (46) and (48) follows

$$h(x) = 2A(x) + 2f(0) - g(0) \qquad (x \in \mathbb{R}) \tag{49}$$

Finally, (46) and (48) imply that

$$g(x) = 3A(x) + g(0) \qquad (x \in \mathbb{R}). \tag{50}$$

One can easily verify that functions (48),(49) and (50) satisfy the functional equation (36) for arbitrary choice of $f(0)$ and $g(0)$. Thus we have proved

Theorem 10. *Functions $f, g, h : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation* (35) *if, and only if,*

$$\begin{aligned} f(x) &= A(x) + c_1 && (x \in \mathbb{R}) \\ g(x) &= 3A(x) + c_2 && (x \in \mathbb{R}) \\ h(x) &= 2A(x) + 2c_1 - c_2 && (x \in \mathbb{R}) \end{aligned}$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function on $\mathbb{R}^2$ and $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

Remark 4. One can easily verify that Problem 10 is a common generalization of Problems 1-9 and the solving of it does not apply Theorems 1-9. Thus, determining the special form of functions g and h in each cases and using Theorem 10, we get a new method to the solution of our Problems 1-9.

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