Historical Aspects in Calculus Teaching

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ABSTRACT: *This paper offers a few examples how to integrate short historical facts into calculus teaching. One the one hand these supplements contribute to an appropriate image of mathematics as a developing science and as a part of our culture whereas on the other hand the time they require in classroom is negligible.*

1 Preliminary remarks

Most people, pupils, students notice mathematics only as a collection of methods and/or problems. To reduce mathematics to this aspect is in my opinion a distorted picture. Mathematics is much more: it is part of our culture such as literature, music, philosophy, arts, a.s.o. The cultural aspect of these subjects has been underlined in school by teaching also their historical development. Therefore I – and many of my colleagues – propose to integrate history of mathematics in teaching too. Even many syllabuses prescribe history as well, but this often remains discounted.

I know many teachers who are interested in history of mathematics. They participate in teacher training seminars, they read the respective chapters in the textbook with interest, but they often admit that they do not integrate history in their teaching, mostly, as they say, because of lack of time. Therefore it is not sufficient to offer in teacher training only historical facts, it is also necessary to offer strategies how to integrate history in teaching, in particular strategies which are not time-consuming.

The German mathematician Otto Toeplitz (1881-1940) proposed and distinguished between the “Direct Genetic Method” and the “Indirect Genetic Method”. The “Indirect Genetic Method” means that the teacher can learn from the history about difficulties which have been caused even to the great mathematicians such as Newton, Leibniz, Fermat, Cavalieri and others and to take this into account in his/her planning of the teaching process without mentioning historical details. In contrary to this the “Direct Genetic Method” proposes in addition to offer historical details as well (often only a few sentences or a single historical problem) explicitly in the teaching at suitable occasions.

2 Differentials

Although all students deal with differential calculus and differential quotient dy/dx they (even mathematics students at the university) often do not know what these “differentials”, the “numerator and denominator of the fraction dy/dx” are. They have only a vague conception of these differentials (“very small”, “infinitely small”) and which difficulties are connected with this notation. In the New Math time in the seventies it was even frowned upon to use dy/dx. Why? Should we avoid this notation? Or should we use it (as physicists and technicians do)?

2.1 How did Newton deal with infinitesimals? How did he argue?

In his “Quadrature of Curves” of 1704 Newton (1643-1727) determines the derivative (or “fluxion” as he called it) of x^3 as follows. (We here paraphrase Newton’s treatment.)

In the same time that x, by growing becomes x+o, the power x^3 becomes $(x+o)^3$, or

$$x^3 + 3x^2o + 3xo^2 + o^3$$

and the growth or increments

$$(x+o) - x = o \quad \text{and} \quad (x+o)^3 - x^3 = (x^3 + 3x^2o + 3xo^2 + o^3) - x^3 = 3x^2o + 3xo^2 + o^3$$

are to each other as

$$1 \quad \text{to} \quad 3x^2 + 3xo + o^2$$

In other – our – words or symbolism:

$$\frac{(x+o)^3 - x^3}{(x+o) - x} = \frac{(x^3 + 3x^2o + 3xo^2 + o^3) - x^3}{o} = 3x^2 + 3xo + o^2$$

Now let the increments vanish, and their “last proportion” will be 1 to $3x^2$, whence the rate of change of x^3 with respect to x is $3x^2$. (Cf. [5], p. 175)

Bishop George Berkeley criticised its approach in his famous tract “The Analyst in 1734. He wrote:

“And what are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them ghosts of departed quantities?”

(Cf. [1], p. 430)

2.2 Leibniz’ differentials

Gottfried Wilhelm Leibniz (1646-1716) introduced the powerful mutation of differentials which has been used subsequently in continental Europe with great success (esp. by the Bernoullis and Euler). Leibniz gave formulas like

$$d(x\cdot y) = x\cdot dy + y\cdot dx$$

which he explained as follows:

$$d(x\cdot y) = (x + dx)\cdot(y + dy) - x\cdot y = (x\cdot y + x\cdot dy + y\cdot dx + dx\cdot dy) - x\cdot y = x\cdot dy + y\cdot dx + dx\cdot dy$$

Now he argued that $dx \cdot dy$ can be disregarded for it is like a grain of sand compared with the globe. In a similar way Leibniz found

$$dx^n = n \cdot x^{n-1} \cdot dx$$

$$\frac{dy}{dx} = \frac{dy}{dw} \frac{dw}{dx}$$

and some more rules of this type.

(Cf. [6], p. 90)

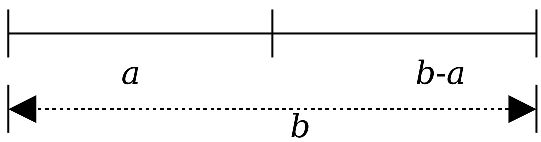
Often Newton and Leibniz are regarded as “the” inventors of infinitesimal calculus. But this is not true. Already mathematicians in the antiquity used argumentations with infinitesimal quantities to calculate areas and volumes (Eudoxos, Archimedes, and later on Cavalieri and others). The merit of Newton and Leibniz was “only” that they developed a general method whereas previous mathematicians found “only” formulas for special types of functions. This situation is also expressed in an anecdote, a saying of Newton:

“If I have seen a little further it is by standing on the shoulders of Giants.”

2.3 Fermat’s Maximum Method

One of these Giants was Pierre de Fermat (1607 – 1665). He developed a method to find the maximum of a product when the sum of the two factors is given:

Given: b, wanted: a so that a(b-a) is a maximum



$a \cdot (b - a) \approx (a + e) \cdot (b - a - e)$ $\qquad (e << a)$

$$ab - a^2 \approx ab - a^2 - ae + eb - ea - e^2$$

$$eb \approx 2ae + e^2$$

Assume $e \neq 0$*; then it is* $\qquad b \approx 2a + e$

For – so argued Fermat – e is arbitrary small (!!), we get:

$$b = 2a \quad \text{or:} \quad a = \frac{b}{2}$$

(Cf. [9], p. 82f)

2.4 Exactification of Infinitesimals

The 18th century brought various applications of the calculus and also a further dissemination (esp. by Euler’s book “Introductio in analysin infinitorum”.) Nevertheless the critics of Bishop Berkeley remained still valid. And the mathematicians were awake to this! What they were looking for was a logical basis of infinitesimal calculus, a basis “more geometrico”, that means an axiomatic basis such as Euclid’s elements which is resistant against critics (in the same way as the Euclid’s axiomatic basis was the solution against the critics of the sopists).

Although many brilliant mathematicians were looking for such an exact basis it tooks a whole century until Augustin Louis Cauchy (1789-1857) formulated the definition of the concept of limit.

This is in my opinion the most impressive example for the “Indirect Genetic Method”: We as teachers have to learn from the history that the development of the limit concept was a very difficult task. This implies that we must not assail our students with this definition within five minutes after writing the headline “Calculus”! We have to learn from history that a longer period, beginning with a aive conception of “limit” or “infinitesimal” is strongly recommended before we can dare to make a step further towards a subsequent exactification. (Cf. [9], p. 76ff, [8], p. 8ff, [2], p. 11ff)

2.5 Nonstandard Analysis

Although Cauchy’s definition of the limit concept is widely used in school and undergraduate mathematics there is another interesting step in this development. Around 50 years ago Abraham Robinson (1918-1974) published his book on Non-standard Analysis (ROBINSON 1966) which most historians regard in some sense as a late justification of Newton’s, Leibniz’ and other’s infinitesimal quantities. The key idea is simple: if our concept of real numbers is not compatible with infinitesimal quantities we have two options:

1. we avoid infinitesimal quantites (and use Cauchy’s limit concept) or
2. we change the real number concept.

Robinson substituted the standard concept of real numbers by the “non-standard real numbers” *.

The Nonstandard model R* contains R as a proper subset and furthermore a set of infinitesimal (but ≠ 0) elements (~ differentials!)

$$I = \{ x \in R^* \mid |x| < \varepsilon \ \forall 0 < \varepsilon \in R \} \subseteq R^*$$

For we want a field we have also to include the reciprocals of the infinitesimals, i. E. infinitely big numbers. The finite numbers are a subset

$$E = \{ x \in R^* \mid \exists r \in R \text{ mit } |x| \leq r \}$$

According to Leibniz’ idea we cannot distinguish between for two numbers whose difference is infinitely small Robinson introduced something like a second kind of “equality”:

$$x \cong y \Leftrightarrow x - y \in I$$

For $x \cong y$ we say: x and y differ infinitesimally from each other.

One can imagine around every real number r a “cloud” of elements of E which differ infinitely from r. More precisely: It holds:

Theorem: Let z ∈ E und $M_z = \{x \in E \mid x \cong z\}$. Then there exists a unique number $r \in R$ mit $r \cong x \ \forall x \in M_z$. The real number r is called the standard part of z; briefly: r = st(z)

In order to do calculus in R* we have to extend the functions which are defined on R (or a subset of R) to functions defined on R* (or respective subsets).

Theorem: To any function $f : \mathbf{R} \quad \mathbf{R}$ there exists a unique function $f^* : \mathbf{R}^* \quad \mathbf{R}^*$ with $f^*|_{\mathbf{R}} = f$. On the other hand $f^*|_{\mathbf{R}} = st(f^*)$ is called the standard part of f^*.

Now we can define the concepts like continuous, differentiable and derivative:

Definition: f is called continuous at p∈ R, if $f^*(p+k) \cong f(p) \ \forall k \in$ I

Definition: *f* is called differentiable at p ∈ R, if:

1) $\frac{f^*(p+k) - f(p)}{k} \quad \mathbf{E} \ \forall 0 \quad k \quad \mathbf{I}$

2) For all k ∈ I all expressions $\frac{f^*(p+k) - f(p)}{k}$ have the same standard part.

As soon as the conditions 1) und 2) are fulfilled, st($\frac{f^*(p+k) - f(p)}{k}$) is called differential quotient or derivative f'(p).

(Cf. [9], p. 85ff)

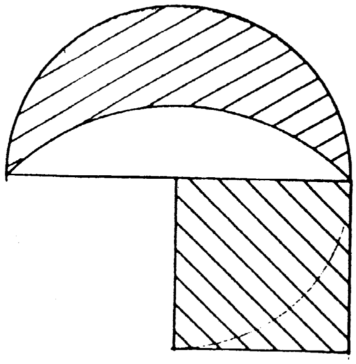
3 Some steps in the history of integral calculus

3.1 Squaring the circle

In German lanuage it is a well known saying “This is squaring the circle” to express that something is impossible. But only very few people (even among mathematics students) know more about the background. In my opinion it is the duty of mathematics teachers to explain details, e.g. when teaching about the circle and the number π.

Squaring the circle is one of the three classical problems of antiquity (beside dubling a cube and trisecting an angle). The Greek mathematicians tried to find a way to construct the side of a square which is of equal area as a circle with given radius. They were encouraged to find such a method for Hippokrates of Chios showed that the moon / the moons in fig. 1 / fig. 2 have the same area as the square /

the triangle respectively. (The calculation can easily be done by 13 / 14 year old pupils.)



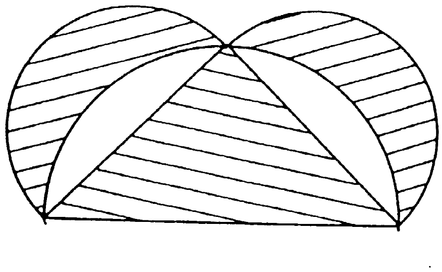


fig. 1 The “lunulae” of Hippokrates fig. 2

For these figures bounded by parts of circles can be squared the ancient Greeks hoped to find also a method to square the whole circle too, or equivalently: a method to construct the number π. As you all know it is impossible only with the help of compass and ruler. But a proof for this impossibility has been given more than 2000 years later.

3.2 The Quadratrix

Hippias of Elis (5th century BC) found a method of squaring the circle with the help of a special curve, the Quadratrix. This curve can be created as follows: Let the segment OC turn to OA uniformly and let CB simultaneously fall down to OA uniformly. At a certain time OX and MN are a snapshot of these two motions. The intersection of OX and MN is a point of the Quadratrix.

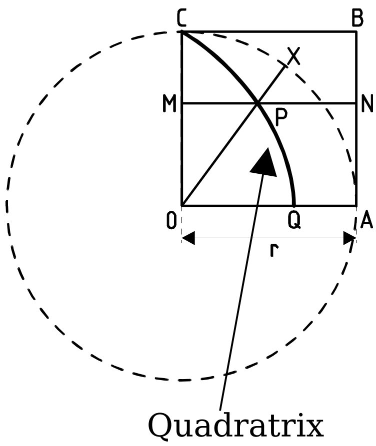


fig. 3

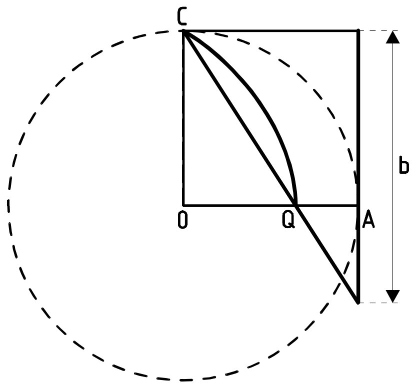


fig. 4

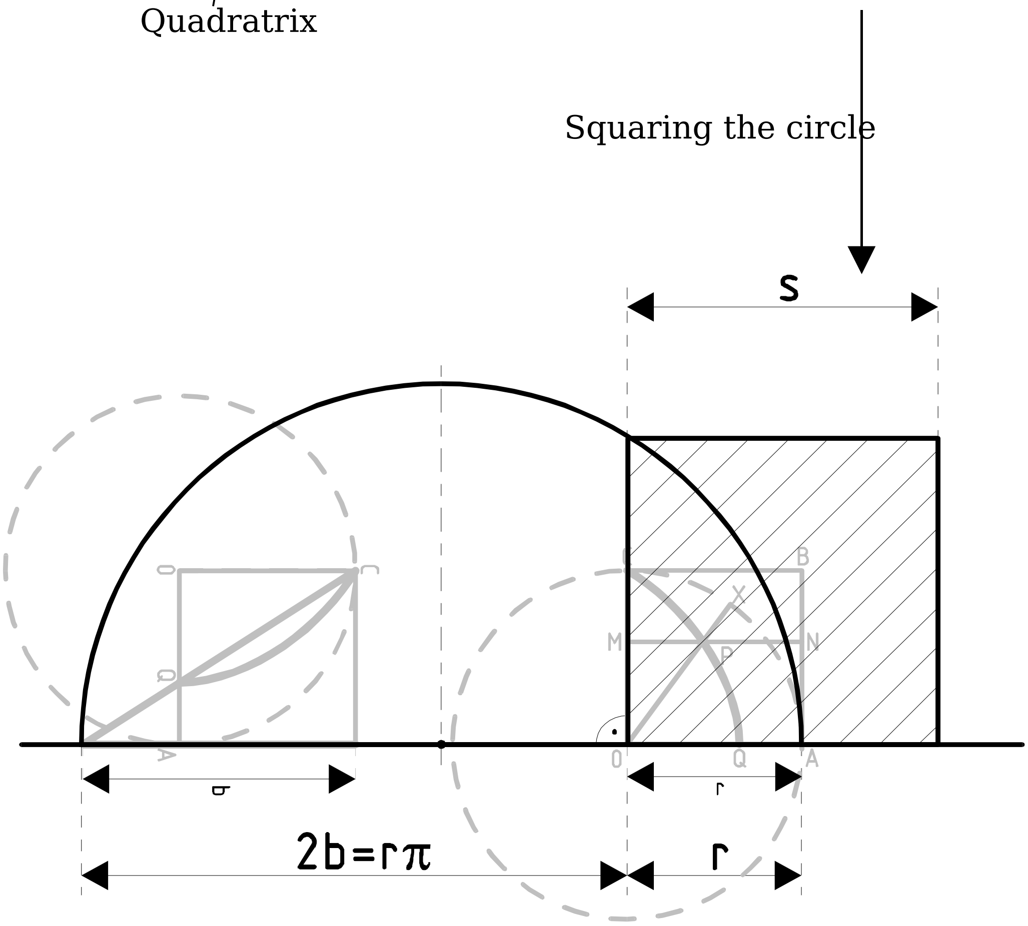


fig. 5

Apart from the proof in fig. 4 that b = 0.5·r·π all steps of the argumentation can be done using basic knowledge from elementary geometry.

But for the Quadratrix cannot be constructed by compass and ruler the ancient Greeks did not regard this method as a solution of the problem.

3.3 Area of a parabola – method of exhaustion

Another problem of the Antiquity using infinitesimal argumentation explicitly is the calculation of the area of a parabola by Archimedes (287-212 BC) using the method of exhaustion, which is usually credited to Eudoxos of Knidos (~370 BC):

If from any magnitude there be subtracted a part not less than its half, from the remainder another part not less than its half, and so on, there will at length remain a magnitude less than any preassigned magnitude of the same kind.

(Cf. [4], p. 307)

Archimedes used this method of exhaustion (this name has been given later in the Middle Ages) and some special geometric properties of the parabola. He showed that

$$\Delta CDA + \Delta CEB = \frac{1}{4}\Delta ACB$$

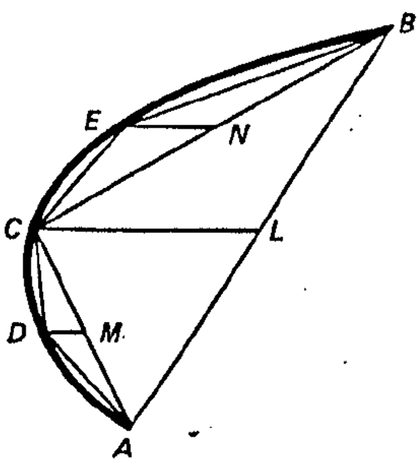


Fig. 6

By repeated applications of this idea it follows that the area of the parabolic segment is given by

$$\Delta ABC + \frac{1}{4}\Delta ABC + \frac{1}{4^2}\Delta ABC + \frac{1}{4^3}\Delta ABC + \ldots =$$

$$= \Delta ABC \quad 1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots \quad =$$

$$= \frac{4}{3}\Delta ABC$$

(Cf. [4], p. 309, [3], p. 61f)

3.4 Galileo Galilei

In the 16th century the natural philosophers began more and more to use mathematics to try to understand the nature and the universe. Galileo Galilei (1564-1642) blended observation and experimentation with mathematical analysis.

In particular he described the motion of the free fall and discovered that the formulas $s = \frac{g}{2}\, t^2$ can be interpreted as the area of the triangle given by the formula $v = g\ t$

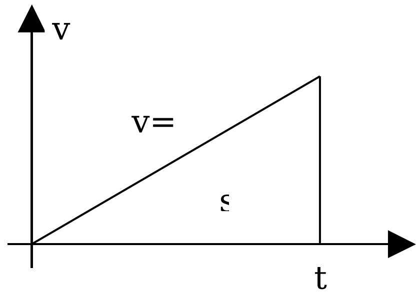


fig. 7

Studying moving objects lead to the problem what the velocity in a given moment is. Questions of infinite divisibility of time and space have already been discussed by medieval scholars.But their new relevance made them more urgent and many mathematicians worked on them. One of them was

3.5 Bonaventura Cavalieri (1598-1647)

Cavalieri studied areas and volumes of curved figures and developed the so called principle of indivisibles: a planar region can be considered as an infinite set of parallel line segments and a solid figure can be considered as an infinite set of parallel planar regions. One of his most famous applications of this principle is the calculation of the volume of the sphere, an easy task suitable for classroom teaching of 14 year old pupils and a good opportunity to anticipate “infinitesimals”:

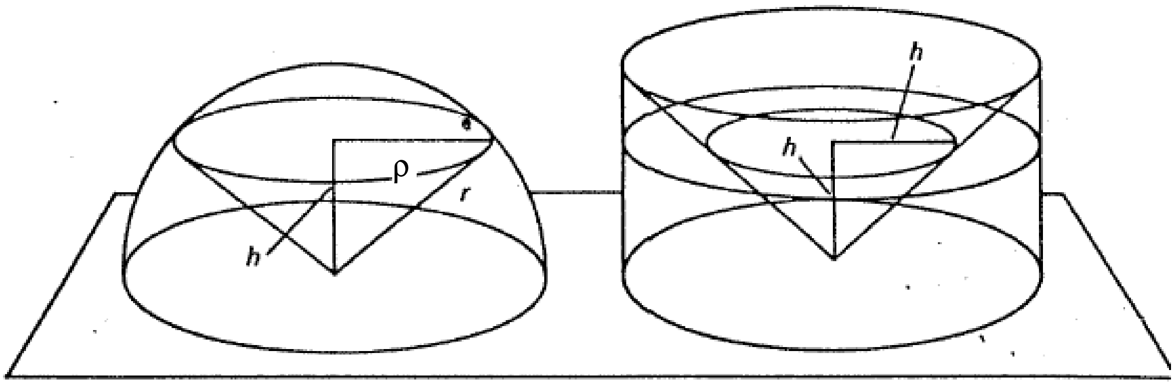


fig. 8

The task is to calculate that both solids, cut in any height by planes parallel to the bases, deliver figures, a circle and a ring, of same area:

$$\rho^2 = r^2 - h^2$$

$$A_{circle} = \rho^2\pi = (r^2 - h^2)\pi \qquad\qquad A_{ring} = r^2\pi - h^2\pi$$

From this one can conclude that the volumes are equal as well:

$$V_{semisphere} = V_{cylinder} - V_{cone} = r^2 \cdot \pi \cdot r - \frac{1}{3}\, r^2 \cdot \pi \cdot r = \frac{2}{3} r^3 \pi$$

(Cf. [7], p.86)

Also with the help of indivisibles Cavalieri gained a result which we in our terminology would express as follows:

$$k\ \ f(x)dx = k\ \ \ f(x)dx$$

Moreover it is interesting to show the students that the conception of indivisibles has also weak points. In particular it allows misinterpretations:

For in the triangle ABC to every vertical line EF correspond a line of same length GH in the triangle BDC and vice versa. So the triangles “consist” of the same set of line segments and therefore they should be of same area.

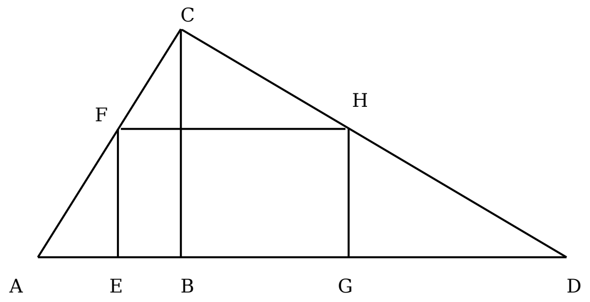


fig. 9

Similarly if one regards a circle as consisting of the set of its radii the a circle with double radius should have the double area. (Cf. [3], p. 63)

3.6 Further steps

Although there are many more people who contributed to the development of infinitesimal calculus I have to come to an end. I want to conclude my talk with a special scientist for three reasons:

1. He spent a long period of his lifetime in Austria
2. He is well known not as a mathematician but rather from physics, in particular from astronomy. (Most people do not know that Archimedes, Torricelli, Galilei, Newton were mathematicians. The physicists claim them for their community and we as mathematicians have to correct this distorsion!)
3. There is a nice story how it happened that he contributed to the development of infinitesimal calculus.

It is Johannes Kepler (1571-1630). In his book “Nova stereometria doliorum vinariorum” (New stereometry of wine casks) he calculated not only the volumes of different shapes of solids, he also explained why he became interested in this problem. When he prepared his wedding he wanted to buy casks of wine. He noticed that the method of estimating the volume of a cask by putting a stick through the bunghole on the top of the cask to the farest point inside the cask do not take into account different shapes of casks. So he started calculations in order to receive the respective amount of wine he had to pay for.

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