Faster and equally fast series of successes and failures

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Abstract. *Many factors have the influence on discovering and understanding mathematics, among others intuition. The abstraction and the schematics in teaching mathematics are being confronted with the vision and perceiving of general, essentially important mathematical structures and the quantitative and spatial relations. Our common sense i.e. our intuition is the author of any ideas, statements or hypotheses, it is he inspiration, the beginning of any discovery and the clue delivering us confidence in reasoning o any type. In the work the examples of stochastic paradoxes are presented. These paradoxes are connected with special relations defined in a set of successes and failures series, that standing against our intuitions appear to be a mean of the mathematical activation.*

Let $u \in (0,1)$, $\Omega_{0-1} = \{0,1\}$, $p^u_{0-1}(1) = u$ and $p^u_{0-1}(0) = 1-u$. Any experiment which model (see [4]) is a probabilistic space $(\Omega_{0-1}, p^u_{0-1})$ is called *Bernoulli trial* or briefly *a trial* and is denoted by δ^u_{0-1}. The result denoted by number 1 is called *success*, and the result denoted by number 0 is called *failure*. The number u is called *the probability of success.*

In this work as a model of a probabilistic many-stage experiment we assume a probabilistic space created with the following rules of stochastic tree:

(R1) the result of a many-stage random experiment δ as an element of the set Ω_δ, at the same time so-called as *elementary event* is represented by a sequence of results of subsequent stages,

(R2) the probability distribution p_δ on the set Ω_δ we define by so-called *multiply rule* that says: if $\omega \in \Omega_\delta$ and $\omega = (a_1, a_2, \ldots, a_n)$, than the pair (Ω_k, p_k) is the model of the k-th stage and $a_k \in \Omega_k$ for $k = 1, 2, \ldots, n$, so

$$p_\delta(\omega) = p_1(a_1) \cdot p_2(a_2) \cdot \ldots \cdot p_n(a_n).$$

Let $m \in \mathbb{N}_1$. Any result of m-times repeated Bernoulli trial e.i. any arrangement of m out of 2 elements (of the set $\{0,1\}$) is called *the series of successes and failures* and is denoted by α. The number m is called *the length of the series* α.

Let α be any stated series of successes and failures of length m, where $m \in \mathbb{N}_1$. Repeating the Bernoulli trial as long as results of m last trials will create the series α is called *the waiting for series* α and is denoted by δ^u_α. The number of trials done in an experiment δ^u_α is called *the waiting time for series* α. This number (mentioned before beginning of awaiting) is a random variable in a probabilistic model of awaiting δ^u_α and it is denoted by T^u_α. The number $E(T^u_\alpha)$ or expected value of the random variable T^u_α - is an average waiting time for series α.

Definition. Let α_1 and α_2 be the series of successes and failures. If

$$E(T^u_{\alpha_1}) = E(T^u_{\alpha_2}),$$

so the series α_1 and α_2 are called *equally fast at the point* u and are denoted by

$$(\alpha_1 \diamond \alpha_2)_u.$$

Definition. Let α_1 and α_2 be the series of successes and failures. If

$$E(T^u_{\alpha_1}) < E(T^u_{\alpha_2}),$$

so the series α_1 is called *faster than series* α_2 *at the point* u and is denoted by

$$(\alpha_1 \lhd \alpha_2)_u.$$

Let α_1, α_2 be the stated series of successes and failures of length m_1 and m_2 respectively. Repeating the Bernoulli trial as long as:

- the results of m_1 last trials create the series α_1,
- or the results of m_2 last trials create the series α_2,

is called *the waiting for one of 2 series of successes and failures* and is denoted by $\delta^u_{\alpha_1-\alpha_2}$.

Let us introduce an event in a probabilistic model of the experiment $\delta^u_{\alpha_1-\alpha_2}$:

$$A_j = \{\textit{waiting } \delta^u_{\alpha_1-\alpha_2} \textit{ will be finished by series } \alpha_j\} = \{\ldots\alpha_j\} \quad \text{for } j = 1, 2.$$

The probability of the event A_j is denoted by $P^u_{\alpha_1-\alpha_2}(\ldots\alpha_j)$.

Definition. Let us consider the waiting $\delta^u_{\alpha_1-\alpha_2}$. If

$$P^u_{\alpha_1-\alpha_2}(\ldots\alpha_1) = P^u_{\alpha_1-\alpha_2}(\ldots\alpha_2),$$

so the series α_1 and α_2 are called *equally well at the point* u and are denoted by

$$(\alpha_1 \approx \alpha_2)_u.$$

Definition. Let us consider the waiting $\delta^u_{\alpha_1-\alpha_2}$. If

$$P^u_{\alpha_1-\alpha_2}(\ldots\alpha_1) > P^u_{\alpha_1-\alpha_2}(\ldots\alpha_2),$$

so the series α_1 is called *better than series* α_2 *at point* u and is denoted by the symbol

$$(\alpha_1 \gg \alpha_2)_u.$$

The below examples illustrate paradoxical properties of relations:

$$\approx,\ \gg,\ \lhd \text{ and } \diamondsuit.$$

Example1. Let us consider a series of successes and failures: $10, 01$ and 00 for $u = \frac{1}{2}$. Here we have

$$(10 \approx 01)_{\frac{1}{2}} \wedge (01 \approx 00)_{\frac{1}{2}} \wedge (10 \gg 00)_{\frac{1}{2}}.$$

Therefore the relation $\approx$ is not a transitive relation in a set of successes and failures (at stated parameter $u = \frac{1}{2}$).

Example2. Let us consider a series of successes and failures: $1101, 1011$ and 0111 for $u = \frac{1}{2}$. Here we have

$$(1101 \gg 1011)_{\frac{1}{2}} \wedge (1011 \gg 0111)_{\frac{1}{2}} \wedge (0111 \gg 1101)_{\frac{1}{2}},$$

therefore in these three series, no one is best (e.i. better than any of the two other). Relation $\gg$ in a set of successes and failures, is not a transitive relation.

Example3. Let $\alpha_1 = 0111$, $\alpha_2 = 1110$. Here we have

$$(0111 \diamond 1110)_{\frac{1}{2}},$$

but in the waiting $\delta^{\frac{1}{2}}_{0111-1110}$ there is

$$(0111 \gg 1110)_{\frac{1}{2}}.$$

From the fact that series are equally fast it doesn't result that they are equally good. At the same time we notice that $E(T^{\frac{1}{2}}_{0111}) = E(T^{\frac{1}{2}}_{0111}) = 16$ and the average waiting time for one of these two series 0111 and 1110 or the average duration time of the experiment $\delta^{\frac{1}{2}}_{0111-1110}$ is 14.25.

Example4. Let $\alpha_1 = 1111$, $\alpha_2 = 1110$. Here we have

$$(1110 \lhd 1111)_{\frac{1}{2}},$$

however in the waiting $\delta^{\frac{1}{2}}_{1111-1110}$ we have

$$(1110 \approx 1111)_{\frac{1}{2}}.$$

It doesn't appear from the fact that series are faster that they are better.

Example5. Let $\alpha_1 = 111$, $\alpha_2 = 0011$. Here we have

$$(111 \lhd 0011)_{\frac{1}{2}},$$

but in the waiting $\delta^{\frac{1}{2}}_{111-0011}$ we have

$$(0011 \gg 111)_{\frac{1}{2}}.$$

Faster series can be "worse" series.

Example6. Let $\alpha_1 = 1100$, $\alpha_2 = 000$. In the waiting $\delta^{\frac{1}{2}}_{111-0011}$ we have

$$(1100 \gg 000)_{\frac{1}{2}}$$

but

$$(000 \lhd 1100)_{\frac{1}{2}}.$$

Better series don't need to be faster series.

Stochastic paradoxes play the huge role in the development of appropriate stochastic intuitions. The student should not be acquainted with the probability calculus as the system of rules, axioms and calculation techniques. He should carry concrete experiments out, to discover and to explain stochastic surprises. Thanks to that the student will be able to move more easily from false stochastic ideas and incorrect intuition to mathematical probabilistic rules.

Literatura

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