Leonhard Euler and continuous fractions*

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Abstract

A famous mathematician Leonhard Euler (1707-1783) was born 300 years ago. The anniversary certainly deserves to celebrate his human and mathematical achievements. In this article we present two typical examples in which Euler used continuous fractions to solve contemporary problems of mathematical analysis. Continuous fractions can serve teachers at a secondary school as a suitable topic for motivation and introduction to sequences and limits.

Keywords: Leonhard Euler, approximation, continuous fraction, the square root, infinity series

MESC: I30, F40.

1 Introduction

According to [1], Leonhard Euler was born in Basel on 15th April 1707 and died in St Petersburg on 18th September 1783. He worked in St Petersburg in the years 1727 - 1741 and 1766 - 1783. In the period 1741 - 1766 he was affiliated with the Berlin Academy and so this year the Humboldt University organized memorial exposition devoted to this anniversary (see Figure 1).

Figure 1: Memorial exposition of Euler at Humboldt University – March 2007

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In this article we present some mathematical results of Euler related to continuous fractions. We believe that results of this type provide a suitable motivation and a proper introduction to the limit processes.

2 Continuous fractions-definition

Leonhard Euler (1707-1783) dealt with continuous fractions in his book *Introductio in analysis infinitorum* and the last chapter in the first part of the book is entitled *Continuous fractions.*

Recall that a continuous fraction is a fraction, the denominator of which is a sum of the whole number and a fraction, the denominator of which is a sum of the whole number and a fraction, and so on. This series of fractions can be finite or infinite. Euler considered two basic types of continuous fractions:

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\dots}}}}} \quad \text{and} \quad a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\dots}}}}}.$$

3 Approximation of square roots

Let x be a positive real number. If $x=\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}}}$, then it is natural to put $x=\dfrac{1}{2+x}$.

We get a quadratic equation $x^2+2x=1$ and its solution is $x=\sqrt{2}-1$.

Euler generalizes this example as follows. If a is a positive real constant, then

$$x=\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\dots}}}}, \text{ hence } x=\frac{1}{a+x} \text{ and } x^2+ax=1,$$

This implies $x=\dfrac{\sqrt{a^2+4}-a}{2}$. Using a similar continuous fraction, we can approximate the square root of every natural number. Starting with

$$x=\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+\dots}}}}}.$$

we get the following equation: $x = \cfrac{1}{a + \cfrac{1}{b+x}} = \dfrac{b+x}{ab+1+ax}$. Hence

$$x = \frac{-ab + \sqrt{a^2b^2 + 4ab}}{2a}.$$

For example, let $a = 2$, $b = 7$. We get

$$x = \frac{-2.7 + \sqrt{2.7.2.7 + 4.2.7}}{2.2} = \frac{-14 + \sqrt{7.36}}{4} = \frac{-7 + 3\sqrt{7}}{2}.$$

Consequently,

$$x \approx \cfrac{1}{2 + \cfrac{1}{7 + \cfrac{1}{2 + \cfrac{1}{7 + \cfrac{1}{2}}}}} = \frac{239}{510}.$$

Hence $\dfrac{-7+3\sqrt{7}}{2} \approx \dfrac{239}{510}$. So we get an approximation $\sqrt{7} \approx \dfrac{2024}{765} = 2,6457516....$ The exact value is $\sqrt{7} = 2,64575131.....$ The difference between the exact value and our approximation is smaller than $\frac{3}{10000000}$.

4 Changing an infinite series to a continuous fraction

Euler has constructed an algorithm how to change an infinite series with alternating signs into a continuous fraction. We can continue as follows. Let

$$x = \frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \dots.$$

Consider a continuous fraction of the form

$$\cfrac{1}{a + \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \dots}}}}.$$

The partial sums of the infinite series are

$$\frac{1}{A}, \frac{B-A}{AB}, \frac{BC-AC+AB}{ABC}, \dots$$

and the expressions in the continuous fraction are

$$\frac{1}{a}, \frac{b}{ab+\alpha}, \frac{bc+\beta}{abc+a\beta+\alpha c}, \dots$$

Comparing the corresponding expressions, we get a system of equations. We explain the procedure on the first three steps:

$$\frac{1}{A} = \frac{1}{a}, \frac{B-A}{AB} = \frac{b}{ab+\alpha}, \frac{BC-AC+AB}{ABC} = \frac{bc+\beta}{abc+a\beta+\alpha c}.$$

Hence
$a = A$
$b = B - A$
$AB = ab + \alpha$
$BC - AC + AB = bc + \beta$
$ABC = abc + a\beta + \alpha c$
The first two equations are simple. If we substitute b into the third equation, we have $A(B-A)+\alpha = AB$ and hence $\alpha = A^2$. We simplify the fifth equation $ABC = a(bc+\beta)+\alpha c$. If $BC-AC+AB = bc+\beta$, then $ABC = a(BC-AC+AB)+\alpha c = A^2(BC-AC+AB)+\alpha c$, which implies $c = C - B$. We substitute now into the fourth equation for b and c. We get $BC - AC + AB = (B - A)(C - B) + \beta$, hence $\beta = B^2$. Euler generalizes these equations:

$$\frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \ldots = \cfrac{1}{A + \cfrac{A^2}{B - A + \cfrac{B^2}{C - B + \cfrac{C^2}{D - C + \ldots}}}}.$$

He illustrates this interesting construction by using the Leibniz´s series:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \ldots}}}}.$$

Analogously, Euler demonstrates his argument on the following change:

$$\frac{1}{A} - \frac{1}{AB} + \frac{1}{ABC} - \frac{1}{ABCD} + ... = \cfrac{1}{A + \cfrac{A}{B - 1 + \cfrac{B}{C - 1 + \cfrac{C}{D - 1 + ...}}}}$$

and illustrates it on the series:

$$1 - \frac{1}{e} = \frac{1}{1} - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + ... = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + ...}}}}.$$

5 Summary

The history of mathematics provides many inspiring examples and approaches to sequential convergence which can help in understanding the limit processes. There are paralells between historical mathematical thinking and the development of mathematical thinking in the mind of students. Continuous fractions served as a tool to approximate the values of irrational numbers with fractions. Even though computers are widely used in our times, in our opinion, for pupils it is important to known some techniques used in the history. The beauty and depth of mathematical thinking of famous mathematicians such as Leonhard Euler can help pupils to understand and appreciate mathamatics and its applications.

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