REMARKS ON EULER SUMMABILITY METHOD

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Abstract. In his clessical monograph (Théorie des opérations linéaires. Monografie Matematyczne, Warszawa 1932) Stefan Banach adresses also to the problem of summability of series. The results by Euler concerning the summability were criticized by 19th century investigators as they did not correspond to the modern standards of rigor.

We conclude that the abstract formalism of functional analysis allowed to treat from the general point of view concrete calculations due to Euler. This demonstrates the diversity and reachness of interaction of mathematical ideas which sometimes have distant (also in temporal aspect) origin.

In 2007, we celebrate the 300th anniversary of Leonhard Euler (1707 - 1783). He was born on April 15, 1707 in Basel and died on September 18, 1783 in Saint Petersburg After graduating from philosophical studies in Basel, since 1735 Euler worked in the Saint Petersburg Academy of Sciences, in 1744–66 in the Prussian Academy of Sciences in Berlin, and spent the rest of his life again in Saint Petersburg. A number of mathematical objects is named after him: Euler's constant, Euler numbers, Euler's substitution, Euler's graph, Euler angles, Euler integrals, Euler's method of broken lines, Euler's line, Euler's circles, Euler's formula, Euler characteristics etc. The only analysis of the mentioned objects would be a sufficient material for a book. Nevertheless, this is merely a part of what Euler did. He published about 900 papers in the following fields: mathematics, mechanics, celestial mechanics, metrology, hydromechanics, hydraulics, ship construction, ballistic, philosophy, 500 of them are devoted to mathematics. The international German-Russian-Switzerland commission publishes "Leonhardi Euleri Opera omnia" during about 100 years. The beginning of the process of publishing was in the first years of the 20th century and the finish will be in 2 or 3 years. L. Euler is one of the creators of modern mathematics.

In May 14–18, 2007, the international scientific conference "L. Euler and modern science" took place in Saint Petersburg.

There were the following sections: the history of Emperor's Academy of Sciences, history of mathematics, history of mathematical education, history of mechanics, history of physics and metrology, history of ship building, history of philosophy.

In his talk on the mentioned conference the author touched a trace of Euler's scientific heritage in the activity of another mathematical genius — Stefan Banach — the leader of the Lwów school of mathematics, the founder of functional analysis.

Modern investigators in the history of mathematics note that, despite of lack of rigor in Euler's works related to analysis, from the point of view of later period of development of analysis when Cauchy and other mathematicians established modern standards of rigor, the rise of new fields of mathematics such as non-standard analysis of A. Robinson allows for finding appropriate background for various calculations and constructions, sometimes very sophisticated. In many aspects this remark concerns Euler's arguments on summability of series. Euler deals with series as they were finite sums and, seemingly, does not pay any attention to a fundamental question whether one even can speak on existence of a sum of a given series. In this connection one can mention reasonings of some historians of mathematics that the critics of Euler's style by analysts of later generations was based on the wrong assumption on existence of a unique approach to substantiation of analysis.

Even before invention of the non-standard analysis the questions of summability of series and their appropriate substantiation were an object of interest in functional analysis. In the classical monograph "Théorie des opérations linéaires" by S. Banach (Monografie Matematyczne, Warszawa 1932; earlier the Polish version [1] appeared), namely, in Chapter V, the problem of summability of series is considered.

Scan in more details, let

$$\begin{array}{ccccc} a_{11}, & a_{12}, & \cdots & , a_{1k}, & \cdots \\ a_{21}, & a_{22}, & \cdots & a_{2k}, & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1}, & a_{i2}, & \cdots & a_{ik}, & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

be an infinite matrix, we say that a sequence $x = \{\xi_k\}$ is summable (to $A(x)$) by the method (which corresponds to the matrix ()), if every of the series $A_i(x) = \sum_{k=1}^{\infty} a_{ik}\xi_k$ is convergent and the sequence $\{A_i(x)\}$ converges (to $A(x)$).

The notions of permanent and invertible summation methods are introduced. Also, a comparison relation for the summation methods is defined. We say that a method B (which corresponds to a matrix $B = \{\{b_{ik}\}\}$ is not weaker that a method A if every sequence summable by A is also summable by B.

A method A is said to be *permanent* if every convergent sequence is summable to its limit by this method. A method A is said to be *invertible* if to every convergent sequence $\{\eta_i\}$ there corresponds precisely one sequence x (not necessarily convergent) such that $A_i(x) = \eta_i$ for $i = 1, 2, \ldots$.

Theorems 10–12 are main results in [1] in this direction. The first one is a characterization theorem for the permanent summation methods.

The proofs of the two following theorems are based on a few technical results. The arguments essentially involve a theorem on extension of a linear functional from a linear subspace to the whole space; this theorem is now referred as the Hahn-Banach theorem.

Theorem 11 asserts that if a permanent method B is not weaker that a permanent and invertible method A, then every bounded sequence summable by the method A is also summable by the method B to the same number.

Theorem 12 states that if A is a perfect method and B is a permanent method not weaker than A then every sequence summable by A is also summable by B to the same number.

The latter theorem is a generalization of a result by S. Mazur [4] which was obtained for the normal summation methods. Here the summation method Ŕ is called *normal* if $a_{mn} = 0$ for $m < n$ and $a_{mn} \neq 0$. It is noted in [4] that every Euler summation method E_k, where k is a positive number, possesses the property that it is consistent with every not weaker permanent summation method. We remind here that in the summation method E_k we have

$$a_{mn} = \frac{1}{2^{km}} \binom{m}{n} (2^k - 1)^{m-n},\ m, n = 0, 1, 2, \ldots$$

We also recall that the methods A and B are said to be *compatible* if all the sequences summable by both methods are bounded by the same number.

References

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[3] S. Domoradzki, *On Euler summability method in monograph of Stefan Banach*, In: L. Euler and modern science, Russian Academy of Sciences, Saint Petersburg, 2007 ,s.111–116. (In Russian)

[4] S. Mazur, *Eine anwendung der Theorie der operationen bei der Untersuchung der Toeplitzschen Limitierungsverfahren*, Studia Mathematica, t. II, 40–50.

[5] W. Orlicz, *Osiągnięcia polskiej matematyki w zakresie analizy funkcjonalnej*, Historia Nauki Polskiej, wiek XX, Zeszyt 1, s. 129–151.

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