Handbook of Mathematics Teaching Research:

Teaching Experiment - A Tool for Teacher-Researchers

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INTRODUCTION

This volume presents the results of the three-year-long International Professional Development of Teacher-Researchers Project: Transforming Mathematics Education with the help of Teaching-Research Methodology (2005-2008), supported by the grant from the Socrates Program of the European Community no. 226685-CP-1-2005-1-PL-COMENIUS-C21. Six teams of mathematics teachers, apprentices in the craft of teaching-research participated in the project and were supported in their work by teaching-research mentors comprising experienced teacher-researchers and academic researchers: one team from Debrecen, Hungary, one team from Italy including teachers from Modena and Naples, two teams from Poland, one from Kraków/Rzeszów and another from Siedlce, and one team from Lisbon, Portugal.

While the term "teacher-researcher" indicates a professional who is a teacher and a researcher, it also has several slightly different meanings that partially overlap with each other. "Teacher-as-researcher" can be understood as an action-researcher in a classroom or a school, or, possibly as the "reflective practitioner" of Schön. All of those approaches are represented in the volume and this variety has contributed to the richness and scope of work done by the teachers of the project and their teaching-research mentors.

The project description mentioned that:

> The central aim of the project has been to engage classroom teachers of mathematics in the process of systematic, research-based transformation of their classroom practice while producing evidence-based innovative instruction and contributing to research knowledge of the profession; it is to initiate, using teaching-research as the leading methodological agent, the transformation of mathematics education towards a system which, while respecting the standards and contents of the national curricula, would be more engaging and responsive to student's intellectual needs, promoting independence and creativity of thought, and realizing fully the intellectual capital and potential of every student and teacher.[1]

Teaching-research is a powerful approach that can move mathematics education dramatically forward to fulfill its promise of education to young children, pupils and students, by incorporating research methods into classrooms, supporting application of research results and, especially, as the generator of research questions which flow directly from teaching practice. Its power depends, however, on balancing and smooth integration of teachers' professional knowledge with the knowledge of the research profession. Members of the project have had many discussions concerning this balance and the process of integration. Echoes of those discussions can be found in the submissions to the volume.

Initially, teachers' work addressed issues and problems of the mathematics component of the PISA International Test, and the results of that aspect of the work are presented in the companion volume. This part of the work served as the preparation for the second phase when TR apprentices designed, with the help of their mentors, classroom teaching experiments, collected the data, observed their classrooms with a new eye of an investigator, analyzed and discussed the data with their team members. Teacher-designed teaching experiments focused on the observed issues and problems in their classrooms of mathematics. Some of the issues for investigation were suggested by teachers' concerns, others by the studied literature or discussions with the academic

[1] Comenius 2.1 Project Description, Section 4.

researchers, mentors of the teams, but always they were connected to the questions that the participating teachers asked about their classroom teaching-learning experience.

The participating teams had different levels of experience with the teaching-research methodology, some have been practicing different versions such as action research or reflective practitioner approach for a significant number of years, and some of them were introduced to the teaching-research methodology during the PDTR. Thus, the significant differences in the presented organization of teacher-researchers' work, in their immediate classroom aims and methods reflect both methodological differences, and the differences in the cultural background of the teams. All submissions are written in English and many of them betray the cultural and linguistic influences from their original languages, enriching the meaning of submissions. The responsibility for the adequate translation of the submissions rested in the hands of national teams.

The submissions to the *Teaching-Research Handbook* have been divided into four parts or categories:

Part 1 "Fundamental Ideas" presents different views on the meaning of a teacher-researcher and of teaching-research;

Part 2 "Elements of Theory of Teaching-Research" contains a few more theoretical submissions relevant for our work;

Part 3 "Instruments and Tools of Teaching-Research" presents the results of different techniques and methods utilized in TR investigations by members of different teams;

Part 4 focuses on teaching-experiments conducted in teachers' classrooms.

While, naturally, the boundaries between different parts are fluid, yet the attempt was made to keep these categories as the guide for the submissions to the book.

We hope that this *Teaching-Research Handbook* provides a good flavor about the work done by the project and stimulates many other teachers and teacher-researchers across Europe and elsewhere to get involved in the development of teaching-research craft, researching their own professional practice, improving understanding and achievements of mathematics of their students.

Words of appreciations are due to all the teacher-researchers, authors of the contributions, their mentors and national team coordinators for the commitment and involvement in the creation of this volume. In particular, special appreciation is due to the typesetter, Magda Michniewicz who spent innumerable hours working on the book organization, and to Agnieszka Legutko for the patient proofreading of the final version.

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RESEARCHING OUR OWN TEACHING

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ABSTRACT
This paper is a commentary on a series of research work conducted by a group of teachers who participated in the PDTR Project in the PL2 Team. All of these teachers aimed at improving their own teaching by a careful observation of their own classroom practice, by trying to identify the critical points, and by introducing changes. They met regularly at weekly seminars to discuss their cases of classroom episodes and to study the selected pieces from the literature on mathematics education or to exchange some ideas about possible improvement of their professional practice. Observation of their students in the classroom, observation of their own classroom behavior and reporting on it at the seminar gave these activities an action research character based mainly on case studies. The present education system in Poland is governed by the central government's curriculum for every subject and a stiff examination system. Teachers are obliged to prepare their students for the exams which are external and mandatory after each education stage. Trying to meet these subject matter demands and at the same time taking care of all key competencies which also require some time allotment is difficult and leaves little time to spare for large scale experimentation. The answer is perhaps to improve teaching practice here and there, so as to avoid teaching old-fashioned skills which are not useful at later stages and sometimes even detrimental to further mathematical development. Along with such subject matter improvements, there is a strong need to improve pedagogical skills much more oriented to the PISA style mathematics and psychological constructivism. The PL2 Team identified several such mathematical subject matter topics which need restructuring from the mathematical point of view, in order to avoid blocking students at later educational stages. They are discussed below.

Researching our own teaching comes from observing our students at work and trying to find sensitive points where our intervention can help. Hopefully, such an attitude should result in a better mathematical development of students and in the long run in our own professional development. The important question for teachers is which treatments to choose from their repertoire in a particular situation in order to improve student's development. There is also a possibility that a new treatment technique and teaching aids are needed and the trial teaching that follows is designed with reference to literature and on the basis of our own experience and ideas. The trials confirm our expectations or not and sometimes give negative evidence of any improvement. It is therefore typical *action research* (Denzin & Lincoln, 1994). We can falsify our conjectures but we can never prove anything for sure.

Some researchers in Poland raise objections to the fact that an activity aimed at the improvement of practice may be called research (Konarzewski, 2000), and such a restrictive position is not rare in Poland. Our position in the PL2 is that we strongly disagree with such statements. A substantial part of research in medicine and technology aims at improvement of practice. We do not think it reasonable to consider it off-limits of “true research.” It also creates an interesting paradox, similar to the famous barber’s paradox: “Is a researcher who is working on improvement of his research doing a research or not?”

A particular design of teaching treatment is always based on an adopted theory and a teaching plan, curriculum or program. This theory may be expressed openly or may remain hidden. A curriculum in Poland is usually a list of topics to be covered. The choice of a teaching style and methods are officially left to teachers. As for the teaching style, there is a strong tradition of the frontal “plenary” style in the classroom.

Didactical and pedagogical analysis of our own teaching often begins with the following questions: (i) What are the emerging difficulties of individual students or groups of students? (ii) Are there any visible reasons for them? (iii) Are these difficulties or learning problems addressed in the literature? (iv) Is the change of treatment possible? (v) What are the alternatives? (vi) Is there any hidden theory which might be changed? (vii) If the change in treatment is applied, what is the impact on students’ learning? Does it help? and (viii) How to record that change?

An example of one of available theories is Krygowska’s Metaphor for planning teaching by extracting concrete actions for students from the final structure of definitions, proofs, and theorems (Krygowska, 1977). Another one is that at the start of each topic teachers explain, i.e. they choose an appropriate metaphor in order to explain the issues in terms that students already know (Sfard, 1994; Bauersfeld & Zawadowski, 1981). There are of course other theories. It seems that some of them are related by a family resemblance in the sense of Wittgenstein but they provide different languages for teachers to communicate classroom events.

Apart from these questions, there is another group of questions related to the mathematical content and its structure. In Siedlce the PL2 group discussed such questions with special care. It turned out that there are certain topics in the curriculum where we *teach* procedures and skills, which we should *unteach* later on. The use of percentages is an example of such a topic. First, we impart the additive model for percentage operations in the early school grades and later on when we try to teach the multiplicative model we encounter difficulties. Many students persist in using the additive model, even though it is difficult to apply it in many problems. Students who are successful use the new multiplicative model more easily but those who stick to the old additive model have a much more difficult task and fail.

Research aimed at the improvement of students’ learning must first identify their difficulties. These difficulties may come not only from the structure of content but also from social circumstances in the classroom or an individual condition of a single student. The design of an improvement plan and its evaluation for a class of about 30 persons is no easy task. Therefore, teachers often focus their attention on individual students recording *case studies* of some individuals.

Collecting sheets of paper with scripts of solutions to given problems yields some information about students activities and constitutes concrete evidence. But it does not give the complete story of what has happened. We can get more information by engaging in a dialogue. Registering such dialogues on-line or recalling them by the end of the day was important but not always feasible. Notes that are too short are difficult for a later interpretation. Longer records are time consuming. A compromise is necessary. It comes with experience.

Usually, there is little access in schools to the literature on mathematics education. Contact with other teachers might be helpful and the regular weekly seminars with discussions and confrontations with other teachers of the PL2 team and also e-mail contacts were essential. It gave professional support and encouragement.

Teachers addressed their teaching treatment to the whole group in the classroom but their evaluation of the treatment was based on a careful examination of individual students. The systematic and detailed records of such examinations in a given format provided necessary feedback.

The communication "teacher-to-student" and "student-to-teacher" placed emphasis on students, their behavior, attitude, perseverance, mathematical skills, and overall advances in mathematical development. The aim was to observe the emergence of mathematical objects. But teachers do not have any direct opportunity to see these mental objects; they may only judge their emergence by observable traces of their existence: word utterances in communication situations, whole sentences, figurative use of the language, metonymies, metaphors, and larger units of text. I agree that "the need for communication – any attempt to evoke certain action by others – is the primary driving force behind all human cognitive processes. Effectiveness of verbal communication is seen as a function of the quality of its focus. But in some discourses focus engendering objects must be created" (Sfard, 2000). This might be called "the discursive construction of mathematical objects" (Sfard, 2000). Sfard considers metaphor as a point of departure for a construction process of mathematical objects ensuring effective communication. And so the effectiveness of communication may be in a sense a symptom of the emergence of mathematical objects. In some circumstances a good metaphor induces the expectations and the need to a construction procedure to ensure effective communication. Such effective communication is often attained by the use of metonymies and so the metonymical expressions could be in some circumstances considered as symptoms of emerging mathematical objects (Kadej 1999, 2000).

Observable traces of students' thinking arise in the course of discussions and various interactions. We shall consider three types of interactions:

A Teacher – Teacher

B Teacher – Student

C Student – Teacher

TEACHER – TEACHER INTERACTIONS

A1

Teachers met at the seminar every week on Friday afternoons. They taught in various grades. This way they had first hand information of what went on at various stages before and after their own grade. They knew what difficulties and interests the students had at earlier stages and what the expectations of teachers at subsequent stages were. Perhaps what was taught at the earlier grades did not prepare them well for a later mathematical development. They may also have acquired some skills and beliefs that were counterproductive at the later stage.

A secondary school teacher ZŁ analyzed the way students approached the percentage problems. They all used the additive model which described the growth or decline in terms of adding or subtracting. At the same time they used the multiplicative model, e.g. in geography when considering scale of a map, or in geometry when considering scaling down and up as central enlargements in a given scale. A phenomenon occurred which was called *imprinting*. Most students used the additive way for describing growth using percentages which they acquired at the earlier stages, when they first encountered percentage problems. The corresponding formula was e.g.

$y = g + \frac{22}{100} g$

A common mistake occurred when they wished to calculate the reverse operation. It was not simply

$x = s - \frac{22}{100} s$

When the multiplicative model is used with decimals instead of percentages these formulas take the form

y = 1.22 g and the reverse formula is simply

$g = \frac{y}{1.22}$

In trial teaching most students persistently chose the additive model in various problems and committed errors, even though they were taught the easier multiplicative model and its advantages in a series of lessons. They chose the way which they first encountered. Because such a didactical phenomenon occurred in various other circumstances we called it *imprinting* at our seminar. We found that imprinting is in effect similar to the didactical phenomenon of *proceptual divide* described by Gray and Tall (1994). Most students use models they first learned and are accustomed to even though such models might be difficult to apply and conducive to errors in sophisticated situations.

A2

Teacher AP noticed that students' achievement depended on their willingness to learn. One of the ways to enhance that willingness was to create a specific ethos in the classroom, a special attitude when teachers went beyond the routine of the narrow curriculum. One of the references to this effect in literature can be found in Bruner (1966). Together with a group of teachers, she decided to organize a "Mathematical Tournament" for schools in Siedlce and vicinity. Every year a special topic is chosen outside of the curriculum. The Tournament begins with a lecture on that topic. After that students get special literature and problems to solve at home. The tournament ends with a final session when the participants are given problems to solve individually. The winners receive special diplomas and symbolic prizes. It turned out that these tournaments were very popular. They became popular not only among students but also among teachers. The success of their students is often considered a confirmation of the effects of their work as teachers.

Last year the special topic was "PISA Style Problems." First of all, it was necessary to characterize what the PISA style was for this effect. As in all other teams, teachers in the PL2 analyzed the common PISA problems, gave them to their students, analyzed solutions, and again produced some refinements and analogous problems for analysis. To put it briefly, the PISA style as analyzed by the PL2 team *used mathematics as a language* – as a specific language to communicate, to explain, to predict, and to generalize. This language is visual. Even algebraic formulas are "read" visually not acoustically as it is with natural national languages. With such a general working characterization of the PISA style a series of problems was chosen. This year the tournament went on until May, 2008.

A similar attempt but on a small scale, one day school event was designed and managed by teacher KS. The topic covered was "Mathematical and Logical Snapshots." The older students acted as judges and organizers and using IT gave a visible and stimulating acceleration to information processing on-line. The benefit of the event was not only mathematical. It contributed to the integration of older and younger students around a common topic.

A3

Annual national and regional conferences of the Polish Association of Teachers of Mathematics are a good opportunity to exchange professional information and present the results of research in a workshop style. The teachers of the PL2 participate not only as workshop leaders but often are also organizers of the conferences. The active members are authors of many initiatives. A number of ideas came out of such encounters between teachers.

A4

Another opportunity for teacher-to-teacher contacts occurs annually by the end of the school year in May or June. At that time an event called Science Picnic of Bis Radio takes place in Warsaw.

Polish ATM has a special stand there. Some teachers go there with their students. The students present their work to public. Usually they present models for spatial geometry and explain some other topics to the visitors who are usually other students, their parents and teachers. The event is popular in Warsaw. Teachers who decide to prepare their students for such a show work for many months in advance of the event. Students are motivated and their teachers meet other teachers and there are interesting discussions. The PL2 provided some ideas there.

A5

Some teachers participating in the PL2 are mentors at the in-service courses for teachers of mathematics at University of Podlasie in Siedlce. They have then an opportunity to repeat their teaching trials in other schools using their student-teachers. It can verify their research-teaching and also the ideas of other teachers of the PL2.

A6

Some papers on mathematics teaching appear in the quarterly *NiM* of Polish ATM. Participants of the PL2 are authors of several contributions there related to their work in the PDTR.

This is a good channel to spread some ideas from the PL2. Some of them repeat the themes already known and only give a fresh confirmation, like the use of graphic calculators but others are rather original contributions like the observation of the imprinting phenomenon and its relation to proceptual divide, or the invention of *trigonometric stamp* as a useful dynamic model for the introduction to trigonometric functions. Such publications are evidence of professional development of their authors and also source papers for others. Commenting on various positions from literature is a regular part at the PL2 weekly seminar.

TEACHER – STUDENT INTERACTIONS

B1

Teacher JC analyzed the way students prepare to an on-line classroom test. She assumed that group work in school time in preparation to the test with discussion among students will have a positive effect (Sfard, 2000). Previously, her students worked individually. The first trial of the new method of work gave positive results. Students seemed much more engaged and their mutual stimulation extended their vocabulary and willingness to learn. The work in that new way is still going on. She decided to look closer at the language games involved (Sfard, 2000; Wołos, 2002; Wittgenstein, 2004).

B2

Teacher AŁ systematically engaged students in the design of a teaching program. It was very clearly stated in her lessons what was already achieved, and what had been achieved so far and what would be next. She did it with one class. She took an ordinary routine with another class.

She noticed that such a thorough explanation of the teaching plan had a stimulating effect. In order to confirm her observations she compared the achievement in tests for these classes against the standardized achievement for the whole school, the city, and the whole county. The results were encouraging for her. But the direct qualitative results were more convincing. Increased interests of students and increased willingness to learn were directly visible.

B3

An experienced teacher ZŁ noticed that his students had unusual difficulties with percentages (as it was mentioned above). Especially with problems in which to use a reverse operation in adding some percent to a given value was essential. Such problems occurred when one wished to find out when the price included VAT and we wanted to know the value without VAT. He could see that multiplicative model is easier to handle. The multiplicative model looks at the change not as adding a percent but as multiplying by a factor expressed as a decimal number. The importance of being sensitive to a reverse operation from the very beginning of the topic was expressed in Krygowska's Metaphor (Krygowska, 1977). ZŁ explained to his students the advantages of the multiplicative model. A short time afterwards he gave them problems to check their attitude to the choice between the two models. It turned out that the common additive model was so strongly rooted that most students preferred that model and of course ran into difficulties and were not able to perform calculations successfully. Only few of them used the new and easier model. And even then most of these few tried to check the result additively. The class divided into those who could *accommodate* the new scheme (just to use the Piagetian language) and then successfully *assimilate*, and those who could not and their assimilation therefore was not successful. The old model and old language was not *syntonic* enough in relation to the new kind of problems (Papert, 1980).

It was noticed at the PL2 seminar that the effect was similar to the proceptual divide (Gray & Tall 1994). Those who were more flexible in their attitude and thinking were successful and chose the new and easier model. Those who stuck to the old model had difficulties related to it, ran into errors and failed.

The initial encounter with a mathematical model is important. If it is too strongly rooted or even fixed, it gives an effect similar to imprinting (in bird biology), and it might be an *obstacle* to a later development.

B4

A primary grade teacher MA exposed her students to an adventure story of a visit on a different planet that is called Duo. The Extra Terrestrials there, the ETs, had a different system of numerals which they called duo. The story about these visits developed slowly during three years. Students wrote letters, received answers and exchanged gifts. At one moment they asked about prices and values and compared. Then they realized that the ETs counted in a different system. MA did the story telling experiment for the first time. It followed an assumption that translations and a controlled dialogue with other systems of numerals might enhance understanding of the number concept. Of course, the decimal system was their native one and skills were to be trained only in the decimal system. But the binary system is so easy that even though the expected gains were not quite clear, she started the story nevertheless and continued it for three years. Apart from some conceptual gains of broadening the number concept by a hairsbreadth, there are other aspects of realizing that there are other possible beings that have a world of their own and are friendly. After three years MA stated that at least the emotional attitude of her students toward mathematics was very positive.

We shall see if the relay is successful the following year when they enter fourth grade and get another teacher.

B5

WG**,** a teacher in secondary school and the so-called technical secondary school with some vocational subjects noticed that his students were interested in mathematics only if there were some applications understandable for them. A gadget or a toy with something special that could be explained with mathematics. The usual treatment of trigonometric functions in secondary school was rather boring for his students. Very many details at the beginning with slow graph drawing step by step: first, trig functions for the right triangles, then extending them to the arbitrary angles, then trig functions on the number line, some trig identities, some problems and exercises. All that without graphic calculators or a computer application for drawing graphs. The students had difficulties with all these details to remember, not to mention structural understanding (Krygowska, 1977). He designed a special trigonometric stamp for drawing the graphs. The stamp produced a graph of sine and cosine functions with one stroke. The pilot studies showed that we should rather begin with cosine function than sine function because the graphic of the cosine is axially symmetric relative to y axis, the vertical one. So getting a cosine graphic first is a much better starting point to trig functions then the traditional starting point with a sine graphic. The trials are pending.

Teacher ICh who has 30-year-experience as a teacher in upper elementary school grades (students aged 10-12) wanted to check her students' skill in solving PISA type problems which were a little unusual. She was especially interested in the way the 12-year-olds would approach the problems usually addressed in PISA test to this particular age group. One of her students solved the problems in an original way even though he had yet no knowledge of quadratic equations he managed to solve it practically. Others were only partially successful.

In another situation during a lesson reviewing fractions in practical situations at some point a good student after finishing reduction of a fraction on the blackboard obtained $\frac{18}{1}$ stopped for awhile, and asked:

„Is the fraction $\frac{18}{1}$ the same as 18?"

He quickly "assimilated" the answer "yes" and went on to the next problem. It was clear, however, that in this particular class a fraction was "not yet a point on the number line." It was still far from structural understanding (Krygowska, 1977) but it was nice to see that such a question was posed by a student and not by a teacher. It was a symptom of a very good communication relation of student to teacher and teacher-to-student communication.

Teacher KS who had a good access to a computer laboratory in a middle school wrote a special program to engage students in an interactive way into shortcuts in algebraic expressions including difference of squares etc. The program itself was self-made and not much elaborated graphically but nevertheless it was pedagogically efficient. The engagement of his students increased as compared against a routine lesson. Good students were happy to find out some small technical faults and the morale was high. Questions were also raised how it worked.

The same teacher used graphic calculators at the beginning of a special unit about functions in middle school. He did it disregarding an older teacher who made some very discouraging remarks. His students mastered the technicalities very fast and their repertoire of graphic representations of functions broadened and their understanding was much deeper than a year before in similar circumstances but without technology.

It is not possible to relate and comment on all incidents in communication that were noticed and recorded in the PL2. The teacher-to-student and student-to-teacher communication is usually interwoven and rooted in specific topic circumstances. The idea of Wittgenstein's *language game* fits well in such a situation. It was introduced to our usual teacher-to-teacher communication vocabulary at the PL2 seminar. It was of course not used in teacher-to-student communication.

STUDENT – TEACHER INTERACTIONS

C

Teachers and students often talk different languages. The expected formalism is often an obstacle for students hampering the communication. One way to build up common vocabulary and understanding is to create a common topic situation around a story or a specific activity or both. Examples of this can be found in many PISA style problems, where given problems are embedded in a story. The "carpenter and a table circumference," the "race car speed" and their refinements are good examples. Longer stories are even better, like the "ET story and a binary system" or the "Drying Melon Story." Expanding such stories while using mathematical language is a somewhat unusual but useful way of arousing interest.

Arranging such situations is easy when students start to make models for spatial geometry. The paper models are especially easy to use for such a purpose, i.e. for arranging a language game for meaning building (Wittgenstein, 1994; Wołos, 2002). In the PL2 we make such models without glue so that they are easy to assemble and easy to dismount and to store for a later use. Some models of spatial shapes are especially good for focusing on the mathematical discourse. These models are the "tetrahedron inscribed in a cube," or rhombic dodecahedrons. There are other paper models; some of them are a little more sophisticated, like Sierpiński Tetrahedron or other simple fractals. While analyzing models, students often use a figurative language loaded with metonymical expressions. To notice such expressions by a teacher and to recognize which of them are symptoms of understanding requires some experience (Featherstone, 2000; Kadej, 2000). One example of such a metonymy is a "triangle" used for a tetrahedron, or a "square" for a cube (as an example of "part for the whole," or a synecdoche).

Teacher AŁ who is also an experienced mentor for university students at practice in school noticed that student-teachers often had too formal a language for their classes. It is not easy to notice critical moments in one's own behavioral habits. She noticed that it was easier for her to notice some small differences of language that matter or other such things when she observed university students who found themselves in a teacher position.

The important questions discussed among PL2 teacher were the curricular questions focused on the structure of mathematics which is taught (Freudenthal, 1983). One such question is about decimal numbers. Should we teach them before or after fractions? It is not such a simple question to answer. It was triggered by a report from Mathematics Education session at the joint meeting of the Polish Mathematical Society and the American Mathematical Society in Warsaw in July-August 2007. It was reported that mathematicians liked to say that "fractions are points on the number line." This expression is of course a figurative statement – a figure of speech. For a mathematician that uttered it, it was probably a *metonymy* of the "part for the whole" type, or "the place for the object in that place." For teachers discussing it at the PL2 seminar, it was rather like a metaphor showing the *focal point* of the long line of topics for which it was only one of the lines of thought (leitfadens). It shows the long way and the final structure or one of the important structures to be *accommodated*. But to achieve this aim we should know that such "points on the number line" are best identified by their "decimal addresses there" – by dividing the numerator by the denominator. Are we ready to define the fraction as a short program "multiply by the numerator and divide by the nominator"? If so, at what stage? What kind of the language game will be used to achieve that aim?

In place of such short metonymical statements some authors (Semadeni, 2008) like to use literal statements describing all these changes of meaning and the whole road map from the very beginnings. Such an approach has some advantages and some disadvantages for successful communication. Too many small details give plenty of opportunity to disagree. Metonymical expressions are often used as tools for avoiding such small disagreements in discussions. If fractions should be conceived as points on the number line, then what is the role of decimal numbers for structural understanding of the real numbers and their visual representation on the number line? (Krygowska, 1977)

Very good and simple examples of *structural understanding* as opposed to the formal one were given by Mostowski (1996, 2000). The use of a calculator from the very beginning of school education would help to bring decimal numbers to the attention of students on an equal footing as fractions. But it would require a better mathematical preparation of teachers and also a special way of recording and visualizing the results. Some proposals to that effect were discussed at the PL2 seminars.

Another question is more general: "What mathematical facts should we show to students and what could be left for discovery learning?" When analyzing students' written papers and their oral expressions we should pay attention to the figurative nature of the speech which is much more obvious in oral speech than in the written text. Digital photography is very helpful. When at all feasible (because of law restrictions or ethical issues) this method of registering of what is going on in the classroom is very practical. A teacher or his assistant can photograph the essential points from a lesson. The time needed for a later analysis is much shorter than by ordinary real time registration. The recorded material is also focused on special points which need attention.

Teachers' own research differs from the typical pedagogical research based on large samples. And most of the latter is focused on such events that can be expressed numerically. Teachers' own observations are usually unique in character and qualitative not quantitative.

Teachers describe as truthfully as possible what they can see in order to analyze it later. A *didactical phenomenology* of the mathematical structure is not the same as the formal or informal mathematical structure itself (Freudenthal, 1983; Denzin & Lincoln,

1994) and the main source of our knowledge of such phenomena are case studies and action research.

Describing teaching situations with numbers, i.e. quantitatively has a strong disadvantage. A lot of concrete information is lost in such a process. Sometimes it is impossible to register frequencies because the items to be counted are unique and not possible to be covered under one category. Nevertheless, such unique concrete cases noticed in a class of 30 students during a lesson are often very useful for comparison, some generalizations by analogy, as in the above-mentioned proceptual divide and imprinting phenomenon. Teacher-researchers are in a constant interaction in relation to the *emerging phenomenon*. They may change their behavior a little and observe the effect. The subject of research is under a constant influence of the observer. On the other hand, a concrete embodiment of a theoretically described phenomenon might be difficult to notice and count.

During a language game teachers observe: (i) concrete interventions of students i.e. "concrete moves;" (ii) repertoire of words and grammatical forms; (iii) figures of style (metaphors and metonymies); (iv) visual symbolizations; (v) concrete acts of integration of all the precedent items to communicate, to explain, to predict, to generalize; (vi) concrete acts of modeling; and (vii) compatibility of the informal language with formal requirements.

Teachers can see, hear and try to record all these items during a language game but they have no direct access to the mental space of particular students. They can only see some traces of the hypothetical mental objects, their symbols and judge the effectiveness of a particular language game arranged with them.

The general conclusion of trials with PISA-like problems in the PL2 team was only a confirmation of the conclusion in PISA Project itself and of some independent authors (Sierpińska, 2007). Students: (1) have difficulties with mathematical argumentation and proofs; (2) do not feel at ease with generalizing. Do we pay enough attention to proofs and generalizing in our daily routine in the classroom?

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RESEARCHING OUR OWN PRACTICE[1]

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ABSTRACT
This theoretical paper argues the value of teachers undertaking research to deal with the problems of their own professional practice. It sustains the claim that research is a fundamental strategy of knowledge production and can be undertaken by professionals to better understand the problems that they face and find ways to deal with them. It discusses the characteristics of research on practice and associates this concept with related ones such as teacher-researcher, action-research, reflection, and academic research. It indicates the main moments of this kind of research and underlines the fundamental importance of assuming an inquiry attitude. It also reviews the most common critiques regarding investigating our own practice, made by scholars of different fields, and discusses several possible quality criteria of this kind of research. Finally, it discusses the paradigmatic affiliation of investigating our own practice and points the scope of this perspective in mathematics education, in Portugal and elsewhere. Thus, the paper contains an agenda of theoretical work to legitimize this kind of research, besides suggesting the need to reflect on the experiences that are carried out in the educational field.

RESEARCHING TEACHERS' PROFESSIONAL PRACTICE

In order to fulfill their mission, teachers act at several levels: conducting the teaching-learning process, evaluating students, contributing to the construction of the school's educational project and to the development of school-community relationships. At all these levels, teachers are faced with problematic situations. As a whole, problems that arise are willingly and sensibly dealt with based on teachers' professional experience but often this does not lead to satisfactory solutions. Hence, teachers' need to engage in research that helps them to deal with problems arising from their practice.

In fact, teaching is much more than a routine activity where one simply applies pre-determined methodologies. It is simultaneously an intellectual activity, a political activity and the management of people and resources. It requires a constant exploration into its practice and an ongoing evaluation and reformulation. Different forms of work that get students to reach optimal results must be tried out. To do so, it is essential to clearly understand students' ways of thinking and the difficulties they encounter. Successful teaching requires that teachers continuously analyze their relationship with students, colleagues, parents and their working context. An active consistent participation in the school life also requires that teachers have the capacity to discuss their proposals. The natural base for this way of working both in the classroom and in the school is research activity in the sense of inquiry, questioning and grounding.

[1] This paper is a revised version of a previous version in Portuguese.

We may thus state that researching professional practice, alongside participation in curricular development, is a decisive element of teachers' professional identity. This is nothing new. Actually, this idea was elaborated 25 years ago by an English educator Stenhouse (1975). This article pays close attention to researching one's practice but keeping in mind the teachers' role as regards curriculum development.

Alarcão (2001) resorts to the above-mentioned ideas of the author to argue that good teachers must also be researchers, developing investigation that is intimately bound to their role as teachers. She explains this idea as follows:

> In truth I cannot conceive a teacher who does not question him/herself about the reasons underlying his/her educational decisions, who does not question him/herself when some of his/her students are underachievers, who does not turn his/her class plans into mere work hypotheses to be confirmed or refuted in the laboratory that is the classroom, who does not critically read the textbooks or didactic proposals that he/she is given, who does not question him/herself about the school's functions and whether these are carried out. (5)

A reflective, inquiring activity is usually performed by teachers intuitively, not in a formal way that is typical of academic research. Actually, because it has specific purposes, teachers' research about their practice does not have to take on features identical to research carried out in other institutional contexts. But teachers' activity will gain a lot if they cultivate a more careful approach in formulating their research questions and in conducting their intervention projects in schools.

Research is a privileged process of knowledge construction. Subsequently, researching one's practice is a fundamental process of the construction of knowledge about this very practice and is therefore a valuable activity for the professional development of those who engage in it actively. Besides the teachers involved, the educational institutions they belong to can also benefit tremendously from the fact that their members are involved in this type of activity, reformulating their working methods, their institutional culture, their external relations and even their own objectives.

We can point out four major reasons why teachers should research their own practice: (i) to emerge as true protagonists in the curricular and professional field, with more means to face the problems arising from this practice; (ii) as a privileged form of professional and organizational development; (iii) to contribute to the construction of a patrimony of culture and knowledge of teachers as a professional group; and (iv) to contribute to general knowledge about educational problems.[2] In other words, problems pertaining to curriculum construction and management, and problems arising from the different levels of professional practice require that teachers have competencies in terms of problematization and investigation, besides a dose of professional common sense and good will. Besides, in certain conditions the knowledge created by teachers researching their own practice may be useful for other professional and academic communities. We will come back to this later.

THE CONCEPT OF RESEARCHING ONE'S PRACTICE

What characterizes researching one's practice?

Researching practice can have two main types of objectives. On the one hand, it may aim above all to change some aspects of the practice, once the need for change is

[2] This argument is subscribed to by Lytle and Cochran-Smith (1990), two authors for whom research done by teachers "makes accessible [to outsiders] some of the expertise of teachers and provides both university and school communities with unique perspectives on teaching and learning" (83). Zeichner and Nofke (2001) also argue that research carried out by professionals upon their practice, far from being a simple process of professional development, represents an important process of knowledge construction.

determined, and, on the other hand, it may seek to understand the nature of the problems affecting this practice so as to define a strategy of action at a later moment.[3]

Let us start with the following question: What are the minimum requirements for an activity to be considered research? A French author, Beillerot (2001) indicates that research must meet three conditions: (i) it must produce new knowledge; (ii) it must have a rigorous methodology; and (iii) it must be public. These are undeniably important conditions.

It is natural to assume that if a certain work simply reproduces what has already been done, without producing anything new, it might be a useful "exercise," but it is not exactly research.[4] "New" here refers to an actor undertaking research. If I take on a problem similar to another already worked on by other people but whose work I know nothing about and I produce solutions that are original (to me), then I am certainly doing research. If I just consciously follow tracks that have already been beaten by other researchers, I may be doing a worthy job but I am not doing real research.[5] Also, to deserve being called research, the work has to involve some form of rigor, that is, it must assume a minimally methodical systematic nature, thus allowing for its eventual reproduction. Finally, research must be communicated so it is appreciated and evaluated. Only by doing so can it eventually integrate the patrimony of the reference group and perhaps of the community at large.

It seems to me that with appropriate adaptations these three conditions may apply to the research that teachers carry out on their own practice. The presence of some kind of novelty in teachers' research is not too problematic, as situations of professional practice tend to be unique and unrepeatable. However, the utmost attention to the specificity of each situation is indispensable. The rigor that should be used is a more complex problem and it is necessary to find a point of balance between the informal procedures that characterize teachers' professional culture and the formal procedures that are part and parcel of academic research. Finally, the question of making it public is not difficult to overcome. There are many opportunities to partake and discuss teachers' research – in their schools, in professional meetings and journals, and in educational meetings and journals.

Lytle and Cochran-Smith (1990) speak of teachers' research as "systematic, intentional inquiry by teachers about their own school and classroom work" (84).[6] According to these authors, research arises from questions or generates questions and it reflects teachers' concern in giving meaning to their experiences, adopting a learning attitude towards their practice. Underscoring intentionality aims to stress that research requires some planning and is not merely a simple, spontaneous activity. Finally,

[3] A similar distinction between teachers' research into their practice steered towards change or towards understanding is assumed by Richardson (1994) when she talks about the possible objectives of what she calls *practical inquiry*.

[4] In this sense, a simple replication of an investigation whose only aim is to corroborate the results of a previous study is not research *per se*. Actually, investigative work does not imply that everything is new – usually there is an "element" of novelty.

[5] Often it is hard to distinguish between what is new and what is *déjà vu*, even concerning a social protagonist. All new situations involve familiar elements and all social situations that we apparently know well always carry something new. Therefore, it is appropriate that researchers take care to highlight what is new (at least for them) in their investigation.

[6] Lytle and Cochran-Smith particularly stress Beillerot's point (ii) (method and rigor). However, in my opinion, points (i) novelty and (iii) public character indicated by this author are equally essential to really consider something research.

signaling the systematic character has to do with procedures of data collection and documentation of experiences and with the way events are analyzed and interpreted.[7]

Unraveling meanings…

Besides characterizing teachers' research on practice, it is important to confront it with other activities that are more or less alike but not equivalent. Therefore, I shall analyze other terms that are sometimes mistaken for synonyms, while other times they are viewed as having distinct meanings.

One of these terms is the "teacher as researcher" or "teacher-researcher" (Stenhouse). Teacher-researchers are teachers who carry out research normally on their practice but sometimes on other matters too.[8] For instance, the teacher-researcher who Silva (1994) speaks of is a mathematician who, on the one hand, does research on mathematics (frontier problems) and, on the other hand, teaches at university (basic subjects such as algebra or calculus). For this teacher the activities of teaching and of researching are located in clearly defined departments. In this manner, the concepts of researching practice and being a teacher researcher largely overlap but do not coincide totally. Another example is given by most of the basic and secondary level teachers who have concluded master theses in Portugal. As Serrazina and Oliveira (2001) point out, only six theses report research carried out by teachers into problems of their practice. All the others refer to problems outside their practice.

Another very close concept to that of researching practice is "action-research." The creation of this expression is attributed to the social psychologist Kurt Lewin at the time of the Second World War. His idea was to promote the advance at the same time of social theory and social changes. Lewin proposed action-research as a succession of cycles involving a description of the problems present in a given social field, followed by the elaboration of an action plan, by putting that plan into practice and by evaluating it, which might, in turn, give rise to a new improved action plan, thus restarting a new research cycle.[9]

The nature and objectives of action-research are characterized in many different ways by other authors. For example, Zeichner and Nofke (2001) state that aside from the "cyclical" perspective, there is another version where the questioning process has an essentially "linear" form. Carr and Kemmis (1986) define action-research as follows:

> Action research is simply a form of self-reflective enquiry undertaken by participants in social situations in order to improve the rationality and justice of their own practices, their understanding of these practices, and the situations in which the practices are carried out. (162)

Many teachers have been involved in action-research. But action-research is far from being confined to the field of education. As Esteves (1986) indicates, this form of

[7] An Australian author, Mousley (1997) considers it difficult to find an exact definition for what research is, stressing that this concept is in constant evolution. However, in consistency with the perspective of these authors, she argues that investigative activity, even when its object is the teachers' practice, involves planned, systematic work, and also attributes a very meaningful role to the theoretical frame.

[8] Some authors present other concepts of what a teacher-researcher can mean. This is the case of Stoer and Cortesão (1999), for whom teacher-researchers are people who act as ethnographers in their classroom. Without questioning that an ethnographer is an important reference for whoever researches his/her own practice, it seems to me that this characterization is too restrictive, for the teachers who research may also take up other references (such as the psychologist, sociologist, philosopher or researcher in education) and look at other objects of study (such as knowledge, students, the school, school-community relations, and so on).

[9] A description of the typical processes of action-research may be found, for example, in Arends (1997) and Collins and Spiegel (1995).

work is also largely used in areas such as social services, communication, health, organizations, rural development and social movements.

Usually action-research involves a matter of immediate intervention, often radical change, which may or may not exist when we research our practice. Action-research also frequently involves teams whose leaders are not even members of the institution or community where the intervention is to take place.[10] Once again, we may say that action-research and researching practice are two very close, partially overlapping concepts, but they are not entirely coincidental.[11]

We should keep in mind that the concept of action-research has a vast history that includes many varieties and has witnessed countless controversies.[12] To some there is only one way of doing "good" action-research, one that follows certain objectives marked by the pursuit of justice and social change. This is not the choice made in the present paper, which seeks to cast a rather vast problematizing view of research and finds it legitimate for the research to assume its own objectives within a vast scope (considering, nevertheless, justice and equity as fundamental values). Basically we have before us two counteracting views of research: (i) a "normative" view, filled with ideological interests – research as a means to certain predetermined ends of social change; (ii) a questioning problematizing view – research as a process that is born within a practice and is not necessarily subject to external agendas. In an ideologically framed investigation, objectives are clearly defined – the doubt remains as to whether these can be reached under the circumstances that exist. On the contrary, when we begin a process of questioning within a practice, from the start we never know where it will take us. In this case, the investigation is also steered by values, but it is not subject to any values – except those of questioning and reflection.

And so we arrive at another expression, "reflection" which is also very close to the notion of researching practice. As Geraldi, Messias and Guerra (1998) state, Dewey characterized the reflective act as one that is not merely guided by an impulse, tradition or authority. To this author, reflecting implies a careful, active consideration of what one believes in or practices, in light of the reasons that justify it and of the consequences stemming from it.[13]

Once again, these concepts overlap partially. No one can research one's practice and not be a reflective practitioner… But probably being reflective is not enough to do research. In truth, the concept of reflective teacher allows for rather diverse interpretations. For some, all human beings are reflective and subsequently all teachers are necessarily reflective.[14] To others, being reflective implies several conditions that vary according to their proponents' theoretical frameworks.[15] Therefore, the degree of proximity between the concepts of researching one's practice and reflecting about one's

[10] Such is the case, for instance, of the ECO Project, an educational project that marked the 1970s and 1980s in Portugal (see Benavente, Costa, Machado & Neves, 1987).

[11] For some authors, to say that teachers perform research in their classroom is the same as saying they do action-research (for example, Arends, 1999, 525).

[12] Esteves (1986), for example, distinguishes two main variations: research-for-action and research-in/through-action.

[13] The concepts of reflective teacher and reflective practices are discussed in another chapter of this book by Oliveira and Serrazina. Other discussions in the Portuguese language on the reflective teacher may be found in the likes of Alarcão (1996), Serrazina (1999), or Vasconcelos (2000).

[14] For instance, this is the stance adopted by Teresa Estrela at the Seminar on Conceptions and Models in Teachers' Pre-Service Education that took place at University of Lisbon in October 2001.

[15] For example, to Olga Pombo (1993), the reference model of reflection is philosophical reflection. In her perspective, reflective teachers are those who interrogate their practice in the philosophers' way.

practice depends most of all on the meaning attributed to "researching" and "reflecting."[16]

Finally, we must distinguish researching one's practice from the common "academic research."[17] As I have mentioned before, they are two types of research that correspond to distinct finalities and must be thought of in different ways. Academic research aims to increase academic knowledge in the fields and subjects established in the respective community – the academic community. Researching one's practice aims to solve professional problems and increase the knowledge regarding these problems, turned not towards the academic community, but to the professional community. These concepts also partially overlap because, on the one hand, the members of the academic community are teachers too and may want to research their own practice and, on the other hand, teachers may want to research their practice as a way of being accepted by the academic community.[18]

Richardson (1994) stresses that researching one' practice "is not conducted for purposes of developing general laws related to educational practice, and is not meant to provide *the* answer to a problem. Instead, the results are suggestive of new ways of looking at the context and problem; and/or possibilities of change in practice" (7). However, teachers' research on their practice, besides providing this type of result and being a requirement for a good-quality professional practice, as I argued at the beginning of this article, implies a series of other potentialities that should not be forgotten. Actually, this research may strongly contribute to the professional development of the teachers implied and to the organizational development of their institutions, and produce important knowledge about educational processes that will be useful to other teachers, to academic educators and to the community at large. It is an undeniable fact that teachers are in a privileged situation to provide a view from within about the school's realities and problems.

Critiques regarding researching one's practice

There have been substantial critiques regarding teachers or other professionals researching their practice. Cochran-Smith and Lytle (1999b) systematize these critiques into three major groups, referring to (i) the knowledge that is generated; (ii) the methods; and (iii) the ends of this kind of research.[19]

The critique concerning knowledge generated by researching one's practice has an epistemological nature in that it questions the reason why knowledge generated by

[16] In the present text, I think it is useful to establish these distinctions between the concepts of (i) researching one's practice, (ii) teacher-researcher (in Stenhouse's sense), (iii) reflective practitioner and (iv) participant in action-research projects. Many authors do not establish these distinctions and consider these terms to be synonyms. Such is the case of Alarcão (2001), for whom these terms are all alike. It is also the case of Richardson (1994), to whom teachers as reflective practitioners and as participants in action-research are some of the varieties of what he calls "practical inquiry." A more in-depth discussion about reflective teachers may be found, in this volume, in Oliveira and Serrazina (2002).

[17] Sometimes this distinction is not made by certain authors who seem to view research by teachers as a "minor" variety of academic research (see Esteves, 1999, 150-152). However, this distinction is greatly adopted by many other authors, such as Richardson (1994).

[18] For example, with a view to obtaining degrees such as a master or PhD. Some authors (like Alarcão, 2001) seem to view researching one's practice and the research carried out with a view to obtain academic degrees as belonging to different worlds. Despite the difficulties that researching practice may encounter in academic contexts (extensively discussed, for example, by Breen, 1997), I do not think these two activities have to be viewed as disconnected. On the contrary, carrying out research on one's practice as the basis for obtaining academic degrees may, in my opinion, contribute seriously to the assertion of this kind of research.

[19] More critique as well as some possible answers can be found in Zeichner and Nofke (2001).

teachers might be considered valid knowledge. As the above-mentioned authors underline, this critique is based on the premise that there are two forms of knowledge about teaching: a formal, theoretical or scientific form, and a practical, craft, situated, tacit or popular form. Response to this critique must be based on an epistemological discussion about the nature of knowledge. The distinction between the so-called scientific knowledge and non-scientific knowledge has been questioned by several authors who balance between pointing out the limits of scientific knowledge, or of the so-called technical rationality (Schön, 1983), and suggesting that post-modern society needs a new type of relationship between scientific knowledge and common sense (Santos, 1987). This issue is not closed, but an increasing number of authors feel that different forms of knowledge can be legitimate in certain reference communities and according to certain finalities and that the idea that there is one form of knowledge universally superior to all others should be abandoned.

The critique regarding methods, besides questioning the lack of clarity and methodological rigor of much research on practice, also questions the proximity between the researcher and the object of research, wondering how research produced by those who are directly implied in the events at stake can be minimally reliable and prejudice-free.[20] This appreciation may be reversed through the establishment, by the respective reference communities, of appropriate standards of quality for this type of research. It is especially important to analyze the conditions that allow researchers to distance themselves from the object under study when it is very close, thus enabling a rational analysis.

The third critique refers to the ends of researching practice, questioning those studies whose objectives have an essentially "instrumental" nature and lack connection to larger social and political agendas. As Cochran-Smith and Lytle (1999b) state, this appreciation is based on the premise that, although this research has the power to fundamentally change the nature of practice and the role of teachers, this power is severely reduced if it is not politically branded or if it is used to solidify educational practices that are harmful to students. In response to this critique, I have already stated that research (both researching practice as we speak of in this context and research in general) can assume different objectives, according to the concerns and interests of its agents. This research can and should be steered by ethical, social and political values, acknowledged in its professional field but it should not be at the disposal of this or that external movement. On the contrary, researching practice must emerge as a genuine process of the actors involved, seeking to develop their knowledge, looking for a solution to the problems they encounter and thus asserting their professional identity.

THE RESEARCH ATTITUDE

Teachers who research can begin with problems related to their students and learning, but also to their classes, the school and the curriculum. This immediately raises the question: If there are multiple possibilities as to where research can start, is there anything permanent about research? Actually, research practice is based on two main conditions: on the one hand, there must be an inclination to inquire, which implies the fields of affect and of attitudes; on the other, one must master a certain *savoir faire*, including the use of different methodological tools.

The vital importance of having an inquiring and reflective attitude in order to research is well underlined by Alarcão. In this respect she recalls Dewey's words: "We

[20] A problem considered, for example, by Caetano (1997).

must be ready to maintain and prolong the state of doubt, which is a stimulus for perfect research, where no idea is accepted, no belief asserted, without finding the justifications for them" (Alarcão, 2001, 7). Stenhouse also stressed the importance of this research attitude, which he characterized as "a predisposition to analyze one's own practice in a critical, systematic manner" (Alarcão, 2001, 3).

Therefore, research is not something that can be done routinely, with no passion or genuine intellectual and affective investment. That is, research cannot be done as a passive employee – it requires a spirit of a social protagonist. Being part of a project without adopting, from the start, a stance of commitment and effort, means representing a secondary role in that project without ever living the real experience of research.[21]

Cochran-Smith and Lytle (1999a) point out a similar idea when they refer to "inquiry as stance," which to them involves generating local knowledge, theorizing practice, interpreting and interrogating the theory and research of others. "Inquiry as stance" implies an ongoing attitude of questioning, while simple *inquiry* is carrying out a project bound in time.

To these authors, working in inquiry communities is social and political. They disagree with the distinction between formal knowledge and practical knowledge in that both may be deeply integrated in teachers' work. They also disagree with the notions of expert and novice. They advocate that (i) both experts and novices have to engage in similar intellectual work; (ii) the expert/novice distinction simply benefits the maintenance of the individual teacher model; and (iii) learning over the life span is essentially based on the relational dimension, thus highlighting the role of the communities and intellectual projects of teachers over time.

Cochran-Smith and Lytle underscore the idea of inquiry as agency. To them the culture of inquiry communities has four major dimensions: (i) time – teachers need enough time in which to work together; (ii) the nature of the discourse – this involved what they call a "rich descriptive talk" and "writing help;" (iii) the dynamics of interpersonal relations – which is quite complex; and (iv) leadership – here closely linked to activist features. Their fundamental idea is that teacher learning should not be considered primarily as an individual professional achievement but as a long-term collective project with a democratic agenda.

THE PRACTICE OF RESEARCHING PRACTICE

Moments of the research

Any research involves four major moments: (i) formulating the problem or questions of the study; (ii) gathering elements to respond to that problem; (iii) interpreting the information gathered so as to reach conclusions; and (iv) disseminating the results and drawn conclusions.[22] Very briefly we will look into some of the issues arising at each of these points.

The formulation of good questions to research is of great importance in investigative work. Questions should refer to problems that concern teachers and be clear and answerable with the available resources. Actually, if the questions do not really

[21] That is why engaging from the start in the elaboration of questions that are to be investigated and in the definition of all the stages of a project is a fundamental requisite in the research process (see Jaworski, 1997, for more on this point).

[22] Arends (1999) speaks of the first three moments discussed here. If we assume that the public character is an essential feature of research, we must add the fourth moment. These moments do not always develop in a strictly sequential manner. Sometimes they may overlap or include complex back-and-forth movements.

matter to teachers they should not be expected to make the necessary affective investment to conduct the research properly.[23]

Teachers are really interested in solving a problem that concerns them or in understanding a situation that intrigues them, not just in doing research for the fun of it. It is preferable that they channel their energies towards issues regarding which they can have tangible results than towards those that are far beyond their scope of action. Issues evolve with the development of the work itself but it is important that this variation is oriented towards greater precision and demarcation. If issues vary erratically, it is likely that in the end there is no minimally plausible answer to them.[24]

As plain and simple as this all might seem, it is precisely in the formulation of questions that many investigations get lost. In certain cases, from the start they are too ambitious and it becomes impossible to answer them in the time foreseen and with the available resources. In other cases, questions lack good formulation at the start and change so radically as work advances that it is impossible to give them a convincing answer. Therefore, learning how to formulate sound questions is a fundamental requisite for researching.

Gathering elements to answer the study questions implies making a research plan which states the working methodology in practical terms. Generally speaking, researching practice resorts to the most commonly used work plans and techniques in human and social sciences and, in particular, in studies in education. However, researching practice has certain prominent features. One of them, its defining trait, is its strong tie to problems pertaining to professional practice. Other that we often find is a collaborative dimension in that several actors intervene and organize themselves as a working team.[25]

The nature of the questions that are formulated determines the nature of the study object and of the data that have to be gathered. Therefore, a study essentially aimed at understanding should have quite a different methodology from a study aimed at introducing immediate changes in one's professional practice. In either case, the study object may be a well defined entity, such as a student, a class, a school, a curriculum, a project, and so on. It can also be a unique property or feature of a vaster object, such as the reasons for the difficulties a group of students has in mathematics, a way to introduce new software in the classroom, the way results in mathematics influence students' school trajectory, and so on. The data that are to be collected may be quantitative (numerical data concerning measurable or at least countable variables) or qualitative (non-numerical data) in nature, depending on the problem of the study.[26]

The most common techniques for gathering data of a quantitative nature are tests and questionnaires, although observation and the analysis of already existing

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[23] The idea that having good questions is a fundamental condition to research is also advocated by authors like Alarcão (2001) and Lytle and Cochran-Smith (1990).

[24] For instance, Tinto, Shelly and Zarach (1994) report a study where two participating teachers began by formulating some questions in a vague manner – what are the reasons for their students' lack of engagement in mathematics classes? These questions made them try a number of changes in their practice – group work, problem-solving through the use of technology, students' writing – which allowed them to formulate much clearer questions to research, regarding these new working methods in the classroom.

[25] The collaborative dimension is extensively discussed in another chapter of this book by Boavida and Ponte (2002).

[26] Obviously this article does not withstand a detailed discussion about research methodologies and techniques. For more information on this matter, there are books on research methodologies in education, such as Altrichter, Posch and Somekh (1993), Bogdan and Biklen (1994), Lessard-Hébert (1996), Lessard-Hébert, Goyette and Boutin (1994) and Ludke and André (1986).

documents (such as students' school processes) are also used.[27] Analyzing quantitative data is usually with statistical techniques, both descriptive and inferential.

On the other hand, the most common techniques for gathering data of a qualitative nature are observation, the interview and documental analysis.[28] Recently the use of personal journals, where researchers register the relevant events that arise in the process of the work and the ideas and concerns that crop up, has also become frequent. To analyze these data a variety of techniques are used, including content analysis and discourse analysis.[29]

In either case, whether regarding quantitative or qualitative data, the most important thing is not to gather a lot of data but to gather data that are suitable for the goals at stake and are trustworthy. To do so the development of a global work plan is essential, to foresee what is going to be done, when and how. It is also important that data are always gathered in the same way, with clear, well-defined procedures, so as to facilitate their subsequent interpretation.

Throughout the whole development of the work, it is essential that researchers (or research teams) assume control over the process. To do so it is necessary to keep in mind the aimed objectives, the finalities of the programmed activities, the roles that have been defined and the calendar. It is not about stiffly following everything that has been programmed but about working around all the adaptations that seem to be necessary, with flexibility but also with a critical eye. The work plan and the records (for example, in the personal journal) will provide researchers with an autonomous space of reality where they can create a distance regarding the events of the day whenever necessary.

Finally, it is important to refer to the fact that the interpretation of the information gathered in order to draw conclusions, and the way results are disseminated, largely depend on the particular nature of each study. Disseminating results and conclusions takes on many forms, from informal conversations with actors close to the researchers (or research teams) to formal presentations at scientific meetings and publications in scientific journals.[30] Dialogue with other actors is essential to keep in perspective what has value and what does not, what is important and what is not, so it is a decisive element for the quality of the research. Sometimes it is good to create the role of a "critical friend" from the start, a kind of project consultant who asks questions (that can be uncomfortable), thus helping researchers to reflect on the strengths and weaknesses of the work under way.

These two activities – interpretation of information and dissemination of results – far from being disassociated cross one another often in unexpected ways. Actually, many times from the start we have an idea of the meetings or journals where we would like to publish the conclusions of the study. It is also common for the work still to be under development and be the object of dissemination, in terms of its objectives and activities but also in terms of its partial results. In these cases, public dissemination starts long before entering the final phase of the project.

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[27] There are books that pay great attention to these techniques, some are even devoted to a single technique, such as Ghiglione and Matalon (1992) or Mucchieli (1979).

[28] Each of these techniques, in turn, has its auxiliary tools. Observation may be supported by grids, the interview by a script and documental analysis by a set of steering categories. For observation techniques, see, for example, Estrela (1986). For interview techniques, see, for example, McCracken (1988), Nunes (1983), Powney and Watts (1987) or Spradley (1979).

[29] For content analysis, see, for example, Bardin (1979). For discourse analysis, see, for example, Gee, Michaels and O'Connor (1992).

[30] Several problems related to reporting and disseminating teacher research are discussed, for instance, by Smith (1996).

At other times, it is when we produce texts with reports of the experiences and papers for presentation at meetings that we further the analysis of one aspect or another. Also, during the presentation of results questions and reflections may arise that take us in an unexpected direction, opening the way for new inquiries and new projects. All this shows how different moments of a research can intertwine deeply.[31]

Quality criteria

The value of teachers (or other professionals) researching their practice depends on meeting certain quality criteria, as consensual as possible for the respective reference community. To this end many criteria have been proposed but we are still far from a consensus, which is understandable given that this is a new field of work that is being developed. What is not appropriate is to judge research carried out by teachers on their practice by the standards of academic research. As these are different activities with clearly different finalities, the criteria for rating their quality are necessarily also different. Let us see what several authors have to say about this.

Anderson and Herr (1999) suggest five quality criteria for teachers' research on their practice. These criteria concern: (i) validity of the results; (ii) validity of the processes; (iii) democratic validity; (iv) catalytic validity; and (v) dialogical validity. To these authors, the validity of the results regards to what extent the actions undertaken lead to the solution of the problem. The validity of the processes is related to the way problems are handled and solved, allowing for the ongoing learning of the people involved and of the organization itself. Democratic validity refers to the way research is conducted with the collaboration of all the parties with interests in the problem under investigation. They speak of catalytic validity when the activity that is carried out promotes reorienting and energizing participants so they understand the reality better in order to transform it. Finally, dialogical validity has to do with the way the research was subjected to a process of scrutiny and analysis by peers.

Another author who looked into this matter is Zeichner (1998) who indicates two main criteria for the quality of researching practice: (i) clarity; and (ii) expressing one's own point of view. Clarity concerns a sound problematization and the use of evidence to base conclusions on. Expressing one's own point of view is associated with the presence of the authors' personal marks and their articulation with the respective social, economic, political and cultural context. In another work Zeichner adds two more criteria: (iii) dialogical quality and (iv) the tie to practice (see Geraldi, Messias and Guerra, 1998). The criterion of dialogical quality raises the question of knowing whether the research promoted debate and reflection among teachers. As for the criterion being tied to practice, above all it defines this type of research.[32]

The conditions we have just systematized as defining this kind of research (see beginning of point 2) provide us with a base for reflecting about its quality criteria. With these conditions as the baseline, it is natural to assume that researching one's practice should: (i) refer to a practical problem or situation lived by the actors; (ii) contain some new element; (iii) have a certain "methodological quality;" and (iv) be public. These conditions are very close to those of Zeichner. Methodological quality may be associated

[31] Practical suggestions based on numerous experiences, regarding teachers conducting research, can be found, for instance in Collins and Spiegel (1995). On the other hand, we find a table of the competencies needed by the teacher who does research in Alarcão (2001), for instance.

[32] After having contributed greatly to the definition of possible quality criteria regarding researching one's professional practice, Zeichner, in a text elaborated with Nofke (Zeichner & Nofke, 2001) prefers not to propose any criteria whatsoever, suggesting that the teachers themselves take on this task.

with the explicit presence of questions and data collection procedures and of ways of presenting conclusions based on the data. This is not very different from Zeichner's "clarity." The dialogical character of research depends on its public nature, one being a natural extension of the other. Actually, as it is related to the way the research was accepted and discussed by the elements of the reference community; dialogical quality is one of the strongest features that grant a project its credibility. The tie to practice seems to be consensual. Also, the condition of expressing one's own point of view that Zeichner points out in a way extends the idea of a tie to practice, and I think it should be maintained as a criterion of authenticity.[33] In this manner, we would have the set of criteria shown in Figure 1.

These conditions seek to adjust to what may be expected of quality research carried out by teachers on their own practice. These or other conditions that may prove more appropriate may constitute an essential reference for that which the teacher community feels to be worthy of attention. Therefore, teacher research may have interest to a larger professional community than that of the actors who lived the process directly.

Criterion	Research…
Tie to practice	… refers to a practical problem or situation lived by the actors.
Authenticity	… expresses the actors' own point of view and its articulation with the social, economic, political and cultural context.
Novelty	… contains some new element, in the formulation of questions, in the methodology used or in the interpretation of the results.
Methodological quality	… explicitly contains questions or data collection procedures and presents conclusions based on the evidence obtained.
Dialogical quality	… is public and has been discussed by actors close to the team and others distant from it.

Figure 1. Quality criteria of researching one's practice

Also, these conditions continue to have some relation to the quality criteria demanded of certain types of educational research. This is not surprising, for within the context of the investigative process there is a common, original trademark. Researching one's practice might be less sophisticated, methodologically speaking, but on the other hand it tends to imply a strong tie to practice, authenticity, novelty and dialogicity.

We should keep in mind that the classical criteria for research in the human and social sciences (validity and reliability) are a heritage of positivism, concerned mainly with the possibility of securing the "certainty" of conclusions. Currently, the notion of certainty has become much more relative. It is understood as being unattainable (even in the exact and natural sciences) and that other values must equally be taken into account. Often the importance of a research lies not in its conclusions but in the questions it poses or in the view it provides of a given reality.

In other words, research is not just about gathering certainties, it is about pursuing various ends – understanding a situation or solving a specific problem, related to our practice or not. Quality criteria in research should be aligned with this diversity of ends and not just focused on the issue of validity and certainty.

[33] There is still the element of novelty that Zeichner seems not to consider relevant but perhaps it is best to maintain so as not to trivialize the idea of research.

When researching practice clearly meets the criteria mentioned in the table above, naturally it grasps the interest of the academic community. In these circumstances, the value attributed to that research takes it beyond the scope of a local research, steered towards the resolution of specific problems, to become something of added value to the whole educational community.

CONCLUSION

Researching one's practice has emerged as a possible fourth great paradigm in research in education, beside the three great "classical" paradigms – positivist, interpretative and critical (Anderson & Herr, 1999; Zeichner & Nofke, 2001). Much is still to be done until this type of research truly surfaces, such as furthering its epistemological grounds, improving its quality criteria and, above all, using good examples to illustrate its worth and its potential as a formative tool, as a tool for educational change and as a form of constructing important knowledge about education.

To this day, the notion of researching practice and the relatively close notion of action-research have been poorly explored in the field of mathematics education, both in Portugal and abroad. However, some studies have been done. For instance, the book by Zack, Mousley and Breen (1997) contains a description of various experiences of researching practice.[34] This book discusses many of the problems teachers encounter when researching their practice. In the introduction Mousley (1997) indicates that this research is a demanding activity that involves a different level of thinking from the plain learning process based on experience. He also stresses that contrary to the simple sharing of experiences, researching practice is threatening to the *status quo* in that it puts into question the official culture of the school and challenges traditional hierarchies and roles.

In *International Handbook of Mathematics Education* Crawford and Adler (1996) discuss the action-research perspective regarding mathematics teachers. The authors make a distinction between what they call "positivist conception of research" and action-research, characterizing the latter as the research that is carried out with the intention of changing professional practice or social institutions through the active, transformative participation of its actors. I find this distinction to be rather limiting because it leaves aside all academic and non-academic research that is conducted according to an interpretative or critical perspective.

The authors suggest that action-research is somewhere along a *continuum* between reflection and ("positivist") research. They declare that doing action-research requires abandoning the traditional cultural rules regarding authority and qualifications to exercise research activity. In this respect, their opinion is in keeping with the proposal presented in this paper concerning the definition of specific working procedures and quality criteria for research that teachers carry out on their practice.

The potential of research for mathematics teachers has also been discussed by D'Ambrosio (1996) and Nelson (1997). Both authors point out two recent changes in the literature in this respect: (i) the influence of the reform movement in mathematics education; and (ii) the application of the same ideas about learning to students and teachers. D'Ambrosio argues that in order to accomplish the new practices recommended by the reform movements, teachers must adopt an attitude of constant

[34] The work gathered in this book derives mainly from the Anglo-Saxon world (Australia, South Africa, Canada, United States, United Kingdom), but it also includes experiences from Latin countries (Brazil, Colombia, Portugal).

vigilance as regards their students' forms of thinking. On the other hand, Nelson underlines the value of an investigative perspective for the development of the teachers' professional identity and also states that an investigative attitude towards students' mathematical thinking may be of great importance in the exchange of experiences between teachers. In her perspective, as teachers do research, these issues move to a self-sustained level of change regarding their beliefs about teaching and learning and about their practice. This author seems to feel that teachers have a lot to gain, in terms of their training, if they draw on the working methods of the academic researchers and if they study the same objects as these do. This perspective is clearly dominated by the academic tradition of research, which is presented as a model for teachers to follow.

Perspectives valuing the role of research in teachers' pre-service education can be found in several other studies. For example, Lampert and Ball (1998) recommend approaching pre-service education by basing it on research to be performed on a *corpus* of digitalized data. Similarly, Comiti and Ball (1996) indicate that in the currently ruling pre-service education in France, the end of the course includes a thesis with a heavy investigative component, although its accomplishment turns out to be rather problematic.

In Portugal, as regards mathematics education, there is relatively little research that fits inside this paradigm and much reflection to be carried out with respect to its potential and limitations (in this sense, see Serrazina and Oliveira, 2001). However, the growing importance attributed to students' and teachers' research, as a form of knowledge construction, has helped shift this topic to the limelight. The idea has been launched. The studies gathered in this book witness much of its potential. Future practice will demonstrate its real scope.

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GOOD PRACTICES OF ACTION RESEARCH IN MATHEMATICS TEACHER TRAINING

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ABSTRACT
This contribution is aimed at the professional development of teachers as action researchers in their own teaching situation. Following a theoretical study, the teacher-as-researcher developed skills further by gaining practical research experience. Such an experience concerned knowledge and skills that are required to positively influence students' developmental process of abstraction. The teacher-as-researcher applied a social scientific research approach and discovered that students' level of thinking at the final examination level cannot be determined on the basis of answers or explanations in a written exam. The research instrument was therefore adjusted, making use of think-out-loud protocols. This intervention resulted in a positive effect.

PROBLEM DEFINITION AND RESEARCH QUESTION

The general problem definition is whether teacher-as-researchers can contribute to the development of the professional skills of mathematics teachers. In this research, the professional skills of the teacher concern: activation and structuring of the development of mathematical concepts (abstraction). The *research question* is: "Are mathematics teachers capable of stimulating the developmental process of abstraction among students by using paradigmatic examples?"

THEORETICAL FRAMEWORK

The transfer of knowledge from experts to novices is a generally recognized problem in education (Brown & McIntyre, 1993). The context, the subject and the practical framework of the subject determine the complexity of this problem, taking into account the choices constantly made by teachers and the decisions made on the basis of experience (Peterson & Clark, 1978; Schoenfeld, 1998; and Shulman, 1987). The pilot study by Ainley and Luntley (2007) which focuses on the practical situation of experts deals with the generalization of this knowledge. Their research results indicate knowledge and skills aimed at improvement of mathematical education on the basis of context-specific practical examples. There is however a certain tension, due to the generalization being based on a pilot study. According to Ainley and Luntley, the results of their research offer new perspectives on analysis of practical situations. Broekkamp and Wolters (2007) place this tension in a broader perspective. They distinguish between four problems; educational research:

(1) results in a limited number of definite results. There are no findings from experiments, the results have little significance and there is great dependence between (pre-)conditions and differentiations (Labaree, 1998);

(2) gives hardly any practically useful results. The relationships with other educational research are often not relevant to the context in which the results apply (Hammersley, 2002);
(3) is often felt to be non-conclusive and impractical. Teachers refer to it as non-applicable in education, irrelevant and unreliable – educational researchers apparently have no idea of the reality of teaching (Gore & Gitlin, 2004);
(4) is often felt to be hardly practically useful. Teachers have negative expectations of educational research; they therefore hardly make any use of it in teaching practice (Burkhardt & Schoenfeld, 2003).

The conclusion is that teachers are rarely interested in participating in educational research so its progress stagnates. A consequence of this negative image of the relationship between educational research and educational practice is the quest for successful links. Pieters and De Vries (2007) advocate the active involvement of teachers and teacher teams in educational research. Scientific studies have shown that the diversity of roles in educational research and teaching practice can fulfill a bridging function (Levin, 2004; Newman & Cole, 2004). That direct link between social scientific research and educational practice opens the door to scientific research by teachers in their own teaching practice with the help of action research.

Action research provides teachers with a possibility to professionalize themselves as teacher-as-researchers by making use of social scientific research approaches (Ponte, Beijaard & Wubbels, 2004). In concrete terms, teacher-as-researchers' process and analysis of data they themselves collected are applicable to social scientific research in the following ways: by involving existing data in research, focusing on problems from their direct environment, working in keeping with the school policy, linking the research to evaluations, analyzing the existing practical situation, and integrating theory from literature in research (Baume & Beaty, 2006). The idea of educational research in teaching situation and consequently improving teaching practice dates back to the early 20th century (Ponte, 2002).

Riedel (1977) defined professional teacher expertise as knowledge required and applied by teachers to function adequately in the classroom environment. This knowledge was based on experiences in their own practical teaching situation and was very diverse. Riedel defined three domains with regard to the professional knowledge of teachers: the ideological, the empirical and the technological domains.

Teachers' professional development needed to be specified within these domains, and each domain therefore had two distinct levels: the actual *degree* of teachers' knowledge development in concrete terms on level one and teachers' *awareness* of their knowledge development on level two.

In this case study, teachers' ideological knowledge domain is based on mathematics-didactic principles. On level one, the emphasis is placed on teachers' own ideas for activating students' abstract thinking patterns. Level two targets the development of theories relating to structuring. In the empirical knowledge domain, level one concerns the application of paradigmatic examples to stimulate students' developmental processes of abstraction. Level two focuses on the development of theories regarding their effects on learning mathematics. The technological knowledge domain accentuates on level one adaptations to the paragraph on rational functions and the educational teaching discussion as a way of realizing the stated objectives. Level two refers to the development of theories regarding the added value of the adaptations and the educational teaching discussion associated with it. The teacher's professional development is described by these three knowledge domains (Table 1).

	Level	Teachers' professional mathematics development
Ideological knowledge	1 2	Ideas for activating students' abstract thinking patterns Developmental theories relating to structuring
Empirical knowledge	1 2	Paradigmatic examples to stimulate students' abstraction Theories regarding the effects of this on learning mathematics
Technological knowledge	1 2	Rational functions and educational teaching discussion Theories of the adaptations and educational teaching discussion

Table 1. Three knowledge domains

RESEARCH METHOD

Research into professional development of teachers to stimulate abstract thinking can be typified as design research (Van den Akker, Gravemeijer, Mc Kenney & Nieveen, 2006). Following a preliminary study, the design cycle of the professional development of the teacher-as-researcher comprised two implementation cycles:

Experiments in the first quarter of the 2006/2007 academic year, followed by adjustments in the second quarter of the 2006/2007 academic year, and follow-up experiments in the third quarter of the 2006/2007 academic year, followed by a publication and justification in the final quarter of the 2006/2007 academic year.

This contribution concerns both implementation cycles comprising four phases in professional development of teacher-as-researchers: the awareness, the design, the implementation and the evaluation phase. The evaluation phase covered all knowledge domains of teacher-as-researchers and the setup of research instruments. Continuing development in the following cycle led to an acceptable, applicable balance with a view to professional development via action research.

Material

On the one hand, the material comprised theory results (ideological knowledge domain) in the guidelines for teacher action (empirical knowledge domain). On the other hand, it concerned the adapted (developed) teaching material for students (technological knowledge domain).

Teacher-as-researchers aimed at activating the developmental processes of abstraction (level one of the ideological knowledge domain). Abstraction, "generalization and compression," is essential for mathematics and for its teaching (Ehrenfest-Afanassjewa, 1960). The process of abstraction evolves in two ways: (1) there is a gradual developmental process based on a wealth of contexts; and (2) there are also situations in which the abstractions are chosen as a starting point, after which the abstraction process begins on the basis of paradigmatic examples. Freudenthal (1978) referred to this distinction using the concepts of *comprehension* and *apprehension.* Comprehension of a number of contexts having mutual characteristics enables the gradual viewing of these characteristics themselves as objects under consideration. However, it sometimes suffices to apprehend the characteristics and properties of that example as such. Such a limitation to a single example then creates exactly the space required by students to be able to direct their attention to the characteristics. These characteristics typify the classification into Van Hiele levels of argumentation (1973). Van Hiele distinguished four levels of argumentation: the zero level, the first, second and third levels. The zero level concentrates on sensory perception and is referred to as the intuitive or visual level. Level one is aimed at describing observations and recognition of characteristics. The second and third levels are deductive, being informally deductive and theoretically deductive. Level two focuses not on the characteristics themselves but on the relationships between those characteristics (the references). The highest level, level three, is typified by the focus on the nature of the relationships between characteristics. In Van Hiele's theory, levels are raised (abstraction) on the basis of reflection, each level based on the internal structure of the previous level.

This classification by Van Hiele and the strength of the use of paradigmatic examples is the principle behind the ideological knowledge domain of teacher-as-researchers. The first level of the empirical knowledge domain comprised: (1) familiarization with the theory; (2) adaptation of the teaching material; (3) implementation of the adapted teaching material in the experiment group and the non-implementation in the monitoring group; and (4) reflection on students' results in both groups. The material (the first level of the technological knowledge domain) comprised an Algebra chapter aimed at refreshing the necessary algebraic knowledge and skills required for handling rational functions. The paragraph concentrated on the simplification, division and combination of fractions. It also dealt with the characteristics of rational functions (zero points, horizontal and vertical asymptotes and perforations). In the adapted teaching material, these characteristics were emphasized in the introductory assignments using paradigmatic examples. The algebra was then practiced in the subsequent assignments, in order to reduce more complex functions into paradigmatic examples: 1) $f_1(x) = \frac{x}{1} = x$ (view to zero points); $f_2(x) = \frac{1}{x}$ (view to asymptotes); and $f_3(x) = \frac{x}{x} = 1 \quad (x \neq 0)$ (view to perforations).

The difficulties with the three functions were gradually increasing. Although all three were intended to emphasize varying characteristics, they could not be regarded separately from one another: each previous function was then necessary in order to understand the next one. The third function could also be seen as being the product of the first two. This form of approach was aimed at the educational teaching discussion.

Participants

Implementation of adapted and original teaching material took place in the pre-university class in two groups. For scheduling reasons, the groups were randomized between two clusters. The experiment comprised 25 students and the monitoring group 16 students. Both groups were taught by the teacher-as-researcher from the beginning of the 2006/2007 academic year.

Data collection and research instruments

Data collection took place in phases, in the three knowledge domains. The ideological knowledge domain was focused on activating students' developmental processes of abstraction (level one of the knowledge domain). In the awareness phase, the teacher's ideals were noted in a logbook beforehand.

The teacher's actions were described in a logbook for the empirical knowledge domain (level one) and reported by an external expert on the basis of an observation list (method triangulation). The teacher's reflection was presented in a semi-structured interview (appendix I). The design phase resulted in the technological knowledge domain (level one) in a series of three lessons of 50 minutes each. The implementation phase comprised two pre-experiment and three experiment lessons. The two pre-experiment lessons were used in order to refresh students' knowledge of algebraic skills (solving quadratic, rational and square root equations).

In order to determine the level of knowledge increase, the two existing groups were subjected to 1 pre-measurement and 2 post-measurements. The pre-measurement was necessary in order to determine the starting level. Directly after the experiment, the first post-measurement was held in order to measure the short-term effect; the second post-measurement was held two months later in order to measure the long-term effect.

Pre-measurement occurred during the 2nd pre-experiment lesson. The students of both groups were given a test (without warning), featuring a final examination assignment from the year before. The students were explicitly asked to give a clear explanation of their answer.

The *short-term post-measurement* occurred on the day after the series of lessons. The students did not know this measurement was to take place and had therefore not prepared for the test. They were given 2 assignments, the first of which was comparable with the assignment of the pre-measurement. In the second assignment, the focus was on more algebraic skills.

In order to also determine the required level of improvement in the longer term, a *long-term post-measurement* was held during a regular progress test. This was held six weeks after the experiment lessons, whereby both groups were once again present in the same room at the same time.

The explanations given by the students in the previous measurements gave insufficient information on the actual level. The research instrument was therefore adapted. This new measurement made use of an answer matrix, which implicitly requested a specification of each Van Hiele level for the explanation.

Data Processing and Analysis

Processing of the observation lists, the logbook and the semi-structured interview occurred on the basis of a predetermined criterion list based on the theoretical framework. The explanations given by the students were assessed in terms of quality. A correct final answer in itself was valued 0; explanations resulted in values of 1-2-3, corresponding with increasing Van Hiele levels. The '0' score was given for an incorrect explanation. The differences between the two groups were mapped out following the various measurements and their significance calculated, using a T test for the average, based on 2 groups with an unknown standard deviation. A significance level of 10% was chosen because of the limited group sizes. There were a number of peaks and lows during the measurements (due to systematic poor efforts by certain students or the lack of a graphical calculator during a measurement, for example). These peaks and lows were ignored in all cases since they were not representative of the group and had a negative effect on the group scores. The student work was used as justification of the assessed results of the observation lists (method triangulation). Subsequently, a member check was staged with the teacher-as-researcher (Van den Akker et al., 2006).

RESULTS

The results in the professional development of the teacher-as-researcher can be shown as follows per knowledge domain, at level one.

In the ideological knowledge domain, the teacher-as-researcher became aware of the level of students' development of abstraction. Evaluation showed that the use of the paradigmatic examples he himself used in the lessons partly had the expected effect. As far as pre-measurement was concerned, there was no significant difference in the 4th year pre-university level of the two groups in neither their answers nor their explanations. The expectation was that the results of the 1st post-measurement (directly after the research lessons) in the experiment group would be better than those in the monitoring group. This was not the case. With the exception of the explanation of assignment 1, the results did not differ significantly. However, think-out-loud protocols of four students afterwards affirmed a raising concept development from the first to the second Van Hiele's level of argumentation by reflective abstraction. There was also no evidence of the expectation that the experiment group would perform better in the 2nd post-measurement. Although the experiment groups gave slightly better answers to the assignments than the monitoring group, the latter group gave slightly better explanations. However, neither of these deviations was significant (appendix II).

In the empirical knowledge domain, the teacher-as-researcher learned to take creative action, due to having increased his knowledge on the use of paradigmatic examples at the deductive level and his goal to improve the level of students' abstract thinking. The intuitive and descriptive levels were detected in the experiment group, although less frequently than the deductive level. In teaching practice, the teacher-as-researcher learned: (1) how to construct pointed questions; (2) to sooner refer back to the use of a Van Hiele level of argumentation; and (3) to support reflection and control in consideration of steps in advance of the learning process.

In the technological knowledge domain the teacher-as-researcher learned to adapt material on the basis of the goals to stimulate abstraction. The material put extra emphasis on the exemplary examples. The assignments facilitated reflection as a new and explicit element. The material has to challenge teachers to start a plenary discussion. Such an intervention seemed to motivate and activate students' learning process effectively. Finally, the teacher-as-researcher learned to adapt the research instruments, in order to gain insight into the individually gained levels of abstraction.

CONCLUSIONS

The conclusions in the professional development of the teacher-as-researcher can be shown as follows per knowledge domain on level two.

In the ideological knowledge domain it was apparent that the theory on learning to abstract is processed in the practical situation. The reality of teaching proved to be consistently complex and to demand more of teachers than mere theoretical knowledge.

In the empirical knowledge domain, the teacher-as-researcher did not arrive at the results he expected. The unexpected questions from students forced him in a direction different to that planned beforehand.

In the technological knowledge domain, the experimental group, where the use of exemplary examples was a central issue, spent less time on their homework than the monitoring group. The results were comparable nevertheless.

DISCUSSION

This case study can be typified as being a pilot study: one teacher, two small groups of students, a brief teaching material at the beginning of the book (to refresh technical mathematical skills) and the initial research experiences of the teacher-as-researcher.

Whether or not to use paradigmatic examples is difficult to measure through analysis of the students' explanations of their answers, even with the use of an answer matrix. Not everything is put down on paper, lots of information stays inside the brains. It is therefore recommendable to use think-out-loud protocols as a research instrument.

Participation was prevalent to distance in this research which has consequences for the generalization of the results. On the other hand, strong involvement of participants is positive, when considering the type of research. Contextually speaking, the subject was a small repetition paragraph with few deviation options for alternative approaches, due to its strongly algebraic, repetitive nature. The question is whether the chosen paradigmatic examples were truly paradigmatic. There was a minimum difference when compared with standard examples, especially in this context in which the management of technical mathematical skills had priority. This pilot study is important for adaptation in the next research cycle (the 'vector' concept as its principle) with regard to the research instruments.

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APPENDIX I

Semi-structured interview questions (beforehand)

1. **Determination of the initial situation**

What is your background and experience as a teacher in mathematics?

1.1. Why did you choose the subject "broken functions"?
1.2. How many hours of study are reserved for this subject?
1.3. What educational material do you use and how is it composed?
1.4. In what way do pupils study the subject-matter?
1.5. What teaching practice do you use during the lesson as a teacher?
1.6. How do you test this subject?

2. **Reflection on the initial situation**

Do you think that the goals you have in mind are being reached?

2.1. Are you content with mathematics at this subject?
What would you like to change about it and in what way?
2.2. Are you content with the educational material you are using?
What would you like to change about it and in what way?
2.3. Are you content with the teaching practice?
What would you like to change about it and in what way?
2.4. Are the students content with the current situation? Do you ever check this?
2.5. Are you content with the way you are testing this subject?
What would you like to change about it and in what way?

Semi-structured interview questions (afterwards)

3. **Reflection on the end situation**

Do you think that the goals you had in mind have been reached?

3.1. Are you content with the developed instructional material you used?
What would you like to change about it and in what way?
3.2. Are you content with the teaching practice?
What would you like to change about it and in what way?
3.3. Are the students content with the new situation? Did you check this?
3.4. Are you content with your own role during the learning process?
What would you like to change about it and in what way?
3.5. Are you content with the way you have been testing this subject?
What would you like to change about it and in what way?

APPENDIX II

SUMMARY OF RESULTS												
Experimental group	**General**			**Explanations**				**Answers**				
	4V	**PR (min)**	**ST (min)**	**PRT**	**PT1 prb1**	**PT1 prb2**	**PT2**	**PRT**	**PT1 prb1**	**PT1 prb2**	**PT2**
$\bar{x}_e$	6.75	14.50	0.05	0.60	0.70	0.59	1.74	2.03	2.06	1.94	3.44
$\bar{x}_e$ without outliers	6.81	15.26	0.05	0.55	0.72	0.59	1.89	2.17	2.11	1.97	3.64
σ_e without outliers	0.93	9.50	0.23	0.49	0.84	0.47	1.08	1.63	1.82	1.40	1.48
n_e without outliers	22	19	19	20	19	19	22	20	19	19	22
Control group	**General**			**Explanations**				**Answers**			
	4V	**PR (min)**	**ST (min)**	**PRT**	**PT1 prb1**	**PT1 prb2**	**PT2**	**PRT**	**PT1 Prb1**	**PT1 prb2**	**PT2**
$\bar{x}_c$	6.61	42.00	0.80	0.64	1.37	0.74	2.23	1.88	2.50	1.92	3.13
$\bar{x}_c$ without outliers				0.71				2.05			
σ_c without outliers	1.09	30.05	1.78	0.61	0.70	0.47	0.71	1.94	1.57	1.63	1.14
n_c without outliers	15	15	15	13	15	15	16	13	15	15	16
Testing on significance	**General**			**Explanations**				**Answers**			
by T-test	**4V**	**PR (min)**	**ST (min)**	**PRT**	**PT1 prb1**	**PT1 prb2**	**PT2**	**PRT**	**PT1 prb1**	**PT1 prb2**	**PT2**
$\bar{x}_e \neq \bar{x}_c$? Significant differ?	p=0.56 No	p=0.00 **Yes**	p=0.11 No	p=.43 No	p=0.01 **Yes**	p=0.36 No	p=0.24 No	p=0.85 No	p=0.50 No	p=0.92 No	p=0.23 No
$\bar{x}_e > \bar{x}_c$ or $\bar{x}_e < \bar{x}_c$? Significant differ?	$\bar{x}_e > \bar{x}_c$ p=0.28 No	$\bar{x}_e < \bar{x}_c$ p=0.00 **Yes**	$\bar{x}_e < \bar{x}_c$ p=0.05 **Yes**	$\bar{x}_e < \bar{x}_c$ p=.21 No	$\bar{x}_e < \bar{x}_c$ p=0.01 **Yes**	$\bar{x}_e < \bar{x}_c$ p=0.18 No	$\bar{x}_e < \bar{x}_c$ p=0.12 No	$\bar{x}_e > \bar{x}_c$ p=0.43 No	$\bar{x}_e < \bar{x}_c$ p=0.25 No	$\bar{x}_e > \bar{x}_c$ p=0.46 No	$\bar{x}_e > \bar{x}_c$ p=0.12 No
Abbreviations:	PRT = pre-test PT = post-test Prb = problem (task)			PR = time spent in solving problems at home (in minutes) ST = time spent in studying the subject at home (in minutes).				$\bar{x}_e$ = mean score of experimental group $\bar{x}_c$ = mean score of control group			

A TEACHING EXPERIMENT

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ABSTRACT
A basic tool or a methodology of classroom investigations called a teaching experiment *is discussed in this contribution, with a special emphasis on delineating the differences and similarities that tool has for researchers and for teachers. Several definitions/descriptions of a teaching experiment primarily from constructivist sources are provided; they are followed by the discussion of research questions and the definition of teaching-research questions, by different approaches to a theoretical framework, the conduct of a teaching experiment and by different approaches to data collection and analysis. The discussion is directed towards beginning teacher-researchers in order to provide a motivation and initial tools to start classroom investigations.*

INTRODUCTION

The *teaching experiment* (TE) is a term very often used in reports and literature of mathematics education; very rarely, however, its nature, role and structure are explicitly discussed. Clearly, the corresponding concept either is not clear or it is not yet sufficiently mature to be formulated. One of the best, it seems, professional references for the discussion of meaning, structure and the role of a teaching experiment is *Handbook of Research Design in Mathematics and Science Education* edited by Lesh and Kelly (2000) where the whole chapter is devoted to this tool. Naturally, classical papers by Steffe (1983) and Cobb and Steffe (1983) referring to their Athens, Georgia teaching experiment give a good sense of what this tool means to a constructivist researcher. Both their papers as well as those of Thompson (1979) and Hunting (1983) emphasize the impact of Vygotsky's notion of the teaching experiment used in Soviet Russia in the 1930s as the motivating factor for its use in North America.

The discussion in this essay is to a large extent motivated by:

(1) The discussion document of the ICMI Study in Kilpatrick and Sierpinska (1998) which proposes that mathematics education has "two separate types of knowledge: the theoretical knowledge of the scientific community of researchers and the practical knowledge useful in applications for teachers and students." The authors of the document suggest that "it might be helpful to reflect on the nature of these two types of knowledge, on relations between them, and whether it would be possible to have unified body of knowledge encompassing both of them."

We do intend to contribute to that quest despite the fact that solely attributing application character to practical knowledge of teachers unnecessarily constrains an existing possibility for the emergence of a new theoretical concept out of teaching practice. The reflections upon the differences and similarities between the two types of knowledge will be indicated whenever a comparison will be natural and helpful in understanding the nature of relationship between them;

(2) Many controversial discussions taking place during the PDTR project focused on the role of a learning theory or research in the development of knowledge of teacher-researchers. Do we familiarize teachers with the theoretical framework with which we work at the very beginning of our collaboration, in the middle, in the end or not at all? That naturally leads to questions of the relationship between a theory and a teaching experiment. Is a theoretical framework necessary for the conduct of a teaching experiment or is it not? The variety of participating teachers' work in the present volume suggests a variety of answers.

The discussion is initiated by providing several definitions of TE encountered in the literature that show different concerns and aims of experimenters. Since our discussion is directed primarily to new and partitioning teacher-researchers in order to provide elementary knowledge needed to start small classroom investigations by classroom mathematics teachers themselves, we follow with the question of an idea formation, of how a research question or a teaching-research question appears, which leads us to the possibility of using a theoretical framework. The need for a theoretical framework (or its absence), and its role are discussed in section 3, while section 4 discusses the structure and conduct of a teaching experiment.

1. WHAT IS A TEACHING EXPERIMENT?

Mathematics education is an experimental science; one might call it an experimental mathematics. A teaching experiment is its experimental tool as well as the methodology for action of teacher-researchers with the help of which one asks and seeks answers to teaching-research questions, inquire into the nature of learning mathematics, of the development of mathematical thinking of our students, into the role of our own interaction in the classroom and many other issues interesting to teachers, researchers and teacher-researchers.

The actual structure of a teaching experiment has never been, according to Steffe and Thompson (2000) formally established nor, as the authors suggest, it should be. This way, its potentially ambiguous nature can acquire different characteristics depending on particular orientation of each experimenter. Thus, for Steffe and Thompson (2000) a teaching experiment is a living methodology designed initially for the exploration and explanation of students' mathematical activity. It involves a sequence of teaching episodes. A teaching episode includes a teaching agent, one or more students, a witness and the method of recording what transpires.

On the other hand, for Confrey (2000), one of the researchers who is credited in the US to have discovered the significance of classroom teaching experiments for the research profession, a teaching experiment is a planned intervention, which takes place over a significant period of time in a classroom where a continuing course of instruction goes on. It involves a dialectical relationship between the conjecture and the components of instruction: (a) curriculum; (b) methods of instruction; (c) the role of the teacher; and (d) methods of assessment.

For mathematics classroom teachers, on the other hand, whose prime motivation for engaging the teaching experiment methodology is the desire to improve their students' learning, a teaching experiment is "the classroom investigation of teaching and/or learning process, conducted simultaneously with teaching and aimed at the improvement of learning in the same classroom, and beyond" (Czarnocha & Prabhu, 2006).

Notwithstanding the differences, all approaches discussed here reflect a common constructivist belief that mathematics can be constructed, or broader, created by the autonomous minds of students, and that in reality there is no much difference between their struggles with mathematics and those of mathematicians and scientists *sensu stricto*.

2. HOW DOES AN IDEA FOR A TEACHING EXPERIMENT ARISE?

How does an idea for a teaching experiment arise? Or, in other words, how does a research question, or rather a teaching-research question (Czarnocha & Prabhu, 2006) get formulated? Again, it depends on who is doing a teaching experiment. For Cobb et al. (2003), "Design experiments are conducted to develop theories, not merely to empirically tune what works." Consequently, research questions in such an experiment will be connected with the development of theories. On the other hand, the teacher-researcher in physics, Minstrell, asks questions: "What is the understanding of my students?" and "What experiences can I put before the students to cause them to have to rethink their present understanding and reconstruct that understanding in order to make it more consistent with a broader set of phenomena?" (Feldman, Minstrell, 2000). Therefore, teacher-researchers' questions will be connected with the development of instruction, which will facilitate student learning towards the pre-determined direction. Or, a question appears through a long process of classroom investigation so aptly described by a mathematics teacher-researcher from the Bronx:

> The teaching-research questions start from quite general questions arising obviously in teaching or in students' exams or other written work, and get refined further and further till one arrives at the essence of the teaching-research question. At the very essence, the teacher-researcher is in completely synchronized oneness with the thinking of the student/s with whom the mathematical obstacle resides. In the process of refining the teaching-research questions from their original crude and general state to their finer refinement, the teacher-researcher is deep in the investigation of student thinking. The style of the classroom and the nature of the discourse at this stage is highly charged with the teacher-researcher always in the questioning mode, the question within herself/himself being, "what is it that bothers them or is not clear to them?" and "what did I say that sparked that particular comment from the student?" or "when I say such and such how exactly are they responding?" Any response is quickly seized and acted upon to discover its origin, and if the student is able to explain the origin, some clarity on the nature of the obstacle becomes less opaque to the teacher-researcher. The teacher-researcher designs exercises and activities and conversations designed to target the specific issues that he/she considers the trouble spots and investigates students' response to the questioning inherent in either the activity/exercise or conversation. The dialogue which has started between the teacher-researcher and the class, from the formation of the TR question in its first form through its various stages of refinement is a process whereby all participants are actively learning about the thinking of the other (*TR Journal*, VP ,April 23, 2006).

After the rediscovery of the intuition of indivisible in mathematical thinking about area by students of the Freshman calculus course (Czarnocha et al., 2001), a natural question arose: is there a way to join such an unusual[1] spontaneous intuition of area by students of calculus with the modern concept of the definite integral, and would such a way improve their understanding of the concept of an area? That question became what we now would call a teaching-research question of the Introducing Indivisible into Calculus Instruction Teaching Experiment supported by NSF Grant #126141 (Czarnocha & Prabhu, 2002).

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[1] The concept of the indivisible (or an atom) of space, although extremely powerful computationally had twice (Archimedes, Cavalieri) appeared in and disappeared from the mathematical discourse. The concept disappeared from the discourse for the second time in the 18^{th} century till it was rediscovered by Czarnocha et al. (2001).

A teacher-researcher from Poland noticed that high school students had serious difficulties with problems which require discovering and formulating theorem. Their knowledge concerning the assumption and conclusion was not well founded. Outcomes of her analysis of textbooks and research are as follows: these mathematical activities are neglected and require a detailed insight in order to change the weakest elements. That inspired her to conduct a teaching experiment connected with discovering, formulating and justifying theorems by high school students (aged 16-17). She designed a diagnostic test. One of the conclusions of that test was that students have particular difficulties with falsifying a proposition: they too often consider the false proposition as true. In connection with that she decided to specify her research questions: What is students' ability to use examples in assessing the logical value of a proposition? What is students' ability to find a counterexample to the manifestly false proposition? (Radoń, Part 4). Hence, the idea for a teaching experiment arises for teacher-researchers directly from their teaching practice. It takes its refined conceptual shape after some time of incubation. For example, it took three semester cycles of the Indivisible Teaching Experiment to find such a formulation of the definition of the limit of a sequence, which while preserving its mathematical depth, was yet contained within the Zone of Proximal Development of mathematics students of the Bronx.

It is important to point out that teacher-researchers' actions can be motivated by their own experiential craft knowledge as well as by the application of a suitable theoretical framework to a given classroom situation. Teacher-researchers decide to conduct a teaching experiment when they want to respond to a learning problem in the classroom in a systematic and creative manner or when they feel the need of constant improvement of their workbench. To sustain such a motivation one needs a *creative* attitude towards professional methods of teaching as well as a good predisposition to using research present in professional literature. Consequently, the discussion of the research questions of the teaching experiment, to which we still return, leads us directly to the issue of the theoretical framework.

3. THE NEED FOR AND ROLE OF THEORETICAL FRAMEWORK

To what degree a theoretical framework is the defining feature of a teaching experiment is one of the questions, which divides the experimenters depending whether they are members of the scientific/academic community or whether they are teachers of mathematics. In judging the significance of these differences we must remember that teachers have at their disposal their own craft knowledge gathered from years of experience, which alone is sufficient to provide the base for the design of a teaching experiment. On the other hand, the knowledge of professional researchers is based on theories, ideologies and research results of the profession; consequently for them a teaching experiment is deeply immersed in the idea of theoretical framework.

It is quite possible to try the classroom effectiveness of an innovative approach developed on the basis of professional craft knowledge without connecting it with any definite theoretical approach. The motivating question: "can I improve learning in my classroom?" has a perfect, well-defined meaning for classroom teachers of mathematics without any theory. Teacher-researchers differ from the researchers of mathematics education in that respect that they do not depend on the presence of a theory in the conduct of a teaching experiment; rather they may trust their own confirmed professional intuition. When is it that teacher-researchers make contact with a theory or research results? It seems that – and we underline the hypothetical nature of the conjecture – since not much research had been done on the development of teacher-researchers, it takes place in two situations:

(1) It happens at the moment when their craft knowledge, as well as the knowledge of professional colleagues concerning an urgent didactical issue in the classroom had been relatively exhausted; only then teacher-researchers need to look for a theoretical approach which can guide them. As long, however, as teacher-researchers' craft knowledge of teaching is sufficient to deal successfully with the classroom learning challenges, the need for a theoretical approach does not arise.

(2) When sufficient number of particular teaching experiments had been conducted by teachers, then a natural desire and need for the organization of acquired information arise in their mental apparatus. In fact, a Chinese colleague[2] from Honk Kong National University having worked with teachers for the last 10 years asserts that only "once sufficient examples (cases) are gathered, generated, experienced, and learned, teachers begin to theorize." It is primarily when teachers want to move beyond the question of particular improvement, when many different instances of similar approaches have been noticed, when they want to understand better and deeper, which means, they want to find connections and relationships of "improvement" with other observed facts in the classroom, then the need for a theoretical framework as an explanatory tool arises. One of us (BM) characterized this process in the following words:

> I think we need to distinguish between the process of becoming teacher-researchers and trying different teaching experiments, and the case when the investigations are performed by accomplished teacher-researchers. In the first case, indeed, a theory does not play a significant role; teachers support themselves with their professional experience and teaching intuition. After a certain moment, however, there appears a need to consult the literature – this is what happened in the Rzeszow-Krakow team: we matured for a long time until we reached that awareness but now all teachers in the team know that theory is part of a teaching experiment.

What is more, the described process is a manifestation of a natural development of any conceptual scheme of thinking. In other words, the process of development of teachers' understanding of the practice leading to theorizing proceeds along the familiar (Czarnocha, Part 2), one would even say, natural process of the human conceptual development. In fact, when we ask the question, "what is a theoretical framework?" the simplest answer is, "it is the network of relationships between concepts, facts and procedures which turned out to be useful in the pursuit of investigations in mathematics education as a profession."

4. THE STRUCTURE AND CONDUCT OF A TEACHING EXPERIMENT

The structure and conduct of a teaching experiment naturally depend on a teaching-research question but in its simplest form their outlines can be seen in the daily work of teachers who are about to start a new section in the curriculum and want to find out how effective their teaching is in facilitating the mastery of the mathematics in question by students. Teachers are aware that pupils come to their class with some knowledge, and consequently, in order to find out how much they contributed to students' learning, they have to make a diagnostic test at the beginning of the section (pre-test), a standard test at the end of teaching that section (post-test), and to compare the results. The simplest comparison is a grade comparison but this in itself does not give much of an assessment of the change of knowledge. More often than not such an assessment is supported by teachers' knowledge of their students from classroom

[2] Personal communication from Prof. Chun Ip Fung, Honk Kong Institute of Education.

interactions, style of working, communicating which together with grade statistics provide a comprehensive classroom assessment (Nunez, Part 4).

A contemporary teaching experiment grew out of a simple tool for teachers, through adapting it to new and relevant variables such as different types of interventions between the pre-test and the post-test, introduction of more refined tools of assessment of knowledge such as interviews, videotapes, student course workbook, or different methods of classroom interaction and/or student motivation, and ultimately, by the change of its research objectives. Both Steffe and Thompson (2000), as well Confrey (2000), emphasize either transformative or conjecture-driven nature of a teaching experiment which allows the experimenter to be actively changing the course of the experiment during its conduct: "Research guided by a hypothesis merely attempts to discover if a given intervention worked or not or, if a given theory was supported or not. But research guided by a conjecture seeks to revise and elaborate the conjecture while the research is in progress (Confrey, 2000).

Hence, in distinction to traditional teaching experiments where the intervention has been pre-planned and resulted in the advance planning manner, here we have the notion of a teaching experiment where experimenters can enter its structure and change its course.

The structure of the Teaching Experiment, NYC Model (TE/NYC) is based on a cyclical methodology which it inherited from the Action Research approach, and it is to a large degree determined by the requirements of teachers' ethics which had been formulated in another article of this volume. *We investigate in order to improve learning in our classrooms and beyond*. Therefore, the teaching experiment cycle consists of (1) design of the instruction/intervention; (2) implementation in the classroom during an adequate instructional period, most typically – a semester or a school year together with the collection of data (although a teaching experiment can also take part of the lesson or a couple of lessons situated properly within the curriculum); (3) analysis after the end of the cycle; and (4) refinement of instruction leading to improvement to be implemented in the next cycle with a similar cohort of students.

The refinement of instruction leading to improvement is critical in the methodology both for the development of students and teacher-researchers, and therefore TE/NYC Model consists of at least of two full cycles. Since one of the ethical dimensions of TR ethics requires that some knowledge gained by teacher-researchers is to be utilized for the experimental cohort of students, we consider the requirement that such two cycles inclusive of improvement are to be conducted within one instructional period, as the most immediate form of satisfying the requirement. Another, more difficult to achieve but fully acceptable here approach is that the experimental cohort is taught in two consecutive instructional periods by the same teacher-researcher (for example, two consecutive semesters in college, or two school years in school).

As mentioned in "Ethics of Teacher-Researchers" in this volume as well as in (Czarnocha & Prabhu, 2006), the requirement of improvement of learning forces the formulation of teaching-research questions, where the first question asks for the state of knowledge relative to a given concept, and the second one, the dynamic one, asks for the ways of improving that knowledge. The first one is a typical question posed by researchers investigating a particular state of knowledge; the second one is a question of concerned teachers. For example, the recent teaching experiment investigating the feasibility of introducing the concept of indivisible into calculus instruction (Czarnocha & Prabhu, 2002) sought the answer to two questions:

(1) What is the process of transformation and development of the intuition of indivisibles into a precise, mathematical concept?

(2) Do the introduction of the Cavalieri-Wallis construction based on indivisibles, the interplay of that construction with the standard Riemann construction, and the instruction that integrates both, strengthen students' understanding of the concept of the definite integral?

The first question asks for a description of the process of integrating students' intuition of indivisible with the modern concept of the integral, while the second question asks whether this process improves students' understanding of the concept. We suggest this organization of questions ahead of a teaching experiment as the compromise between theoretically-based and practically-based knowledge; in distinction to standard factual research questions, we call such a related pair of questions, the teaching-research question. Note the difference with the two questions of the conjecture-driven experiment formulated by Confrey (2000): "The conjectures that we envision as essential in this type of research have two significant dimensions. First, the conjecture must have a mathematical content dimension" that answers the question: "What should be taught?" "Second, there must be a pedagogical dimension linked to the content dimension that answers the question: How should this content be taught?" (235). Teachers in school do not have a lot of choice as to what should be taught, given state and national curricula, although certain aspects of these questions can be investigated. But teachers need to know from the point of view of their teaching, what students in their class know, and what needs to be done so that they understand, learn and master relevant piece of mathematics. Hence while the TE NYC Model agrees with the conjecture-driven approach on the number of teaching-research questions, it disagrees on the content of the first question. NYC Model is positioned almost in the golden middle between researchers investigating the state of affairs and teachers concerned with improvement of understanding and mastery of mathematics in their classroom and beyond.

Formulation of such pairs of questions has an important implication for the conduct of a teaching experiment. Since we are not only curious about the state of students' knowledge relative to a mathematical concept or a problem, but we are also interested in the details of the process of improving or developing that knowledge, teacher-researchers can not act solely as observers to the events taking place in the classroom or as the implementers of the prepared curriculum, but they also need to intervene whenever students have difficulty in reaching necessary understanding and this intervention might and often is moving beyond the method of instruction decided a priori. Teacher-researchers as teachers must adapt their actions to the actual needs of students as they present themselves in the classroom; as researchers, they have to take such action as part of the investigation process and data. Let us just note that a similar requirement imposes itself upon interviews with students about mathematics. Such interviews with students which can have a formal character of conversations, or a character of short conversations with single students about a problem at hand in the classroom are often used by teacher-researchers in order to gain deeper information about students' understanding and mastery of the subject they teach (Quevara, Part 4). However, the responsibility of teachers, as opposed to the responsibility of researchers, is that having learned about any deficiency in students' knowledge they have the responsibility to improve that knowledge during the same conversation. These movements from researcher-to-teacher-to-researcher, or from teacher-to-researcher-to-teacher characterize the dynamics of the classroom teaching-research.

5. DATA COLLECTION AND INTERPRETATION: AN EXAMPLE

A proper choice of data collection is of fundamental importance for teacher-researchers who generally do not have a lot of time to devote to sifting through the collected data and to developing its analysis. Moreover, the role of data is two-fold: on the one hand, it has to provide summative knowledge to researchers about the nature of the whole experiment, but on the other it has to inform teachers about the next best steps of instruction. It has to be then relatively easy accessible. The classroom data can be of two kinds, statistical data supporting quantitative statements and qualitative data supporting qualitative knowledge; naturally, it is useful and desirable if the conclusions from both kinds of data can be related. Their analysis needs to be supported both by mathematical knowledge as well as by pedagogical craft knowledge of teacher-researchers. Moreover, the two separate parts of a teaching-research question require their own answers, one describing the state of knowledge where the quantitative approach might be more useful, and the second dynamic one, needs data which can evidence changes in the process of learning as well as interventions which might have caused them. Consequently, we collect both scores on tests, homework as well as the analyses of errors made by students. If we want a better picture of how students think, solve the problems and understand mathematics, we have to create data that records these components. Mathematical essays, paragraphs of explanations, or transcripts of mathematical conversations conducted with students either during class or during more elaborate interviews conducted outside the classroom are especially good for such qualitative analyses.

Amongst the tools utilized by the Polish teacher-researchers, there was a didactic intervention whose goal was to develop amongst students a habit to explain their own difficulties through the instruction: "If you don't know how to respond to the question, describe the nature of your difficulties, write what exactly causes your difficulty." Students who earlier would not even start responding to the problem and returned blank pages, now tried to describe reasons why they could not solve the problem, which often provided insights for teachers about the causes of students' difficulties. Also, in many cases it motivated the students to start looking for answers by themselves.

One of the critical elements of teachers' investigation is the ability to observe. This important method of pedagogical investigations (Łobocki, 2006) focuses on the behavior, reactions and natural events within the scope of a research question; registering observational data in the most objective manner possible, data classification in the light of one's own knowledge, theories, experience…Teachers are interested in what students feel, think, what their motivations for actions are, what the level of students' conceptual development is, but all of it is based only on the external behavior. The teaching-research notebooks are very helpful here.

Still, how to choose a set of data which gives maximum information with the least amount of work is the main question here. It seems that this choice is governed by two options: (1) formulation of the question with the utmost precision concerning the mathematical concept in question followed by the choice of relevant data motivated by teachers' craft knowledge; and (2) coordination of the classroom situation with a theory of learning which had proved itself useful in the professional discourse followed by the formulation of a question with the help of theoretical constructs of the theory which determines the choice of data and its analysis.

Example: Mathematical Essays as Data

The calculus teaching experiment (Czarnocha & Prabhu, 2002) had investigated, as one of its important sub-questions, the trajectory of teaching and learning the concept of the sequence limit. Since the new integrals were defined with the help of this concept, understanding of the limit concept was indispensable for grasping the role of CW and PW integrals. From the past research as well as from the pilot experiment, experimenters knew that the central issue in learning the concept lies in abandoning the intuitive, spontaneous understanding of the limit which is usually expressed by a phrase similar to: "as *N* increases, the terms of the sequence become closer, and closer to the limit," and accommodating one's understanding to the agreement with a phrase that might say "doesn't matter how close neighborhood epsilon of the limit *L* we choose, there will always be only a finite number of terms outside that neighborhood." During the class, students participated in several activities involving graphing of a sequence, choosing different values of the epsilon, and finding the index beyond which all the terms of the sequence were within the desired distance. As the data collection item designed to see the effect of instruction, a special theme for written mathematical essays was assigned where students were supposed to discuss the relationship between the two definitions of the limit of the sequence $\{a_n\}$,

(1) the Weierstrass definition: For any $\varepsilon > 0$, there is a number N, such that $L - \varepsilon < a_n < L + \varepsilon$, for all $n > N$; and

(2) The sequence a_n has a limit L if the terms a_n can be made as close to L as one wishes for n sufficiently large.

It was experimenters' hope that the effort needed to explicate the terms of one definition with the help of the other would provide an occasion for students for explicit description of their own understanding of the concept, which then can be properly assessed.

A fast glance at 20+ essays revealed that the central difficulty is located in the understanding of the phrase "for *n* sufficiently large." In the following fragment we see a student who quite correctly identifies ε, epsilon, as the measure of closeness for the terms (values of the sequence), displays familiarity and understanding of the role of, what later will be called, epsilon-neighborhood, but totally misunderstand the relationship of *n* with ε:

Student 1: Making the terms as close to zero as one wishes" in the example, we have zero as limit, a person can pick a number, epsilon, as close to zero as one wants and still find an infinite number of values closer to zero. *N* is the number into the sequence that a person picks as the epsilon.

The last sentence "*N* is the number into the sequence that a person picks as the epsilon" betrays confusion about the relationship in question. Finally, student 3 states directly the source of his confusion when he says:

Student 3: Conceptual definitions are a way of telling the "non-math" mathematical world what your proofs are and where they come from… For *n* sufficiently large? If one spoke to an individual without much math experience that individual would still be lost… It is hard to explain what is for n sufficiently large… a better explanation in English could have been, the larger *n* becomes, the closer to zero the value becomes.

From this student's statement we find out not only the correct identification of the difficulty in understanding the phrase ("It is hard to explain what for *n* sufficiently large is"), but also we get implicit suggestions that this spontaneous change of the phrase into the dynamic definition - the larger *n* becomes, the closer to zero the value becomes may be motivated precisely by the absence of clarity as to "what is for *n* sufficiently large" that is precisely by the absence of the *n* – epsilon coordination.

Notwithstanding a detailed analysis of student utterances, it has become clear that the phrase "for *n* sufficiently large" constitutes the main obstacle for students' understanding precisely because of its ambiguous character. Despite the fact that the "conceptual definition" quoted above has been one of the main tools for the discussion of the limit by several notable textbooks of the reform, it had to be eliminated from the instruction and substituted by another approach to enable students' grasp of the definition. Our teaching-research question concerning the limit had been answered in the negative by a relatively small and compact set of data– this was not the way to do it, and the conclusion could be implemented still in the same semester of the course. More penetrating analysis of student responses conducted after the end of this cycle of the teaching experiment revealed absence of mastery of elementary logic to be the responsible culprit for the situation.

CONCLUSION

We have shown how standard classroom environment of daily teachers' work can serve as the site of a teaching experiment, which might bring forward new knowledge of the process of learning, in particular, of mathematical development. We analyzed the degree to which a teaching experiment arises out of the daily work of concerned teachers in the classroom as one of the basic similarities and differences between the research and teaching-research methodologies, offering a possible route of integration into one body of knowledge, as suggested by the authors of the ICMI study. The formulation of two parts of a teaching-research question was offered as a compromise between two different types of knowledge, the researcher knowledge and the teacher knowledge discussed in the ICMI document of Kilpatrick and Sierpinska (1998).

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RESONANCE: A KEY WORD IN MATHEMATICS TEACHING-RESEARCH

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ABSTRACT
In this paper a model of cognitive dynamics is proposed, for interpreting classroom as well as teacher training processes, nowadays supported also by some recent neuroscience results. Core relevance in the model is assigned to a basic resonance dynamics, assumed to work at the root of all the modulations, from perception to abstract thinking, and interferences characterizing knowledge construction. On the basis of this theoretical framework, teachers are seen as "resonance mediators," i. e. experts who favor the resonance process in their students; correspondingly, any teacher training process has to accomplish this result. Finally, we will briefly examine a school episode, showing how a teacher, playing this role, acts effectively both on students' understanding and on their motivation.

1. INTRODUCTION

Many cognitive theories have been developed in recent years. For a general frame, concerning in particular mathematics education, we refer to Tall (2004). Among other things, Tall notices an emerging strand of researches into brain activity, where a lot of experimental data have been collected from those concerning innate numerical competences to those based on Brain Imaging Techniques, applied to subjects engaged in elementary arithmetic tasks (Dehaene, 1997; Butterworth, 1999).

In this paper a particular model[1] of cognitive dynamics is employed, in order to draw some consequences for teachers' role and for their training strategies. The model is presented in depth in Iannece and Tortora (2007a, 2007b), Mellone (2007), and is outlined in Guidoni et al. (2005a), so we will present here only some of its essential features. This model has been developed mainly on the basis of our classroom experience but nowadays it is supported also by the above quoted neurophenomenological studies.

The importance of focusing the attention on a cognitive model arises when we recognize how teachers' beliefs about the essence of mathematics, the ways of knowledge transmission and the ways people learn and understand are relevant in teachers' behaviors. Very often these beliefs are just embodied in daily practice and partially unconscious but notwithstanding they determine automatic small and large scale decisions in the conduct of class activities. For instance, teachers can act as knowledge transmitters just because they are merged in the Piagetian milieu, where three basic assumptions operate: scientific concepts – inserted as they are in a structured

[1] We use the word "model" in a weaker sense than the one used in hard sciences, but more suitable for a quite unformalized domain like math education.

cultural system – are the only ones that allow valid knowledge; they are obtained by substitution of the originary everyday concepts; the reasoning of any educated mind follows the formal rules of logic. Therefore, a first task in any teacher training is to let these beliefs emerge, in order to be compared and discussed (Mason, 2002). Then, usually only as a consequence of a shared agreement, a necessity can arise to modify these beliefs, and to assume new views that turn to be more effective in designing, managing and assessing learning environments and didactic actions.

This paper is arranged as follows: in the second section we present our model of the understanding process together with some related results from neuroscience; in the third section we derive our views about the role of teachers and some strategic lines for teacher training; the fourth and last one is devoted to an illustration by means of an example of the potential impact and the possible outcomes of our assumptions.

A distinctive feature of our cognitive modeling is the key word *resonance*, used as a powerful metaphor for learning processes. We borrow it from physics,[2] where it denotes the increase of the amplitude of enforced vibrations, taking place whenever the frequency of the source is approaching one of the proper frequencies of the receiving system. We can see resonance phenomena in a lot of fields of common experience, sometimes with catastrophic consequences, like when a bridge can be destroyed by virtue of a prolonged sequence of quite weak earthquake waves. Or everyone who has been on a swing knows very well how movements of the legs can expand or extinguish the swing amplitude, depending on their accordance with the swinging frequency.

Much in the same way, we say that a learning process, which is always the result of a complex interaction between different variables, is driven by a resonance dynamics when different cognitive items (an idea, a mental construct, an image, an action, etc.) are simultaneously activated whenever one is evoked, producing by reciprocal interference a mutual reinforcement effect. But, as we will see in the sequel, what is even more important for educational goals in mathematics is to put in resonance: (1) the actual potentialities of individual cognitive structures; (2) the patterns supplied by codified cultures; and (3) the constraints of the real world. Finally, the circumstance that in evolutionary theories of the brain, the term "resonance" appears also as a key word in modeling human cognitive behavior, which, as we will show in the next section, does not seem to us a simple coincidence.

To activate resonance, or, even better, to guide children to consciously look for resonance, is one of the necessary conditions that allow them to be in the mood to succeed in understanding. This is a hard task for any teacher, while the task of research is to investigate how this can be accomplished at best. The importance of the pivotal role of teacher-researchers – whatever this means – is quite evident.

2. OUTLINE OF A COGNITIVE MODEL

In every didactic action we recognize very schematically at least four basic "model-ingredients:" (i) a realistic, even rough, model of "natural" cognitive dynamics; (ii) a global, epistemologically founded view of mathematics as an internally structured scientific discipline; (iii) a modulated view of the variety of interferences of mathematical thinking with other cultural fields (mainly scientific and technological ones), and with everyday culture(s); and (iv) a model of cultural transmission in

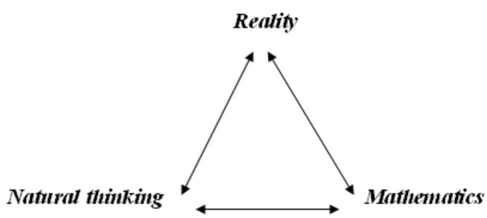
[2] The word resonance appears not only in physics but in many other contexts with different meanings. In an educational context, for example, it is employed by Comiti et al. (1996), as a measure of the responsitivity of teachers to students' interventions.

knowledge areas, in particular, scientific ones. Such ingredients, obviously crucial to teaching profession, are clearly correlated to each other: in particular, the basic framing role assumed by i) with respect to other aspects is clear.

In order to characterize our cognitive modeling, we feel it necessary to root it within the complex landscape of the cognitive theories available nowadays. In particular, we draw attention to the fact that many critical aspects of cognition have been variously regarded in the course of time within different (often reciprocally contrasting) cognitive theories and/or epistemological positions. Actually, our basic research finding is that most of such aspects appear relevant in interpreting experimental teaching/learning evidences: and this directly implies their reciprocal complementarity. For example:

It is now quite common to refer to Vygotsky's views about the crucial role assumed by a natural language in the development of knowledge, since the earliest age (Vygotsky, 1934). However, such a role is only in a minor part an automatic, passive one: careful observations of cognitive transactions show that an early, active adult mediation plays a key role in fostering resonances and preventing dissonances ("misconceptions" appear as the result of missing/wrong/misleading mediations between developing cognition and culture): Apart from the "stages" machinery, some insights by Piaget appear to be crucial to outline features of cognitive dynamics. Assimilation, accommodation, and (temporary) equilibration lively define the main modes of any resonance process.

The point is to correlate such views within a comprehensive dynamical model: and the resonance dynamics frame, as described in Guidoni et al. (2005b), actually lends itself to account for many crucial correlation aspects as we want to show. So, we assume that any true learning in scientific/mathematical field is a result of the process of resonance between individual cognition, social culture and reality structures along cognitive paths efficiently addressed and controlled in their meaning-driven dynamics. It requires, at any level, also resonance between various "dimensions" of natural thinking (Guidoni, 1985): perception, language, action, representation, planning, interpretation, looking for sense, etc. We can schematize our view by means of a triangular schema, resembling the famous Chevallard triangle but with different variables on its vertices.



We want to stress: (a) the complexity of every "vertex," in particular of the "natural thinking" because of its multidimentional and time-dependent character; (b) the two-way direction of the arrows that marks the impossibility to uniquely determine the thinking process. Looking for resonance is a very useful and flexible tool for teachers in driving a didactic action and in reflecting on it, and for children in understanding how they understand. Let us go into some clarifying details.

In a Vygotskian perspective, we know that the roots of a large number of culturally sophisticated concepts, like those of mathematics, can be found in the ways children spontaneously face the complexity of the surrounding world, far before they are involved in school contexts. For example, very early on it is necessary to correlate things and properties changing in time: a typical cognitive strategy is to give causal explications to these phenomena, trying to interpret, often forcefully, a variation of a property as a cause or an effect of the variation of another. Here, we clearly recognize the "natural" cognitive root for the dependance of a variable from another that is the mathematical concept of function. Of course, things are all but easy: a natural thought strategy cannot go very far in managing mathematical complex situations, where abstract structures come into play typically neglecting semantical counterparts. In this case, a search for resonance guides teachers in addressing the development of natural strategies toward a goal of reification and nominalization; just like in the sense of Vigotsky.

Again, let us suppose that children work with numbers, their operations and properties. For natural numbers (natural not by chance!) it is easy to see a correspondence between the things around us, our perceptions and actions, and finally the mathematical structures. So, a teacher can utilize the correspondence between reality and our natural ways of thinking to build suitable mathematical structures, again according to a Vygotskian evolutionary perspective.

But the problems arise when new kinds of numbers come into play. Trying to root again multiplication by zero (see Section 4, below) or between negative numbers in the reality can be the first spontaneous attempt both for teachers and for learners, but the circumstance that the correspondence with reality, already observed for natural numbers, does not work anymore could provoke irreparable consequences in children's minds, like severe separation between intuition and mathematical knowledge, if not the first refusal of being involved in active learning. All this can be avoided if teachers wisely guide their students in the complex play of resonance. This time the need for a rich and sound mathematical structure must be invoked, pointing for instance to natural strategies of generalization or to a natural ability to structure a new game. Reality comes only afterwards, perhaps when we are able to recognize that the multiplicative structure of integers offers a powerful model for a more complex physics phenomenology.

A more detailed example will be discussed in Section 4 below. For other examples, see Iannece and Tortora (2007b) and Mellone (2007), and also some of the papers contained in this book and in the PDTR *PISA Handbook*, e.g. De Blasio et al. (2008) and Pezzia et al. (2008): In all these examples, referring to various school grades, several positive outcomes of our resonance model can be observed, among others: (a) the ability to autonomously utilize one's informal knowledge to support the construction of formal knowledge, and, in the opposite direction, the ability to give a sense to formal knowledge interpreting it within informal contexts; (b) the ability to select linguistic tools according to specific objectives; and (c) a marked growth of self-esteem.

All these examples and, more generally, the goal of promoting resonance between the mathematical constructs and natural cognitive structures suggest deepening of knowledge of the latter. We will try to do this in the next section using some recent neuroscience research results.

The development of human brain, prerepresentations, mirror neurons

Leron (2004) says that "Human nature [can be seen] as a collection of universal, reliably-developing cognitive and behavioral abilities – such as walking on two feet, face recognition or the use of language – that are spontaneously acquired and effortlessly used by all people under normal development. Common sense is a cognitive part of human nature, the collection of abilities people are spontaneously and naturally good-at."

Many research streams deepen the Leron's notion of human nature, showing the existence of other universal behavioral abilities, like the ability to formulate hypotheses (Changeux), or to control the coherence of argumentation (Houdé), or to use metaphors (Lakoff & Núñes). And there are still a lot of things to discover, as it is easy to recognize just observing, as we do everyday, the learning process of students of every age. Using, as we do, an extensive meaning for the word "natural," our construct of resonance appears well expained by Vygotsky's dychotomies between natural/scientific concepts and natural/higher psychic functions (Vygotsky, 1934).

Some models of the working and cognitive behavior of the brain, recently devised on the basis of experimental neurophysiology data, highlight how our synaptic structures develop according to a continuous learning process. We mainly refer to Changeux (2000) for these results. Today, they offer new experimental confirmations to Vygotsky's hypotheses about the social nature of learning and the evolutionary character of concepts and of the psychic functions mediated by culture: these ideas are part of our own modeling.

According to Changeux, our brain is characterized by a marked "structural plasticity", due to a continuous interaction with the external environment and to a likely continuous internal reworking (e.g. dreams, thoughts, and imagination). It develops according to two distinct but related processes: on a biological scale, it is the outcome of a Darwinian selection of the most advantageous representations of the external world (for this idea, see also Edelman, 1987), on an individual scale, it changes according to a never-ending learning process. Spontaneously, the neurons generate impulses and transitory synaptic connections are activated, giving rise to "prerepresentations" of the external world. In a sense, as foreseen by many authors, like e.g. Neisser (1981), our brain does not simply receive information from outside but throws its own interpretative schemata in. In this way, knowledge origins as a result of selection and stabilization of prerepresentations, guided by a "cognitive relevance" principle, similar to that studied by Sperber and Wilson (1993) in communication theory. The relevance is marked by a correspondence with reality: "The answer coming from outside is decisive. It constitutes a test of how the prerepresentation fits in the environment."[3]

Therefore, the spontaneous activity of our neurons can be seen as a natural aptitude to explore and modelize the physical world. Recalling also Galilei's words: "*Ma io stimerei prima la natura aver fatto le cose a modo suo e poi i pensieri degli uomini atti a capirla*,"[4] a possible answer can be found to the ancient question about the more or less innate nature of mathematical concepts, in the sense that the innate numerical abilities (Dehaene, 2000; Devlin, 2002) can also be seen as the result of an epigenetic selection of neural networks, stabilized as the most effective to mankind survivance.

A noteworthy amount of recent neurophenomenological results concern the primacy of the perceptual-motory brain system also in the processing of higher functions, with a central role assigned to the so-called mirror neurons (see, e. g. Kohler et al., 2002; Gallese & Metzinger, 2003; Gallese & Lakoff, 2005): For these results we mainly refer to Rizzolatti and Sinigaglia (2006) where many of them are collected and supported by experimental data. The basic hypothesis is that the roots of any cognitive process stand in the motory cortex that is to say in our *actions*. "Processes that are

[3] Changeux, 2000, 65; our translation from the Italian edition. A lot of data supporting these assertions can be found there.

[4] "I would suppose that nature [comes] before made things and then men's thoughts capable to understand them," our translation from Galilei (1964).

usually considered of a higher order, like perception or recognition of the acts of other people, imitation and also gestural and verbal forms of communication, can be referred to the motory system, where they find their neural background."[5]

The physiological bases of this integration are the so-called "canonical neurons" and "mirror neurons." The former codify motory acts, i.e. not just movements in themselves but movements having a specific goal (see Iannece & Tortora, 2007a, for more details). In other words, there is an area (F5) in our brain where a kind of *vocabulary of actions* is stored. Some of these canonical neurons are excited when we act for a particular goal, as well as when we just observe an object on which an action can be done (e.g., seeing a cup on a table stimulates the potential – complex – act of picking it): So, what happens is that the vision of an object generates in our brain a prerepresentation: "A brain that acts is, first of all, a brain that understands," in the words of Rizzolatti and Sinigaglia (2006):[6]

The mirror neurons are so called because they reproduce actions of people in the brain of other people: e. g., they are excited when we move our hand to pick something as much as when we see someone moving their hand for the same goal. Due to them, our vocabulary of actions can be viewed as our cognitive budget to understand and interpret actions of other people. Now, this budget is strongly modified by a learning process, therefore, the things that we understand "without effort" vary according to how our own experience and knowledge develop. The notion of elementary (effortless) tasks is crucial. They are accomplished by means of innate brain circuits (typically those deputed to treat sensory-motor information), automatically activated when the exigence of guaranteeing survivance occurs: to drink when thirsty, to escape in front of a danger, to recognize a face, etc. On the other hand, an effort is necessary when other areas and functions (e.g. memory and imagination) of the brain are involved. It is more or less the same Vygotskian distinction between elementary and higher psychic functions. Of course, most of the tasks involved in the learning of mathematics are of the second type.

In an interesting experiment Houdé et al. (2000; see also Iannece et al., 2006, for details) show that activating emotions can be crucial even for complex reasoning,[7] since effortless (perceptual) strategies are inhibited in favor of rational ones, more suitable for the specific goal (of a logical type, in the case studied): This confirms that two different kinds of "reasonings" are both natural for us, although the former is more "spontaneous" (in the sense of Vygotsky), while the latter requires an effort. We claim in Guidoni et al. (2005a) that in mathematics education, like in any complex learning environment, the key is not that of inhibiting a kind of reasoning strategy in favor of another, but to put them in resonance, i. e. to consciously pass from a cognitive dimension to another in a continuous reinforcing game. Houdé himself says: "*Le cerveau de l'homme est una sorte de jungle où le competence du bébé, de l'enfant e de l'adulte, sont à tout moment susceptibles de se télescoper, d'entrer en competition, en même temps qu'elles se construisent*" (2000).

In eveyday life, our way of understanding is a process of continuously projecting outside prerepresentations looking for a feedback from the external world. In our opinion, the same thing happens within an abstract cultural context: attempts and errors, conjectures validated or refused are the ingredients of the dynamics by which our

[5] Rizzolatti & Sinigaglia, 2006, 122; our translation.
[6] *ibidem*, 3, our translation.
[7] According to Damasio (1994) there is a very strict relation between rationality and emotions, at the level of brain circuits.

brain works in order to understand. Therefore, in any education context, it is necessary to favor this process: our resonance model and our didactic choices are made following these assumptions.

Coherently, we look at the formal structures of mathematics as at one of the two principal cultural tools by which the phenomena of the external world are interpreted and described (another one being physics): Today, the prevailing view of mathematics, within most curricula, emphasizes its *a priori separateness* from other scientific areas, thus conflicting with natural cognitive processes, and causing many students' difficulties. On the contrary, if mathematics is perceived as an *a posteriori abstraction* coming through a modelization process, its "cognitive" resonance stimulates students' motivated interest toward its structural development and allows them to reach quite high levels of formalization.

We are not saying that mathematics should be reduced to a mere language for physics nor physics to a simple field of examples of mathematical structures. We are saying that it is necessary to recognize that: a) the same cognitive process underlies both disciplines; b) both can be viewed as *discourses* (in the sense of Sfard, 2000), characterized by different rules but complementary in their role of cognitive reconstructions of the reality; c) this complementarity is a precious resource from an educational point of view. In this sense, the cultural constructs of physics are a bridge between the perception and the more abstract notions of mathematics: for example, the physical concept of motion fills the gap between the variety of experienced movements and the symbolic mathematical treatment. We take it a step further: introducing abstract structures as linguistic tools to describe and reason about things is the strongest way to motivate even their autonomous disciplinary development.

But what does it mean to put individual cognition and reality structures in resonance? We suggest that attention should always be paid to all those models and strategies that have been developed by mankind as a whole, and are developed by each individual in order to interpret and manage often unconsciously the daily experience. We are thinking of Rizzolatti and Sinigaglia's "vocabulary of actions" or the "schemata" described in Lakoff e Núñez (2005): When observing everyday human actions and reasonings, it is easy to recognize how such models and strategies are generally employed, and also how complex and sophisticated they can be.[8] But what is essential for educational purposes, is to observe them at work in students' behaviors.

In the first section we have already proposed the example of the relationships between two changing variables. But also the order relation, the direct proportionality (Guidoni et al., 2003) are all models that are employed very early, independently from one's linguistic tools, and the same happens for thinking strategies like the dychotomy concrete/abstract (in children's words: "by truth or by fiction"): see Iannece and Tortora (2007b) for more examples.

3. TEACHERS' ROLE

In Section 1, we have recalled how, according to Piaget, the scientific concepts are constructed. This view assigns to teachers the role of knowledge transmitters. On the contrary, in our model of cognitive dynamics, where the focus is on the integration

[8] For example, when crossing the street with heavy car traffic, a human being (but also, say, a cat) must uncounsciously and rapidly activate sophisticated controls of distance and speed, and of their variations, in order to avoid danger.

(resonance) among natural thought, real world and culture, teachers play a completely different role which we call *resonance mediation* (Guidoni et al., 2005b):

"Pick them up where they are then find a path which guides them to the place you want them to reach." According to this famous Wittgenstein's mot, teachers must recognize in every class context the "space of cognitive configurations" (let us use again a metaphor from physics), in order to design possible learning trajectory paths, drawn on all the available resources. In general, they will adopt teaching strategies that are not imposing but supportive of potentialities. Teachers should create, on a local level, the many possible links between individual cognition, social culture and reality structure through the use of dynamics of abstraction and de-abstraction (modeling and de-modeling) with graduality, coherence, flexibility and competence.

In other words, what teachers have to do is first of all to make explicit and to foster all natural models and strategies of their students. This means, for instance, that, when recognizing proportional thinking as a spontaneous strategy for interpreting real phenomena, the right thing to do is not to reinvent *ex novo* the corresponding mathematical notion, but to favor the development of the language which appears appropriate to express it, exploiting suitable learning environments (Iannece & Tortora, 2004):

According to the above-mentioned Houdé's words, many ages always coexist in our minds. Therefore, teachers could and should exploit the synergy of all those cognitive dimensions, rather than being cast down for the so frequent *cognitive regressions* of their students.

A critical awareness and a responsible assumption of such a role surely make teachers reflexive and also in some cases turn them into true researchers. In working with in-service teachers, where teachers and students are simultaneously involved, this awareness and this assumption of role are supported by the immediate and long term interaction with the cognitive processes of learners. Several years of research on our part in the formation of school teachers, based on didactic strategies gradually validated, have convinced us that the guided collective participation in modelization or problem solving processes makes up a privileged entrance into the world of the combined acquisition of knowledge and professionalism. It is important to highlight that the cognitive dynamics put into play by in-service teachers (and by pre-service teachers in training as well) in substance correspond to what takes place in class; likewise the crucial role played by a meta-cognitive attitude is analogous both on an individual and group scale. We have also noticed that it is important in all situations to alternate auto-directed work of manipulation and interpretation either individually or in small groups (including substantial homework) with collective guided work of comparison and analysis of partial results, yet leaving to the individual the final systemizing of results and interpretation of the processes being adopted.

As said above, two kinds of activities appear as critical keys for both teachers' formation and students' learning: modelization processes from every day experience contexts and word problems. For word problems, we refer to Guidoni et al. (2003) and Tortora, (2001), where some of our views are reported; a similar analysis can be found in Mason (2001a). Here, we want to say a few more words about the modelization process.

In mathematics education literature, not to mention other fields, the meanings assigned to the word "modelization" considerably vary (see, e.g. Mason, 2001b and Verschaffel, 2002). Therefore, it is necessary to begin with an explanation of what "modelization" means for us (contrasting it for example with Verschaffel's definition). The data on the functioning of the brain show that our way of interaction with the external world lays on powerful, often unconscious, neural mechanisms for interpreting it: so, in a very general sense, a "model" is nothing more than a linkage between the things that happen and the brain that tries to understand them. According to this, we interpret modelization as a very complex process, neither deterministic nor one-way, where the formal structures are seen as one of the different correlated ways into which the cognitive reconstruction of external world structures takes form. This process cannot be reduced to guiding students toward abstraction, through a standard hierarchy of multi-representations (actions, words, graphs, and so on) whose top is identifiable by the algebraic formulation of a physical law. Due to the subtended cognitive dynamics, what is most effective is a continuous shifting from one cognitive dimension to another in a mutual progressive enhancement. And the process itself requires very lenghty didactic paths, even extended over the whole school curriculum.

A systematic resort to modelization processes in mathematics education, because of their resonance with natural learning strategies, enhances also students' motivation allowing them to actively participate in the construction of culturally validated theories of course within the cognitive and linguistic bounds of every age. At the same time both teachers and students can distinguish between a substantial continuity of the natural and scientific ways of organizing knowledge, and an essential discontinuity in terms of systematicity and inner coherence (a distinction which recalls again Vygotsky's dychotomy between a natural and scientific thought):

Therefore, it is important to make the first moves in the abstraction process starting from perceptual-motory experiences, which allow the involved notions to develop better and to be transferred to other situations. The choice of the contexts to be explored is always addressed by some conditions. Some contexts are surely privileged, like, e. g., *motion* (Balzano, 2007), *springiness* (Guidoni et al., 2003; 2005b), *shadows* (Boero et al., 1995), since they contribute to approaches characterized, since the beginning of the cognitive path, by direct manipulations guided by reflection on what is being observed. A good context should be at the same time simple enough to allow for an exploration not too rigidly guided, and complex enough to demand a careful, previous individuation of interacting systems and of pertinent variables, and to allow formulating non trivial conjectures.

We conclude emphasizing once again that collective and guided modelization activities bear an intrinsic value, independently from the mathematical content that they allow to build: a cognitive, a metacognitive and also an emotional value, inasmuch they are resonant with the natural way of functioning of our minds, and provided they are accompanied by the awareness of the development of our thought processes.

4. SOME EXCERPTS OF A CASE STUDY

Here we present a brief account of a class episode, in order to illustrate how the construct of resonance can help teachers in designing and managing class activity, as well as the interaction teacher-tutor in the training process. We refer to situations faced by some teachers of our team that are presented in De Blasio et al., (2008), where more details of the activity can be found.

Teacher (Nicoletta) with a mainly pedagogical background wanted to explain to a third-grade class, the role of zero in the multiplication of natural numbers. She tried to put reality and disciplinary structures in resonance, using the "linguistic" mediation of an action procedure. This worked very well until the zero was not involved:

As usual for me, I looked for "real" stories to support children in their construction of meaning for the operation. So, I proposed a movement activity where children were required to go back and forth on the number line then I said: "Go forward 4 steps 0 times"

Children's reactions were very interesting:

Anna: I am not able to do this, I can't move... What is the sense of the words "4 steps zero times"? I would never ask anyone to make 4 steps zero times.

Giovanni: 4×0 is not really a multiplication! A multiplication needs repetition of an action; it needs the 'times.'

Alessia: The word 'multiplication' is obtained by putting together two words: 'multi,' that means a lot and 'action.' Then multiplication means to carry on a lot of actions. But when there is zero there is no action, so we have to choose a new name... Perhaps we can still decide to call it multiplication but with a different meaning.

Of course, the children encountered serious difficulties in extending the familiar model of multiplication of natural numbers to this new "strange" situation. Till now, the cognitive strategies based on actions, strongly embodied according to Rizzolatti and Sinigaglia (2006), supported the conceptualization of the multiplicative structure, and, coherently, Anna and Giovanni tried to refer to those strategies, in order to understand. On the contrary, Alessia felt a dissonance between teacher's request and the "action" meaning of multiplication and, rather than forcing the motor-perceptual metaphor, displayed a different cognitive behavior which turned out to be resonant with a typical mathematical process: to enlarge the meaning of an operation. To solve the problem, Alessia resorted to a direct resonance between the mathematical structures and her own cognitive resources – the aptitude to change the rules in a game, in order to respect coherence constraints, – while neglecting reality, the third pole of our triangle. This kind of reasoning, of course more refined than the previous one, was still natural (Iannece et al., 2006), in fact it was spontaneously activated.

Alessia's intervention opened a way toward a higher level of shared understanding. Nicoletta, though at first bewildered, was able to recognize this opportunity by virtue of her participation in the project and of her exposition to teachers' formation activities, as described in her diary:

In a sense, my students were trying to convince me about the uselessness of my searching for a concrete situation that constituted a metaphor for this operation. To my great surprise, the students revealed a natural aptitude to change their point of view, jumping from reasoning supported by observation to logic argumentation. Alessia even analyzed the structure of the word "multiplication," looking for the sense of the operation.

Analyzing with mentors my students' reasoning, I became aware that not all mathematics can be discovered starting from observation of the reality. In fact, there are some rules that can be justified only by the necessity of an internal coherence of mathematics as a discipline.

But a still greater surprise was that my difficulties in leaving concrete motivations for mathematics rules were not shared by my students: for children the acceptance of a sort of game rules led to a generalization of already established meanings.

The sequel of the activity designed and guided in collaboration with my colleagues in the project and with my mentors successfully brought me and my students into encounter with a problem that I had never solved before: why is it impossible to divide by zero?

A new story begins…

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COLLABORATION AND LEADERSHIP

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ABSTRACT

This paper aims to contribute to the discussion about the key elements of collaborative work. The first part deals with the nature of such work, analyzes the difficulties of its management, and the importance of leadership. The second part, presents an example of a collaborative group within the Association of Teachers of Mathematics (APM). It is a group with self-identity that has been working in cycles with specific goals and purposes. The experience of this group suggests that the most serious barriers to the formation and sustainability of collaborative groups concern elements such as developing common goals, availability of time, support, creativity, and shared leadership.

INTRODUCTION

Nowadays, teachers work in a world marked by economical, political and organizational change. Change also affects the nature and the context of the curriculum and teachers' role. However, educational change requires a simultaneous and articulated action between the professional development of teachers and the development of school cultures (Hargreaves, 1998; Sowder, 2007). The work of collaborative groups may also be a key element in such processes of change.

The first part of this paper deals with the nature of collaborative work, emphasizing the importance of developing projects, and analyzing the difficulties of its management and the importance of leadership. The second part, presents the case of a collaborative group within the Association of Teachers of Mathematics (APM) in Portugal, to which I belong. Drawing on this example, this paper aims to contribute to the discussion about these processes, in order to enhance our understanding of how the work of collaborative groups can contribute to the professional development of its members and of their organizations.

COLLABORATION AND LEADERSHIP IN THE CONTEXT OF PROJECTS

Collaboration may develop from an idea of an individual or of a group of people facing a professional need. However, for collaboration to take place, a common goal is needed that unites all the members in a common activity in order to reach it. Collaboration is a favorable context for the reinforcement of trust which is necessary for innovation, as well as to improve effectiveness, reduce overload, create opportunities to learn and foster continuous improvement (Boavida & Ponte, 2002; Hargreaves, 1998). Furthermore, reflection and collaboration promote learning (Hargreaves, 1998; Schön, 1993; Sowder, 2007). The development of collaborative work within a project may promote the professional development and enrichment of its participants. It may create a dynamics that mobilizes the capital of knowledge held by teachers and support them in their actions.

As I suggest elsewhere (Nunes & Ponte, in press), sharing experiences, analyzing and reflecting about practice can promote teachers' professional development and the development of new professional and school cultures. However, some important issues need to be considered when we expect teachers to become reflective and to research their own practice. For teachers it is easier and enriching to do it with someone supporting them in the context of a collaborative relationship. It is difficult to overstate the importance of collaborative groups in the process of professional improvement. However, as Sowder (2007) indicates, it is important to emphasize that collaboration does not lead to improved results unless people are focused on the right issues.

During a collaborative work teachers are not alone dealing with an issue but they work together with other teachers that face the same problem. This creates an empathy that provides energy to overcome frustration, isolation and the feeling of not being able to deal with the problems. With collaboration it is possible to build an action strategy, where everyone has roles and responsibilities and works towards the same goals. The structure of collaborative work is quite demanding, since all participants are necessary. This work often does not demand a complex logistics but requires a real experience, sharing time, concerns, knowledge, expectations, and dedication. This sharing enables building better strategies to overcome difficulties and professional problems, a feeling of trust that turns a problem into an added value (Boavida & Ponte, 2002; Hargreaves, 1998; Sowder, 2007).

Collaboration does not necessarily mean that everyone has exactly the same power and the same role. Absolute mutuality is rarely achieved. What is critical is that all participants feel comfortable in their roles and are attentive to the needs of others and open to negotiate the understandings that emerge from the collaborative effort (Hargreaves, 1998). This is not an easy process. However, tensions that arise in collaborative relationships may help to keep these relationships alive and dynamic. In collaborative processes, there are no easy and safe answers. But what is problematic may provide an environment for further learning as all members try to understand themselves and others (Boavida & Ponte, 2002; Hargreaves, 1998, Sowder, 2007).

If we assume that all projects are built on personal interests and expectations, on former experiences of the team members, any project is unique because it arises from the participants' concerns about a particular situation within a social context, developed in a given space and time. We also have to consider the complexity of the situations and the uncertainty that motivates collaboration in a project. During the work of a project – in a context of professional problem solving – there are systematic reflection and collaboration. It is also important to establish some networking between schools, teachers, and researchers from different fields. Results from research can support change and innovation, enhancing trust and helping teachers to deal with the problems that emerge from their practice (Krainer & Koop, 2003; Sowder, 2007).

Some authors (e.g., Field, Holden, & Lawlor, 2003; Hargreaves, 1998; Krainer & Koop, 2003; Sowder, 2007) point out that one essential element of collaboration is leadership. Most especially, leadership may help mobilize different actors to work together, create collaborative working dynamics, stimulate new initiatives, and promote the professional development of teachers, with effects on the quality of students' learning. Collaboration involves sharing leadership and control over decisions about what group members will do, how and to what extend they will participate in decision-making (Krainer & Koop, 2003; Sowder, 2007). Leadership requires listening to the colleagues' experiments and preserving professional relations. These can occur in a formal way, such as subject group working sessions, or informally, when a colleague asks for an opinion, requests a piece of advice or shares a personal experience, a task or a classroom episode (Field, Holden & Lawlor, 2003). Each individual contribution of the group members makes the collaboration stronger. According to Stewart (1997), the leadership process has, necessarily, to be shared and symbiotic. Effective management of interpersonal relationships requires that leaders know how to develop their role within the group, make decisions, express themselves and listen, ask for and give information, stimulate discussions, know how to mediate and share, know how to encourage and facilitate communication, create a climate of trust, and solve possible conflicts (Boavida & Ponte, 2002; Field, Holden & Lawlor, 2003; and Stewart, 1997).

According to Sergiovanni (2004), leaders are like orchestra conductors and managers of meaning. They identify what is important and give the other participants a sense of direction and of purpose through the articulation of a compelling worldview. The vision (Bryman, 1996) and leadership expertise and skills, in association with personal experiences and creativity of colleagues (Field, Holden & Lawlor, 2003) are key elements for a successful project of professional development of teachers, thus contributing to a creation of a new professional and school culture (Sergiovanni, 2004). We can say that there is a strong connection between collaboration and leadership, namely there is no collaborative work without leadership and no effective leadership without collaboration (Hargreaves, 1998; Krainer & Koop, 2003; and Sergiovanni, 2004).

However, the process of establishment of a collaborative group is often complex. Sometimes, it begins from a personal project of one individual that, little by little, gets attention and attracts partners. It often happens in school environment that when successful, such slow processes lead to the creation of collaborative groups or, simply, to partnerships between two people. It requires an insightful work by leaders, as the way they deal with the group's potential collaborators and negotiate the collaboration contract may determine the success of the alliance. Achieving this first step allows the development of new projects and the definition of a working agenda with well defined goals. But how does a group maintain its vitality and working dynamics? How to assure the sustainability of a collaborative group through time? To answer these questions we need a deeper understanding of the dynamics of collaborative groups and their leadership processes.

THE STUDY GROUP

To be a teacher today requires constant learning and capacity to investigate and reflect on one's own practice and on students' learning. These issues are quite present in recent research on teachers. In 1999, the Working Group for Research (GTI) of the Association of Teachers of Mathematics (APM) decided to create a study group focused on the topic "teacher as researcher." This was an answer to the interest of a group of teachers of different school levels to analyze and reflect on the situations they faced in their professional practice. The group considered itself, since the beginning, as a collaborative group, with its own identity. Its members are mathematic educators and teachers with experience of research in education. The group has developed up to now three working cycles, lasting for about three years each. In each cycle the group developed research projects and wrote papers about them. The first cycle ended in 2002 and resulted in the publication of the book *To Reflect and Research about Professional Practice* (GTI, 2002). The second cycle ended in 2005 with the book *The Teacher and the Curriculum Development* (GTI, 2005). At the moment the group is finishing the collection of papers that will compose the third book to be published in September 2008.

At first, the work of the group concentrated on selecting, reading and discussing theoretical papers that helped to define the different research projects of its members. After this stage different projects were discussed in the group, which helped to outline strategies for their development, and an agenda that allowed achieving the main goal of each cycle – publication of a book. It was a moment when sharing individual experiences, reflecting within the group and supporting individual members were essential features of the group activity. Subsequently, the working sessions essentially consisted of a discussion and reflection focused on each of the papers composed for different books, which tested the individual or collaborative group work of each member of the group. In the process of paper elaboration remarkable work, collective and collaborative in a team, developed on two levels. The first level, in work developed by teams, each author worked with a partner and, in the case of individual projects, they had a support of one or more members of the group for suggestions on their version of the paper, before it was submitted for a general discussion. On the second level, involving all the members of the group, all the papers were discussed several times and successively improved. The development of different kinds of work in the three cycles of the study group created unique opportunities of sharing, and of personal and collective enrichment.

In the first cycle of the collaborative work the group focused on the notion of teachers researching their own practice. Different members wrote papers that presented their professional experiences while researching issues emerging from their own practice, as a way to understand their actions better and, if needed, to improve them (GTI, 2002). The various investigations, carried out individually or pairs, constituted an opportunity of professional development. Such papers addressed students' learning, students' relation to mathematics, and the underlying classroom dynamics.

This is the case, for example, of the papers by Segurado (2002) and Pires (2002). The paper by Segurado (2002) illustrated how the students of the fifth and sixth grade addressed and engaged in investigative tasks, and what that may reveal about their knowledge and skills. The study took place over a school year and included four investigation tasks. She assumed the role of researcher and teacher of the class. Students demonstrated even better understanding of the meaning of investigative tasks showing different mathematical knowledge. She also observed that her students developed superior skills such as exploring, conjecturing, proving, justifying and arguing, as well as autonomy and capacity to communicate.

Pires (2002) presented a project that she developed with a group of teachers with the eleventh grade, aiming to explore the extent, the potential and the difficulties associated with carrying out different tasks in the mathematics classroom in secondary education, in teachers' and students' view. The teachers selected, prepared and performed in the classroom four types of tasks: exercises, problems and modeling tasks and projects. She stressed the importance of the subject group working as a reflective-action group, in order to create rich and stimulating learning environment, try new materials and optimize resources, particularly technologies. In general, the teachers concluded that for profound learning, it is necessary to offer students different tasks that involve processes of discovery that requires understanding of the issues. Finally, they concluded that it is necessary to put more emphasis on the problems, without reducing their degree of difficulty, and to implement modeling tasks and projects with balance, and to attach superior skills.

In the second cycle of the study group the papers that constituted the book showed in such a way that curricular questions are particularly difficult and the curriculum management becomes most complex, even more as teachers today have to deal with a large social and cultural diversity of students (GTI, 2005). However, it is perceived in the collective papers that these questions may be considered when the various actors of the educational process – especially students and teachers – are involved and participate actively. The entire study focused on the subject of curriculum and curriculum management still in the view to properly research teachers' own practice. The book describes experiences lived by the authors in the management and development of the curriculum with its students, who looked to bring contributions to the understanding of the curriculum issues. It focused in particular on the role of various tasks and strategies in curriculum development, teachers' role in mathematics literacy, and curriculum development in its relation with the mathematical competence.

Good examples are given in the papers by Ferreira (2005) and Rocha and Fonseca (2005). The study described by Ferreira (2005) illustrates a strategy for teaching elementary arithmetic operations over the four years of the elementary school, with particular attention to the division, starting with problem solving. Rather than start by teaching the algorithm and then solving exercises of application, she proposed to students problems involving different situations, which supported and reinforced their strategies. She concluded that students were able to solve relatively complex problems long before what is usually assumed.

Rocha and Fonseca (2005) deal in particular with the moment of discussion in class arising from the conduct of investigative work by students from the tenth grade of secondary school. The authors documented how students gradually participated in a more productive way at times of discussion and concluded that these moments involved two fundamental processes – the confrontation and the defense – and allowed further work, took the students and teacher to get involved in mathematical reasoning, formulating new problems and new conjectures, and enhancing the process of justification/proof.

The third cycle is not finished yet. As of today, the papers that will shape the third book are still being worked on. They document different realities of school projects and dynamics of several school mathematic subject groups. In the seven different projects developed we can identify descriptions of ways developed by the group members to create collaborative working dynamics and, on the other hand, to keep the energy sustaining the subject group that, as always, identifies itself as collaborative.

With these experiences the group aims to contribute to the research, disseminating the work of collective projects in schools, looking for understanding how schools are organized and face the problems that arise. Moreover, it is also important to understand how these projects can change the realities of schools and what the role of leadership processes is. For example, Almiro (in press) describes a project that he developed with eight other mathematics teachers from his school. The project had a central goal – to overcome and reflect on some of the difficulties they faced in their practice in mathematics communication. This theme had increasingly been a concern of this group because they identified that many of their students, particularly in secondary education, had enormous difficulties in interpreting documents and in producing written mathematical texts.

There were some strategies that the group decided to use that may have had some influence on the improvement of the solutions of the students. On the one hand, they paid attention when doing the interpretation of the paper, suggested underscoring the most important contents, diagrams or notes in the margins, which contributed to the understanding of the essential aspects of the problems. Moreover, they considered regularly giving students tasks that called for the production of written papers. Supported by the results of one-year-work Almiro believes that the completion of regular tasks of this kind led the students to look more naturally to the production of texts in mathematics class and may have been a significant factor in the improvements recorded in this work. Finally, the author stresses the possibility that the students had to rewrite their answers, confronting themselves with their previous solutions and the possibility of redoing. This seems to have been a strategy of satisfaction of the students, which have brought some improvements to their performance.

The correction of the students' written productions was one of the most labored aspects in this project. It was a difficult and complex matter. And, although it did not go very deep in identifying the difficulties expressed by the students, all of the work that was done on finding solutions using the literature, building a table of descriptors, applying it to a set of answers, evaluating its effectiveness, and identifying difficulties in its implementation was certainly a significant moment for the teachers' development in this area.

Today, a year after the Mathematics Communication Project, another project is being developed at the same school, whose theme focuses on investigative tasks and demonstrations, two aspects where the group felt they needed training. This mode of facing the profession capabilities has led to the creation of an environment of dialogue and cooperation with the sharing of ideas and experiences, successes and anxieties, the clash of views, concerns and knowledge and that seems to have, so far, a satisfaction of all and an essential contribution to the personal and professional growth of all.

Reflecting deeper over the dynamic work of the study group along the three cycles, we realize that it has the capability to adapt to the needs of its members. Also, the group allowed its renovation according to each member's interests and availability, enriching and revitalizing its dynamic. Likewise, the environment of cordiality and collaboration in the different cycles of this work is considered by all participants as very useful and productive, factors that allowed the existence of a genuine reflection on the practice on two levels, within the individual pair and within the group (GTI, 2002, 2005).

Nevertheless, the sustainability of this group depends also on its leadership. The group does not have a formal leader and everyone assumes leadership roles, in particular, in their research projects. Analyzing the dynamic of the group working sessions we can clearly see that their management is shared and the members of the group recognize one of its members as the person that legitimizes many of the group decisions with the GTI coordination and, therefore, with APM. After the third round of studies the time comes to make a balance and reflect on the future of the group. New projects will depend on the determination and interests of its members, but also on the dynamics of leadership, particularly on the ability to generate new energy and find new focus for research and a common agenda.

CONCLUSION

The most serious barriers to the formation and sustainability of collaborative groups involve the definition of common goals and the availability of time. In the collaborative culture of the GTI study group it has been possible to identify the key elements for collaboration as common goals and agenda, commitment, communication, willingness to share and reflect, trust, partnership, respect, and leadership (Boavida & Ponte, 2002; Field, Holden, & Lawlor, 2003; Hargreaves, 1998; Krainer & Koop, 2003; Stewart, 1997; and Sowder, 2007). These elements contributed to the professional development of the members of the group, in particular as they reflected about their professional practice and shared their concerns. Also, the work developed by the group contributed to our knowledge about teachers' professional practice, bringing them together and involving them in research.

The research made by the members of the study group shows that teachers' professional development in the context of school projects must be systemic to be effective and to improve student learning particularly by researching their own practice. But other things are necessary as the dynamic interaction of curriculum, assessment, school organization, materials, support and leadership (Hargreaves, 1998; Sowder, 2007).

This reflection shows that educational change needs the combination of collaboration and leadership (Hargreaves, 1998; Krainer & Koop, 2003; Nunes, & Ponte, in press; and Sowder, 2007). This is a key issue for educational research. It is important to continue exploring the culture of collaborative groups and their dynamics to contribute towards better knowledge of the goals, processes and issues faced by teachers when they work collaboratively and how these processes can help change school and the professional culture.

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ETHICS OF TEACHER-RESEARCHERS

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ABSTRACT
The paper examines the integration of researchers' ethics with teachers' ethics in the context of the teaching-research methodology classroom. It is shown that the integration of the two principles treated as ethical imperatives of equal strength leads to a series of constraints upon the classroom teaching-experiments as well as upon the acceptable research questions. Two component teaching-research questions are defined as better corresponding to the nature of teaching-research, and are offered as the path of reconciliation between two separate kinds of knowledge: theoretical knowledge of researchers, and practical knowledge of practitioners.

INTRODUCTION

The discussion of the teacher-researchers' ethics is motivated by three independent sources:

(1) The editors of an important ICMI study *Mathematics Education as a Research Domain: A Search for Identity*, which aims to lay the foundations for the scientific identity of the educational research profession, observe that it may contain two, apparently separate types of knowledge: "the theoretical knowledge for the scientific community of researchers and the practical knowledge useful in applications for teachers and students." They suggest that it might be helpful to reflect on the nature of these two types of knowledge, on relations between them, and on whether it would be possible to have a unified body of knowledge encompassing them both. The comments below as well as in other papers by the author in *TR Handbook* oscillate around the relationship between the two.

(2) One of the few comments on the relationship between research and teaching comes from a teacher-researcher, Marian Mohr (1996) who offers a very interesting and relevant perspective, the exploration of which will bring us closer to the questions stated above. According to Mohr, "Teacher-researchers have assumed that what they do differs from others ideas of both teaching and researching." Teacher-researchers see themselves as "doubly bound to ethical behavior both as teachers and researchers. How students are treated is a measure of the quality of both teaching and researching."

Consequently, in order to formulate the principles of TR ethics one needs to be aware of both research and teaching ethical principles, and properly formulate the relationship between them. This way we might get a proper set of conditions with the help of which we will be able to define harmonious integration of both based on the principle of equality of ethical imperatives. Creation of such a compromise will contradict Wong, who, in an article in *The Educational Researcher* (quoted in Mohr, 1996) asserts that researching and teaching are in conflict because they require a different kind of knowledge and generate a different kind of inquiry. If indeed the two types of knowledge and inquiry are so substantially different, then of course, the ethics of teaching research would be contradictory; and yet at the same time, since practicing "teacher-researchers do feel bound to both research and teaching ethics," the possibility of successfully resolving the contradiction between the two seems quite real.

(3) The ethics of the teachers/researchers' interface is at present fraught with the absence of such an equilibrium and is of concern to us here. In fact, in the same ICMI volume Bishop (1999) points out to an "assumed power structure [within the academic researcher/teacher interface] which accords the researcher's agenda and actions greater authority than the practitioner's. The increasing moves to involve teachers in research teams are to be applauded, but currently serve only to reinforce the existing power structure."

A very pointed confirmation of that particular power structure within the interface can be seen in the same volume where Wittmann (1998) in his excellent remarks about a didactic instrument called "a teaching unit" wants to convince the academic profession that this didactic tool is worthy of research attention of academicians. However, he notes that there is a problem with the teaching unit, namely, it is a standard tool of good teachers, who till now were the only ones paying attention to it. Therefore, since "the design of teaching has been considered as a mediocre task normally done by teachers and textbook authors … why should anyone" – he asks – "anxious for academic respectability stoop[1] to designing teaching and put him- or herself on one level with teachers?" (96). Having thus defined the extent of the social, and possibly, class gap separating the profession (the term "stoop") and demonstrating how that gap hampers the investigative interest of academic educators, he now has to extricate the teaching unit from its grip. The author continues "that teachers take part in design can be no excuse for mathematics educators to refrain from this task. On the contrary: the design of substantial teaching units … is a most difficult task that must be carried out by the experts in the field. By no means can it be left to teachers, although teachers can certainly make important contributions within the framework of design provided by experts."

We can see several elements in the statement which define the relationship between the two components of mathematics education profession: we can see the assessment of the activity of designing instructional sequences by teachers as "mediocre" within the academic profession; we can see the word "stoop to" characterizing the actions of academic researchers which "put them on one level with teachers;" we can see the process of recognition of its didactic value and the prescription to take away the ownership of the tool from teachers, so that its scientific improvement can be made under sole responsibility of research community experts. That process can be seen as the act of dis-appropriation of teachers as a profession from its professional tool with the help of power structure within the mathematics education community, which distinguishes "experts" from mere teachers.

Clearly, if we believe that the ethical research imperative and ethical teaching imperative are of equal value in the class where teaching-research investigations take place, then Wittmann must be violating some ethical components in his argument. Consequently, a natural question arises: what is the ethical principle that Wittmann's process is violating?

[1] - bend body: to bend the top half of the body forward and downward
- walk or stand bent over: to walk or stand with the head and shoulders bent forward and downward
- do something unethical: to act in an unethical or self-degrading way
- condescend: to do something reluctantly and with the attitude of somebody who considers such action unworthy (Encarta dictionary, MSN)

TEACHERS' VOICES

Among teacher-researchers of mathematics who have expressed their views on the ethical values of their classroom work, the fundamental theme, which comes out in sporadic comments is the primacy of teachers' responsibility to children in class. Pawłowski (2003), one of the few teacher voices exploring the ethics of classroom research, offers the following suggestions on the matter:

> Professional investigators have as their responsibility to explore a research problem in its entirety, to grasp as many of its aspects as possible, to continuously doubt and to continuously document the investigations. Teachers, responsible for children, students in their classroom, have to as their primary responsibility use the best methods available to them. All doubts that teachers are entitled to, must be solved with the help of the criterion of their responsibility to children, taken in the best of faith; their right to a planned meandering or to conduct control measurements is strongly restricted. Teachers have no right to conduct "negative" experiments directed to show that some configuration of factors leads to worse results. If teachers learn from [experience] that some didactic procedure is better than a standard one and know about it (because of course they have the right and responsibility to rely on their own memory), but did not yet collect an adequate documentation to provide the evidence for their observations, nonetheless they do not have the right to return to the method recognized as less successful – to return only in order to document their observations.

Teacher-researcher Jim Minstrell develops the theme further stating thatThe more immediate of the two is the improvement of their teaching practice. "That is, when teachers engage in research on their teaching, they do so to get better at what they do. The second purpose is to seek an improved understanding of the educational situations in which they teach so that they can become a part of the knowledge base of teaching and learning" (Feldman and Minstrell, 2000).

Teacher-researchers from related Fairfax County Public Schools Teacher-Researcher Network agree with Minstrell stating that their primary responsibility is to their students. Pawłowski (2003) agrees with that view as well:

> The main goal and decisive criterion for teachers must be a correctly diagnosed well-being of students for whom they are responsible. Only in the second instance, the usefulness of observation from a general, objective point of view can be the goal. It is the objective investigative constraint, and not, as one might think in a simplified manner, the limitation concerning only teacher-researchers, who are involved in other tasks. That is why didactic investigations are governed by a rigid system of values, which must be respected during any such investigations independently of who conducts them.

We would like to pause and reflect on the last statements of Pawłowski, because of the seriousness of their implications for the ethics of classroom research. For if we accept Pawłowski's statement that the ethical considerations of classroom research constitute objective constraints upon it, and they need to be fulfilled not only by teacher-researchers but by any investigators or teaching-research team doing action research in the classroom, then we have a strong criterion according to which the program of Wittmann above can be judged as non-ethical. Elimination of teachers from the framework, within which the design of instructional units is determined, eliminates or downgrades the concern for the well-being of students, for which, in a given classroom, solely teachers are responsible. Consequently, the program of action concerning teaching sequence units discussed above can be rejected on purely ethical grounds governing the mathematics classroom. Since teachers have the sole responsibility for the well-being of their students independently of any research design, every design of classroom interventions such as teaching units must be developed with and approved by these very teachers. To achieve it, one can not leave it in the hands of academic experts – teachers must be members of the team with their experience as an essential source of knowledge in the unit's design in order to guarantee the intellectual well-being of students in the experimental class.

Of course, contrary to views of many academic researchers who are convinced that teachers' participation in research negatively "changes the relevance and meaning of results" because, for example, "teachers' result is attached to their school and is therefore not research," research can be done with adequate requirements of generalizability. But the research questions need to be asked in a different way, and the organization of a teaching experiment needs to undergo a serious re-definition.

Teachers know that what they experience in one classroom is not very far removed from what is experienced in another, even quite far removed. What we do not know yet exactly is how to characterize different classroom domains where a similar experience can take place. Jim Minstrell informs us that

> My "rule of thumb" has been that if about 10% or more of students exhibits a similar sort of thinking, then I need to acknowledge and describe the conceptions and reasoning they are using, and I need to design instruction to address that thinking. These findings have been generalizable beyond my classroom. Although one might think that there would be as many different ideas as there are students in the classroom, this is not the case. Usually, there are between two and eight approaches to thinking exhibited by the class when confronted with a particular situation. When we present similar sort of situations, I see the same behavior replicated in the classrooms of other teachers. And, in most cases, the lessons that work to perturb the problematic thinking in one classroom also work in another. Thus, these findings are considered generalizable.

In a similar vein it was interesting to observe the interaction of teacher-researchers from one of the Polish PDTR teams, R-K with mathematics teachers from Warsaw, 250 km north of Rzeszów, who, upon learning about the teaching-research on the problem of solving and understanding the concept of an equation done by Rzeszów teachers, performed similar work in their classrooms, and, of course, found similar issues and similar ways of addressing them. Hence, when properly analyzed, one of the first issues in need of correction is the generalizability of teaching-research done by teacher-researchers. It is not as wide as researchers would like, that is, most probably it does not work "for all classrooms," but it is not as narrow as "teachers' own school." Thorough investigations are needed to determine the scope and nature of teaching-research generalizability of results. In the next section we will investigate in detail several other constraints upon the teaching-research work implied by the teachers' ethics.

The responsibility for the intellectual well-being and development of students constitutes an independent ethical principle of teachers' profession in its collaboration or cooperation efforts with the academic profession. That principle is complementary to the basic research ethics of education researchers, i.e. the responsibility to the field (e.g. Ethical Standards AERA or AARE Code of Ethics), and as such it assumes an equal ethical value as that of the professional research integrity principle e.g. expressed by the point 1 of the Section I, Guiding Standards of AERA "Educational researchers should conduct their professional lives in such a way as not to jeopardize the research results."

One could say that the teacher ethics principle discussed above can be similarly phrased as "teachers should conduct their professional life so as to not jeopardize the intellectual well-being of students in their class." One of the main goals of this paper is to show that if the consequences of the teachers' ethical principle are explored to their utmost, then the nature of teaching-research and of a teaching experiment are determined to a substantial degree.

RESPONSIBILITY TO CLASSROOM STUDENTS *AND* (BUT NOT *OR*) RESPONSIBILITY TO THE FIELD – TEACHER-RESEARCHERS' ETHICAL DILEMMA

Is there a possibility of a compromise between the two responsibilities? How to conduct classroom research so that it is 100% for the benefit of children in the class the teacher-researchers? How to teach so that every moment of it is the investigation of learning? Below we have some, possibly incomplete as yet, answers to these fundamental questions, phrased in the context of methodological differences between research and teaching-research.

One of the ways in which the quality of teaching and learning can be maintained by teachers in the classroom is through an investigation of teaching process in order to improve learning. That emphasis on the improvement of learning as a necessary component of classroom investigation has been aptly expressed by Jim Minstrell in Feldman and Minstrell (2000). This emphasis differs from the aims of collaborative teaching-research as expressed, for example, by Raymond and Lienenbach, (2000) who state in the section Goals of Collaborative Action Research:

> A primary goal of the collaborative action research identified by many is to bridge the gap and strengthen the relationship between universities and schools. Collaborative research between university researchers and classroom teachers present opportunities for a more action-oriented approach to teacher enhancement. As teachers are encouraged to reflect upon and systematically examine aspects of their classrooms, they are likely to make changes based on observations that lead to the improvement in their classrooms.

In other words the classroom improvement of learning might possibly be a by-product of researcher-teacher collaboration, whose main goal is to bridge the gap between university and schools. For Feldman and Minstrell (2000, above), on the other hand, the mere *likelihood* of classroom improvement is not enough for classroom teachers; instead, it is their primary goal with reflection and examination of the classroom as the tools for that improvement. It is a very important though subtle change of emphasis which can govern the nature of any teaching-research collaboration.

The difference in research interests between educational researchers and teacher-researchers resulting from incorporating teachers' ethical principles, is the difference in research interests, which, of course, is also natural; while educational researchers' primary concern is the investigation of teaching and learning processes in their generality, the task of teachers committed to the best quality of instruction is the investigation of a particular learning issue. Consequently, we obtain significant differences in the formulation of research questions. Cobb and Steffe (1983) assert that the primary interest of experimenters engaging in a teaching experiment lies in "investigating what might go in children's heads" and in "hypothesizing what a child might learn." Czarnocha (1999) responds that "In contrast to the interest of the experimenter, the teacher's interest here is to find means and ways to foster what students need to learn in order to reach a particular moment of discovery" or mathematical understanding. Hence investigative acts of teacher-researchers are geared to better the instruction and it is because of that that they undertake classroom research. Doing it simply to investigate students' thinking without using it to better the teaching and learning process in the classroom is in disagreement with the teachers' ethical principle. A PDTR colleague, an Italian teacher-researcher offers a following compromise between the two different interests:

> I imagine that the goal for the team teacher-researchers/educational-researchers (ER) is a harmonious whole between 'might learn' and 'need to learn.' I see them as 'relative concepts' and the balance between them is a common compass for TR and ER. In other words: what might go in

children's heads while they are working with topics that they need to learn? Or: How do children's different ways of thinking (often underground, not expressed verbally) influence learning of what they should need to learn?

Note that in the compromise proposed here the question "what might go in children's heads?" has a correctly limited scope defined by that which they "need to learn".

Nature of teaching-research questions

To fulfill both the responsibility of teachers and the responsibility of researchers, TR/NY City model proposes to establish two teaching-research questions for each classroom investigation, the first one investigating the ways of improvement of learning a particular concept, or a procedure, the second question concerning the assessment of the state of affairs concerning to the issue in question – a research question more in agreement with the standard approach of educational researchers. For example, one can ask how students understand the symbol of equation – a standard research question assessing the state of affairs; and one can ask how to improve student understanding of the symbol of equation – a standard improvement question characteristic for TR work.

What is equally interesting is that the second research question about the improvement necessitates the first one: it is difficult to talk about improvement and to assess its needed degree without knowing the state of affairs before the improving intervention. Formulation of such pairs of questions, which separate the static question about the state of affairs and the dynamic question of the best route to effect the change of improvement brings mathematics education endeavor closer to the structure of classical physics, and in particular, its first two laws of dynamics, where the first one asserts the state of affairs without the presence of force (or interaction), while the second law formulates the dynamical principles of change of the system, which give its trajectory. It is worth noting that a recent interest in the Hypothetical Learning and Actual Learning trajectories quite neatly coordinates the formulation of the two teaching-research questions, providing, quite possibly a new, complete and scientifically grounded metaphor for the classroom teaching experiment.

ORGANIZATION OF A TEACHING EXPERIMENT (TE)

Teachers' ethics formulates strong conditions upon the length of a teaching-research experiment, which require classroom ingenuity of teacher-researchers to satisfy (Czarnocha, Prabhu, 2006). Teaching experiments need to be situated within the regular cycles of work so that the students who are the subjects of investigation are also its first beneficiaries. In simple cases, one can state as a guide that a minimum of two cycles of TE per unit of the classroom instruction, a semester or a year, are needed to fulfill this ethical requirement. Two cycles assure the refinement of instruction based on the qualitative or quantitative analysis, and hence its improvement through incorporation of results of investigation. However, one can bypass this simple condition in special circumstances, which still allow for the satisfaction of the ethical principle. For example, if teachers teach the same cohort of students the next unit cycle in the school then they have an opportunity to introduce the results of research after one unit and still to fulfill the requirement. Or as a TR apprentice in one of the TR teams of the Socrates Project describes her way of dealing with the problem of a control group through the parallel class taught by the same teacher (Łaszczyk, 2007). The ethical problem that teachers encounter here is that, in reality, it is impossible to have two parallel classes and to implement new instruction one believes in, only in one of them and not in both. In other words, she did not want a control class, whose students were used as an object of comparative assessment, not to receive the benefit of a teaching experiment conducted in the parallel experimental class. She divided the teaching experiment into parts within the year and having received the confirmation or rejection of her hypothesis in one class in a given part of the curriculum, she immediately introduced an improving technique to the other class, but only for that part. This way she was able to assess the effectiveness of the innovative instruction in its components, and at the same time satisfy the ethical principle.

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PART 2
ELEMENTS OF THEORY IN TEACHING-RESEARCH

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METHODS AND TOOLS TO PROMOTE A SOCIO-CONSTRUCTIVE APPROACH TO MATHEMATICS TEACHING IN TEACHERS

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ABSTRACT

This paper offers a short introduction on teachers' role in socio-constructive teaching and a synthesis of the main directions suggested by research in this field, and then deals with methodology used by and for in-service teachers involved in a teacher training program, aiming at a refinement of their skills in orchestrating mathematical discussions. Particular attention is paid to the description of a specific tool for teacher education that we constructed starting from teachers' transcripts of their own classroom processes and by going through multiple analytical comments. This tool stresses and magnifies teachers' attitudes, conceptions and knowledge as emerging from the process: teachers are thus led to critically reflect upon their own action in the classroom and possibly adjust it. In this paper discussion focuses on the main problems concerning teacher education, highlights the use of this tool, and ends up with reflections on its formative efficacy.

INTRODUCTION

The rapid increase in scientific and technological research led to a need for a different mathematical education, pointing to modeling and problem solving rather than to the learning of mathematical facts, and open to argumentation and cooperative work in the classroom. As a consequence, conceptions about the mathematics to be taught and the related teaching modalities have changed, as shown by the USA Standards (2000), by the International P.I.S.A. assessment test (2000/03) and also by the recent proposals made by MIUR-UMI for mathematics teaching (Anichini et al., 2001-2004).

Current research in mathematics education points to *socio-constructive* teaching as the most suitable to raise students' interest in mathematics and help them develop a conception of the discipline, closer to real life and fitting with the current educational needs. This is particularly true for compulsory school, where a conception about the discipline is initially shaped. In this type of teaching, students are supposed to explore situations purposefully created to raise specific mathematical notions: the latter are constructed within a teacher-students interactive process, aiming at objectifying mathematical concepts and facts by reflecting upon the exploratory and investigative processes that brought to them.

In this perspective, teachers come to play a multifaceted and complex role. They need to play a wide range of roles (provoker, maieutic agent, orchestrator of discussions, model, etc.) in the classroom and thus need to face a number of unpredicted and not easily manageable situations.

In particular, teachers carefully: (1) plan teaching sequences able to foster students' conceptual constructions; (2) create an environment that favors students' mathematical exploration and formulation of conjectures; (3) choose suitable communicative strategies that support students' interaction and sharing of ideas. Moreover, with relation to specific mathematical issues, they need to be able to predict students' reactions to particular questions, their possible thinking processes and, most importantly, need to face threads of mathematical discussions that deeply differ to what they expected. For this reason, teachers need to be able to foresee the problems that can arise in the classroom even before getting to solve them.

This stresses the importance of issues related to *teacher education and professional development*, which have been widely explored by researchers in recent years. This is also confirmed by the space devoted to this theme at the most important international congresses; by the increasing number of research studies on teachers' role (Sfard, 2005; Adler et al., 2005); and, significantly, by the recent implementation of the 15th ICMI study *The Professional Education and Development of Teachers of Mathematics* (Ball & Even, 2008).

Researchers agree on the fact that in order to implement a socio-constructive type of teaching, not only should teachers get a fine and in-depth knowledge of the so-called "pedagogical content knowledge" (Shulman, 1986), i.e. notions related to learning difficulties, psycho-pedagogical issues and epistemological obstacles of the mathematical contents to be taught: most of all, they should get to know interactive and discursive teaching models (Wood, 1999).

Regarding the latter issue, back in the 1990s some scholars highlighted the macro-effects on usual classroom activities caused by sudden and out-of-control micro-decisions made by teachers (Artigue & Perrin-Glorian, 1991).

In view of a reduction of these phenomena, several scholars stressed the importance of a critical reflection by teachers on their activity in the classroom: this might help them develop an attitude characterized by constant self-control in action as well as in the awareness of the possible consequences of their actions (Mason, 1998; Jaworski, 1998, 2002; Lerman 2001; Shoenfeld 1998). Mason, in particular, proposed the study of the *discipline of noticing* (2002); he claimed that the skill of consciously grasping things comes from constant practice, going beyond what happens in the classroom. He recommended the creation of suitable social practices in which teachers might talk about and share their experience (Mason, 1998).

In the last decade, several studies have been carried out both *with* and *for* teachers, in order to lead them to acknowledge the incidence of their choices (both actions and omissions) in the development of a mathematical discussion and, at the same time, make them aware of their way of being in the classroom. These studies show many different approaches, but generally they all aim at fostering teachers' critical review of their own conceptions of mathematics and of its teaching, so that they can become aware of the complexity of the classroom work, as well as acquire new and more appropriate models of behavior (Borasi et al., 1999; Ponte, 2004; Potari & Jaworski, 2003).

This trend of studies provides a framework to our investigation, although we also deal with issues concerning the renewal of the teaching of algebra through a linguistic and constructive approach in the sense of early algebra (for the latter theme, see Malara & Navarra, 2003; Malara, 2008).

Our research experience with teachers made us aware of the difficulties they meet in both designing and implementing a socio-constructive type of teaching. We have observed how, despite the good intentions, in the development of mathematical discussions, often teachers do not devolve problems to students: in this way, students are not aware of the fact that a solution should emerge from a collective investigation, though validation and critical merging of everyone's contributions. Teachers tend to talk to single students individually and do not coordinate peer-to-peer discussions: anxious to get to a conclusion, teachers often ratify the validity of productive interventions without getting the classroom to discuss and validate them. Moreover, teachers tend to let interesting contributions drop, if they diverge from the plan they have previously outlined, or rather are not able to recognize potentialities of certain students' interventions (Malara 2003, 2005; Malara, et al., 2004).

For these reasons, in accordance with Wood (1999), we deem important that both pre-service and in-service teachers, analyze their own or others' didactical processes, undertake new modalities for teaching mathematics and reflect and share ideas about their own actions.

OUR METHODOLOGY

Our studies focus on the design and experimentation of innovative teaching pathways and they have always been realized in a strict collaboration with teachers, following a traditional Italian model (Arzarello & Bartolini, 1998; Malara, 2002) which shows analogies to the co-learning partnerships model implemented by Jaworski (2003).

Our first studies focused mainly on students' learning difficulties, although we always kept an eye on curricular innovations; later, the setting up of postgraduate schools for teacher training and the international trends in research led us to focus on teachers' problems and their education, with the aim to identify instruments and methodologies that may help them acquire the skills needed to implement a socio-constructive type of teaching, meeting the current cultural needs.

Our hypothesis is that critical and reflexive observation and analysis of socio-constructive classroom processes[1] are a necessary condition for teachers to become aware of the new role they should play in the classroom, of the processes enacted by a collective mathematical construction and the related variables. We also believe that this activity should be supported by the study of mathematics education theoretical results, which can strengthen teachers' knowledge of both the discipline and its teaching, support or modify their beliefs, and make them aware of the incidence of theoretical studies on their own professional development (Malara & Zan, 2002).

For this reason, our current studies focus on the *analysis of classroom processes* that develop along planned sequences implemented by teachers. Our main aim is to lead the involved teachers to get a higher and finer control of their own behaviors and communicative styles and observe the incidence of critical analysis discussions on both classroom processes and students' behaviors and learning. Moreover we aim to design tools for teacher education that may be used in both postgraduate schools for training teachers and in distance education (Martellotta, et al., 2006; Malara & Navarra, 2007; Malara, 2008).

To reach these aims, we carry out a complex activity of critical analysis of classroom processes' transcriptions and reflect on them, looking at the relationships

[1] In Italy, due to recent changes in the modalities of judicial proceedings, nowadays it is widely recognized in the legal field that the training of Public Prosecutors and Defense Lawyers should include the study of paradigmatic examples of trials. The aim is to explicitly show the effects of moves, decisions, actions undertaken during a trial and allow trainees to master a range of models and attitudes that may suit their role in different situations, by examining different cases (see for instance Carofiglio, 2007)

between knowledge constructed by students and teacher's behavior in guiding them to achieve such constructions.

The activity develops along four subsequent phases, focusing on: the class teacher's autonomous reflection; teacher and researcher's joint reflection; shared reflection of teachers involved in the same teaching sequence; teachers and researcher(s)' interactive reflection.

In the first phase, concerning individual interpretation of what happened in the classroom, teachers are asked to transcribe mathematical discussions and comment upon critical or productive points. Teachers are thus forced to critically re-examine the process evolution with a particular focus on their own ways to communicate with students (asking questions, giving directions, making decisions etc).

Transcriptions of recordings, enriched with these initial comments made by teachers, form the kernel of the *process diaries* (later on simply referred to as 'diaries').

In the second phase, researchers read the diary analytically and write both local and general comments. Diaries containing these new comments are then sent to teachers by e-mail and later jointly studied by teachers and researchers. The latter guide teachers to reflect on specific issues, by asking them to explain the meaning or reasons underlying some interventions, point to potential strategies to overcome dead-ends or rather explain some subtle facts of the emerged mathematical issues. Researchers also encourage global reflections on what has been done and highlight meaningful points in the evolution of the mathematical construction.

This joint analysis helps teachers spot their habits, stereotypes, conceptions and to disclose possible gaps or misconceptions in their mathematical knowledge. This is a particularly important moment for teachers, who become aware of the validity of their choices and teaching actions, by reflecting on wrong choices, omissions, misunderstandings, etc.

The third phase allows teachers to share what happens in their classrooms; it helps teachers to express their doubts or seek reasons underlying common teaching and learning problems. In this phase, teachers become aware of the diverging nature of the teaching processes they have carried out, and reflect upon their own ways of interacting with students (interventions/silence, talking turns, reintroductions, timings).

In the fourth phase, teachers and researchers collectively reflect on the points emerged from previous phases. Reviewing critical points of their own experience leads teachers to individually evaluate the global efficacy of their own actions, to make explicit obstacles, deviations, mistakes as well as new knowledge about their role. Researchers are thus enabled to observe how differently the experience influences each teacher and how the single personalities impact on the enacted educational process.

MULTI-COMMENTED DIARIES

Within recent projects[2] we made an interesting, although demanding and time-consuming, change to our methodology.

Due to the mentors' cooperation as well as to participants' diverse locations, and the need to share the analysis of currently studied processes, we decided that an ongoing analysis and discussion of the diaries, i.e. transcriptions commented upon by teachers, should be carried out by at least three people: the mentor assigned to the teacher (M1); the coordinating mentor (M2); and the head of the project (M3).

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[2] The projects are the Comenius Professional Development of Teachers Researchers (PDTR) Project and the National 'Master in Science Education' Project (MDS).

Diaries are thus enriched with a multiplicity of written comments (sometimes added independently on one another, sometimes following a hierarchical order), which reflect a wide and various range of points of view and interpretations, highlighting crucial points of the process as well as critical elements in the teacher's behavior. Not rarely following these comments teachers feel the need to get back to the diaries to mend their ways providing motivations or rather to clarify specific points, by explicitly pointing out hidden processes, behaviors of certain students etc. in order to support the reasons underlying their own decisions.

Multi-commented diaries become a complex investigative tool for both teachers and researchers. With particular relation to the examined processes, they can be characterized as: (i) *formative tool* for teachers, enabling them to develop skills and sensitivity, and therefore improve the global quality of their own teaching action; (ii) *diagnostic tool* for researchers, enabling them to identify malfunctions in the teaching action, to formulate hypotheses and enact interventions to fix them; it also helps researchers to identify points for further research; (iii) *evaluation tool* for both teachers and researchers, providing elements to empower their interventions in the respective fields (teaching activity and design of local and/or general training and formative interventions).

More generally, they turn out to be a rich source for the production of materials to be used within laboratory-based training activities.

In order to have an idea of the materials generated by this collaborative work, the initial excerpt from a diary is reported in the Appendix: the diary refers to a teaching experiment carried out in a sixth-grade class with a young temporary teacher. The excerpt can be immediately put in context and easily read, thus giving an idea of how a diary is structured. It was chosen because it contains a wide range of comments that highlight problems recurring in all diaries, although with different nuances. Moreover the teacher, who did not make comments at the beginning, added some reflections on the basis of mentors' comments: the reflection process she went through is thus well documented.

Features of comments emerging from the diaries

Our investigation on the types of comments led us to identify five, interconnected key areas. Some of them raise some points for further research and open new perspectives for teacher education:

(1) General cultural and/or educational issues (e.g. conceptions of arithmetic and algebra, conception of teaching and students, conceptions about meaningfulness and centrality of certain topics).

(2) Mathematical and educational-mathematical issues (e.g. sequences: what are they, how to teach them, how to represent them, what critical teaching points do they raise?).

(3) Bifurcation between theory and practice (e.g. difficulties in realizing what has been studied or planned, and in working on the basis of relational thinking).

(4) Linguistic issues (massive use of operative linguistic expressions coming from the received model of teaching; difficult balance between colloquial language and language of scientific teaching; scarce attention to word paraphrases in view of an algebraic translation).

(5) Management of classroom discussions (dialogues mainly between teacher and student; widespread prompting; yes/no questions; lack of attention to the development of 'social intelligence' in the classroom).

Two issues seem to be crucial and dramatic at the same time: *the teacher's language* in communication, often imprecise, not correct, full of slang expressions and rich in not always appropriate metaphors; *the conception of mathematics*, too often operative, as 'calculate' and 'find' often prevail over 'represent', and 'doing' over 'reasoning' and 'reflecting.'

Examples

In order to give examples, we present here excerpts from multi-commented diaries, with relation to some of the comments' categories illustrated. Excerpts refer to teaching experiments carried out with sixth-grade classes within a teaching sequence concentrated on the study of numeric and figural sequences that can be modeled algebraically. The teaching sequence was designed by a group of teachers involved in the mentioned projects, after a theoretical study of papers from the international literature concerning issues related to the teaching and learning of algebra, with a particular focus on generalization and algebraic modeling, and more generally to teachers' role and the relationship between theory and practice.

The sequence developed along the exploration of five situations mainly concerning linear sequences, but it ended with a situation,[3] in which students were supposed to carry out a simultaneous exploration of two sequences – linear and quadratic – together with a comparison of their progression. The teaching sequence's main objectives were to lead students to acquire a functional view of sequences and be able to construct algebraic representations, by modeling the relationship between ranking (or place) number and correspondent term of the sequence. Crucial mathematical points in the teaching sequence were the identification and representation of correspondence laws in general terms. These were connected with the enactment of different numeric representations of the sequence terms and their coordination, the recognition of structural analogies, the identification of variables and their naming through letters, the condensation of analogous arithmetic formulae in one single representation, the transformation of arithmetic formulae in order to recognize that they are identities, and the awareness of the role played by arithmetic properties in transforming formulae.

Example 1: General cultural and educational comments referring to an excerpt from a discussion, showing how the teacher paid little attention to students' interventions she viewed as trivial or not plausible

The teacher asked students to explore the sequence with initial terms 4; 11; 18. The class had already identified the sequence's recursive generating law. The teacher wrote the following table on the blackboard and opened up a discussion to introduce the class to the study of a representation for the general correspondence law.

Sequence ranking number	Sequence number	Operations made to jump from the place number	'Mathematical recipe' to construct the number
1	4	4	
2	11	4 +…	
3	18	4 + … + …	
4	25		
5	32		

3 It is the 'Apple trees' task from the PISA International test 2000. It was selected by teachers within specific meetings concerning the study of the test.

Teacher: Operations made to jump from the first number. So what shall we do? Christian. You see we start from 4. In the second place, how do we get to 11? We make 4 +…

Christian: Eeeeeeeem….

Teacher: How do we get to 11? We make 4 + … ?

Christian: Three?

Teacher: To 11? Sabrine. **(1)**[4]

Sabrine: + 7.

Teacher: We make 4 + 7. What about the third place, Sabrine, we make?

Sabrine: 4 + 7 + 7.

Justice: Wouldn't it be better to make 4 × 2? **(2)**

Teacher: What about the fourth place?

Sabrine: 4 + 7 + 7 + 7.

Teacher: What about the fifth?

Sabrine: 4 + 7 + 7 + 7 + 7.

Teacher: What if we had a sixth place?

Sabrine: 4 + 7 + 7 + 7 + 7 + 7.

Teacher: Correct. So, now we find…

Andrei: I didn't get it. What do I put in the first place?

Teacher: Well, there is 4 in the first place.

Andrei: I put 4 × 1. **(3)**

Teacher: Well, but there is no '×' there. The first place is 4 **(4)**

Comments:

[Authors are labeled as: T: teacher; M1: reference mentor; M2: coordinator of mentors; M3: head of research.]

(1) M3. This fragment of discussion highlights two points for reflection: the issue of the whole class participation. Christian with his doubts and his answer clearly shows he is 'elsewhere;' the teacher's behavior, since she 'moves forward' without paying attention to the student.

(2) M1. Why doesn't T comment upon Justice's intervention? M2. I wondered the same too. Perhaps Justice's intervention was lost in the mass of interventions. Transcriptions are important for this too, because they enable the teacher to reflect a posteriori. **M3** I agree. Justice grasps a regularity but doesn't express it correctly, instead of saying 4 + 7 × 2 he packs everything in 4 × 2. T should have clarified this. She also missed the chance to introduce the multiplicative operator that allows for an objectification of the 'number of times' (that you need to add + 7 to the first term to get the considered number) as 'second factor' of the multiplication in the abbreviated representation of the additive expression which gives the number.

(3) M2. Also this intervention might have been investigated. What is Andrei's background thought? Why does he think about the product of 4 and 1? **M3.** Again we encounter a badly expressed intuition. The student probably wants to 'fill the gap' he sees in the representation of the first term as compared to the others. Here T misses the chance to change the representation of the first term, 4, into one that fits with the situation, for example writing 4 as 4+0 and getting back to the class posing the problem to find a representation for the first term, similar to the other ones.

(4) M3. This intervention by T suggests that she excludes the possibility of representing 4 in another way, thus showing little algebraic farsightedness. It would be extremely appropriate to encourage these intuitions, although imprecise, trying to redirect them. In this case one might work towards a representation of 4 in terms of a general rule, for instance, if the correspondence is modeled according to the law: "the term in place n (n-th term) of the sequence is given by 4 + 7 × (place number – 1)" this immediately leads to a representation of 4 as 4 + 7 × (1 – 1) i.e. 4 + 7 × 0.

Teacher's reflection:

All these remarks made me think I am really close-minded which I didn't realize before. I don't know whether this is a matter of attention, of being used to seeing things in different ways, of fear to get out of the scheme to be followed or the one I thought I should follow.

Example 2: An educational-methodological comment referring to a typical action taken by the teacher: the spoon feeding

The teacher deals with the transition from the recursive law to the general one.

[4] This and the following numbers in brackets refer to specific comments which are reported immediately after the excerpt of discussion.

Teacher: The first place is 4. Then the second place I give 4 + 7, in the third 4 + 7 + 7 and so forth. Let's see if I can use multiplication. How can I get to 11? I can make 4 + 7 but I can also make 4 +… **(1)**

Teacher: Biagio?

Biagio: 4 + (7 × 1).

Teacher: Correct. Because we saw that in order to get to 11… Biagio, you explain.

Biagio: Because making 7 × 1 is still 7. Therefore, you make 7 × 1.

Teacher: Therefore, we saw that to get to 11 we must make 4 + 7, but as you say, 7 equals…**(1)**

Biagio: 7 × 1.

Teacher: Hence saying 4 + 7 or saying 4 + (7 □ 1) is the same. So what do we put in the third line Biagio?

Biagio: 4 + (14× 1).

Teacher: Careful **(1)**

Biagio: 4 + (7 × 1) + (7 × 1).

Teacher: You do this. Any other idea?

Riccardo: 4 + (7× 2).

Comment:

(1) **M2.** I would advice T not to ask questions that 'invite students to complete the sentence,' 'prompting' the answer. This strategy does not pay. It reminds me of an episode quoted by Brousseau (if I am right, taken from the comedy *Topaze* by Marcel Pagnol) in which a preceptor is giving a French lesson to his student. Relatives assist quietly. The theme is 'understanding from the context whether a certain word is in the singular or in the plural' (in spoken French -s in the plural is not pronounced). The word is *moutons* (muttons). The student has no idea and the preceptor is afraid his relatives might express a negative judgment. For this reason he walks around the student, who stays still, with the pen in his hand, whispering 'moutons;' then, not being successful, he starts to raise his voice 'moutons'… 'moutonss'… 'moutonsss.' Finally, the student brightens and writes a –s at the end of the word. The surrounding gets calm. End of quote. **M3.** OK. Brousseau talks about 'Topaze Effect' (see Brousseau, 1984 or 1997).

Example 3: Mathematical comments referring to both refinement and coordination of different laws that represent one single sequence

Situation 1

The teacher assigned the exploration of the arithmetic progression generated by the operator +7 starting from the term 4. The class had got to identify two 'rules,' summarized by the teacher on the blackboard as follows:

	4	11	18	25	32	39	46
(1)	+7	+7	+7	…			
(2)	7×2–10	7×3–10	7×4–10	7×5–10	7×6–10	7×7–10	7×8–10

Law 1: you need to add 7

Law 2: you multiply by 7 then take away 10 **(1)**

Comment:

(1) **M3.** The two laws are expressed very roughly. Attention, the first one is clearly a recursive law; the second one is general, although the variable is tacit. In order to compare them one needs to put them on the same level. The first law needs to be transformed into a general one. In order to do this, one needs to explicitly state the number of times that the operator +7 is applied, starting from the first term. The first law thus becomes

4 ; 4+7 ; 4+ 2×7 ; 4+ 3×7 ; 4+4×7 ……….. .

Students might also be led to notice that 4 = 4+0 = 4 + 0×7, embedding the first term into the general scheme. One might also try to proceed in parallel with the two representations:

law 1: 4 + 0×7 ; 4+7 ; 4+ 2×7 ; 4+ 3×7 ; 4+4×7; ……….. .

law 2: 7×2–10; 7×3–10; 7×4–10; 7×5–10; 7×6–10; ……….

to invite students to see the terms as correspondently equivalent and lead them to find out of the underlying reason in the fact that, since 7×2 = 14 then 4 = 7×2-10.

Situation 2

The teacher compared the representations of the sequence's terms following the two identified laws (situation 1). From the analysis of cases in the classroom, they got to the following conclusion:

Place	Number	Rule 1	Rule 2
1°	4		$7 \times \boxed{2} - 10$
2°	11	$4 + 7 \times \boxed{1}$	$7 \times \boxed{3} - 10$
3°	18	$4 + 7 \times \boxed{2}$	$7 \times \boxed{4} - 10$
4°	25	$4 + 7 \times \boxed{3}$	$7 \times \boxed{5} - 10$
8°	53	$4 + 7 \times \boxed{7}$	
12°	81	$4 + 7 \times 11$	

In rule 1 the changing number is the third, i.e. the second factor. This number changes after the place, i.e. it equals the one in the previous place.

In rule 2 the second factor is changing again, but this time the place equals to the number preceding the second factor. **(2)**

Comment:

(2) **M3.** The rules are not very clear from a linguistic viewpoint. A better statement would have been 'according to rule 1, when a term of the sequence is represented, the third number changes, i.e. the second factor of the written product. This number changes *in correspondence* with the place number and it equals the place number minus 1. One remark: why wasn't the first case completed? (Students have seen it, it is not appropriate to throw away fine and valuable interventions like this one). Another remark: I regret to notice that rule 1, although identified, remains not expressed and not objectified. Rule 1, generated by a re-writing of the sequence terms through successive multiples of 7, needed to be made explicit, at least verbally, and students should have taken charge of this task. We would have probably got formulations like 'the sequence number at a certain place is 4 plus 7 for the first place number minus 1' (faithful translation of the procedure) or rather ''the sequence number at a certain place is given by the product four plus the place number minus 1 times 7" (mixed relational-procedural formulation) and others; the teacher might get to express it in the evolved relational form: ''the sequence number at a certain place is the sum of 4 and the product of 7 times the place number minus 1). I purposefully avoided using terms like preceding or antecedent of the place number, of little help in the algebraic translation. This variety of verbal formulations already leads to interesting problems to be discussed in the transition to the algebraic formulation (in this, Brioshi's metaphor works perfectly): these problems not only concern the representation of the variable (place number) but also the use of parentheses. I'd like to remark that rule 1 is given by expressing the second factor as a function of the place number (working well for the algebraic translation). Rule 2 was expressed by giving the place number as a function of the second factor (not working for the algebraic translation).

Example 4: Comments related to linguistic issues in the construction of an algebraic representation of the progression, within a discussion which was heavily influenced by the teacher's language

The example concerns a delicate moment of the study of the sequence in play: the transition to the algebraic representation of the correspondence rule in general terms. Analyzing how the first terms of the sequence are generated and representing them through the first term, the students individuated two 'rules' to which such terms fit. They then went on to tackle the generalization problem and worked collectively on the meaning of the term '*n*-th.' The teacher tackled the question with students, by summarizing on a table all the results they got to. In the study of the 30^{th} place case, there is a mistake: within the generalization process, the number preceding 30 is swapped with the number following it. Below we report only the beginning part of the discussion, which, with some omissions, focused on the search for a formula that might represent that correspondence. The excerpt highlights the teacher's inadequate language and the type of difficulties implied in the students. (Due to space limitations, we do not report the entire discussion which in itself could constitute an object of a paper).

Place	Number	Operations…	Rule 1	Rule 2
1°	4			
2°	11	4 + 7	4 + (7 × 1)	7 × 3 – 10
3°	18	4 + 7 + 7	4 + (7 × 2)	7 × 4 – 10
4°	25	4 + 7 + 7 + 7	4 + (7 × 3)	7 × 5 – 10
30°			4 + (7 × 31)	
n				

Teacher: I want to know: if I'm at place *n*, that we said – remember? – it was a place at a certain point, without knowing what point it was. Well, I want to know what is the rule that allows me to find this number at position n **(1)** [I point to the *n*-th term on the blackboard] right?

Teacher: Good, so, let's find the rule. **(2)**

…

Teacher: How can we find the formula we will need? **(**Don't look at me; look at your sheet and the blackboard! How can you find it? **(3)** Andrea?

Andrea: Well, if we know… last time we said *n*-th stands for any place **(4)**

Teacher: Question: number *n* means a number at any place **(5)** without telling you what number it is – this is the difficult part! What formula do I write for the number at the *n*-th place? **(6)**

Sergio: Well, I think you can't find it because *n*-th is a number you don't know.

Andrea: As you said, *n*-th stands for a number at any place, so I say like Sergio, if the place is not definite, we will never know what number it is! **(7)**

Teacher: Correct, I agree too! If I don't tell you, at the 3rd, at the 4th, at the 100th, at the 7003rd place, we won't be able to know. But if I tell you that this number ... about this number, instead of telling you the place number, I tell you it is at place n, can I make a calculation ... can I write a formula to find this number? **(8)**

Stefano: I think so... we don't know what number is n and even though we don't know it we can find a formula.

…

Comments:

(1) **T.** I now realize I have used wrong terms thus inducing students to give the answers they gave and I desperately "fought." When I said "the rule to find this number at position *n*" students understood I wanted to know the value of a_n. Perhaps I should have said "the rule to find one sequence number, knowing its position."

(2) **M2.** My suggestion is to lead the class to discover, and highlight with arrows, relationships, repeated numbers, 'local' regularities. Many of these might not be productive, but help students get used to carry out full explorations. For instance, the same sequence, proposed in another class, led some students to identify a relationship between numbers in the first two columns and represent it with 11 = 2 × 7 – 3, 18 = 3 × 7 – 3, 25 = 4 × 7 – 3, and so forth. Arrows might link the various fours with the first term of the sequence, numbers 1, 2, 3 of the fourth column with the place numbers of the first column, shifted one line down, etc. These latter arrows might highlight the fact that 31 is wrong and it should be substituted for 29. **M3.** I agree with T. The use of the term 'to find' is misleading: it induces the idea the *n*-th term of the sequence can be determined by calculations. Here the term 'to represent' has to be introduced (cf. Theoretical frame and glossary); it can also be used for highlighting the analogies between the representations of the terms in the examined cases for both rules (I agree with M2 as to his suggestions). The fact that T.

did not recognize the mistake in the case of the 30th term shows that T. did not control the situation. Recognizing and analyzing the mistake could have facilitated the shifting of the attention on the representations.

(3) **M1.** T. is tormented by the idea she should get to the formula written in algebraic language. I keep thinking that in this phase the objective is to lead students to grasp the relationship between place and correspondent number, and clearly express this relationship**. M2**. Why not making them express things in natural language, by describing the shape of columns 3 and 4: "I get the number by adding to the initial number as many 7 as ..." or in any other way. The paraphrases proposed by students can then be compared and the most suitable can be chosen to be translated into algebraic language for Brioshi.

(4) **M1** Very well Andrea, that "any place" is gold!

(5) **M3.** More than 'any' term which triggers the idea of variability, it would have been more appropriate to point out that this is a number we don't want to specify, 'indeterminate' (and this term, while focusing on the element, somehow fixes it)

(6) **M1** I often wondered why we shouldn't put, in the mathematical recipe's column, the – mental or not- operations done to identify the factor that multiplies 7, starting from the number at the given place. In this way, students would have grasped the regularity, the re-iteration of a procedure, getting closer to the construction of the formula. **M3.** The construction of the formula must be guided; it can be determined by identification in the studied cases of invariant parts (4+ 7×…) and variable parts (place number – 1).

(7) **M2.** Approaching the use of letters is very complex; it requires lengthy times, different strategies, comparisons and explorations and involves continuous and unpredicted evaporations. Intuitions of different meanings co-existing in the interventions made by Sergio and Andrea are inevitable and physiological. The (real or presumed) need to conclude the worksheet and get to the rule might have imposed on the teacher speed that can hardly go together with this complexity. We are fully immersed in algebraic babbling, and learning a new language, with its meanings and rules necessarily requires settling (metaphor: sedimentation of solid substances dispersed in liquids).

(8) **T.** I understand now why they could not answer! We didn't understand each other! As I said before, the verb "to find" put them on the wrong track! Perhaps I should have said "find a representation of the place number *n* that makes us understand that this number is in the sequence." Too complicated! I don't know… **M1**. I agree on the damages caused by the term "to find." M2 I agree on representing too, even more if also this term (glossary) becomes one of the keywords of the class' cultural baggage and therefore it gets a negotiated and shared meaning (again Glossary).

M3. Finally, Good T! Representing, yes, representing is the key term.

SOME FINAL REMARKS

Commented transcriptions of classroom-based processes support teachers' a posteriori reflections on how the activity was carried out and managed and require them to reconstruct it critically, through an interpretative effort that is highly formative. These diaries enable researchers and mentors to test teachers' consistency with their teaching practice, declared beliefs and reference to the theory at stake (both from mathematics and from mathematics education). Besides, they require an explicit *fine* analysis of micro-situations that show teachers both consistencies and inconsistencies of their teaching action. Moreover, a global analysis of comments leads class teachers to elaborate on the activity with significant follow-ups in their teaching practice, and enables mentors and researchers to detect, in a deep and extensive way, teachers' cultural backgrounds and attitudes.

The sharing of commented diaries and the analysis of the diverse whole-class discussions that arose from one single problem situation, led to an objectification of the reasons that determined them. Moreover, by comparing their own actual implementation of a certain step of the teaching sequence to other colleagues,' teachers identify important distinctive elements and reflect on both efficacy and limitations of their own work (for instance, hurried and decisive interventions, little attention paid to listening, lack of understanding of potentially fruitful interventions, scarce ability to orchestrate voices, and difficulty to contain leaders or minimize the effects of tacit alliances, etc.).

All this leads teachers to acquire an increasingly deeper awareness of their own way of being in the classroom, a better control of their behaviors, and stimulates them to think about, and enact in a fine way, changes to their teaching modalities.

It is now appropriate to make some remarks on the validity of our methodology as well as on what emerges from the described processes. Due to the line-by-line comments made by two or three researchers to teachers' diaries, the transcription is analyzed under a multiplicity of viewpoints, ranging from content-related aspects (setting up of the problem exploration, mathematical aspects developed, attained or missed objectives, etc.) to communication and language-related aspects (Questions' formulation, operative guidelines given, etc.), to issues related to the control of students' participation (number and type of interventions) as well as the agreed didactical contract (induced attitudes in students and socio-mathematical norms in the classroom). The analysis we carried out led to "freeze" a wide range of remarks and in-depth comments that revealed and amplified different aspects of teachers' professionalism (their mathematical and pedagogical knowledge, their conceptions about teaching and relating to students, the style of the agreed contract, their hurries and digressions, their affectivity). It is a real "radiography" of teachers, facing which they experience a moment of a healthy crisis, very often followed by a positive reaction leading them to challenge themselves and act towards a fruitful professional development.

Besides, multi-commented diaries offer extensive material to construct prototypes of activities for critical reflection, useful in both laboratories and training activities concerning the specific taught discipline, useful to novice teachers, in distance teacher education, to the construction of learning objects, in the formative training of mentors and supervisors, to provide them with models of analysis of interactive and discursive educational processes.

These tools were used with interesting results in SSIS (pre-service teacher training) courses, in-service teacher training courses, e-tutoring, etc. Observation of teachers tackling tasks of this type clearly shows the main objective, which is to put teachers in a situation, in which they might critically link three crucial points: their own conception of the mathematical topic at play (and of mathematics itself); the conflict caused by the meeting-clash with the teaching modalities used by colleagues and by the results they attained; a mediation between these two crucial points produced by a collective sharing and a dialogic relationship with researchers.

Analyzing the comments made by researchers, the authors' epistemology appears clearly, as some particular types of comments prevail. Both common and diverse points of view in the produced comments help teachers, the former by strengthening comments themselves and the latter as their complementary nature provides enrichment.

This methodology obviously strictly depends on teachers' involvement and commitment. For this reason, it cannot be used during inset courses with significant results. However, our hypothesis is that it is possible to spread curricular and methodological innovations in schools and society, through a dialogue involving participants in the projects as well as colleagues and parents.

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APPENDIX

INITIAL PASSAGE OF A MULTI-COMMENTED DIARY

We report the initial passage of a diary referring to the process of implementing the teaching sequence in a sixth-grade class, carried out by a young temporary teacher. Due to the variety of comments it contains, this excerpt provides evidence of the complexity of a diary. In particular comments refer to:

a) general aspects of issues related to teachers' beliefs about the usual type of mathematical activities, and the degree of relevance of the activity proposed to students, in terms of the 'minimum curriculum' expected by both school and parents;
b) representation-related aspects, that involve mathematical issues about potential infinity and generalization;
c) didactical-methodological issues regarding didactical contract, relationship with students and testing students' actual understanding and learning;
d) linguistic issues linked to both the teacher's communicative style and the attitudes induced in students.

The teacher proposes a topic concerning the exploration of a sequence, given the first three terms (it is the arithmetic progression with initial element 4 and step 7). The activity is aimed to determine a general representation of the sequence. The excerpt reported here deals with students' appropriation of the task. Analytical comments, which provide the structure of the diary, are reported in the footnotes in order to preserve the discussion flow. (Authors of comments are labeled as: T: teacher; M1: reference mentor; M2: coordinator of mentors; M3: head of research.)

Teacher: Today we are going to do something different.[5]
Some: Great!
Riccardo: What are we going to do?
Teacher: We play a game. Have any of you played the games you can find on puzzle magazines?
Chorus: Yes!/No!
Teacher: It is a magazine where you find crosswords and other games and puzzles.
Teacher: I write some numbers and then I put some empty spaces, which means there will be other numbers: 4 11 18 __ __ __. Ok[6]?
Try and think about what the numbers should be.[7] We cannot put some random numbers; we should try to explain why we put those numbers.

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[5] **M2.** The 'diversity' of this activity is presented as a motivating aspect. This is partially true, but it can also represent a sort of distracting factor ('Great!' in the subsequent intervention induces this suspect). On principle, I consider neutral approaches that actually introduce students to a 'permanent sharing' and do not emphasise the episodic nature of the activity, more productive. **M3.** I agree, even though I see this 'opening' as a support to the teacher, who is aware she is undertaking a different, somehow risky activity and far from usual ones. **T.** It was actually meant to be a way to capture everybody's attention, including those who usually don't follow or find it difficult to stay focused. But it might have been a sort of unintentional psychological crutch.

[6] **M1.** I would have clarified some conventional items: empty spaces (I imagine dashes) that will be filled by numbers, dots to indicate that numbers might be a lot, infinite. This would have been useful to distinguish between the concrete signs on the blackboard (with its space constraints) and the abstract nature of mental images. **M3.** I agree. I would have put at least 4 terms in the sequence, commas between place marks, dots at the end to indicate the sequence's indefinite length. These tricks might avoid limited visions, misunderstandings and overcome a view of finite sequence, inducing the idea of infinite sequence. **T.** True. I actually took for granted that the written expression at the blackboard was clear to anyone: dots instead of missing numbers. I didn't think about pointing out the difference between the infinity of actually missing numbers and the finiteness of the blackboard.

[7] **M1** I would have solicited the class to identify what could connect the three numbers, what they share, that is: why were those three numbers chosen? **M2.** I agree. Putting the activity in context might favor clarity in

Lorenzo: Perhaps I got it … I know it, I know it
Teacher: Lorenzo, since you said "I know it, I know it, I know it", say it aloud.
Lorenzo: 4 + 7, 11; 11 + 7, 18; 18 + 7, 25.
Teacher: So which numbers do you add?[8]
Lorenzo: 18 + 7, 25; 25 + 7, 32; 32 + 7, 39; 39 + 7, 46. (On the blackboard, the teacher writes in columns the following equalities: 4 + 7 = 11; 11+7 = 18; 18 + 7 = 25; 25 + 7 = 32; 32 + 7 = 39; 39 + 7 = 461 + 7 = 18)
Teacher: Etcetera. Does anyone disagree?[9]
Antonio: How did he do that?
Sabrine: I didn't understand.
Teacher: Didn't you? So, Lorenzo explain to Sabrine what you did.[10]
Lorenzo: I summed 4… it's a chain. 4 + 7 is 11. 11 + 7 is 18. 18 + 7 is 25. 25 + 7 is 32. 32 + 7 is 39. 39 + 7 is 46. I kept adding 7 to what I got.[11]
Sabrine: Why do we need to add that 7?[12]
Teacher: Why do we need to add that 7, she's asking? Who's going to answer?
Andrei: Me, me!
Teacher: Andrei.
Andrei: Because in this sequence a number goes ahead by 7; because it might go ahead by 10 in another one.
Teacher: Did you understand?[13]
Sabrine: No.
Teacher: Laura, try to explain.
Laura: Sabrine, the teacher gave some numbers: 4, 11 and 18. Try to count how many numbers you have from 4 to 11.[14]
Sabrine: 18.
Laura: So: 5, 6, 7, 8, 9, 10, 11. They are 7 numbers. From 11 to 18 how many do you have? There are seven numbers more, I tell you. And the teacher put some dots and you must discover the number that going ahead you were adding 7 more.
Sabrine: Ah! I got it.
Teacher: Did you understand?
Sabrine: Yes.[15]

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the didactical contract. **T.** Right, I didn't even think about it. For sure, due to my scarce preparation to the activity, I did not reflect upon these possible aspects and went straight to the core of the problem I proposed.

8 **M1.** Better than this; which numbers might be included and why? **M2.** Ok. It is important to make explicit that statements have to be justified.

9 **M3.** Is it taken for granted that the rule uttered by Lorenzo is clear to everyone? Is it rather an interlocution? The teacher might have better asked the class explicitly 'what rule did Lorenzo follow?' **T.** True. My question was meant to provoke their reaction. I would have later asked them to explain and justify both affirmative and negative answers, as it actually happened.

10 **M1.** I would have asked Lorenzo to explain what led him to construct the numbers following the given three numbers in that way.

11 **M3.** Lorenzo expresses what he did starting from 4 (first term). In making a summary, he forgets to mention the starting point. T. should have intervened to invite him to be more precise. Example 'What do you mean by 'what I got', and 'where do you start adding 7?' **T.** True. I think I have preferred to let them carry on with their interaction in that moment. Students speak fast, overlap their comments without letting classmates finish their sentences. It's sometimes difficult to intervene and still let them follow the thread of the arguments they are trying to express with effort.

12 **M2.** Sabrine brings up a sore point and opens a door in the direction anticipated by M1 in previous comment. Great.

13 **M3.** This intervention is not very appropriate. What the student says is not clear and T should have invited him to be more precise.

14 **M2.** Laura does her best to lead Sabrine to understand that there is a '7 units step' between one number of the sequence and another one.

15 **M1.** Perhaps it would have been good to test, with an example, if Sabrine really understood. **M2.** I agree. **M3.** The dialogue sounds sterile. You can't accept a 'yes.' You need to make sure that the student understood and give her the chance to make the intuited procedure more solid. It would have been enough

Teacher:	OK, so what is the rule?[16]
Riccardo:	That you must find out how many numbers you need to get there.[17]
Teacher:	Well, give me a nicer rule.[18]
Riccardo:	Now, you count how many numbers there are from one to the other and you go on like that.
Giuseppe:	Depending on the given numbers, we must find the number which we go on to.
Lorenzo:	Depending on the given numbers, we must find the instruction.
Giuseppe:	And what is the instruction?
Teacher:	In actual fact, this is the thing. When we must find a rule or explain something to someone, we need to make ourselves understood. Maybe everything is clear in our head, but what we say is not always that clear to others as well.[19]

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to ask the student what she would have put after number 18. **T.** True, I should have verified. I usually do it, when I see some students having doubts; I assign them more exercises or ask them to give another example. I think that, in that moment, but also along the whole experience, I made a mistake of being in a hurry to finish everything I had planned. I cut off that issue, because I saw it as trivial (but I should have thought about them) and I felt I had already wasted too much time.

16 **M1**. I would have posed the question as: may we talk about 'rule of behavior'? **M2** T. should try to lead the class to a representation of the relationship between numbers in the sequence. **M3.** I agree. The teacher could take the chance to intervene, focusing on a term and its successive nature and working on verbal expressions of the relationship that links them. The term 'rule' is jargon, relationship is a better one. **T.** I used "rule" because I thought this term was more straightforward and clear to everyone.

17 **M3.** There, where? Is the task uniquely about gaps to be filled, for this student?

18 **M3.** Jargon expression, the term 'nice' is used as a synonymous of 'clear.'

19 **M3.** This intervention by T. is relevant and makes the contract clear. **T.** Thanks. This is to me the hardest problem we have to tackle when we want to get students to interact.

AN APPROACH TO PROOF IN ELEMENTARY NUMBER THEORY FOCUSED ON REPRESENTATION AND INTERPRETATION ASPECTS: TEACHER'S ROLE

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ABSTRACT
This paper reports some results of the study we carried out together with two teachers involved in the PDTR Project, about the construction and implementation of an introductory path to proof in elementary number theory (ENT). This work is part of a wider project of didactical innovation aiming to introduce a more significant approach to the teaching of algebra, within traditional school settings. Some of the results coming from the analysis of activities proposed within this path, will be presented, with particular focus on the role played by the teacher during collective discussions, as a guide to the enactment of fundamental skills for the development of proofs in elementary number theory, such as being able to translate, to interpret and to anticipate. Our main research aim is to point out how teachers' attitudes and actions impact on students' learning. In order to achieve this, we will highlight the connection that links the approach chosen by teachers and students' awareness of and approach to the construction of reasoning by means of algebraic language.

1. THE IMPORTANCE OF MEANING IN THE TEACHING AND LEARNING OF ALGEBRA: THE ROLE PLAYED BY PROOFS IN ENT

Among the numerous studies on the teaching and learning of algebra, a fundamental role is played by those that focus on the issue of how to manage meanings in algebra. A considerable number of research studies analyze in-depth the role played by algebraic language as an instrument that stimulates and supports the development of reasoning. Among these, we quote, for instance, the work by Arzarello et al. (1994, 2001), who propose a model for teaching algebra as a game of interpretation and highlight the need for the promotion of algebra as an efficient tool for thinking**.** Many researchers share a similar vision of the approach to the teaching of algebra. Among them, Bell (1996) in particular states that it is necessary to foster among students the use of algebraic language as a tool for representing relationships, and to explore aspects of these relationships by developing those manipulative abilities that could help in the transformation of symbolic expressions into different forms. Similar remarks are also found in Wheeler (1996), who asserts the importance of ensuring that students acquire the fundamental awareness that algebraic tools "open the way" to the discovery and (sometimes) creation of new objects.

The concept of symbol sense, as proposed by Arcavi (1994, 2005), well represents this view. The author highlights the importance of fostering a different and deeper vision of algebra among students, suggesting that in order to achieve this view, students should be led to develop particular attitudes. Some examples of these attitudes are listed: the ability to know when to use symbols in the process of finding a solution to a problem and, conversely, when to abandon the use of symbols and use alternative (better) tools; the ability to see symbols as sense holders (in particular to regard equivalent symbolic expressions not as mere results, but as possible sources of new meanings); the ability to appreciate elegance, conciseness, communicability and power of symbols to both represent and prove relationships).

These objectives, proper of any type of teaching, seem to be unreachable for most students, if they receive traditional teaching, which stimulates the development of syntactic skills, but does not foster a new vision of algebra, as a tool for generalizing, communicating, understanding situations, arguing or producing actual proofs. Students can actually achieve this view, if they are given the opportunity to "get a touch" of the power of algebraic language. This can only happen if, within traditional algebra teaching sequences, some time and space are devoted to activities aimed at favoring this type of achievement.

Kieran (2004) also stresses the importance of devoting much more time to those activities for which algebra is used as a tool but which are not exclusive to algebra (global/meta-level activities according to Kieran's distinctions). These activities help students develop transformational skills in a natural way, since meaning supports manipulations (Brown, 2004).

We believe that among these activities an important role might be played by proofs in elementary number theory. We actually claim that activities on proof in elementary number theory would both provide students with the opportunities they need to progress gradually from argumentation to proof, and help them appreciate the value of algebraic language as a tool for coding and solving situations that are difficult to be managed through natural language only (Malara, 2002).

The idea that proof is one of the main activities through which students might be helped develop a mature conception of algebra is also stressed by Wheeler (1996), who states that activities of proof construction could be "a counterbalance to all the automating and routinizing that tends to dominate the scene." Moreover, Selden and Selden (2002) argue that elementary number theory is "ideal for introducing students to reasoning and proof" because it makes students deal with familiar objects and reduces the level of abstraction required.

This specific research field has not been completely explored, as maintained by Zaksis and Campbell (2006), who state that "the idea of introducing learners to a formal proof via number theoretical statements awaits implementation and the pros and cons of such implementation await detailed investigations" (10).

2. COMPONENTS THAT INFLUENCE AN APPROACH TO PROOF IN ENT

Many different competencies are required of students who have to face proof problems in elementary number theory. In particular, they have to: know the meaning of the mathematical terms in the problem text and interpret them correctly by reference to it; translate correctly from verbal to algebraic language; be able to interpret the results of the transformations operated on the algebraic expressions in relation to the examined situation; and control the consequences of their assumptions.

Not forgetting the importance of syntactic skills, the three key-sentences that might represent essential competencies to the complete development of a proof in elementary number theory are: (a) being able to *translate* into algebraic language, (b) being able to *interpret* algebraic expressions, and (c) being able to *anticipate* results of the operated transformations. These three competencies have also been highlighted by research as key aspects in the development of algebraic reasoning in general.

More specifically, being able to translate from verbal to algebraic language and being able to operate syntactic transformations, starting from expressions within a proof can be viewed as particular cases of what Duval (2006) calls "transformations of representations." The author defines as registers of representation those semiotic systems that envisage transformations of representation and includes both algebraic and natural language among them. He remarks that a critical aspect in the development of learning in mathematics is represented by changes of representation, be they either from one representation register to another (conversions) or within a single register (treatments): conversions allow for the enactment of further transformations of representation, whereas treatments allow for an explicit statement of the properties of the transformed object.

The importance of interpretative processes, as key aspects in the use of algebra as a tool for producing reasoning, was highlighted in the model proposed by Arzarello at al. (1994, 2001). The authors stress the importance of using *conceptual frames*[1] and changes from a frame to another and from a knowledge domain to another, as fundamental steps in the activation of the interpretative processes. In the authors' view, learners get to use symbolic expressions as a *thinking tool* when: (1) they operate transformations on them that highlight not initially evident properties; (2) when, without appealing to transformations, they interpret the expressions in a new frame.

Finally, the concept of anticipating as key element to the production of thinking through transformational processes was highlighted by both Arzarello et al. and Boero (2001). The latter defines anticipating as "imagining the consequences of some choices operated on algebraic expressions and/or on the variables, and/or through the formalization process." In order to operate an efficient transformation, the subject needs to be able to foresee some aspects of the final shape of the object to be transformed, in relation to the target. Arzarello et al. stress that the ability to produce anticipations strictly depends on changes in the frame considered in order to interpret the shape of the expression.

Our studies enabled us to point out how important is to enact these three components for a good production of proofs in elementary number theory, the role each of them plays in the proving phase and the mutual relationships existing between them (Cusi & Malara 2008).

Difficulties encountered in making students develop the essential skills we have outlined, as well as in helping them become aware of the meanings that algebraic language can transmit, if used appropriately, enable us to highlight the crucial role played by teachers as models students should refer to. Before presenting our research study, we will complete our theoretical framework with an excursus on part of the literature dealing with the role played by teachers in the collective construction of knowledge.

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[1] Defined as an "organized set of notions, which suggests how to reason, manipulate formulas, and anticipate results while coping with a problem."

3. CONSTRUCTION OF MATHEMATICAL MEANINGS THROUGH INTERACTION: THE ROLE OF THE TEACHER

Promoting a meaningful learning of the discipline is one of the main objectives of mathematics teaching: this becomes more attainable, if students become aware of the importance of an approach to the discipline, associated with a constant search for the sense of the activities they are to deal with. In order to construct mathematical meanings, students are supposed to be able to reflect at the same time as they "act." Research suggests that, in order to promote this type of reflection, specific activities should be implemented at school, including whole-class mathematical discussions. Different scholars, who dealt with these themes, associate the introduction of moments for discussion among classroom-based activities, with objectives that concern students' learning and link back to the different theoretical approaches used to analyze the activities. Through moments of collective discussion, students have an opportunity to: (a) propose their own ideas and arguments supporting them, examine their own strategies and reflect upon them (Schwarz et al., 2004; Wood, 1999; Martino & Maher, 1999); (b) listen to others, understanding their points of view and reflecting upon their ideas (Wood 1999; Martino & Maher, 1999); (c) express their disagreement with others' ideas and argue about it (Schwarz et al., 2004; Wood, 1999); (d) construct links, develop tools for representation, define and communicate what they have learned (Anghileri, 2006); (e) develop a creative energy that favors building up concepts through a constructive process that involves processes of conjecturing, proving and refuting (Richards, 1991); (f) set, through the expression of the sense attached to the examined mathematical objects, an explicit relationship between students' conscious knowledge and existing objective knowledge (Bartolini Bussi, 1996); (g) view the classroom-based activity as an object for reflection, so that a different attitude towards the activities may be developed (Cobb et al., 1997); and (h) set socio-mathematical norms that will regulate future discussions, as well as the very approach to mathematics (Yackel & Cobb, 1996).

Different terms have been coined in the past years to refer to this fundamental activity, which stimulates exchanges and reflection: *collective reflections* (Cobb et al., 1997), *critical dialogues* (Schwarz et al., 2004), and *conceptual discourse* (Anghileri, 2006). Some authors proposed their own definition for classroom-based discussion, reflecting (a) the fundamental features it should have: "It is a purposeful talk on a mathematical subject in which there are genuine student contributions and interaction" (Pirie & Schwarzenberger, 1988, 460); or (b) the objectives it pursues: "the scientific debate that is introduced and orchestrated by the teacher on a common mathematical object in order to achieve a shared conclusion about the object that is debated upon (e.g. a solution of a problem)" (Bartolini Bussi, 1996, 17).

Arcavi and Schoenfeld (1992) highlight the difficult task teachers tackle, when they try to find a balance between the need to work drawing on students' ideas, and that of not distorting the underlying mathematical aspects. The authors assert that, in order to be able to fulfill this task, teachers' competencies should not only concern their discipline, but pedagogical, cognitive-analytical and communicative aspects as well. Based on this reflection, a central aspect of the analysis of classroom-based discussion activities can be introduced: the role played by the teacher in organizing them. If we look at the social aspects of classroom-based discussions, it turns out that teachers should be able to teach the art of communicating (Sfard & Kieran, 2001), as well as to create a good context for interaction, by stimulating and regulating argumentative processes (mediating argumentation in the words of Schwarz et al., 2004) and, most of all, focusing students' attention on the need to listen to others carefully, so that they might decide whether what they say makes sense and, possibly, criticize or give suggestions (Wood, 1999). If the aim is to favor these attitudes in students, it is essential for teachers to play a role of authentic participant, besides being a moderator of the discussion (Richards, 1991). This means they should be able to alternate their interventions with moments when they are a simple listener, not giving any judgment (Pesci, 2002).

If we look at the mathematical content of the discussion, teachers are in charge of determining the direction this content should take in the different phases of the discussion, "filtering" students' ideas, so that they can focus on those contents teachers view as more relevant and meaningful (Gamoran Sherin, 2002). Besides this, teachers act as the mathematical community representatives and are in charge of the quality of the discussion (Yackel & Cobb 1996; Bartolini Bussi, 1996). It is through their reactions to students' interventions that teachers implicitly evaluate the solutions they propose (Yackel & Cobb, 1996), leading them to become aware of the finest forms of reasoning (Anghileri, 2006). These goals might be attained through precise methodologies to approach discussion that teachers should be able to manage and develop in a flexible and dynamic way (Anghileri 2006; Schwarz et al., 2004). In a paper analyzing in-depth scaffolding practices, Anghileri (2006) includes the processes of reviewing (focusing students' attention on aspects of the activity that might promote an understanding of the underlying mathematical ideas) and of restructuring (encouraging students to reflect and clarify what they have understood so that mathematical meanings might be developed), among the fundamental interactions that should be supported during whole-class discussions. Moreover, the author examines which methodologies can favor a fruitful development of these processes. Among these, for instance, she includes rephrasing students' speeches (to highlight the processes involved in the solution and re-describe the efforts made by students to clarify the underlying mathematical concepts) and the use of probing questions (through which one can investigate on students' statements with the aim of making them clarify what they said and refine their thoughts).

3.1 Interaction in the Teaching of Proof

The role played by teachers during mathematical discussions becomes more complex when mathematical proof is the object of discussion. Contrasting points of view have been expressed about the role of a collective construction of knowledge within the learning of proof. For instance, Balacheff (1991) remarks how, sometimes, interaction may be an obstacle to learning within proving processes, because it favors the development of those *argumentative behaviors* which contrast with the achievement of an awareness of the specificity of mathematical proof, as compared to an argumentative approach, supporting one's conjectures. Similar issues are highlighted by Bussi (1994) who nevertheless points out how the relationship between deductive reasoning and social factors leads to consider the approach to proof not only as an individual construction. More recently, Martin et al. (2005) also draw attention to the importance of whole-class discussions and the role they play in the learning of mathematical proof. The authors stress that teachers who become experts in mathematical language, are in charge of favoring the inclusion of students in the process of class-based negotiation of a conjecture, as well as in the process of collective construction of the proving reasoning. As a matter of fact, only students who are given this opportunity will get to learn how to carry out a formal proof. Referring to the teaching of proof, the authors underline the value of strategies that characterize collective discussions, such as, for instance, re-voicing (summarizing and repeating students' statements), use of rebound questions (students' statements are immediately sent back to the class in the form of a question by teachers) and coaching (persuading and encouraging).

4. HYPOTHESES AND RESEARCH AIMS

Our work looks at innovation in the teaching of algebra, with particular reference to proof in elementary number theory (Malara & Gherpelli, 1997; Malara 1999). The main educational objective we try to fulfill is to foster among students a new, more flexible and deep vision of algebra (by favoring the development of the main components that Arcavi attributes to symbol sense) and among teachers an innovative conception of the teaching of algebra.

In previous paragraphs, we highlighted the complexity of a teaching practice aimed at deepening the sense of the mathematical activity and, in particular, of an approach to the teaching of algebra through proving activities in the field of elementary number theory. We believe that this complexity may be a challenge for teachers and provide a perfect formative context, in which they might achieve a new awareness and refine their methodologies for both teaching and class management. For this reason, we decided to work with two teachers involved in the PDTR project, to construct with them, and experimentally test in their classes, an introduction to proof in elementary number theory in higher secondary school (we will present this in the next paragraph).

We strongly believe that it is not possible to improvise a type of teaching in which students are encouraged to use algebraic language to produce their reasoning. Therefore, we believe that teachers can make the proposed approach productive, only if they have reached an initial level of awareness about the issues we have illustrated so far. For this reason, activities of study and analysis preceded the construction of this pathway: during these activities, teachers understood the need to stimulate the enactment of the three key components in the development of proofs in ENT, and became aware of the difficulties students can encounter as well as of the peculiar and essential role played by teachers.

Our research aimed to highlight: (1) the attitudes of conscious teachers, and the choices they made during the class activities proposed within the teaching sequence; and (2) the effect of their approach on students, from the point of view of both awareness shown and competencies acquired.

5. RESEARCH METHODOLOGY

The research study, on which this paper draws, is characterized by the need to control and analyze both complexity and interaction of two different contexts: (1) the class context (with reference to mathematical construction, concerning both individual actions and group actions); and (2) the context of a training activity for teacher-researchers (in both interaction and reflection processes, involving individuals and the whole group). The first context (the class) was characterized by the alternation of different groups of students over three years, and was analyzed through the double lenses of the role played by teachers and students' interactions, both with peers and with teachers, during discussions. The teaching sequences we proposed in these three years were characterized by a complex work involving individual tests, working activities in small groups and moments of collective discussions. For this reason, in order to carry out an analysis, we decided to collect students' written productions and to audio-record both small groups and whole class activities. During the activities, I was in the classroom as a mentor and a silent observer, taking notes, in order to facilitate the subsequent work of transcription and analysis. Each transcript was analyzed from different points of view.

Transcripts concerning small group activities were analyzed with reference to: (1) the incidence of the three key components in the development of proofs in ENT, with a particular focus on the highlighted interpretative aspects; (2) the link between types of approach proposed by teachers in previous activities and types of approach chosen by students. Besides highlighting the role of the three key components in the class discourse, the analysis of transcripts of collective discussions required a particular focus on the choices made by teachers as well as on their behaviors.

The second context (formative training of teacher-researchers) required a constant monitoring of the evolution of the two involved teachers, concerning both reached awareness and competencies in managing activities to be developed in a research context. This monitoring occurred through: (a) regular formative meetings with reflection on activities – either carried out or to be yet implemented; (b) individual analysis of some transcripts and successive exchanges between teachers and mentor; (c) questionnaires about reflections on one's formative path, proposed to teachers at the end of each year.

The work we present here refers to the first context of analysis.

6. AN INTRODUCTION TO PROOF IN ELEMENTARY NUMBER THEORY: METHODOLOGY FOR CLASS-BASED WORK

The teaching sequence is structured into 5 different gradual phases of work, characterized by the following activities: (1) translations from verbal to algebraic language and vice-versa; (2) study of the relationship between properties of a given formula and properties of the variables it contains; (3) analysis of the truthfulness/falseness of statements concerning natural numbers and justification of the given answers; (4) exploration of numerical situations, formulation of conjectures and related proofs; and (5) construction of proofs of given theorems. The path (about 20 hours) was articulated through small-group activities (8 groups were audio-recorded), followed by collective discussions (audio-recorded) on the results of the small-group activities.

Whole-class discussions, referring to work conducted in groups, always developed in 3 essential moments: (a) collection of single groups' productions: groups' spokesmen presented to the class the answers they gave to the particular task analyzed. In this phase, teacher only wrote on the blackboard what students reported, summarizing their speeches, if necessary; (b) exchanges and discussion about productions: the teacher guided a discussion and analyzed the single answers through students' remarks, she had previously solicited. In this phase, teachers' role was fundamental, since their interventions were aimed at guiding the discussion so that critical points, as well as positive aspects, of each strategy might be pointed out. During this phase, teachers tried to stimulate students to set the basis for reaching learning and educational objectives, related to the particular activity; and (c) concluding synthesis by teachers: teachers summarized the main remarks made about each activity they analyzed. In this phase, they tried to point out the relevant moments of the discussion with the aim of making explicit those aspects that might be not completely clear to students.

7. CLASSES AND TEACHERS

Students involved in the teaching experiment this paper describes, belonged to the second grade 10 classes of a secondary school in Reggio Emilia: one from a socio-pedagogical course and the other from a social sciences course. Mathematics had a minor role in these courses: it was taught only 4 hours per week and the curriculum was more limited than that at schools with scientific or classical courses. An important consequence was that many of the students attending this school were neither interested, nor inclined to mathematics. The two classes involved had already had a traditional approach to algebra, during the previous school year. We chose to propose this teaching sequence to them, in order to highlight problematic aspects related to possible conflicts between an existing view of algebraic language and the new awareness the activities were meant to stimulate. Moreover, due to the difficulties at syntactic level, many of these students showed at the beginning of the school year, we thought that proposing this type of activities to them would have been a good research context to verify whether it was possible, though an approach like this, to facilitate little steps towards a vision of algebra as a tool for generalizing, reasoning and proving.

The two teachers involved participated in the European project PDTR. They enthusiastically approached research for the first time. They were expert teachers (who had been teaching for 15 years), willing to go beyond the rigid schemes that a type of teaching mainly based on transmission models produced in them. During the first year, their "being teachers" prevailed over being researchers. We let them discover the limitations of some of their choices (the activities they proposed turned out to be "too much structured" and aimed at guiding students to a sort of mechanical learning). In the second year, due to the novelty of the proposed teaching experiment (on the principle of mathematical induction), we designed the activities together, letting the teachers organize the management of lessons and whole-class discussions. At the end of the second year, the teachers, having analyzed with us weaknesses and strengths of the approaches they adopted, became aware that their attitudes had an influence on students' engagement and learning. The current, and third, year was devoted to restructuring the sequence proposed during the first year, and successively implementing it. Teachers approached the activities in this sequence with higher awareness, due to our work on both sharing theoretical studies and reflecting (individually and collectively, in itinerary and a posteriori).

Activities analyzed in this paper refer to the class in the social sciences course and their teacher.

8. EXCERPT FROM A WHOLE-CLASS DISCUSSION

This is an excerpt from a collective discussion, referring to a task proposed to groups of students during phase 2, "Analysis of variable writings." We chose to carry out an in-depth analysis of this discussion because it reveals how, starting from the first activities (it is the second phase of work) the teacher tried to lead students to develop those skills that play an essential role in the actual proving phase. The task asked students to say for which natural numbers n, the expression n^2 represents an odd number. During the collection of single groups' productions phase, all groups agreed to claim that n^2 is odd if n is odd, but only two of them feel the need to formalize the property of being odd (by saying explicitly that n will be in the form $2x+1$). The excerpt presented here refers to the subsequent moment of exchange and discussion about productions, which started when the teacher asked a student to justify her answer. We broke down this excerpt into four distinct moments: (1) phase of verbal argumentation; (2) towards a formalization of the property; (3) phase of proof of the property; (4) moment of reflection upon the importance of choosing a certain representation.

Extract from the whole class discussion
(1) Phase of verbal argumentation[2] **1.** T: Let's see if the condition you have set can be convincing. Why did you write $n=2x+1$?[addressing A.] **2.** A: Because we have seen that an odd number to the second power gives an odd number. **3.** T: How did you see this? **4.** A: Because odd times odd is odd! **5.** T: Then there is the multiplicative concept here. They say: if I multiply odd times an odd, where do I find factor 2? **6.** B: I don't find it. **7.** T: So, it is odd.
(2) Towards a formalization of the property **8.** T: And you, Z., how did you see it? **9.** Z: I said that squaring an even number, you get an even. Adding 1 to an even, you get an odd. **10.** T: Hold on. Here I read n^2. Why are you saying "I add 1"? **11.** Z: $2x+1$ squared gives an odd number because: 2x squared is $4x^2$ then there is plus 1. **12.** T: $(2x)^2$ is $4x^2$. **13.** T: You are saying $(2x)^2=4x^2$. $(2x+1)^2$ *is* $4x^2+1$? **14.** Chorus: No!
(3) Phase of proof of the property **15.** T: Let's get back to what Z. is saying. I can't say that $(2x+1)^2$ is $4x^2+1$. But I want to convince you that $(2x+1)^2$ is odd. What can I do? **16.** O: Let's solve it! [T writes $(2x+1)^2=4x^2+4x+1$] **17.** T: Now there is "+1" … This quantity here is the problem [points to $4x^2+4x$]. **18.** P: Let's make the total: we take out $4x$. **19.** T: Do we really need to take out $4x$? **20.** O: It's enough to take out 2. **21.** T: Why 2? **22.** O: Because then we get an even number, plus 1. [T writes $2(2x^2+2x)+1$] **23.** Z: But $4x^2+4x$ is the same as $2(2x^2+2x)$! **24.** T: Yes, it's the same thing. **25.** Z: Ah, I see why! Because taking out 2 you see you get an even.
(4) Moment of reflection upon the importance of choosing a certain representation **26.** P: Is it wrong to take out $4x$? **27.** T: Is it wrong? [addressing the class] [Silence] **28.** T: Looking at $4x(x+1)$ can I still say it is even? **29.** Chorus: No! **30.** T: Why not?! 4 is a multiple of 2, guys! **31.** G: Here you can see very well it is a quadruple! **32.** T: For what I should have got, it was enough to take out 2. Taking out $4x$ I still reach the objective, but it is more disturbing. Here, I highlight that the square of an odd number is the successive of four times a number, thus I highlight something more than I was interested in...

[2] In the excerpt reported here, A, B, O, Z, P and G indicate 5 students involved in the discussion while T. stands for the teacher. Chorus means that the sentence was uttered by a group of students in the class.

8.1 Analysis of the excerpt: incidence of the three key components and role of the teacher's interventions

This paragraph deals with an analysis of the four moments of the proposed discussion, and highlights: (1) weaknesses and strengths of the discussion, with reference to the three key components in the development of proofs in ENT (being able to translate, to interpret, to anticipate); and (2) the role played by the teacher as a "stimulus" to foster an approach to algebra as a tool for thinking, and at the same time as a "model" and "guide" in the production of reasoning.

Phase of verbal argumentation: A. enacted the frame "factorization of a number" to make explicit to the class the justifications at the basis of the answer (line 4). We notice that despite the attempt to formalize the answer, A. only proposed a purely verbal argumentation. The teacher immediately set herself in the same frame as the student and repeated the reasoning proposed by A.'s group to the rest of the class, pointing out the relationship between the fact that 2 is not in the factorization of n and the fact that n^2 is odd (lines 5 and 7).

Towards a formalization of the property: Z. referred to the additive representation of odd numbers to justify her answer, trying to coordinate the frames "odd" and "polynomials" while she was 'mentally' manipulating the expression $(2x+1)^2$ (line 9). Although the student showed good anticipatory thinking (she grasped the idea that the objective was to transform the expression until it got to the form "an even number plus 1"), she encountered some difficulties at the level of syntactic transformations, probably because she tried to proceed only verbally. When Z. made a mistake in calculating the square of a binomial, the teacher echoed her, repeating what the student said, in the form of a question asked to the whole class (lines 12 and 13).

Phase of proof of the property: Once she amended Z.'s mistake and underlined the objective of the syntactic manipulations carried out (line 15), the teacher, following O. (line 16), got to construct the expression $4x^2+4x+1$.[3] At this point, the teacher decided to guide the activity, playing the role of an "investigating subject." She actually remarked that "+1," as Z. mentioned, was in the determined expression, but she pointed at the remaining binomial $4x^2+4x$ as a "problem to be solved" (line 17). In this way, she let the class guide the activity, although she remained the point of reference for the discussion. Through this technique, the teacher acted almost as an implicit "activator of anticipatory thinking." After P. showed she did not enact a correct anticipatory thinking (line 18), the teacher echoed P.'s proposal, sending it back to the class as a question (line 19). At this point, O. enacted the correct anticipatory thinking and suggested that 2 might be taken out (line 20). The teacher asked her to justify her idea, so that she could make it explicit to the whole class. The comment by Z. (line 23) showed that the student did not interpret in-depth the objective of the manipulation within the frame "odd:" she actually showed she did not understand the sense of taking out a factor 2 from $4x2$ and $4x$. The teacher decided to echo her (line 24), simply repeating that the student's statement ($4x^2+4x$ is the same as $2(2x^2+2x)$) is right. At that moment Z. realized that taking out 2 is a way to make explicit the fact that the expression $4x^2+4x$ is even (line 25).

Moment of reflection upon the importance of choosing a certain representation: Despite Z.'s remark, P. did not understand the different effects of the manipulation he

[3] In an a posteriori reflection we carried out jointly with teachers, we pointed out the "haste" shown by the teacher in developing the square of the binomial $2k+1$ without letting students directly contribute to this result. The teacher justified her choice commenting that her objective at that moment was not to test students on this skill, but rather to lead them to the relevant moment of analysis and interpretation of the obtained expression.

suggested (taking out *4x*) on the explicit statement of the property of the final expression (line 26). Again, the teacher asked P.'s question to the class (line 27), getting only silence back. At this point, the teacher changed her role commenting upon P.'s intervention to synthesize the sense of the manipulation they carried out (lines 28, 30 and 32). This phase is equally important within the discussion because the teacher underlines how the different obtained expressions highlight different aspects of the initial expression's properties.

8.2 The teacher's role in the whole discussion

In these simple initial activities some essential aspects for the future approach to proof appear: being able to *translate* in algebraic language stems from the need to justify the obtained solution, being able to *interpret* originates from the need to understand the sense of the manipulations produced, starting from the initial expression, being able to *anticipate* is a condition necessary to decide which manipulation is more appropriate with relation to the objective. The teacher might have decided to keep the discussion at a purely verbal level, but this would have not allowed for the explicit mention of all these aspects that many students seem to have grasped. She acted as a guide and, at the same time, as part of the class group in the "research" work, leading students towards a step-by-step maturation of the first "germs" of the essential competencies they need to acquire in order to face the successive phases of the teaching sequence.

9. POSITIVE EFFECTS OF TEACHER'S ROLE: A GROUP OF STUDENTS AT WORK

The excerpt we present here is taken from work in groups referring to phase 4 of the teaching sequence, i.e. "Analysis of statements." Students were requested to determine truth or falseness of a series of statements, and justify the related answer. This excerpt was chosen because we believe it highlights the influence of the teacher's work on students' awareness. The work carried out in groups immediately after the discussion analyzed earlier, provided evidence of students' achievements. In this excerpt, three female students (C., Z. and G.) tackled one of the tasks they were given, i.e. the analysis of the following statement: "The sum of a natural number and its square is always an even number."

This transcript was split into two phases: (1) reaching a conviction about the truth of the statement and supporting verbal argumentation; and (2) proof of the statement and explicit claim about one's awareness of the importance of representation. After that, we will analyze the excerpt in order to point out: (a) weaknesses and strengths of the discussion with reference to the three key components in the development of proofs in ENT (being able to translate, to interpret and to anticipate); and (b) influence of the work previously carried out by the teacher, on the choices made by students as well as on their approach to the problem under exam.

Extract from group work
(1) Reaching a conviction and supporting verbal argumentation **1.** C: Any natural number is x, its square is x^2 [they write $x+x^2$]. **2.** Z: The sum is $2x$. **3.** G. Here you have to take out x… **4.** C: Taking out x we get $x(x+1)$ … it's false! It's true only if x … It's false because we don't know whether x, the natural number, is even or odd … And then we write: if x equals… **5.** G: Hold on because it might be wrong! Because I'm thinking … Choose any odd number, for instance 3. 3 times 3 is 9, plus 3 is 12: it's an even number. 7 times 7, 49 plus 7 56! **6.** C: Yes, because we get two odds, and odd plus odd is even! **7.** G: Hold on…look at this: $x(x+1)$ …$x+1$ is even …ah, no: it is any number. **8.** C: 2+4 **9.** Z: 16+4 … So it's true!
(2) Proof of the statement and explicit claim about one's awareness of the importance of representation **10.** C: True, because,… we use the rule: if … **11.** Z: If $x=2x$ **12.** C: then it would be (Z. and C. together) $2x+(2x)^2=2x+4x^2$, the n we take out 2 and it becomes $2(x+2x^2)$. **13.** G: Why $4x^2$? Ah, no, sorry! **14.** C: You get an even because it is multiplied by 2. **15.** C: If you rather have $x=2x+1$, then $(2x+1)+(2x+1)^2=2x+1+4x^2+4x+1$ **16.** Z: …they add up! **17.** C: $6x+2+4x^2$. You take out 2: $2(3x+1+2x^2)$…and it is verified because there is 2.

9.1 Analysis of the excerpt: incidence of the three components and students' attitude

Reaching a conviction and *supporting verbal argumentation*. Students showed they started to acquire a symbol sense, by explicitly showing their need to formalize the statement immediately, to be convinced of its truth. Z. then tried to make explicit the property of being even (line 2) but she clearly showed her difficulties in translating from verbal to algebraic language and in keeping control of the problematic nature of the expression she gets to ($x^2+x=2x$), since the latter highlights properties that differ to those expressed in the statement.

After that, students broke down the expression under examination and tried to interpret the obtained expression. C., however, interpreted this expression as a product containing x as factor and deduces, wrongly, that the product is even, only if x is even (line 4). The gap in C.'s interpretation of the expression was due to the fact she did not enact the 'successive of a number' frame, which might have helped her highlight the evenness of one out of the two factors. After that, when G. replied to C.'s wrong argumentation by proposing counterexamples (line 5), C. shifts her attention onto the expression x^2+x and enacts the 'even-odd' frame to interpret it again, this time correctly, by proposing a verbal argumentation, supporting the fact that if x is odd, the considered sum is as well (line 6). Supported by further numerical examples, referring to the case x even, students became convinced that the statement was true (lines 8 and 9).

Proof of the statement and *explicit claim about one's awareness of the importance of representation*. In this second phase, C., not satisfied with her previous verbal argumentation, showed her need to generalize her reasoning by means of algebraic language. The preceding enactment of the 'even-odd' frame encouraged students to carry on with the proof, distinguishing between even and odd cases. Despite the gaps shown by Z. in the control of the conversions she made (line 11), students C.

and Z. correctly performed the substitution of *2x* for *x* and the subsequent treatments of the produced expression. It is noticeable that once they got to write $2x+4x^2$ and $6x+2+4x^2$, they felt the need to make explicit the property 'being even' by taking out 2 (lines 12 and 17).

9.2 Global analysis of the excerpt with reference to students' skills and to influences of the work done by the teacher

We believe that the above short dialogue can represent an initial display of symbol sense, because the work students carry out in groups is guided by a good mastering of the coordination between verbal and algebraic language, besides the three fundamental components for an approach to proof in ENT. The three students (in particular C.), besides feeling the need to generalize as a tool for granting the truth of a statement: (1) can move from a numerical example to a verbal argumentation, to the correct management of an algebraic representation of the terms in use (being able to translate); (2) tackle the produced expressions trying to interpret them, and they mainly succeed (being able to interpret); and (3) show anticipatory skills in the choice of treatments to transform expressions, so that the properties to be proved might be highlighted (being able to anticipate).

This excerpt can be viewed in parallel with that concerning the collective discussion analyzed earlier. During the activity, students put into practice what they had collectively constructed in the previous phase, showing awareness of what the teacher tried to highlight (in particular the fact that among different equivalent expressions we can obtain from a given one, there is always one formulation which makes the properties we want to emphasize more explicit). The three students' approach to the justification of the answer, reflected what the teacher traced during that discussion: (a) phase of conviction: initial argumentative approach to the justification of the answer; (b) phase of proof: drawing on the algebraic representation as a tool for generalization; and (c) syntactic manipulations aimed at stressing the property to be proved and at an explicit claim about one's awareness of the correctness of the obtained expression (in the following table, these remarks are sketched).

<table>
<tr><th>Teacher's approach during the collective discussion</th><th>Students' approach during work in groups</th></tr>
<tr><td>(1) Phase of verbal argumentation</td><td rowspan="2">(1) Reaching a conviction about the truth of the statement, verbal argumentation supporting one's remarks and first attempts to formalize the property under examination</td></tr>
<tr><td>(2) Phase of formalization of the property</td></tr>
<tr><td>(3) Phase of proof of the property</td><td rowspan="2">(2) Proof of the truth of the statement and explicit claim of one's awareness of the importance of a certain representation</td></tr>
<tr><td>(4) Moment of reflection on how important is the choice of a representation</td></tr>
</table>

We believe that this approach stemmed from the positive influence of the teacher's role in the previous phase's discussion. The fact that students were able to propose the trace previously followed by the teacher, showed an actual maturation of those competencies she tried to develop in them since the very beginning. A final remark is needed for the fact that the students tackled the task of verifying the truth of a statement as an actual proof, although they had never worked on proving activities. In our opinion, this supports the efficacy of the teacher's approach, from the initial phases of the teaching sequence, as we strongly believe that the three competencies we emphasized, should be enacted starting from the early phases.

10. CONCLUDING REMARKS

We believe that the analyzed examples may provide evidence of the results our approach produces on both the students' level of awareness and the teacher's professional development. The choice of illustrating examples drawn from initial activities of the teaching sequence, aims at emphasizing the importance of working since the very beginning with the objective of guiding students to develop fundamental competencies for the construction of proofs in ENT (being able to translate, being able to interpret, being able to anticipate). In fact, the teacher is not satisfied with the fact that all students can "see" the link between properties of the variable and properties of the expression which contains it: she does not simply rely on the intuitive level; she rather proposes a formal approach to the problem, supporting the intuition itself. This encourages students not to be satisfied with their intuition, when they tackle the analysis of the truth of a statement, and lets them choose a similar approach to support their answers. Since the very beginning, students show they have grasped the power of algebraic language as a tool for generalization, and even for proving properties.

The teacher's methodology to approach collective discussions, and her attitude towards the class are both remarkably important. As a matter of fact, she acts as a guide and a model for the group, but, at the same time, she lets students guide her, by playing the role of a listener and a "provoking subject." In this way, she stimulates a collective construction of knowledge, without forcing anyone. The approach adopted by the teacher is to tackle the problem of justifying one's statements through a research attitude, which we can find in the approach proposed to students in the subsequent work in groups. In this way, students learn how to translate hypotheses into algebraic language, how to transform an expression to find the range of its possible interpretations, how to interpret the results of these syntactic manipulations, and how to select the expression that most fits the thesis.

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REMARKS ON THE ROLE OF THEORY IN THE WORK OF TEACHER-RESEARCHERS

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ABSTRACT
A short review of the meaning and role of learning educational theories is provided through personal narrative remarks of the author, a teacher-researcher from the Bronx, NYC. The remarks based on classroom experience lead to a re-examination of educational theories in teaching-research. The remarks are coordinated with the comments of new teacher-researchers who graduated from the PDTR project and reflected upon the issue in their practice. Vygotsky's Zone of Proximal Development is discussed as an example of the theoretical framework that fits well into classroom teaching practice of concerned teachers; it is followed by an argument presenting a theoretical notion of schema as a theory particularly useful for teaching-research practice.[1] *In many contributions to this book one can read about other educational theories and their use by the PDTR Project, some of which are discussed below.*

INTRODUCTION

Bruner (1996) introduces an interesting dichotomy in the characterization of mathematical presentations: a narrative and a paradigmatic presentation, each arranging the experience in a different, irreducible, yet complementary manner. While the narrative mode pays attention to the sequence of actions in time, dwells on particular episodes for the illustration of an idea, and is highly contextual and personal, the paradigmatic mode is highly impersonal, categorical and hierarchical in its expression.

The aim of the present remarks is to contribute to a discussion on how the role and need for an educational theory arise out of teaching practice of teacher-researchers. Therefore, the narrative mode of presentation is taken here as better conveying the subtleties of this experiential process, which is often difficult to grasp through the paradigmatic mode characteristic for standard research papers. Moreover, since the developmental process of teacher-researchers is not yet well understood, the narrative story-telling approach has a chance to provide glimpses into TR practice for any professional interested in the development of a teaching-research profile. Consequently, the presentation below is created by integrating the thoughts of an individual teacher-researcher, the author of this contribution, together with statements of participants of PDTR found in this volume of collected articles.

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[1] The remarks below are enhanced if read in conjunction with two other entries by the same author: "The Ethics of Teacher-Researchers," and "A Teaching Experiment," (Part 1), where pertinent issues concerning the learning theories are also extensively discussed.

NARRATIVE

I would like to present these remarks as thoughts of a mathematics teacher, who, as a professor of mathematics at the unique bilingual Spanish/English Community College in New York City teaches arithmetics, algebra, precalculus and calculus courses in a bilingual context. When did the need for a learning or educational theory arise for me? How did the group of teacher-researchers, my colleagues in the Bronx, managed to incorporate it into their classrooms? What are the differences and similarities in the implementation of educational theory between teacher-researchers teaching mathematics and mathematics education researchers? How a theory can arise out of practice of teacher-researchers? Finally, what is a theory in scientific research, and whether our work should be guided by such scientific research? These are the questions I would like to discuss in this contribution.

The contribution is seen as thinking-in-action, i.e. as a process of formulating the role of a scientific theory by a reflection upon the work of teacher-researchers. And all participants of the teaching-research endeavor are invited to construct that formulation together.

WHAT IS AN EDUCATIONAL THEORY AND ITS ROLE?

For Shoenfeld (1998) models and theories should support prediction, have explanatory power, and be applicable to broad ranges of phenomena, while Dubinsky & McDonald (1999) add that a theory should help organize one's thinking about complex, interrelated phenomena, serve as tools for analyzing data, and provide the language for communication of ideas that go beyond superficial descriptions.

While of wide scope, these characterizations do not address fully the work of teacher-researchers for whom a theory can fulfill an excellent role of "a guide in the design of instruction, understanding student responses or classroom interactions, and as a provider of the medium through which classroom observations and reflections can reach more abstract formulations;" while definitely helping in the organization of "one's thinking about complex, interrelated phenomena" of mathematics classroom. The role of theory becomes significant here, especially if teacher-researchers are interested in the improvement of learning in the classroom. A good educational theory can help predict to a certain degree whether its conclusions, when applied to a particular classroom setting, can improve learning in this environment – one of the main tasks of contemporary teachers in mathematics classrooms. The possibility of prediction of the state and nature of learning mathematics can be of invaluable help for classroom teachers in their choice of a didactic approach: (i) How will it change my students' grasp of particular mathematical concepts; (ii) Along which paths should I facilitate students' learning in order to improve understanding of a particular concept?

In distinction to theories in natural sciences, the use and application of theories in mathematics education depends significantly on the creation of learning environment in the classroom by teachers, which allows for their proper and successful application. Consequently, the ability to predict improvement of learning on the basis of a theory depends on the degree to which classroom teaching is coordinated with the tenets of this theory. This coordination is the primary task of teacher-researchers.

These decisions are reflected in the reports of teacher-researchers in the present collection. Thus, for Hungarian colleagues Koi and Toth (Part 4), important theoretical assertions which emphasize the path of learning from a concrete mode to an abstract one; and Bruner's theory of enactive–iconic–symbolic pathway to learn mathematics became an important guide for the design of instruction. Moreover, Bruner's theory provided a strong motivation to reassert Hungarian teacher-researchers' conviction that the manner in which mathematics is presently taught in a Hungarian school is seriously deficient precisely because of downgrading the concrete activities in mathematics classrooms. This deficiency is particularly powerful in teaching geometry.

On the other hand, the work of Verhoef and Posthuma utilizes the research by Kramarski et al., (2002) on the development of metacognitive strategies amongst mathematics students. The authors point out that due to recognized difficulties of students in the use of these strategies, an additional approach of critical questioning in a collaborative setting was needed to support their development. Consequently, we have here an example of a need to supplement a given theoretical approach by the teaching-research work in order to be able to utilize the chosen research in the classroom context.

EXAMPLE: ZONE OF PROXIMAL DEVELOPMENT

For me, a learning theory is an important component of my TR work because it allows for making fast connections with other concepts, ideas or ways of doing things, which might be useful to better understand my particular classroom observations.

I do remember a moment when – while pondering the significance of the increase of essay writing coherence demonstrated by the mathematics/English as a Second Language class (Czarnocha & Prabhu, 2000) – I understood that the effects of the language/mathematics teaching coordination need not be a purely statistical phenomenon but represented the impact of the algebraic structure upon the linguistic one and this realization led me to a theoretical structure called Zone of Proximal Development (ZPD) (Vygotsky, 1987), with all its riches, connections or analogies. The concept of the Zone of Proximal Development discusses the difference between the conceptual distance students can cover in solving a particular problem or construct a particular concept on their own, and the conceptual distance that students can cover with the help of a mentor or a peer group. The sequence of assignments or classroom questions given to the class can be designed with the help of simultaneous classroom investigations into the scope of ZPD's of students helping them reach the maximum of their intellectual possibilities at a given moment (Czarnocha & Prabhu, 2006). The series of precise questions on the part of teachers, designed to facilitate students' understanding of a particular mathematical concept can extend students' capabilities much beyond the original knowledge.

This particular take upon the theory in TR is of special importance in the bilingual mathematics education when teachers have to coordinate learning mathematics with learning a foreign language. Our classroom discoveries in this area (Czarnocha & Prabhu 2000; Baker & Czarnocha 2002) suggest that students' ZPD in mathematics can be differently positioned in relations to students' ZPD in foreign languages and as such both of them can provide help to each other in the process of learning both areas. A theoretical construct (a germ of a new theory) that emerges here is a relative ZPD, i.e. the intersection of ZPD in mathematics with ZPD in foreign languages. Such intersections can justify a claim that learning mathematics can help in learning foreign languages as well as a claim that development of language in mathematical context can be helpful in learning mathematics. It is an example of deriving general scientific hypothesis from classroom practice.

The use of language as a medium of mathematics instruction has been widely implemented in many mathematics classrooms and reports of its impact can be found, among others, in the work of Italian and Catalonian teacher-researchers in this volume. Their work shows the importance of taking into account the complexity of the classroom *in vivo*, where one theoretical point of view such as for example, the socio-cultural framework of Vygotsky can not give full justice to the teaching issues which are widely encountered. While admittedly the framework of Vygotsky points to important components of the classroom as a social organism, yet it misses the importance of an individual student's cognitive development as a socially independent process, which might have to be supplemented by the developmental cognitive framework of Piaget, applicable to learning of individual students. Similarly, of course, the sole application of the Piaget approach to the classroom ethos misses the role of social structure mediated by the language and group work. Consequently, we signal here a need to support classroom work not solely by one theoretical framework but by several which address in a smooth, consistent way different instructional aspects of classroom teaching. An example of the approach which addresses both social and individual cognitive efforts is the resonance model of learning proposed by Tortora, Iannece, (Part 1) in this volume, who assume "that any true learning in scientific/mathematical field is a result of the process of resonance between individual cognition, social culture and reality structures along cognitive paths efficiently addressed and controlled in their meaning-driven dynamics."

MATHEMATICS TEACHER-RESEARCHER AND EDUCATIONAL THEORY

If one takes into account teachers' ethical principle of responsibility for the quality of learning in their classroom ("The Ethics of Teacher-Researchers," Part 1), then the role of theory in the TR work is given immediately as one of the two essential tools with the help of which, the quality of classroom work can be maintained and increased, if, of course, one can well coordinate the theory with the events of the classroom. Mellone and Pezzia, teacher-researchers from the Naples team (Part 3), discuss that process in the framework of the resonance model:

> According to this vision, the goal of didactic mediation should be to create conditions that allow resonance between an individual way of understanding and a cultural systematization of discipline. The main aim is to make disciplinary structures resonant with students' minds (and bodies): the resonance experience is crucial if we want cultural models to be really perceived as useful tools. … So the other useful tool for teachers is the choice of the representation that can effectively, time after time, play the role of the sensitive to resonance semiotic mediator. Therefore, a careful planning of teaching work is fundamental in order to choose the situation sensitive to the resonance of cultural tools.

Another important role of learning theory as a tool for the increase of teachers' classroom awareness for the classroom work of teacher-researchers is expressed by colleagues from the Italian and Catalonian teams of the PDTR. Indeed, a systematic use of theory in the design of instructional strategies and in analysis of student responses to and interactions with the theory changes the relation between teachers and students. Some teaching/learning episodes that can be understood with the help of an applied theory in a particular classroom may be differentiated from those which can not. This calls for a careful reflection on the part of teacher-researchers; it leads to a realization that the complexity of classroom teaching and learning may require several compatible theoretical frameworks, precisely in order to understand the limitations of one theoretical approach in the context of that complexity. Each theory raises our awareness concerning the classroom issues, which can be coordinated with it; other issues need to be looked upon from a different theoretical angle to enable full understanding of the classroom's complexity.

THE CHOICE OF THEORY

Which learning theory to choose, what kind of theory and how do we make this choice? The remarks of Kadej (Part 1) from the Siedlce team touch upon this subject:

> Research aimed at the improvement of students' learning must first identify their difficulties. These difficulties may come not only from the structure of content but also from social circumstances in the classroom or an individual condition of a single student. The design of an improvement plan and its evaluation for a class of about 30 persons is no easy task. … An example of one of available theories is Krygowska's Metaphor for planning teaching by extracting concrete actions for students from the final structure of definitions, proofs, and theorems (Krygowska, 1977). Another one is that at the start of each topic teachers explain, i.e. they choose an appropriate metaphor in order to explain the issues in terms that students already know (Sfard, 1994, Bauersfeld & Zawadowski, 1981).

Recently, the same team has found it useful to utilize the procept theory of Tall and Grey in addressing student difficulties with learning percent as a multiplicative rather than additive structure:

> In trial teaching most students chose the additive way in various problems persistently and committed errors, even though they were taught the easier multiplicative model and its advantages in a series of lessons. They chose the way which they first encountered. Because such a didactical phenomenon occurred in various other circumstances, we called it *imprinting* at our seminar. We found that imprinting is in effect similar to the didactical phenomenon of *proceptual divide* described by Gray and Tall (1994).

The foregoing quote suggests that the choice of theory depends on the nature of the problem encountered and on the nature of the teaching-research question. One of the roles of a theory is to guide teacher-researchers in data production and collection, so that they respond to the posed teaching-research questions.

Since my personal interests are connected with the development of students' independent thinking, observations of this process and of difficulties experienced by my students in areas of critical thinking, mathematics in bilingual context and in calculus, I am very much drawn to the concept and theory of mental schema (Czarnocha, Lisbon presentation, PDTR, 2005). The term "schema" has many different connotations whose excellent history can be found in Marshall (1995). Here we understand schema as a *network of relationships* between different concepts and procedures. If we understand the process of thinking as the ability to make such connections between different, sometimes seemingly unrelated components, then the schema theory looks like a useful approach for those amongst teacher-researchers for whom their students' learning independent thinking is of paramount importance. And then a question arises for us, teacher-researchers: what are the mental processes, through which the formation of such schemas takes place, and how to use that knowledge for improvement of learning? Together these questions lead to a very important theme within the theory of schema development. Of interest for us here are the processes through which such a network of relationships is created, developed, made stronger and more robust. In fact, developing the TR/NY City model led us towards the term "thinking technology," which is a careful integration of the principles of ZPD, a component of the socio-cultural approach with the schema theory, a component usually associated with the development of individual cognition.

SCHEMA THEORY AS A THEORY OF TEACHING-RESEARCH

My professional work in education, and writing professional development grants especially, made me aware that other teacher-researchers find it equally convenient to use that notion. As she describes her problem-centered math curriculum, Lampert (2001) notes that "in contrast to the familiar topic-by-topic approach, I worked on constructing lessons for my students to investigate a number of different but related topics, and to investigate them repeatedly in different problem contexts" (259-261). Jaworski's student Ben describes his own thinking as a system (or schema) he observes in his mind: "I feel that in my head I have a system of mathematics. I don't know what it looks like but it is there, and whenever I learn a new bit of mathematics I have to find somewhere that that fits in" (Jaworski, 1994, 157, teacher Ben). The same concept of mental schema is often used to analyze the development of teaching proficiency by teachers of mathematics. Carpenter et al. (1988) found that (American) teachers' knowledge is usually fragmented and not organized in a way that enables them to understand and further develop children's thinking. Simon (1995) and Simon and Tzur (1999) suggest that teachers – in order to develop the mathematical-didactical know-how necessary for their practice – should follow a learning trajectory from fragmented knowledge of the topics in the curriculum towards "knowledge packages," i.e. knowledge that articulates multiple interconnections among these topics (Ma, 1999; Stigler & Hiebert, 1999). Korthagen and Kessels (1999) suggest that teachers' learning trajectory may be described as a process of reflective abstraction from initial Gestalt or holistic experiences with concrete instances of mathematics instruction first to a schema and then to a theory, i.e. to a logical ordering of relations in the schema (Korthagen & Kessels 1999).

Observing my own development as a mathematics teacher-researcher I find that these authors' description fits exactly my own self-observations and reflections.

Finally, let me add that an error analysis of the 2006 Polish mathematics final high school examination (matura) revealed that graduating students had difficulties in noticing and utilizing relationships between different pieces of information in the text of the examination problems in at least 8 out of 21 questions in the exam. The candidates did not know whether to take a positive or negative root while solving a problem or converging sequence problems; they did not see the significance of a condition stated in the problem for the solution of that problem; they could not decide from the analysis of a geometric drawing which sides of triangles are similar to each other; and they had difficulty understanding the relationship between the two components of mathematical induction, etc. This indicated the weakness of their mental schema and suggested directions for necessary teaching-research didactic interventions.

Consequently, we can see here that the same theoretical framework of schema is able to affect both teachers' and students' development. It provides a tool that can help analyze the intense moments of teachers and students "thinking together" in the classroom, which often take place in the work of teacher-researchers. This is the primary reason why we suggest the notion of schema as one of the fundamental tools for teaching-research. A simple example of such a moment of "thinking together" in the context of student-teacher interaction, which reveals students' thinking process can be found in Czarnocha, (1999). While students work with the schema of square root, teachers work with the schema of students' thinking about that subject; their interaction shows intense coordination between the two which facilitate students' final mastery of the problem. The example is taken from an intermediate algebra class; it occurred during a review of the domain of $f(X) = \sqrt{X}$.

The teacher asked students during the review:

Teacher: Can all real values of X be used for the domain of the function $f(X) = \sqrt{X+3}$?

Student: No, negative X 's can not be used. [The student is confusing here the general rule which states that for the function $\sqrt{X}$ only positive-valued X can be used as the domain of definition, with the particular application of this rule to $\sqrt{X+3}$.]

Teacher: How about $X = -5$?
Student: No good.
Teacher: How about $X = -4$?
Student: No good either.
Teacher: How about $X = -3$?
Student [after a minute of thought]: It works here.
Teacher: How about $X = -2$?
Student: It works here too. [A moment later she adds:] Those X 's which are smaller than – 3 can't be used here.
Teacher: Right. How about $\sqrt{X+5}$?
Student: X 's smaller than – 5 can't be used.
Teacher: How about $\sqrt{X-1}$?
Student [after a minute of thought]: Smaller than 1 can't be used.

We can see here a teacher whose understanding of the schema relationships of the square root function allows the student to recognize her misconception, understand it and correct it autonomously. The success in such a classroom endeavor depends on the ability of the teacher to construct questions with gradual increase in the cognitive steps needed by the student to reach the process of discovery.

CONCLUSION

Theory can play several special roles in the work of teacher-researchers, many of which are utilized in the present volume. This contribution brings together the roles which are important from the point of view of designing classroom instructional strategies and analyzing student responses to them, as well as of helping students reach the maximum of their potential, especially in the development of independent mathematical thinking. The contribution points out to the need for using several different learning theories to account for the complexity of a single classroom.

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MATHEMATICAL TESTS
THEORETICAL ASSUMPTIONS

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ABSTRACT

Test is a word that has different meanings for different people. In teaching, this word is associated above all with the assessment of students' skills and knowledge. One task (sometimes extended) or a set of tasks can constitute a test. In practice, teachers designing tests often compile them of ready made tasks. They rarely create tasks themselves. The ability to construct (context-dependent) tasks proves to have a special significance for teacher-researchers. Firstly, it is because ready-made commercial tests aim at testing students' acquired knowledge, while in research the focus is on students' thinking. Secondly, teacher-researchers intend to test their particular hypothesis on the profound cause of students' difficulty; for this they need tasks that address the difficulty and test the hypothesis most precisely.

The goal of this paper is to share some experiences of the process of mathematical test construction and task designing in the work of teacher-researchers.

THEORETICAL TESTS ASSUMPTIONS

Verifying and assessment of students' achievements is very important and is one of the most difficult pedagogic skills. Okoń claims that "verifying school achievements – in spite of its unquestionable significance – is a nuisance for students, a source of stress and neurosis and a constant nightmare for teachers, who proceed fast and get down to control achieved results with concern and reluctance, and it is also a problem for educational authorities" (Ciżkowicz & Ochenduszko, 1986).

Tests are one of the methods of control qualified as one of the main tools of didactic measurement of school achievements. What does the word 'test' mean? It comes originally from Latin *testari* meaning 'to testify.' It is a conventional term adopted in different fields of science and practical activities for assessment of try-outs, which different subjects are submitted to in order to achieve a necessary recognition of their properties. Bradefild defines a test as a "special kind of measurement tool which helps obtain students' answers and consider them as an indicator of their knowledge, skills, attitudes, etc." On the other hand, according to Wesman, "a test consists of a certain number of tasks designed to be solved by examinees. Some tasks from a test are scored as indivisible units; others are divided with regard to the goals of scoring. Each task from a test gives a single piece of information concerning the person who has solved the test" (Hołubowski, 2006). Whereas Niemierko (1975) states that a test is "every set of tasks, designed to be solved in the course of one school lesson and adopted to a certain teaching material in such a way that teachers can determine from their results how well students acquired that part of the material.

Delving into the test theory we should mention its properties. As a tool for didactic measurement it should have the following features: (i) objectivity – its results agree with the real level of students' achievement in a given area of a program of studies; (ii) accuracy – its results are consistent with students' achievement results obtained by applying other procedures of assessment of the same knowledge and skills; for instance with external examinations results; (iii) expediency – simplicity in use; and (iv) reliability – its results are the same when we construct a different version of the test using the same plan and curriculum content. The test plan is a quantitative comparison of test tasks taking into consideration the curriculum requirements appropriate for different school grades and teaching goals. Curriculum content of the test contains a qualitative comparison of requirements and goals with a specification of activities, which are verified by individual tasks from the test.

The mathematical tests are of course constructed accordingly, depending on purpose, goals, conditions in which they are carried out, examinees, etc. And they can be divided in the following way by:

(1) Aim of measurement: (a) *formative test* (to test a certain stage or just a small part of …) usually applies to a small part of the content. It is used to check a degree to which students mastered some facts or skills. It also provides information on the effectiveness of didactical tools applied by teachers. The test is carried out during the course – not at the end. It gives teachers a possibility, after the result analysis, not only to control students' knowledge and skills but also to obtain information about existing weaknesses of students. The results of this test, after a deep analysis by teachers, can influence the improvement of effects of their further work with students; (b) *summative test* (for one semester, one year) consists of a larger number of tasks and applies to a wider range of material. The test is a chance for an objective assessment of students. It provides information about the semester or the whole year work.

(2) Relation to teaching results: (a) *check-up test* consists of tasks verifying if required parts of the material, selected according to the importance of the tested content, have been acquired. It relates to the curriculum. The results provide information if students meet the program standards of achievement or if they have some weaknesses in mastering certain activities; (b) *multi-stage check up test* consists of groups of tasks. Each group aims to verify a given school grade. It also verifies how the content of subject curriculum is acquired by each student; (c) *test of necessary achievements* checks only the knowledge and skills necessary for further education; (d) *selective test* consists of tasks that not all students can solve in a correct way. It serves to compare students' achievements. The results of other students are the point of reference to the results of given students. These tests are used in selective goals e.g. during exams, competitions, i.e. in situations in which we want to choose the best students.

(3) Manner of construction: (a) *standardized test* undergoes a process of trials, improvements and normalization. When standardizing a test teachers should select an appropriate time and conditions to carry it out and assess the level of difficulty. Standardization is conducted on the group of about 500 people before the actual testing. This test allows comparing the results of each student with the results of other students or with the curriculum requirements in an objective way to a certain degree; (b) *informal test*, also called *teacher test*, is usually intended for an occasional application. It is used by teacher-constructors themselves for their own goals. Teachers determine the test content and circumstances during carrying it out, and they interpret the achieved results themselves. That is why this test does not need to be standardized and does not need the instruction of testing.

(4) Range of its application: (a) *test for many users* is constructed to be useful for other people – not only for its author. This test always has to be standardized. These tests are usually published and made accessible for general use. They often include closed problems and are accompanied by a precise instruction. (b) *teacher's test* is the author's test prepared by given teachers and used only by them.

(5) Kind of activities of the tested students: (a) *oral test* in which students orally answer the questions and give comments to some of them; (b) *written test* in which students give answers to the test tasks in the written form; (c) *practice test* in which students demonstrate a way of solving a problem or carry out an experiment or exercise (Szaran, 2000).

During the test construction teachers should pay special attention to the fact that the test should be adapted to the intellectual level of examinees. It should have a precise instruction of application and result interpretation. It should be essentially correct and should have accessible and interesting content. It also should be fitted in time in an appropriate way. Elements of theory concerning tests, which are mentioned in this article, are reflected in the mathematical texts constructed by teacher-researchers.

MATHEMATICAL TESTS IN TEACHING-RESEARCH

The ability to construct mathematical tests plays an important role in the practice of teacher-researchers. Beside control concerning comparison of the actual state of students' achievements with the state required by the subject curriculum, teacher-researchers should check in what way their experimental activities influence examinees and if they yield intended effects. In their work teacher-researchers use informal, diagnostic and verifying tests very often. They are constructed in order to: (i) recognize the weaknesses of students and their origins, in order to make the so-called preliminary diagnosis; (ii) examine the changes that took place in students when the instructional strategies (corrective procedures) were carried out, overcoming the weaknesses recognized during the diagnosis; (iii) verify the effects and effectiveness of the instructional strategies concerning a given research problem (the final diagnosis).

The range of content examined by teacher-researchers is usually restricted to a certain area. Research often concerns knowledge, understanding of concept(s), and some skills, e. g. formulation of equation understanding and the ability to solve equations; the skill of justifying and reasoning as introduction to proving mathematical theorems; and development of chosen skills of solving tasks with context.

Such a test can be constructed by one person or a group of people interested in a valuable test creation. The effort of a bigger group of people allows for posing many questions, putting different ideas, modifying them and making their unbiased selection.
Designing, carrying out and analyzing the results of a *diagnostic test*, and also *a test serving to improve the diagnosed difficulties* can be put in certain frames that:

(1) determine the main goal of research – what we would like to find out after carrying out such a test;

(2) formulate detailed goals – questions which we would like to answer after carrying out a test;

(3) construct the test according to the research goals. This stage includes the choice of tasks, questions, the arrangement of the tasks on the sheet, and design of the form in which the test will be carried out;

(4) submit the test to a preliminary verification by a group of a few students in order to: (a) determine if the instructions are understandable and straightforward; (b) verify if the solutions given by students answer the basic questions (detailed goals).

(5) carry out the test according to the worked out instruction: (a) *summon the research goal in order to direct the observation*. Answers to the questions: "Why students should solve this task? What would I like to find out about the process of thinking, reasoning and skills of student?" can be helpful in this task; (b) *make individual work with the test possible for students – in the case of diagnostic test* (in other cases different forms of work would be more suitable). When students do not understand something they can ask teachers. Teachers have to answer, but they are not allowed to help solve the task. They should encourage students and confirm that they have to make attempts to solve the tasks themselves. When students give up, they have to write down what they do not understand, what they do not know, and what is difficult for them; (c) *observe how students solve the tasks by teachers, who write down students' questions and comments before, during and after solving the task*. Recording with a tape or video recorder can be helpful; (d) *carry out interviews with students after they have solved the test on the subject, using questions like:* "How did you get the result? Who has got the same result? Who was reasoning in the same way? Who was reasoning in a different way but got the same result? Where did the difficulties occur? Do you have any questions concerning the task? Recordings of such interviews and photos of solutions written on the blackboard can be helpful during the preparation of detailed notes or even the transcript. It is worthwhile to carry out individual interviews with chosen students, asking them questions: "How do you get the result? Are you sure this is a correct way of solving the task? How would you convince me that you are right?"

(6) quantitative analysis – a numerical balance sheet of the students' results, introduction of codes for students and conventional definitions concerning the ways of reasoning, solutions or lack thereof.

(7) qualitative analysis of student' results: (a) answering the question: "What have I found out about the way of students' thinking?" (b) writing down the conclusions and surprising, non-standard solutions, gross mistakes, or typical often reoccurring mistakes; (c) writing down doubts, which we would like to explain, and questions which we would like to answer.

(8) reformulate the diagnostic tasks (if necessary).

Mathematical tests play a very important role in the work of teacher-researchers, similar to the one occurring in the work of ordinary teachers. However, there are some fundamental differences. Teacher-researchers – in opposition to ordinary teachers – in order to assess students' work use a measurement scale different from school grades. Control of knowledge and skills acquired by students and demonstration of their weaknesses are part and parcel of every teacher's work. Teacher-researchers additionally try to reconstruct the way of thinking of examinees and the sources of their failures. An analysis of solutions and mistakes as well as students' justification of their failure to make further attempts to solve the tasks also serve this aim. According to this rule, students encountering difficulties, which make it impossible to carry on solving the task, are encouraged by teacher-researchers to comment on the difficulty and try to give a reason for such a situation. Thanks to this, teachers can get valuable information on the way students' think and also to find out what the reasons of students' failures in solving a task are.

In a classical assessment, mistakes show to what degree students mastered a given skill or acquired given knowledge. Teacher-researchers, during the mistake analysis, ask themselves many questions concerning, among others, students' thinking, communication on the teacher-student level, applying teaching methods and forms, and didactic tools. They look for answers to these questions, because it is the key knowledge for designing further proceedings aiming at the correction of diagnosed difficulties. While designing the corrective steps, teacher-researchers can also use tests (in the meaning given by Niemierko, 1975), which are elements of a certain planned set of instructions. When planning the instructional strategies different tasks should be chosen concerning the content and questions. These tasks should incline students to creative thinking and to a reflection on the choice of solving methods that can be used for solving similar problems.

EXAMPLES OF MATHEMATICAL TESTS IN TEACHING-RESEARCH

As mentioned at the beginning of this article, special attention will be paid to the stage of mathematical test construction by teacher-researchers. We will discuss the process of creation of three basic tests. The first two were the diagnostic test type. The third one was created in order assess to what degree students' difficulties diagnosed earlier were overcome. All three tests were created as a result of the Rzeszów-Kraków team work, which dealt with the understanding of the concept of equation and the skills of solving equations by students aged 14-17.

Diagnostic tests

Teachers' interest in equations originated from the study of the PISA task entitled "Apple trees" (PISA, 2000). After carrying out research, it turned out that students had difficulties, among others, with the following: (a) connecting two algebraic expressions (describing two given values) $8n$ and n^2 by the equality sign; (b) solving the equation $n^2 = 8n$ and presenting the solution as a set of numbers; (c) taking up action while encountering an equation of unknown type to them; and (d) correct interpretation of the achieved solution.

It provoked teachers to engage in this problem. They decided to concretize and specify the goals and scope of research and to create an appropriate test, which could be used for the diagnosis of the above students' difficulties. The diagnostic test entitled "List of Equations" came into being in order to examine the difficulties b), c) and d).

Diagnostic test "List of Equations"

The goal of the following test is to answer the question: "How do students deal with equations that are not typical for them?" We would especially like to answer the questions: (i) do students reflect on the choice of an effective method before beginning to solve an equation? (ii) what methods are chosen by students to search for all the numbers fulfilling a given equation? (iii) how do students proceed when they can not find a procedure for solving a given type of equations?

Our assumption, based on the day-to-day classroom experience, was that students facing an equation would try to assimilate it to the equations that they solved previously and apply the same procedure they used. The procedure being remembered formally, as manipulating symbols, also assimilation of the new task would be made on the symbolic surface and the same formal procedure forcefully applied. This strategy is reinforced by students' conception of learning: they are expected to learn the correct solving procedures. Test problems normally used for assessment are "safe" for this strategy to be successful. In order to uncover what is under the surface students must be confronted with non-standard problems, such that cannot be solved using procedures known to the students, but can be solved easily by other knowledge that the students possess.

The test called “List of Equations” included both equations known to students (of the first degree with one unknown), and equations not typical for them, which they had never came across during their whole education (concerning the type of the equation and number of solutions). The latter, included only variables and operation symbols that students had been learning and applying. They were also simple in the sense that there was no superposition of operations. In each case the solution was easy to guess. We hypothesized of course that examinees in all cases would apply their formalistic strategy instead of ordinary reasoning.

Diagnostic test “List of Equations”

Give the solutions of the following equations. Present the way of arriving at the solution.

1) $x + 1 = 2x$
2) $x^2 = 2x$
3) $x^3 = x^2 \cdot 3$
4) $x + x = x$
5) $x - x = 2$
6) $x + 2x = 3x$
7) $x + 2 = x^2$
8) $x^2 = 2^x$
9) $x^y = y^x$
10) $\frac{1}{x} = 2$
11) $\frac{1}{x} = \frac{1}{7}$
12) $x^3 + 1 = 0$
13) $x^2 + 2 = 0$
14) $(x - 1) \cdot (x + 5) = 0$
15) $(x+1) \cdot (4-x) \cdot (2x-1) = 0$
16) $\frac{2}{x} = 2^x$
17) $x^0 = 1$
18) $x \cdot y = 0$
19) $\frac{1}{(7-x)} = 0$

Results of the test confirmed our hypothesis and suggested a new one: the concept of equation is not fully formed in the mind of the examined students. The following new questions were posed: (1) how do students understand: (a) equation; (b) identity equation; (c) solution of an equation, a set of solutions to an equation; (d) equality sign in the equation? and (2) how do students solve equations, especially of the type $a(x-b)(x-c) = 0$?

The second diagnostic test entitled “Leszek and Jarek” was created. Its purpose was to examine in a more detailed way students’ difficulties with the understanding of equation, creating an equation to a certain task, solving equations (in particular ones created by student), and interpreting the solution. The test was composed of two sheets. Students, after having solved the tasks from the first sheet and collection of the works by teachers, received the second sheet. The first sheet included two similar context problems. Students should not have difficulties with creating equations to both. But as problem 1 had a unique solution, problem 2 led to an identity equation, which made it impossible to answer the final question. The second sheet contained the same problem 2, but this time an equation and standard transformation procedure were presented. Here, students should assess correctness of the solving procedure and interpret the latter identity equation.

Also, the two sheets included problems on solving equations of the type $a(x-b)(x-c) = 0$. But as the first sheet asked just to solve it, the second sheet provided hints and further questions generalizing the method or reversing the path equation – solution.

Diagnostic test "Leszek and Jarek"
SHEET 1
Leszek tells Jarek: "I thought about a number. Then I added number 5 to it. I multiplied the result by 2 and then subtracted 4 from the product. I received the treble of the number I had thought about at first. What number did I think about?"
Do you know what number Leszek thought about? How could Jarek be able to solve this riddle?
…………………………………………………………………………………
Exercise 2
Jarek posed a riddle to Leszek: "I thought about a number. I added 10 to it. I divided the result by 2 and then I added a half of the number I had thought about at first. I received a number by 5 bigger than the one I had thought about. What number did I think about?"
Do you know about what number Jarek had in mind? How could Leszek solve the riddle?
…………………………………………………………………………………
Exercise 3
Here is an equation:
$x\,(x-3)=0\cdot$
Does this equation have a solution? If yes, what is the solution?
…………………………………………………………………………………
SHEET 2
Exercise 4
Jarek posed a riddle to Leszek: "I thought about a number. I added 10 to it. I divided the result by 2 and then I added a half of the number I had thought about at first. I received a number by 5 bigger than the one I had thought about. What number did I think about?"
Do you know about what number Jarek had in mind? How could Leszek solve the riddle?
Searching for the solution Leszek wrote:

$$(x+10):2+\frac{1}{2}x=x+5$$

$$\frac{1}{2}x+5+\frac{1}{2}x=x+5$$

$$x+5=x+5$$

Does Leszek's equation describe Jarek's riddle in a correct way?
Help Leszek solve the riddle.

Exercise 5
Write at least 5 different equations the solution of which is number 3.
…………………………………………………………………………………
Exercise 6
Here is an equation:
$x\,(x-7)=0$
a) Is number 0 the solution of this equation?..
b) Would a number different than 0 be a solution of this equation? ....
Exercise 7
1. Give an example of equation that does not have any solution.
....
2. Give an example of equation that has exactly one solution.
....
3. Give an example of equation that has exactly two solutions.
....
4. Give an example of equation that has exactly three solutions.
....
5. Give an example of equation that has infinitely many solutions.
....

Detailed goals of the test were formulated in the following way:

Goals	Task Number
▪ Do students notice the necessity of using algebraic language to solve the task? ▪ For what purpose do students use algebraic language? ▪ Do students notice that the whole task can be represented as an equation? ▪ Do students use an equation while searching for the answer? ▪ Can students interpret the equation solution in the task context?	1, 2, 4
▪ Do students know that an equation can be solved by using a procedure? Can they use it? Can they interpret the result?	1, 2, 3, 4, 6
▪ How do students understand the notion of solving an equation?	3, 5, 6,7
▪ How do students solve an equation in the form $a(x - b)(x - c)=0$?	3, 6
▪ Can students give an example of equation knowing its number of solutions or its set of solutions?	5, 7

Test as a corrective means

It should be noticed that further tests of teacher-researchers, which allow to: (i) diagnose the state of knowledge; (ii) examine the way of students' thinking; and (iii) answer the research questions; constitute a sequence. Each consecutive test is created on the basis of conclusions after the analysis of the preceding one. Tests mentioned previously are of the diagnostic character. This section presents a test entitled "Windows." This is an example of a test used to correct difficulties diagnosed earlier.

Conclusions of the analysis of students' written solutions and oral answers to the diagnostic test entitled "Leszek and Jarek" inclined the teachers from the Rzeszów-Kraków team to plan procedures aiming at the elimination of the diagnosed difficulties. One of these procedures was a set of tasks whose goals were: (1) correction of the understanding of the concept of equation and solving an equation; (2) developing the skill of solving equations of the type $a(x-b)(x-c)=0$ by "reasonable guessing" or by using the theorem $\underset{x,y\in R}{\forall}(x \cdot y = 0 \Rightarrow x = 0 \vee y = 0)$; (3) developing the skill of identifying the identity and contradictory equations and interpreting their solutions.

Test "Windows" aimed at the first goal. If we want to develop the *concept* of equation and its solution we should play down the solving procedure and highlight the very sense of equation as an open sentence. So we decided to replace the letter x representing the unknown (or variable) by a window to fill in. Thus, students should not be tempted to manipulate symbols or apply known procedures, but try to fill in the windows to obtain a true sentence. This way they will recollect their knowledge on multiplication by 0, which is needed in solving equations of the type $a(x-b)(x-c)=0$. The "equations" with windows are followed by ordinary ones; thus, what was learned would be applied to similar and extended situations, as well as to reversed problems (path equation – solution reversed).

Test "Windows"

I. a) Complete:

$\square \cdot 5 = 0$ $\quad\quad \square \cdot 0 = 0$

$-3 \cdot \square = 0$ $\quad\quad 0 \cdot \square = 0$

b) Give numbers which fulfill the equations:

$x \cdot 5 = 0$ $\quad\quad x \cdot 0 = 0$

$\boxed{x-2} \cdot 5 = 0$ $\quad\quad (x-3) \cdot 0 = 0$

$3 \cdot \boxed{x+2} = 0$ $\quad\quad 0 \cdot (x+7) = 0$

II. a) Give numbers which fulfill the equations:

$x \cdot (x-5) = 0$

$-x \cdot (x+1) = 0$

$(x-2)(x+3) = 0$

b) Give an example of equation whose only solutions are numbers 4 and -6.

c) Give an example of equation whose only solutions are numbers 2, -3, $\frac{2}{3}$.

II. Complete:

$\square \cdot 1 = \square$

$\square \cdot \square = 1$

$\square \cdot \bigcirc = 1$

Give numbers which fulfill the equations:

$x \cdot 1 = 0$

$x \cdot 0 = 1$

$x \cdot 1 = x$

$(x+1)(x+2) = 6$

When teachers become aware of the detailed goals of a test as a corrective means, they are able to design students' work with a test, control interactions in the classroom, and control the discussion about particular tasks in a correct way. Thanks to this, teachers can find out which abilities are mastered by students to a sufficient degree, and which require additional procedures. It can be helpful to distinguish between some concepts, procedures, and elements of mathematical language, which one should pay attention to during the work with the test.

The test entitled "Windows" comprises such: (1) Concepts: (a) linear equation; (b) higher degree equation; (c) equation with two unknowns; (d) task solution; (e) letter as unknown; (f) identity equation; (g) contradictory equation. (2) Procedures: (a) solving the task; and (b) verification if a given number fulfills the equation. (3) Elements of mathematical language: "infinitely many solutions;" "lack of solution;" "amount of numbers fulfilling the equation;" and "amount or numbers, which are solutions of the equation." The form in which the test is carried out should be chosen individually, adequately to a given group of students. Decisions concerning the form choice belong to teachers' competencies.

CONCLUDING REMARKS

Tests created by teacher-researchers have many benefits for both students and teachers themselves. Students' work is assessed by teachers who pay attention not only to correctness, but also to the way students think and to the sources of occurring difficulties. It can give a new view of students' mistakes and contribute to a better understanding of their troubles and difficulties. Changes in the way teachers communicate with students are also evident. Teachers become observers of their own lessons; frequent discussions with students about their problems develop better relations between teachers and students. Students appreciate the fact that their individual needs are noticed. Verifying school achievements with the method proposed by teacher-researchers is neither a nuisance for students, nor the source of stresses or neurosis, because they are more aware that their mistakes provide valuable information for teachers about their own teaching workshop. Activities of teachers that follow are directed to noticing the difficulties in order to explain, consider, and discuss them over. Students, thanks to it, can feel that they are better understood, they are more important, and motivated to mathematics learning. On the other hand, teachers have a chance to improve their teaching workshop. Research based on the tests serves as a means, thanks to which they can get closer to the goals appointed by themselves.

Also teachers' personality undergoes a change. They are less and less inclined to ascribe errors made by the students to their laziness or lack of attention during class; more and more inclined to look for reasons for those errors in their own teaching, and they try to change it according to what they have learned from students. Teachers become less of an expositor of ready-made knowledge, and more of an active participant in the complex teaching-learning process.

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AN ANALYSIS OF STUDENTS' MATHEMATICAL ERRORS IN THE TEACHING-RESEARCH PROCESS

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ABSTRACT
This study attempts to analyze students' mathematical errors in the teaching-research process from teachers' perspective. Students' errors are inevitable in learning mathematics; they ensue from mathematics itself or are results of teaching. Teachers cannot be afraid of errors, but should create such situations in which students reveal their errors so that teachers are able to methodologically correct them. Two examples of students' errors are described in this article, together with their hypothetical causes, suggested corrective methods – actions that teachers may take in a given situation.

1. INTRODUCTION

In the teaching-research process teachers have to thoroughly analyze students' errors, attempt to understand the errors, explain what they consist in, and find what causes them. Depending on the conclusions of such an analysis, teachers should select corrective means and methods in order to deepen their students' understanding of mathematical concepts, improve their reasoning methods and to perfect their skills. In order to achieve that teachers need certain knowledge about errors and the methods of response to errors.

2. ERRORS IN LEARNING MATHEMATICS ARE INEVITABLE

2.1 Mathematics creates an internally coherent structure and some concepts are built on the basis of other concepts, therefore learning mathematics is difficult, requires regularity and (self) control. A seemingly small gap in comprehension or knowledge creates further misapprehensions that are built one upon another, and which after some time are revealed in an error avalanche. An unrevealed error, which is rooted in the mind of students, is therefore a major threat to the construction of students' mathematical knowledge. A revealed and clarified error may be extremely useful both for students and teachers (Krygowska, 1988: Booker, 1988).

Mathematics is an abstract science and uses a specific language. Since the onset of mathematics education we try to create abstract concepts of natural numbers, operations, and geometric figures in children's minds. Students who begin such education live and function in the real world, often think and act in a concrete way, when the first mathematizations of real problem situations as well as the first interpretations of abstract mathematical objects in reality take place at this first educational stage. This is when numerous difficulties and linguistic errors appear together with a simultaneous attempt at precision in mathematical language and the usage of a natural language that is comprehensible for children. Also, first changes of the meaning occur – of the terms taken from reality into mathematics (e.g. circle and perimeter; area). This first transition from the real world to the world of mathematics is extremely important for the entire mathematics education and the establishment of the conceptions on what mathematics is and what its role in life and education is (Krygowska 1988, 1977; Pellerey, 1988). "To understand how errors emerge and how they might be overcome on the simplest level of fundamental procedures is priceless for teachers and students, since it enables them to master the procedures of a higher level" (Booker, 1988).

Various contradictions are rooted in mathematics itself. Mathematics education requires overcoming these contradictions. The common juxtapositions are: (i) attempts at algorithmization versus creative and conscious actions; (ii) natural thinking of every day life vs. formal reasoning based on accepted conventions (e.g. veracity of an alternative or implication); (iii) abstraction of mathematics as a science vs. connections of mathematics with the real world; (iv) what was considered true vs. can turn out to be an error (for example the famous Cauchy Theorem on boundary continuity of converging sequence of continuous functions); (v) a rejected error vs. can become a source of development of new knowledge. These and other contradictions create numerous misapprehensions and might be the reason for numerous students' (and teachers') errors (Krygowska, 1988a; Rouche, 1988; Pellerey, 1988).

What seems simple and obvious in mathematics does not necessarily have to be simple and obvious in mathematics teaching. It was noted among others by Freudenthal (1988) and Thom (1974) who analyzed the 'new math' imposed on educational systems in the 1960s that focused on the concepts of the set and function. Educational systems: mathematical-philosophical-pedagogical concepts, curricula and text books also have an impact on many of students' (and teachers') errors.

2.2 Role errors play in teaching and learning mathematics

39th Meeting of the International Commission for the Study and Improvement of Mathematics Teaching (CIEAEM) in 1987 in Sherbrooke, Canada, was devoted to the role errors play in teaching and learning mathematics. It was attempted to perceive errors that appear in mathematics teaching from philosophical, moral, psychological, mathematical and didactic perspectives. While seeking to define an error it was agreed that an *error takes place when a person chooses the false as the truth*. When the actual result does not correspond with the assumed aim (we talk then about an *erroneous result*); when the undertaken actions are not compatible with the accepted procedures (we talk about *erroneous actions*). There are also *erroneous conceptions* (approaches), which might significantly hinder the problem solving and generate irrational actions of a person.

It is impossible to create a sensible error typology, since the error diversity is as rich as human life. The same error might also be analyzed from very different points of view. The analysis of error causes and the application of these analyses in the process of mathematics teaching planning and learning are more interesting for the teaching process than as an error classification. When introducing new concepts or procedures, knowledge about errors informs teachers what to focus on, what to clarify, how to negotiate the comprehension of new terms in order to avoid a certain type of errors, and how to positively use the occurring errors. However, also in this approach to errors there is a need for some terminological negotiations, a distinction between mistakes and *"essential" errors* and *erroneous conceptions* deeply rooted in the mind of students (Krygowska, 1988a). These erroneous conceptions are associated with epistemological obstacles (Brousseau) and with pragmatic errors (Pellerey).

A *mistake* is a result of the lack of concentration control or weak memory. We make a mistake when we incorrectly apply a formula or theorem, which we know (or should know) from theory acquired earlier.

An *error* reveals inadequacy of knowledge and is closely connected with imagination and creativity in a new situation, and is caused by an insufficient mastery of basic facts, concepts and skills. Such an error in learning something new is called *normal* by Duverney (quoted in Rouche, 1989).

2.3 Mathematical errors and didactic errors

In teaching practice and mathematics lesson analysis we use the terms *mathematical errors* and *didactic errors*. A mathematical error is made by a person (student, teacher) who in a given moment considers as true an untrue mathematical sentence or considers an untrue sentence as mathematically true. Didactic errors refer to a situation when teachers' behavior is contradictory to the didactic, methodological and common sense guidelines.

Mathematical errors are discussed: (i) *in defining mathematical concepts and application of definitions* (omission of essential characteristics in a given class of objects, or inclusion of inessential characteristics into the definition), examples: "a cube is a solid which has six congruent faces;" "parallelogram is a quadrilateral with opposite sides parallel and of equal length;" (ii) *in theorem understanding and application* (using the hypothesis without testing the proposition), examples include: "the sum of indefinite geometric sequence 2,4,8,16,.. with a term a_1 =2 and q=2 equals -1 because $s=a_1/(1-q)$;" (iii) *in mathematical method*, examples: too quick, unjustified generalizations made on the basis of observing a few particular cases; justifying the theorems referring to any triangle for equilateral triangle; (iv) *in algebraic expression and formula transformations*, examples include (degenerated formalism): $2a - a = 2$; $x \cdot x = 2x$; $(a+b)^2 = a^2 + b^2$; and (v) *in using mathematical language*, examples include: interchangeable usage of terms: "digit" and "number;" understanding the term "any number" as a concrete number, which you can freely choose; "cube's side" instead of "edge;" "odd function is a function that is not even," and errors associated with figure comprehension in geometry.

Teachers' didactic errors associated with teaching a class: (i) *incoherent structure of teaching content*, examples: division of two decimal numbers is not preceded by division and multiplication of a decimal number by any power of 10; solving equation sets of the first degree with two variables is not preceded by an equation of the first degree, with two variables and its interpretation on the coordinates on the plane; (ii) *unsuitable selection of examples used in forming a concept*, examples: only one height of parallelogram is discussed (between the longer sides); only one height is discussed in an acute-angle triangle – the one that is parallel to the side of a sheet of paper; (iii) *unsuitable selection of problems used for aim realization*, examples: only simple problems and tasks are solved, and there are no problematic problems; too difficult examples are chosen to illustrate a method; (iv) *underestimating the necessity to master the basic skills* by students, such as correct calculations, representation and comprehension of the data on graphic representation of geometric figures; (v) *inappropriate teachers' response to students' error*, e.g. irritation due to numerous recurrences of errors that have been clarified in class; (vi) *inaccurate selection of methods for subject realization*, e.g. teachers employ a presentation method, when students could discover some theorems themselves by solving accurately selected problems; (vii) *class work is based on the activity of selected students* (the best or average ones), e.g. when the same students solve problems all the time; those who are able to, while the rest of the class copy the solutions from the board thus reinforcing their belief that they themselves are not able to solve the problems, or are bored because they are able to solve such problems easily.

3. TEACHERS SHOULD UNDERSTAND STUDENTS' ERRORS, CONTEMPLATE THEIR CAUSES AND METHODOLOGICALLY CORRECT THEM

3.1 Teachers' response to students' errors

Teachers are afraid of students' errors. Teachers' fear of students' errors is often manifested by teachers asking students many questions in order to navigate them to a correct answer and to avoid an error. A revealed error requires explanation and its correction takes time. Beginning teachers are often afraid that they might be surprised by students' errors and that they might at first overlook them or not to know how to respond to them. It is useful then for teachers to speculate about possible students' errors while preparing the lesson ("predicted errors").

Students' errors influence the grading process and result in a lower evaluation of students' knowledge or skills. A brilliant idea how to solve a problem followed by errors in its implementation (erroneous actions) cannot be graded high. (The best example of such an approach is a common comment among teachers "Fantastic, but you only get a C").

Experienced teachers might be annoyed by persistently recurring students' errors. Teachers have to be prepared to explain something "not 7, but 77 times" in numerous ways, before the students' reasoning and actions change according to expectations. It is especially true for errors ensuing from students' previous knowledge, which has been sufficient to a certain degree but which fails in some situations (epistemological obstacles and erroneous concepts), as well as errors resulting from erroneous conceptions. Even if students correct such errors themselves a change in their conception does not necessarily occur and after some time they will make the same errors again.

Teachers are afraid of students' errors also because the errors might inadvertently result from their teaching. Students' errors might reveal lack of skill or comprehension which should have been mastered earlier on in the teaching process. Freudenthal (1989, 109) believes that "students who make errors always do so with the teacher who teaches them; at least partially the error's role is connected with the teachers' role in the learning process." Similarly, according to Booker (1989, 101) "the origins of many errors are rooted not so much in students but in the manner children are introduced to mathematics."

3.2 Teachers' strategies to deal with students' errors

Teachers have to *accept students' right to err*, especially when students face a new, unusual situation. A familiar action scheme cannot be immediately applied in an unusual situation, it has to be either adjusted (accommodation) or a new scheme has to be formed in order to solve a problem.

Teachers should try to *understand students' errors*, try to understand the way students think, because "children do not make errors in mathematics thoughtlessly; they either believe that what they are doing is correct, or are not at all sure what they are doing" (Booker, 1989, 99). It is then worthwhile to often ask students a question: *"why* *so?"* If the errors result from inattentiveness then students will correct them quickly; if students are not sure that what they wrote is correct then they will quickly erase or cross it out; if students are sure they are right they will defend their opinion, often with determination. If the question is asked not only when students make errors, but also when everything is correct, it forces a moment of reflection and creates an opportunity to justify directly formulated conclusions. If despite the question, *students still do not recognize their errors,* teachers' encouragement e.g. "check your general reasoning in this concrete case," "give a counterexample," "estimate if it is possible," often leads to contradictions and might facilitate error recognition (Krygowska, 1977, 1988).

However, students do not always recognize their errors immediately, teachers then might use other students to correct and explain the mistakes. It is not only about having other students give the correct solution, but also about having them explain what was incorrect and why. It also teaches good students how to explain errors, what arguments to use in order to convince others that problems might be solved differently. Sometimes an error analysis needs to be postponed, and an immediate and efficient reaction is not always possible. It requires reflection, conversation and sometimes a piece of advice from more experienced teachers.

To err is human (*errare humanum est*) as the Latin saying has it; however, the important thing is what conclusions are drawn from errors, how we learn while erring (*errando discimus*). It is easier to overcome difficulties when they arise and are not solidified yet than to unlearn erroneous reasoning habits or skills, which have become incorporated into thinking and acting of a given person. Hence significant conclusions emerge for teaching mathematics: (1)– control (but do not evaluate) early enough during the first stage the comprehension and correctness of performed actions, in order to capture and correct normal errors occurring then; and (2) teach students how to control themselves if they make errors and how to correct them, to teach them how to "master errors."

A systematic self-control (auto-control) *teaching* consists in teaching students certain strategies, certain actions which increase their chances for a correct final solution. They do not guarantee that students will not make errors or that they will recognize errors in their solution and will be able to correct them, but they facilitate correctness control and error recognition (Polya, 1945; Krygowska, 1988; Ćwik, 1987; Turnau, 1990; Dybiec, 1996). Such strategies include: (S1) step by step control; (S2) solving problems by using various methods; (S3) checking a general solution in a given case; (S4) checking data specification; (S5) making a graphic representation; (S6) estimating the solution by an approximate calculation; (S7) estimating if the solution is sensible and if it is possible; and (S8) referring to the real context.

4. EXAMPLES OF ERROR ANALYSIS IN THE MATHEMATICS TEACHING-RESEARCH

A collection of errors analyzed from various points of view can be found in each of the above-mentioned works. I shall add two more examples to this collection. They took place during the work of the Rzeszów-Kraków PL1 teacher team in the project "Transforming Mathematics Education through Teaching-Research Methodology PL-Comenius-C21."

4.1. Example 1.

Thirteen-year-old students received the following problems to solve from their teacher:

1. The square area equals 9 cm^2. What is the length of the side of this square?
2. Solve the equation: $x^2=16$

Students gave only positive solutions in the first problem $a=3$ and in the second problem $x=4$. They were surprised that in the first problem (in which they most often drew a square) the answer was correct, while in the second problem it was not. They had never learned how to solve quadratic equations before. When asked about another solution of this equation, they were unable to give it (Migoń, 2007).

This error can be called a mathematical error – students considered as true a false statement – the solution to the equation $x^2=16$ is 4. Depending on what a reasoning manner the students used to achieve the result, they could make various errors along the way: in comprehending the concept "to solve an equation," in application of the theorem on root extraction or in comprehending the instruction "to solve an equation."

4.2 Example 1 – analysis

When analyzing this error we distinguished *hypothetical origins* (H) *of this error*:

(H1) the geometrical context influence of problem 1. Students become familiar with quadratic equations of this type first in geometric situations and have no awareness that such equations have two solutions, one positive and one negative in the algebraic context;

(H2) students know that when $x=4$ then $x^2 = 16$, so they use a reverse operation, write the symbol of equation they solve $x^2=16$ / $\sqrt{}$ and obtain $x=4$; they act analogically to the situation with the equation of the first degree marked with one variable and perform reverse operations;

(H3) when students were asked about other solutions to the equation, they did not look for another root in the set of negative numbers. They have much more experience in operations on positive numbers than on negative numbers;

(H4) students are happy that they found a solution and they have no awareness that the equation is solved only when all the numbers that meet the requirements of the equation are given;

(H5) students do not know what set might contain solutions to the equation of a given type, e.g., the solution to a linear equation can be one number, any real number or such a number might not exist (empty set); the solution to a quadratic equation can be one number, two numbers or such numbers might not exist in the number sets that we are familiar with.

Can we talk about the teacher's didactic error here? What did the teacher want to find out about his students (in a diagnostic test) when he put together these two problems? Students had not yet learned how to solve equations of the second degree but they were familiar with negative numbers, they were able to perform operations on negative numbers (they knew that the product of negative numbers is a positive number), they were familiar with the root extraction of natural numbers, and they knew what it meant to solve equations of the first degree with one variable. It might be said that they possessed sufficient knowledge. They only faced an unusual situation. The teacher's intention was to create an opportunity to discuss with the students – how the role of letters in mathematics depends on the situation context.

4.3 Example 1 – suggested corrective methods

How to teach equation solving:

(M1) constantly and consistently remind students what an equation and its solution are and what it means to solve an equation;

(M2) contrast the concept of equation $x+1 = 2$ with inequalities: $2+1 >2$ and $(-1)+1<2$, make students aware that the equality is true for x=1 because 1+1 =2;

(M3) having introduced operations on whole negative numbers, solve riddles with students, e.g. "I'm thinking of a number, I multiplied the number by the same number and as a result I obtained 16. What is the number I have in mind?" and search for positive and negative solutions;

(M4) write down the solutions of quadratic equations in the problems with geometric context in the following manner: (a) when calculating the length of the side of the square $a^2= 9$ $a= -3$ or $a=3$ but $a>0$ as the side of the square, so $a=3$; (b) when applying Pythagoras Theorem calculate the length of the polygon diagonal, when the lengths of the legs are known as $a=3$ and $b=4$

$c^2 = a^2 + b^2$ $c^2 = 25$ $c= -5$ or $c=5$ because $c>0$ as the length of the leg, so $c=5$;

(M5) use consistently and with full awareness the theorem: for all real numbers $\sqrt{x^2} = |x|$;

(M6) interpret the solution of the same equation in various number sets e.g. $x^2=2$ does not have a solution in the set of natural numbers, it does not have a solution in the set of whole numbers and in the set of rational numbers but it does have a solution in the set of real numbers. Emphasize that the equation $x^2=2$ does not have a solution in the number sets we know, e.g. in the set of rational numbers (it does not suffice to say that there is no solution);

(M7) teach solving equations of the type $x^2=4$ by application of theorems about equivalent equations $x^2-4=0$ that is $(x-2)(x+2) =0$ hence $x-2=0$ or $x+2=0$ and finally $x=2$ or $x=-2$;

(M8) solve riddles with students in which each real number is the solution, e.g.: (a) what number should be written into the square in order to obtain the equation □+0=□–0 or □•0=0:□.; (b) think of a number, add the same number to it, add 10, divide the sum by 2, subtract the number you thought of originally, you obtained 5 in the solution. What number did you have in mind? (c) solve riddles with students in which they guess the rule when knowing two number sequences, just like in "number machines:" student 1 says numbers that 'enter': 7, (subsequently 10 then 14), student 2 in turn says numbers that 'exit:' 50, (subsequently 101 and 197). Students give the numbers until the other students guess the rule. In this example: student 1 says any number (x), student 2 gives the square number to the number chosen by student 1 and adds number 1. The rule is as follows: any number that is made a square number and increased by one. (When x is a number that 'enters' then the number that 'exits' is $y=x^2+1$);

(M9) "Create equations" – (apply a reverse procedure to solving equations) students themselves have to create equations, with a set of solutions given, e.g. number 4. Students give examples of such equations: $x+2=6$; $x-4=0$; $x:4 =1$; (during the first stage they write equations and check if number 4 is the solution) only afterwards they realize that when applying theorems about equivalent equations they are able to write many numerous equations which have the same set of solutions, e.g.: $2x=8$; $2x+1=9$; $3x+1= 9+x$, (Legutko et al., 1991).

In the situation described in example 1, the teacher could have applied the above-mentioned means, perhaps except M2 and M8.

4.4 Example 2

In class during a mathematics lesson fourteen-year-old students had to solve the problem:

A rectangle whose one side is twice as long as the other side has perimeter that equals 30 cm. Calculate the area of the rectangle.

Students began solving the problem individually. A student drew in his notebook a rectangle:

Below he wrote the formulas: *2a + 2b; a·b* and waited, as if he finished the work.

Observer: Why aren't you carrying on, when you started off well?

Student: Because when we solve problems with rectangles we draw a frame and we use two such formulas but I never know when to use which.

Errors revealed in the student's honest answer can be considered mathematical; they refer to the misapprehension of such basic concepts as: rectangle (as a geometric figure), the area of a figure and the perimeter of a figure, as well as the misapprehension of mathematical language: the convention of the graphic representation of geometric figures and algebraic expressions (formulas for area and perimeter). The second part of the student's statement "we use two such formulas but I never know when to use which" reveals also didactic errors ensuing from teaching: eagerness to generalize methods (procedures) of calculating the area in the algebraic format with the application of formulas without an illustration when and how to apply these formulas.

4.5. Example 2 – analysis

Hypothetical causes (H) of these errors**:**

(H1) A misapprehension of the conventions of graphic representations of geometric figures – rectangle is customarily drawn as a broken ordinary line consisting of four segments, with two sides parallel but we conceptualize it as a figure with its inside. In the student's language it is a 'frame' – empty inside. The student describes what he sees in the figure. One may say with a large dose of certainty that in lower grades (4-5 years earlier) students cut rectangles out of a sheet of paper, and painted over the inside of the rectangle; however, in higher grades students encounter both in text books and in classroom such rectangle figures as the one he drew in his notebook. In his statement the student revealed his conflict between an abstract concept of the figure and its graphic representation.

(H2) The student's other conflict refers to the area of a figure: how to calculate the area of the rectangle when the 'frame' is 'empty inside'? The perimeter and area of the rectangle are numbers which are obtained after inserting numbers into the formula. But what do these numbers mean for the student? Most likely, he does not associate the length measurement with the number of length units in case of the perimeter; nor "filling the figure with square units" with the number of area units in case of the area.

(H3) "Confusing the concepts of the perimeter and area of a figure." If students do not have a well formed concept of the perimeter and area of a figure, it often happens that they calculate the perimeter when they are supposed to calculate the area, and they calculate the area when expected to calculate the perimeter. Errors of such type are made by several percent of students (aged 12-16) and even more often by weaker students (Legutko, 2006). Similar percentage of students in the "carpenter" problem (PISA 2003) calculated the area of the flower bed in order to estimate the length of boards needed to fence a flower bed.

(H4) The student's schematic approach to formulas in learning mathematics "because when we solve problems with rectangles we use two of such formulas." The student did not mark on the figure the letters symbolizing the side lengths nor did he write what the letters meant next to the formulas. When teaching about the area of figures, teachers usually demonstrate how to mark the rectangle area and prove the formulas for the area of triangle, parallelogram and trapezoid. However, teachers move on to exercises in formula application too fast. In mathematics education (in Poland) too little emphasis is put on the formula comprehension: what they help us calculate, if we can calculate the area and perimeter of a figure without a formula, what given letters in the formula mean, how to determine the unit in the solution on the basis of a formula, and how to evaluate the solution correctness.

(H5) Systematic approach to solving problems, which sometimes is formulated by students themselves in a form of good advice in the following way: "draw a figure, write formulas, transform them, substitute data, calculate and you get the result."

4.6. Example 2 – suggested corrective means

How to teach geometric figures, their area and perimeter?

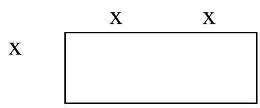
(M1) Discuss with students the role of graphic representation in mathematics teaching. A graphic representation is to facilitate our imagination, to contain essential characteristics of the concept represented according to a certain agreement, e.g., equal segments are equal in the graph, parallel segments are parallel in the graph, and right angles in the plane figure are right angles in the graph.

(M2) Form the concept of a geometric figure. Clearly distinguish real objects with a shape of geometric figure, the graph of a figure and geometric figure as abstract mathematical concepts. When reviewing the figure characteristics do not be satisfied with students' statements: "rectangle has four sides, opposite sides are parallel and equal, and has four right angles." In accepting such constitutive features of a rectangle, we sometimes forget that it is a plane figure and it has the inside.

(M3) Form the concept image of the area of a figure as a result of measuring the figure with area units. Students should associate the area of a plane figure with the land area, (e.g. a ploughed plot, mowed lot or a painted surface) and with defining (estimating) its size with the use of area units. Perhaps it is necessary that the situations connected with the area should be paradigmatic examples for students. Students should have area units "in their eyes," as it were, – cm^2 as a square with a side length 1cm; dm^2 as a square with side length 1dm; m^2 as a square with a side length 1m, are as a square with a side length 10m and so on. The basis for conceptualization of the figure area concept are empirical experiments with measuring the figure; a selection of an area unit, filling the figure with a chosen unit, counting the units and giving the result – the number of area units together with the area unit. Students realize in the empirical experiments that the result is given approximately with a certain margin of error. Useful methods to calculate the units are expressed by means of formulas: therefore, the area of the rectangle is the number of unit squares equal to the product of rectangle side lengths with a common apex, measured with the same unit as the area unit. This individual, empirical experience of students cannot be substituted by teachers' description of how the measurement process is carried out or by the teachers' analysis of the process on a graph.

(M4) Are the formulas indispensable for calculating the perimeter of polygons? I think not. It suffices to comprehend and to often refer to the concept of the perimeter of the polygon – that it is the sum of the lengths of all the sides (and to rationally carry out the calculations). Memorizing (without comprehension) "the formula for the perimeter of the rectangle" might be evaluated as a superfluous didactic action. However, when introducing students to the early stages of algebra: teachers very often (in Polish education) illustrate on the example of the perimeter of the rectangle operations on expressions and the applications of the distributivity of multiplication over addition $a+b+a+b = a+a+b+b= 2a+2b=2(a+b)$ and the product of the expressions a and b by references to the area of the rectangle. Problems in introducing students to the early stages of algebra: the usage of letters to name objects and teaching students to conduct operations on algebraic expressions is an important and difficult task in mathematics teaching. I am not going to discuss this issue and errors associated with it here; however, teachers need to be aware that students' difficulties with formulas in calculating the area and the perimeter of figures might be rooted in their comprehension of algebraic expressions.

(M5) The above-mentioned observed student revealed an active approach to the problem: he made an effort, manifested a willingness to solve the problem, copied the actions which are most often undertaken in such situations, wrote down the necessary formulas and waited. What for? He waited either for teacher's help or for some other student, not necessarily him, to present the solution on the board. What was missing in his attitude? A rational approach to the problem. If he asked and answered his own question, "what do I need to know in order to carry on the instruction in the problem?" then he might have first looked for the length of one side, for example like this:



if he marked a shorter side with x, the longer side with $2x$, then the length of all sides would be $6x$ and he could easily calculate the length of the shorter side and then the longer side from the equation $6x =30$. The application of traditionally used letters a and b to mark the lengths of the sides and to write down that $b=2a$, and then the perimeter $2a + 2b = 2a + 2(2a) = 2a + 4a$ would require more advanced skills in the transformation of algebraic expressions.

In the class situation described in example 2, another student suggested a solution to the problem. Had the observer not asked the student, the entire incident would have most likely been overlooked by the teacher.

In this case it was hard to expect a quick brief reaction from the teacher; perhaps the concretization S1 (what is "the drawn frame" to a rectangle?) and S5 (write down what you know about the lengths of the sides in the given rectangle) would have helped the student to move on with the problem solution. Would it clarify for the student his difficulties associated with geometric figures and their area? I doubt it.

5. CONCLUSIONS FOR TEACHING-RESEARCH

Students have the right to err. An error might teach a lot both students and teachers if it evokes a reflection. Teachers cannot be afraid of students' errors. Teachers should try to understand students' errors; sometimes students' errors reveal more than a correct answer. A conversation with students often clarifies more than a long analysis of their written creations and a long search for the error origins.

Errors have to be corrected methodologically: (a) try to make students aware of their errors; and create such situations in which students can discover their errors themselves. First ask: "*why so?*" ("Look at what you did. How did you get there? Are you sure of that?") (b) If questions are not helpful, then the next action teachers should take is to *create contradictions*, *contrasts* or *to give a counterexample*; (c) if students do not correct their errors themselves, teachers can use the *help of other students*. An error analysis and correction with the participation of good students can be educational for others, also for good students (Freudenthal, 1989), and (d) sometimes it is possible (or even necessary) to postpone the error discussion for the next class. It is important that students correct their errors themselves. Errors can not play their role when they are quickly corrected by teachers or other students.

It is better to prevent errors than to correct them (even methodologically). Teachers should not save time on forming the concepts in mathematics teaching and should control the concept comprehension particularly in the early stages and clarify the arising ambiguities and difficulties. When forming the skills in the first stages, teachers should control (without grading) if all simple actions are carried out properly and in an adequate order. It is exactly this moment in the teaching process when the space is created for what Krygowska (1988a) calls "an error provocation" – creating such a situation in which students' errors will be revealed and an opportunity will be created to clarify them.

It is significant for the error prevention to systematically teach students self-control (auto-control). It requires knowledge, skills, patience and understanding on the part of teachers.

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PART 3
INSTRUMENTS AND TOOLS IN TEACHING-RESEARCH

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THE JOURNAL OF TEACHER-RESEARCHERS

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ABSTRACT
This paper focuses on a journal of teacher-researchers. Two different perspectives are presented: in the first one, Vrunda used the journal as a navigating instrument of her own teaching experiment. In the second one, Cláudia reports how she used the journal as a research instrument in researching her own practice. The journal shapes the experiment and research since it adds precision to them and it informs the practice of teacher-researchers. It answers the fundamental questions of a teaching experiment and research on our own practice – what was achieved, how it was achieved and why it was achieved, why some results were positive and some were not. It illustrates the evolution of a didactic contract between teacher-researchers and the class as a whole, and individual students. The teaching-research journal combines a personal viewpoint as it attempts to shape the social environment to be conducive to learning. Viewed longitudinally, the journal demonstrates the path of learning followed by teacher-researchers.

INTRODUCTION

In the last decade, research in education has evolved, and with it the research methodology. The need to understand problems that emerge from teachers' practice and to improve students' learning has generated the need, on the one hand, to create professional development possibilities for teachers to be teacher-researchers. On the other hand, the demands on teachers have increased dramatically. Varying students' mathematical backgrounds in any given classroom, the familiarity and mastery of different concepts that students have to acquire as pre-requisites to the topics teachers plan to address in their own classroom, the variety of ways in which students learn, etc., all pose significant new responsibilities and potentialities for teacher-practitioners. The goal of teachers in the classroom becomes to create a learning environment for every student in the classroom regardless of their prior mathematical knowledge, while also managing various elements of the current mathematical topic, and in this task, teachers or teacher-researchers are significantly assisted by a journal.

In this paper we focus on a journal of teacher-researchers, through two different experiences: the first one from Vrunda and the second one from Cláudia.

A THEORETICAL BASIS OF A JOURNAL

A journal can be used to record reflections on one's teaching practice, the most significant classroom episodes, difficulties experienced during the learning process, assessment of students and feedback. Evidence recorded in the journal may follow a structure that Smyth (1989) outlines as the process of training reflective teachers in four stages. The first focuses on a narrative and objective description of the teaching practice that reflects regularities, contradictions, significant and insignificant events. The second stage concerns the interpretation aiming to discover the principles and theories behind the practice through dialogue with oneself and with others. The third relates to the confrontation of teachers' practices and theories with alternative conceptions and practices. The fourth stage highlights the reconstruction, leading to integrate new knowledge. The teacher-researchers' journal follows the model of Darwin's notebooks in the careful recording of the situation, which later helped him formulate the Theory of Origin of Species (Wilson, 2006). Teacher-researchers' work closely integrates their own craft knowledge with learning theories, and the journal reflects this coordination. But more importantly, the journal can be used to reflect and learn from practice, following the principle that education is not an immutable reality, but something that will be consolidating and adapting (Nunes, 2004).

Adler and Adler (1994) suggest that the records, as described above, must contain reference to different factors such as, for example, participants, routines, rituals, time, space, or a social organization. Therefore, in preparation for a class description and further reflection, it will be important to prepare a script containing items such as: (1) structure of the class (main stages, their sequence and duration; end; links with the previous lesson and the next lesson); (2) work proposed (tasks proposed: presentation of concepts; exploration of situations, review of notions or concepts, clarification of concepts or processes, consolidation of meanings, organization of data, application of rules, techniques or processes, formulation or verification of conjecture, establishment of conclusions, consolidation of mathematical knowledge and nature of the tasks – exercises, problems, and investigations); (3) origin of the tasks (from the textbook, the working sessions with other teachers, and the teacher authors); (4) role of the teacher (exposes/explains, questions, directs, discusses, supports, manages, and encourages); (5) students' role (discuss, listen, observe, ask, answer, doubt, and argue); (6) class environment and interactions (such as, how do students participate in class, which tasks are performed, such as intervening and what kind of requests are addressed to the teacher, how do they work with peers; Nunes, 2004); (7) formulating hypotheses about learning processes based on teacher-researchers' mathematics expertise, for example, what is the epistemological difficulty with the concept of irrationals, and what path does history of development of the concept in conjunction with learning theories offer as a possible way to alleviate it?

THE ROLE OF A JOURNAL IN A TEACHING EXPERIMENT

Teacher-researchers in conducting a teaching experiment for the improvement of learning in their classroom, cultivate an appropriate learning environment, which takes into account the complex territory that must be navigated (Brown, 1992; Lampert, 2001). Each teaching experiment is a scientific investigation of a set of questions called the teaching-research questions. These are determined prior to the start of a teaching experiment' based on past teaching experience with the topics, or, questions may arise from the on-going teaching experiment through observation of learning difficulties of students (examples are provided below). The original set of questions may branch into many more questions as a teaching experiment proceeds. The process of answering these questions happens in the learning environment, viz, in the actual classroom interaction, and in the teacher-researchers' preparation for it. The preparation, progress, observations, notes, etc., of teacher-researchers are recorded in the journal of teacher-researchers. The navigation of the teaching experiment possible via the journal, as well as the power and impact of the journal are illustrated by the examples below.

Example 1.

From particularity of classroom to creating tools for broad applicability. In summer 2004, the teacher-researcher discovered that her students had difficulty comparing the fractions $\frac{1}{3}$ and $\frac{1}{4}$. This started the immediate quest to understand how to create the environment in which this difficulty would not exist. Through her quest recorded in the journal, the gradual process of discovery began. First, a simple geometric comparison of two equal line segments, one divided into thirds and another into fourths, received positive response from students, and fellow teacher-researchers. Encouraged, the two line segments showing relative sizes of 1/3 and ¼ was extended to a didactic tool, called Fractions Grid (FG) shown below in Figure 1.

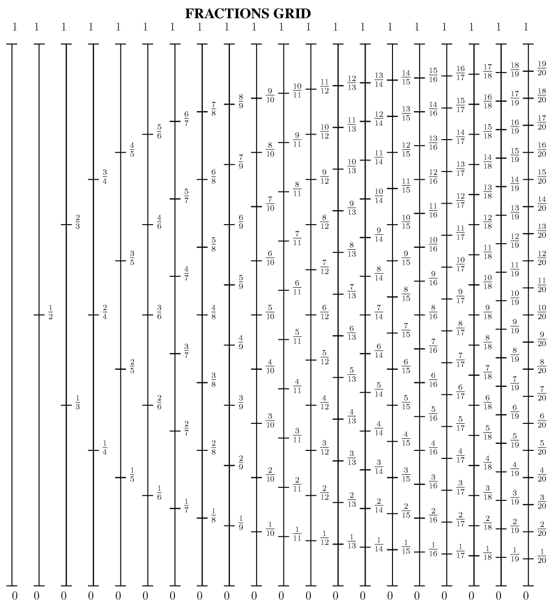

Figure 1. Fractions Grid

A didactic tool has now been created, observation of patterns and relative sizes of fractions is now possible by students; however, its use by students is still not easy for other operations on fractions, where students remember pieces of formulas and rules, incorrectly. The FG is explored further, and in the following journal entry (Figure 2) from the teacher-researcher's preparation for class, a close coordination with geometry is created.

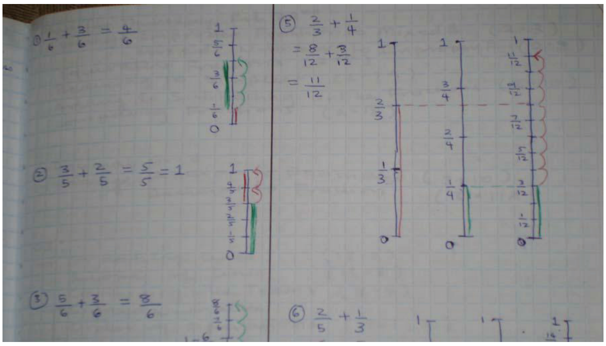

Figure 2. Fragment from a journal (PBL, 2007).

Note various elements that are demonstrated in the journal entry. The problem is 2/3 + ¼. 2/3 is labeled on the first grid and 1/4 is on the second grid. 2/3 is shown to be equivalent to 8/12 and 1/4 to 3/12. The problem is now reduced to 3/12 + 8/12. The arrows indicate the movement from 3/12 by adding 8 pieces each of size 1/12.

The particularity of the classroom contains difficulties of mastering concepts that are broader in scope than just the particular classroom. Teacher-researchers when confronted with particular difficulties of their students have a possibility of learning from the literature how others have successfully dealt with their particular dilemma, and tailor the "solution" to their class and in so doing, the empirical trajectory which is recorded in their journal serves to further inform the profession of the lessons to be learned from the solution adopted/adapted by teacher-researchers.

Preparation of this approach gets teacher-researchers ready for a renewed effort to address their students' difficulties. Continuing in the cyclical manner of trying out a prepared lesson in the classroom, learning from students' responses, assessing the learning via short quizzes, the teacher-researchers' journal entries build toward the beginning of an instructional sequence. Fractions Grid which in summer 2004 was hand-drawn, now (a) has several electronic versions; (b) is embedded in instructional sequences called "Story of Number" and "Story of Number in Abstract;" (c) has the characteristics of a central conceptual structure (Case Okamoto, 1996); and (d) assists in the design of instruction which is in accordance with the development of concepts along the concrete/enactive-iconic-symbolic stages as recommended by Bruner (1960). The excerpt in figure 2 shows an entry from the teacher-researcher's journal, in the process of the resolution of the difficulty of students in the classroom.

2. When supplementing students' intuitive ideas with concepts from advanced mathematics in summer 2004, it was noted in the journal, "Students and even teachers of mathematics tend not to see the difference between ratio and fraction." The existing contemporary textbooks usually treat the concept of ratio as a mere comparison of unlike related terms. The damage to learning occurs when students add fractions in the way ratios are added, and the opportunity to notice and construct the needed learning and to compare the operations on fraction and ratio does not exist given the limited contemporary academic scrutiny of both concepts in depth. By means of a regular reflection and study of the issues recorded in the journals, the teaching-research team will conduct the first cycle of a teaching experiment in fall 2008 utilizing historical development of the concept of ratio and fraction with the illustrations of Farey sequences and middle term sequences. Farey sequences are illustrated on the already developed FG, providing a visual component to facilitate learning.

3. Framing a didactic contract

Journal entries allow for the investigation of the impact of a didactic contract upon students' learning, and also of the extent of didactic contract fulfillment by self-assessment of one's own learning. Supplemented by the lab book[1] of teacher-researchers, the journal and the lab book contain the empirical evidence for the resulting success and non-success in the teaching experiment. In our context, a didactic contract is an agreement that arises between students and teacher-researchers directly, as a result of the classroom work that directly contributes to enhanced learning of students (Prabhu & Czarnocha, 2004).

The daily interactions that shape the didactic contract are recorded by teacher-researchers in their journal, for example, the modifications that need to be made to the existing instructional sequence, because of students' expressed difficulties, the kinds of questions asked by students that reflect on their comfort with the existing instructional sequence, the reasons why the class pace had to slow down or speed up, are all recorded in the journal and affect the teacher-student emergent contract.

The snapshot of the events, the interactions and dynamics conveyed through the teaching-research journal demonstrate the rationale for the didactic contract that emerges. Reporting on the studies from the 1950s Mason states that change is only effective when it comes from those who are to change. The journal of teacher-researchers in its progression, records how those who were to change (teacher-researchers as well as students) either changed causing a transformative action, or did not change, and hence resulted in the existing learning outcomes. The change pattern emerges, if any, and on whose part and how. For example, a problem-based-learning[2] teaching experiment combined the approach learnt at Community Colleges as sites of Global Citizenship[3] to investigate the research question: how could students develop independence of learning and accept responsibility for their own learning leading to an equitable classroom environment with the major portion of the task not shouldered by teacher-researchers?

The teaching-research question investigated the development of independence of learning by means of a scheme in which students could view their own global competitiveness and the role of mathematics proficiency in their role as global citizens. The mathematical concepts of fraction, ratio, and percentage formed the content of the course. The documented occurrence of events, such as who submitted homework, how the homework was completed, what difficulties were encountered and how they were resolved, what the content of the class participation was, and if the existing opportunities were utilized and if so, how, provides insights into the extent of the fulfilled didactic contract by each student. Several conceptions – those of the teacher-researcher and her students find predominance, and through continual negotiations, a didactic contract arises.

Improvement of learning in the environment of a teaching experiment (TE).

The journal provides the evidence of the path of the TE, responsibility undertaken by teacher-researchers in an attempt to create an environment for learning, in partnership with their students. What were the scaffolds created by teacher-researchers

[1] The lab book of teacher-researchers contains the empirical evidence, i.e. excerpts of actual students' work, the consequent interventions to adress the difficulties in learning, etc.

[2] Problem-based learning was developed at the University of Delaware and is described at [http://www.udel.edu/pbl/]

[3] Community Colleges as sites of Global Citizenship are an initiative of Salzburg Seminar, [www.salzburgseminar.org] at which the teacher-researcher was a Fellow in summer 2006.

based on their reading of the class and what were the results of the use of those scaffolds? Were the scaffolds successful in achieving the desired outcomes? How the need to use scaffolds was discovered by teacher-researchers and what supports did they use in creating the situation needed to eliminate the difficulty that had arisen? Did improvement of learning occur in the classroom, and did teachers adapt their teaching methods to the difficulties that arose in the classroom? What role did teacher-researchers play in the shaping of the environment, how could the environment have been shaped differently to bring about greater improvement of learning, is a question that can be investigated based on the journal of teacher-researchers.

In a teaching experiment (NSF-ROLE #0126141) in Calculus classes, based on an innovative curriculum that linked historical development of mathematical concepts with the intuition of contemporary students, it was found that the curriculum was ineffective, i.e. could not be tested owing to the scanty preparation of the students. However, supplementing the instruction by carefully designed Just-in-Time strategies of building needed reinforcement, clarification of misconceptions, and creating environment for connections to be made and strengthened, resulted in high achievement among community college students of the Bronx. The necessary scaffolding and its timing in the conduct of teaching the course were provided by the Just-in-Time interventions which were based on comments in the journal reflection. Appropriate administering of the J-i-T interventions was a direct result of reflection upon journal entries and positive impact learning.

Connection with theory

Theory guides the teaching experiment (i) implicitly – in the design of the instructional materials; (ii) explicitly – in providing guidance, when the learning difficulties cannot be diagnosed; and (iii) in decisions taken in the process of teaching.

The spiral nature of the curriculum allows theory and practice to actively impact the other. The understanding of theory is enhanced through practice, i.e. the problems/students' difficulties arising in practice, force new explorations of how the theory can be applied to alleviate the difficulties. This application is evident in the designed instructional sequences, and finds its way into the journal through the daily classroom recordings. The dominant theories that are evident in the journal are the Zone of Proximal Development, Theory of Schema and Bruner's theory of Instruction.

In two teaching experiments (in India and the Bronx) the Fractions Grid[4] acts as "the teaching and learning of structure, rather than simply the mastery of facts and techniques, [and] is at the center of the classic problem of transfer" (Bruner, 1960).

Reflections

The routes that a teaching experiment can take are numerous, and the journal of teacher-researchers reveals the route taken by the experiment based on the choices, actions, and dynamics of the participants of the teaching experiment. The journal illustrates some of the ways the energies of the participants were directed and may explain the causes of absence of anticipated successes in the outcomes, the discrepancy between what was intended and what was accomplished. In some sense, the journal is a group progress report of the concerned teaching experiment, providing explanatory power to the formulated results, implications for further research and the ways of

[4] Fractions Grid is a didactic tool developed by the teacher-researcher when teaching bright high school students, mainly from the Bronx, shown in Figure 1.

redesign in the next iteration of the teaching experiment.

The discoveries made by the participants of the teaching experiment, both teachers and students and the environment that brought about those discoveries are evident from the journal. Why the schema of the concepts in question did not form completely; which aspects of the schema were particularly resilient to the interventions of teacher-researchers; and what were the moments of understanding that were possibly achieved in the most difficult parts forms the content of the teaching-research journal. The structure of the journal as outlined by Smyth thus provides for a critical reflection upon one's own practice and also an increase in the discoveries made and knowledge contributed to the profession. In shaping the didactic contract, or designing instructional sequences or didactic tools, the growth of teacher-researchers occurs as a by-product of bringing about improvement in learning.

TEACHER RESEARCHING HER OWN PRACTICE

Research on teachers' professional practice in which the reflective component has a decisive role in all phases of the work is a promising approach for research, innovation and educational change. According to Ponte (2002), research on the practice, aims to solve professional problems and increase knowledge on these issues, principally with respect to, not the academic community, but the professional community. This research takes an interpretative perspective, which seeks to understand the world from the point of view of the participants. Also, it is vital to investigate our professional practice given the need to realize the repercussions and implications that our practice has on the evolution of students' learning. During the research it is important to use the journal as a tool that helps teacher-researchers taking notes and reflecting on their own practice.

This research focuses on what pupils think about assessment and how they respond to innovative practices (Nunes, 2004). Students were in one of my (Cláudia) seventh-grade classes. This study uses qualitative and interpretative methods of analysis and it is an investigation about my professional practice, carried out in collaboration with another teacher, Sofia, who also taught seventh grade. The data collected includes my teacher-researcher journal.

In preparing the research-journal at the end of each lesson, it was important to take a set of notes of the teacher and students' interactions that could later be developed, when organizing their records and their weekly thoughts. In any of the situations, it was important that the journal described, in the most faithful way possible, the most significant episodes of the lesson, the difficulties experienced during the use of the assessment instruments, the feedback from students and teachers' reflections on her own practice.

The example presented here represents one of the records made in the journal during my research. It reports an episode of a class which I began by reviewing the concept of power with natural exponent and introduced the rules of operations with powers.

Teacher: This is one of the operations rules with powers: "To multiply powers with the same basis, there is the same base and add up the exponents." That is $2^3x2^3=2^{3+3}=2^6=64$. … Questions?
Sónia: Could it be $2^3x2^3=4^9$?
Teacher: How much is 4^9?
José: *262144*.
Teacher: We had seen that $2^3x2^3=64$. What do you think? $64=262144$?
Sónia: No!

Teacher: Class, have you heard the question of your colleague? It is a good question! Whenever you have doubts calculate the value of each of the powers and thus you can verify whether or not there is equality.

Sónia: Now I understand! [Journal from January 13th-17th]

During this episode Sónia asked the teacher a question revealing that she did not understand the rule formulated. Without answering the student's question the teacher chose to formulate a new issue and open discussion of the collective class.

Reflecting on this class episode, I was concerned not only about the way I managed the class, but about the process I used to help students overcome their difficulties.

> Despite my belief that the lesson was somewhat traditional, the use of this strategy of questioning resulted in the conduct of all the reasoning that led us to one of the rules of operations with powers. However, I could address the discussion by asking for the expression of what 2^3x2^3 means and what 4^9 means, focusing the teaching on the meaning of a new operation. [Teacher reflection, Journal from 13-17th January]

During the investigation there were many other moments of reflection. The focus of teachers' thoughts was their latent concern to regulate their practices, according to the progress of their students learning.

> I think that students were committed to finish this task, but I felt that some students only did it when I reminded them the assessments criteria and instruments. Students showed some difficulty in communication and organization the reasoning processes. I am sure that with the continuation of this type of work the students will improve and develop these and other skills. [Teacher reflection, Journal from 20-24th January]

Reflections

During the investigation the teacher journal was a powerful instrument for reflection and regulation of the teaching and learning processes. The results of the study show that students actively engaged in learning and enjoyed the assessment methods. The instituted culture of assessment reduced their feelings of anxiety towards assessment practices. The feedback given to students throughout the school year helped them learn. Students reinforced the underlying formative and regulatory function of assessment, as they showed support for the continuing implementation of this assessment process. However, students' responses to the different evaluation tools and methods varied, perceiving research reports and portfolios as more meaningful and useful than synthesis assessments. In addition, the results support the idea that the use of a diversified, consistent and clear assessment process leads to a change in the ways students deal with assessment practices. Students acknowledge the efforts to establish an environment of trust and confidence achieved throughout the assessment process and view the feedback they got as a means to help them learn and improve their skills. Finally, the study suggests that experiences of similar evaluation processes, in an atmosphere of dialogue and mutual support between teachers, students and parents can contribute to an improvement in students' conceptions towards assessment and mathematics.

CONCLUSIONS

Two different points of view have been presented here. In the first one Vrunda presented the journal of teacher-researchers as a navigating tool for a teaching experiment. The learning that occurs in the duration of the teaching experiment is significantly assisted by the teaching-research journal. The plan for the class and the actual outcome of the class which are both recorded in the journal provide continuity in the delivery of lessons, but more importantly provide consistency in addressing student difficulties, and the journal has records of how those learning difficulties were addressed by teacher-researchers' classroom and out-of-classroom approach.

In the second point of view, Cláudia, showed a small example of how the research journal was used as a potential instrument for collecting data in researching her own practice and for reflection about the teaching and learning processes.

The journal directly contributes to research knowledge of the profession, and sometimes the contribution provides precise didactic tools, such as the Fractions Grid. While answering the fundamental questions of a teaching experiment and researching on our own practice – what was achieved, how it was achieved and why it was achieved, we are able to better reflect on why some results were positive and some were not, and how further improvement could occur. The teaching-research journal combines the personal viewpoint as it attempts to shape the social environment to be conducive to learning, as in the case of framing the needed didactical contract. Viewed longitudinally, the journal demonstrates the path of learning followed by teacher-researchers.

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ONTO-SEMIOTIC TOOLS FOR THE ANALYSIS OF OUR OWN PRACTICE

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ABSTRACT
The analysis of educational practice is an important part of professional development, as it permits reflection on the various issues that condition the teaching and learning processes. The complexity of these processes requires the integration and application of theoretical tools that facilitate carrying out a systemic analysis. The perspectives required to analyze mathematical practice in the classroom are diverse, but can be usefully grouped in three categories: 1) the epistemic perspective, the goal of which is to provide adequate tools – what we shall refer to as "onto-semiotic" tools – for analyzing the objects and mathematical processes (personal and institutional) that intervene in an instructional process; 2) the normative perspective, which should furnish us with the tools for analyzing the socio-mathematical and social norms that regulate the study process; and, 3) the axiological perspective, which provides us with the tools to evaluate the efficacy of teachers' own practice. In this paper, we focus our study on the epistemic perspective and use the notions drawn from the onto-semiotic approach to mathematical cognition in analyzing a textbook task.

1. THE ONTO-SEMIOTIC APPROACH

The onto-semiotic approach to mathematical cognition tackles the problem of meaning and the representation of knowledge by elaborating an explicit mathematical ontology based on anthropological (Bloor, 1983; Chevallard, 1992), semiotic and socio-cultural theoretical frameworks (Ernest, 1998; Presmeg, 1998; Sfard, 2000; Radford, 2006). It assumes certain socio-epistemic relativity for mathematical knowledge since knowledge is considered to be indissolubly linked to the activity in which the subject is implied and is dependent on the cultural institution and the social context of which it forms a part.

In Figure 1, we represent some of the different theoretical notions of the onto-semiotic approach for mathematical knowledge (Godino, Batanero & Roa, 2005; Font & Ramos, 2005; Font & Godino, 2006; Godino, Contreras & Font, 2006; Godino, Font & Wilhelmi, 2006; Godino, Batanero & Font, 2007; Font, Godino & D'Amore, 2007). Here mathematical activity plays a central role and is modeled in term of systems of operative and discursive practices. From these practices emerge different types of mathematical objects that are related, building cognitive or epistemic configurations between them (the hexagon in Figure 1), such as: (i) language (terms, expressions, notations, and graphs); (ii) situations (problems, extra or intra-mathematical applications, exercises, etc.); (iii)concepts, given by their definitions or descriptions (number, point, straight line, mean, function, etc.); (iv) propositions, properties or attributes; (v) procedures (operations, algorithms, techniques); and (vi) arguments used to validate and explain the propositions and procedures (deductive, inductive, etc.). Finally, the objects that appear in mathematical practice and those that emerge from these practices depend on the language game in which they participate (Wittgenstein, 1953), and might be considered in terms of the five facets of dual dimensions (the decagon in Figure 1):

Personal-institutional. Institutional objects emerge from systems of shared practices within an institution, while personal objects emerge from the specific practices of an individual. “Personal cognition” is the result of individual thinking and activity when solving a given class of problem, while “institutional cognition” is the result of dialogue, agreement and regulation within the group of subjects belonging to a community of practices.

Ostensive-non ostensive. Mathematical objects (both at personal and institutional levels) are, in general, non perceptible. However, they are used in public practices through their associated *ostensives* (notations, symbols, graphs, etc.). The distinction between ostensive and non ostensive is related to the language game in which they take part. Ostensive objects can also be imagined by a subject or be implicit in the mathematical discourse (for example, the multiplication sign in algebraic notation).

Extensive-intensive (example-type). An *extensive* object is used as a particular case (a specific example, i.e. the function $y = 2x+1$) of a more general class (i.e. the family of functions $y=mx+n$), which is an *intensive* object. The extensive/intensive duality is used to explain a basic feature of mathematical activity: the use of generic elements (Font & Contreras, 2008). This duality allows us to focus our attention on the dialectic between the particular and the general, which is a key issue in the construction and application of mathematical knowledge.

Unitary-systemic. In some circumstances mathematical objects are used as unitary entities (assumed to be previously known), while in other circumstances they are seen as systems that can be decomposed in order to be studied. For example, in teaching addition and subtraction the decimal number system (tens, hundreds, etc.) is considered as known, or as unitary entities. However, in the first grade, these same objects have to be dealt with as systemic and complex objects to be learned.

Expression-content: These are the antecedent and consequent of semiotic functions. Mathematical activity is essentially relational, since the different objects described are not isolated, but they are related in mathematical language and activity by means of semiotic functions. Each type of object can play the role of antecedent or consequent (signifier or signified) in the semiotic functions established by a subject (person or institution).

These dualities, as well as the objects, can be analyzed from a process-product perspective, which leads us to the processes in Figure 1 (Font & Contreras, 2008). In an onto-semiotic approach, the intention is not to provide a definition of a “process” from the outset, as there are many different types of processes: we can speak of a process as a sequence of practices, in terms of cognitive processes, metacognitive processes, processes of instruction, processes of change, social processes, etc. These are very different processes and, perhaps, the only characteristic that many of them may have in common is the consideration of the “time” factor and, to a lesser extent, the “sequence” in which each member takes part in the determination of the following step. For this reason, in the onto-semiotic approach, instead of offering a general definition of the process, we opt for the selection of a list of processes that can be considered important in mathematical activity (those included in Figure 1), without making any claims to include all the processes implicit in mathematical activity, nor for that matter even the most important, since, among other reasons, some of the most important (for example, the process of understanding, the solving of problems or modeling) rather than being processes, should be considered hyper- or mega-processes:

Figure 1. An onto-semiotic representation of mathematical knowledge.

In earlier works using the onto-semiotic approach to mathematical knowledge (D’Amore, Font & Godino, 2007; Font & Contreras, 2008; Font & Godino, 2006; Font, Godino & Contreras, 2008; Godino & Batanero, 1994; Godino, Bencomo, Font & Wilhelmi, 2006; Godino, Contreras & Font, 2006; Godino, Font & Wilhelmi, 2006; Godino, Font, Wilhelmi & Castro, 2008), the authors proposed five levels for analyzing the study processes: (1) analysis of the types of problems and systems of practices; (2) analysis of configurations of mathematical objects and processes; (3) analysis of the didactic trajectories and interactions; (4) analysis of the system of norms and metanorms; and (5) evaluation of the didactical suitability of the study process.

Applying level 1 to a study process leads to describing the sequence of mathematical practices. Carrying out a practice mobilizes distinct elements, that is, an agent (institution or person) that develops a practice and a medium where it is developed (there may be other agents, objects, etc. in this medium). Due to the fact that an agent carries out practices oriented towards solving situation-problems, it is necessary to consider, among other aspects, mathematical objects and processes that make these practices possible; which is done in level 2. The final part of this second level of analysis is to describe the complexity of the mathematical practices, taking into consideration the diversity of objects and processes, as well as their typologies.

Given that the study of mathematics usually takes place under the direction of a teacher and includes interaction with other students, the didactic analysis should progress from the situation-problem and the mathematical practices necessary for solving it (level 1) to the configurations of mathematical objects and processes that make these practices possible (level 2), and from there to studying the interactions between teachers and students. In our case, and given the large diversity of didactic interactions that occur in any study process, for level 3 we focus on the interactions regarding easily identifiable semiotic conflicts (in the sense that their identification admits an easy triangulation of perspectives, at least those given by the three authors). In level 4, we will consider which mathematical practices and interactions are conditioned and supported by a set of norms and metanorms that regulate the actions and that are to be analyzed.

The four levels of analysis described above are tools for descriptive and explicative didactics, as they serve to understand and respond "What happened here, and why?" However, they do not evaluate the pertinence of the mathematical instruction process, nor determine guidelines for improving the design and implementation of this process. Mathematical didactics should not be limited to mere description, but should aspire to improving the orchestration and development of study processes. Thus, there is a need for criteria of suitability or appropriateness that permit evaluation of the instruction processes carried out and "guide" their improvement. In this model, level 5 attempts to provide such an evaluative perspective.

These levels are the result of a theoretical synthesis of various partial analyses that have been developed in the research area of mathematics education. For example, level 4 is proposed for integrating aspects of the analysis of socio-mathematical norms developed by socio-cultural perspectives in mathematics education (Civil & Planas, 2004; Cobb & McClain, 2006; Font & Planas, 2008; Stephan, Cobb & Gravemeijer, 2003; Yackel & Cobb, 1996). The levels of analysis proposed by the onto-semiotic approach have been designed for the development of a complete didactic analysis that permits describing, explaining and evaluating study processes. However, further analysis of some of the levels is strongly affected by the type of episode, which in some cases may even become an obstacle. As for level 5, in order to evaluate the didactical suitability of a study process (in accordance with the idea of didactical suitability developed by Godino, Bencomo, Font & Wilhelmi, 2006), a broad longitudinal analysis is needed, which the analyses of levels 1, 2, 3 and 4 applied to a brief classroom episode do not provide. This does not exclude the possibility of carrying out a partial evaluation of the suitability of a study process, keeping in mind, for example, the suitability of the interaction observed in the application of level 3. As for level 4, since the norms are inferred from regularities observed in the study process, their identification in a brief episode may be seen as questionable; despite this, a plausible inference of norms and metanorms can be developed, keeping in mind data obtained when levels 1, 2 and 3 were applied and assuming that these data are local.

In this paper we propose to apply only levels 1 and 2 using one task taken from a textbook.

2. VORONOI DIAGRAMS AS A CONTEXT FOR REFLECTION

As a context for reflection, we shall use the following task taken from *Geometry with Applications and Proofs* (Goddijn, Kindt & Reuter, 2004, part. I, 5) published by the Freudenthal Institute.

> Task 1. In the desert
> Below you see part of a map of a desert. There are five wells in this area. Imagine you and your herd of sheep are standing at *J*. You are very thirsty and you only brought this map with you.

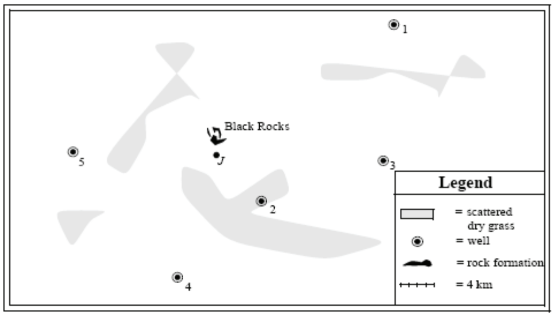


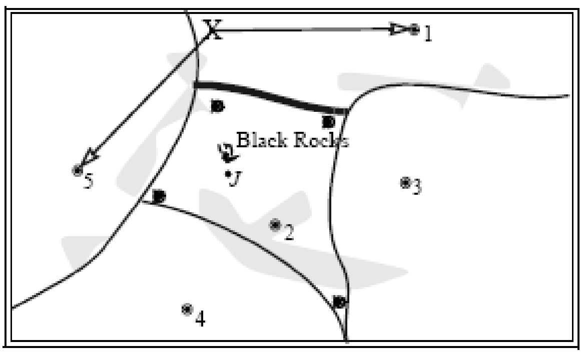
Figure 2. In the desert

1 a. To which well would you go for water?
That choice was not difficult. Of course, you would go to the nearest well.
b. Point out two other places from where you would also go to well 2. Choose them far apart from each other.
c. Now sketch a division of the desert in five parts; each part belongs to one well. It is the domain around that particular well. Anywhere in this domain that special well must be the nearest.
d. What can you do when you are standing exactly on the edge of two different domains?
e. Do the domains of wells 1 and 5 adjoin? Or: try to find a point which has equal distances to wells 1 and 5 and has larger distances to all the other wells.
f. In reality the desert is much larger than is shown on this map. If there are no other wells throughout the desert than the five on this map, do the domains of wells 3 and 4 adjoin?
g. The edge between the domains of wells 2 and 3 crosses the line segment between wells 2 and 3 exactly in the middle. Does something similar apply to the other edges?
h. What kind of lines are the edges? Straight? Curved?

In this exercise you just partitioned an area according to the *nearest-neighbor-principle*. Nowadays, similar partitions are used in several sciences, for instance in geology, forestry, marketing, astronomy, robotics, linguistics, crystallography, meteorology, to name but a few. We will revisit those now and then. Next we will investigate the simple case of two wells, or actually two points, since we might not be dealing with wells in other applications.

Solution proposed by the authors of the book (Goddijn, Kindt & Reuter, 2004, I-91):

1: In the desert
1 a. 2
b.
c. See figure.
(It does not need to be exact, that is for later.)
d. Choose to which well you go.
e. Yes. See point *X*. The arrows are of equal length and the distances to wells 2, 3 and 4 are larger.
f. Yes. Very far to the southeast.
Figure 3



g. No. Not for 3 and 4. (It does hold for the extensions of the edge.)
h. Straight. (This will be investigated thoroughly later.)

The goal of the authors of the book *Geometry with Applications and Proofs* is to enable students to build a mathematical Voronoi diagram on the basis of the task above and the tasks below. A Voronoi diagram of a collection of points S is a partition of an area into cells, each of which consists of the points closer to one particular point than to any others.

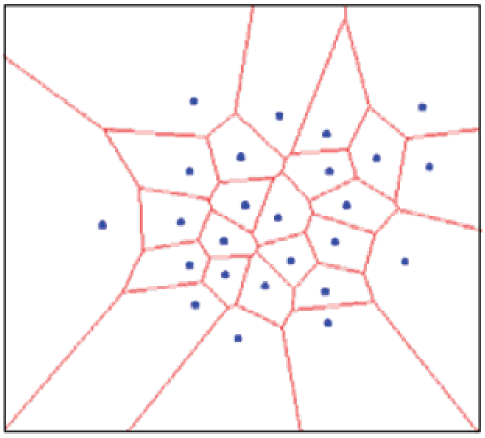


Figure 4. Voronoi diagram

Voronoi diagrams are part of what is known as discrete geometry. Situations like an archeologist attempting to identify the parts of a region under the influence of different Neolithic clans, a meteorologist estimating precipitation at a gauge which has failed to operate, an urban planner locating public school in a city, a physiologist examining capillary supply to muscle tissue, would appear to have little in common. However, all of these problems, together with many others, can be solved by approaches from a single concept. This concept is a simple but intuitively appealing one. Given a finite set of distinct, isolated points in a continuous space, we associate all locations in the space with the closest member of the point set. The result of this procedure is a partitioning of the space into a set of regions, it is known as Voronoi Diagram.

The authors of the book propose the introduction of elements of discrete mathematics in the syllabus of non-university teaching.[1] The method is as follows. Teachers set a series of problems which students must try to solve; they must also justify their responses (normally in groups). Particularly important aspects of their performance are their examination of the task, their justification of the possible answers, their representation, their reappraisal of the solution given, and the arguments they use. As the solutions are pooled together, as well as solving the problems, the unifying concepts are gradually constructed. These concepts are related to each other and are applied to exercises, and are then used to solve more complex contextualized problems.

This book assigns a major role to contextualized problem situations and clearly aims to generate new mathematical objects. The project carried out at the Freudenthal Institute “Realistic Mathematics Education” (Gravemeijer, 1994; De Lange, 1996) proposes a focus on mathematics teaching and learning which conceives the discipline as a human activity like any other, and therefore considers that “knowing mathematics” is the same as “doing mathematics;” as a result, the solution of “realistic problems” should

[1] The discrete algorithms used in computer sciences and computer-based modeling of various phenomena have shifted the emphasis in mathematics today towards discrete mathematics. This has led to calls for its incorporation in mathematics syllabuses. It has even been suggested that certain parts of discrete mathematics are sufficiently elementary to be included successfully in non-university teaching.

be an important part of its study. One of its basic principles is that, for a mathematical activity to be significant, it must depart from the real experience of students (Freudenthal, 1983). Other important principles are that students must be given an opportunity to reinvent mathematical concepts, and that the teaching-learning process must be highly interactive. According to De Lange (1996), there are basically four reasons for including contextualized problems in the syllabus: (a) they facilitate the learning of mathematics; (b) they develop competences; (c) they develop competences and general attitudes associated with problem-solving; and (d) they allow students to see the utility of mathematics for solving situations in other areas such as everyday situations.

Today mathematics is seen as a science in which method clearly predominates over content. For this reason, great importance is given to the study of mathematical processes, particularly the mega-processes "problem-solving" and "modeling." The desert task we discuss here is the first in a series of activities that aim to teach the modeling process. The modeling process is normally considered to follow the five following stages: 1) observation of the situation; 2) simplified description of the situation; 3) construction of a model; 4) mathematical work with the model; and 5) interpretation of results in the situation.

The modeling or mathematization process can also be understood as the result of two other processes: *horizontal* mathematization and *vertical* mathematization. Horizontal mathematization leads from the real world to the world of symbols, and makes it possible to treat a set of problems mathematically. Vertical mathematization is the specifically mathematical treatment of situations.

3. TASK ANALYSIS

In line with the onto-semiotic approach, here student practice is understood as the reading of the task and its subsequent resolution.

In order to carry out a mathematical practice, an agent must have basic knowledge, both to carry out the practice and to interpret the results as satisfactory. If we consider the components of the knowledge that the agent must have in order to develop and evaluate the practice that permits solving a problem (e.g. propose and solve a system of two equations with two unknowns), we can see that a certain verbal (e.g. solution) and symbolic (e.g. x) language must be used. This language is the ostensive part of a series of *concepts* (e.g. equation), *propositions* (e.g. if the same term is added to the two sides of an equation, an equivalent equation is obtained) and *procedures* (e.g. solution by substitution) that will be used in making *arguments* to decide if the simple actions that make up the practice, which is understood to be a compound action, are satisfactory. We will then consider that when an agent carries out and evaluates a mathematical practice, it is necessary that it activates some of the elements mentioned above (or all of them): situation-problems, language, concepts, propositions, procedures and arguments. By articulating these types of objects, we obtain the configuration in Figure 2 (hexagon in Figure 1). The hexagon in Figure 1, referred to as the "epistemic configuration" in the EOS (Figure 5), is a tool that allows us to see the structure of the objects that facilitate the practice that hypothetical students will have to undertake according to the solution envisaged by the authors (for each specific student, we will have a different cognitive configuration). Below we will apply this tool to see which of the objects are active in the hypothetical resolution of the task:

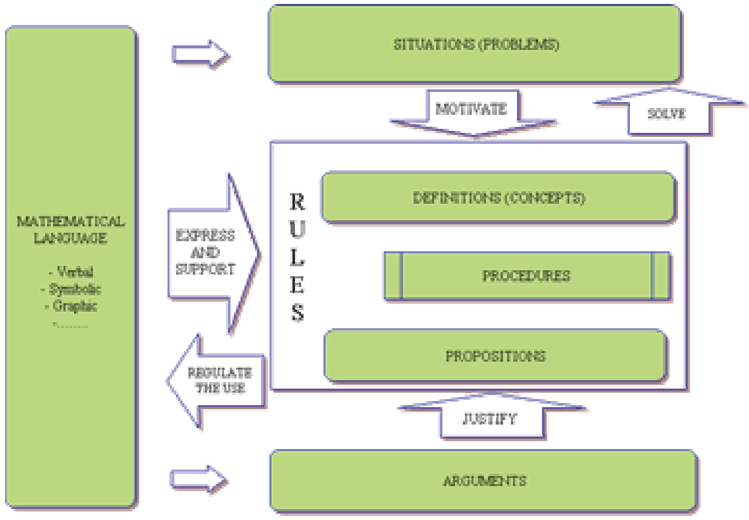


Figure 5. Epistemic configuration

3.1 EPISTEMIC CONFIGURATION OF THE TASK

Situation-Problem: The “In the desert” task – given there are 5 wells, the map has to be divided into five parts in such a way that each contains a well, and so that any place within each domain is closest to this particular well.

Language:[2] (1) terms and expressions: map legend, scale, map, point, region, area, far apart, near, nearest, furthest, division of a region, contains, domain, border, equal distances, larger distances, middle, line segment, cut, intersect, straight lines curves; (2) Graphic representation; (3) Symbolic representation: *J*, 1, 2, 3, 4 and 5.

Map.

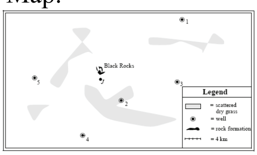


Figure 6.

Sketch showing the division into domains:

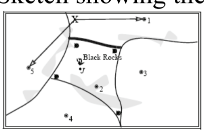


Figure 7.

[2] The sentences are considered representations of definitions, procedures, properties and arguments, and are not included in the “language” category because the definitions, procedures, properties and arguments represented by these sentences appear in the following categories of the epistemic configuration.

Concepts (Definitions): (1) previous or implicit concepts: line segment, end of a segment, greater and lesser (order relations), distance, intersection, belonging to a set, area surrounding a point, border, enclosed domain, unenclosed domain, mid-point, straight line, curve; (2) emerging concepts: a) the domain of a point is formed by the set of points in the plane so that their distance to this point is less than their distance to any of the other points; b) nearest neighbor.

Properties: a) given point *J* and an additional two points, then the point nearest to *J* is that which forms the shortest segment; b) the joining of all the parts is the whole; c) two non-neighboring domains have an empty intersection; and d) the points on the border are equidistant from the centers of the domain.

Procedures: a) estimation of lengths; b) sketch of the division of a region; and c) use of instruments including ruler and compass

Arguments: a) I would go to well 2 as it is the nearest; b) using a compass we can find the points asked for; c) graphic construction; d) I would go to either of the two wells given that they are located at the same distance; e) No, for example the domain of wells 3 and 5; f) Yes, if we extend the line map further south; g) No, for wells 3 and 4; h) They are straight lines.

3.2 PROCESSES

The task given to students is an extra-mathematical situation the resolution of which allows the emergence of, among other things, a new mathematical object: the partition of an area according to the *nearest-neighbor-principle.* The authors intend to present a situation of an extra-mathematical context which is understood by the student as a particular case of a mathematical object. In this case, the particular is extra-mathematical and the general is a mathematical object.

The detailed analysis of the activity needed to solve the task shows that many of the processes in Figure 1 are put into play. The task we are analyzing is divided explicitly into two parts: 1) the statement of the problem; and 2) the commentary that follows the problem. If we look at this from the perspective of the processes considered in the onto-semiotic approach, the statement of the problem aims to generate a process of personalization (in the sense that students construct, among other things, a mathematical object "partition of an area according to the nearest-neighbor-principle"). On the other hand, the subsequent commentary tries to institutionalize this mathematical object, in the sense that it is something known by all the class, that is to say, it comes to "exist" as a mathematical object in the classroom.

Commentary

In the commentary that follows the problem, in seeking to achieve this institutionalization, the authors first generate a hidden process of idealization (the desert becomes an area) and then they establish the particular-general relationship between the idealized situation and the mathematical object. "In this exercise you just partitioned an area according to the nearest-neighbor-principle." The way of presenting the general to students is the result of an additive abstraction, as it consists of a coming together of different elements in the same group. "Nowadays, similar partitions are used in several sciences, for instance in geology, forestry, marketing, astronomy, robotics, linguistics, crystallography, meteorology, to name but a few." We then go on to a process of particularization by saying, "Next we will investigate the simple case of two wells," and one of explicit idealization, when the wells are converted into points "or actually two points, since we might not be dealing with *wells* in other applications."

The statement of the problem

From the outset we assume that hypothetical students can understand the wording of the problem. In other words, we assume an initial process of "communication" (in the sense that "she understands the wording used by others in presenting mathematical problems"), and we shall not analyze this any further. Therefore, we assume that hypothetical students understand what is signified (the process of signifying) in the cartographic representation and its legend, the terms that appear on the map and, above all, that she understands the overall text. This supposition implies that we will not analyze with detail the process of signifying.[3] Moreover, we suppose that students read initially all the items of the text and they are to solve them point by point.

Question a:

"To which well would you go for water? That choice was not difficult. Of course you would go to the nearest well."

Students have to undertake the process of "communication" (understand the task). To respond to this question, students are expected only to undertake a process of "enunciating" a statement or, more specifically, they are expected to respond "I would go to well 2." To be able to give this response, students, in our opinion, have to put into operation, both explicitly and implicitly, certain concepts (line segment, end of a segment, greater and less distance); certain properties (given point J and an additional two points, the point closest to *J* is the one that forms the shortest segment); certain procedures (estimation of lengths) and an argument (thesis: well 2 is the nearest well to J. Argument: Point 2 is the one that is closest to J as can be observed by inspection). We observe that the authors of the book do not consider important that students justify their choice of well 2. Perhaps they consider the answer evident and of course they use the compass or a graduate ruler.

Question b

"Point out two other places from where you would also go to well 2. Choose them far apart from each other."

Students have to undertake the process of "communication" (understand the task). In this question, students are expected to respond simply by undertaking a process of "representation" and "materialization." Specifically, they have to make an ostensive representation such as the following:

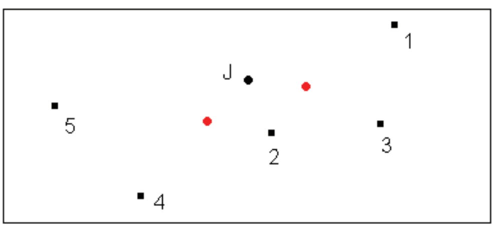


Figure 8.

[3] An example of how the process of signifying would be analyzed is the following contribution to the Topic Group 27 "Mathematical Knowledge for teaching," México: Godino, J. D.; Rivas, M.; Castro W. F. y Konic, P. (2008); "Epistemic and Cognitive Analysis of an Arithmetic–Algebraic Problem Solution," http://tsg.icme11.org/document/get/391

In order to do this, they must formulate an explicit or implicit argument based on: (1) a visual estimation of the fact that the distance that separates the red points from well 2 is smaller than that which separates them from the other wells; and (2) that if the distance between the red points were greater, then there would be another well, other than number 2, that would lie closer to one of the two red points. It is surprising that the authors do not ask students to make explicit the arguments for justifying their representation, given that students are assumed to have this previous knowledge and so they could give this response without any difficulty. For example, students can produce as answer also a sentence such as "each pair of points belonging to a circle centered in well 2 with a radius smaller than the minimum of the distance between the wells no. 3 and no. 4" and they can support this argumentation with a representation where the circle appears.

Question c

"Now sketch a division of the desert in five parts; each part belongs to one well. It is the domain around that particular well. Anywhere in this domain that special well must be the nearest."

Students have to undertake the process of "communication" (understand the task, although in this instance it is not so obvious that the understanding can be assumed). In this question students are expected to respond basically by undertaking a process of "representation" and "materialization." Specifically, they are expected to make an ostensive representation such as the following:

(a)

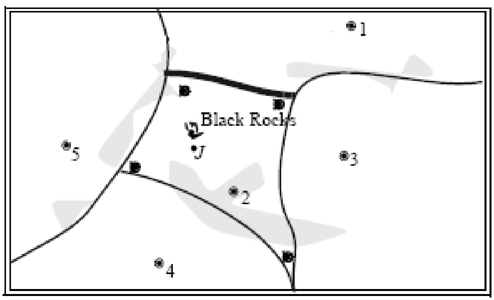
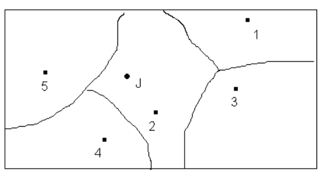


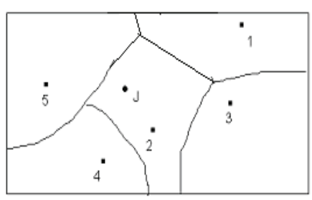
Figure 9.

However, students may offer other valid ostensive representations such as:

(b)



(c)



Figures 10 and 11

As a consequence of (1) students' exploration; and (2) the interaction with teachers and other students, they can refine their initial partition (for example, c), giving a strip of different representations supported by their new considerations through which they arrived at their last representation (for example, a).

To be able to provide any of these representations, students, in our opinion, have to put into operation, both explicitly and implicitly, certain concepts (intersection, belonging to a set, area surrounding a point, border) and a process of "conceptualization" (process of implicit definition) must occur that allows the emergence of a new term (domain) and a new concept: the concept of domain (given a set of points on a plane, the domain of a point is formed by the set of points in the plane so that their distance to this point is less than their distance to any of the other points); certain properties (the joining of all the parts is the whole, two non-neighboring domains have an empty intersection, the points on the border lie at the same distance from the two wells); certain procedures (sketch a division of a region in the plane) and an argument (trial and error?).

Question d

"What can you do when you are standing exactly on the edge of two different domains?"

In the previous question students, at least implicitly, had to use the property that states that the points on the border lie at the same distance from the two wells. This question seeks to produce a process of "conceptualization" (process of implicit definition) that allows the emergence of a new term (edge/border) and a new concept: the concept of a border understood as the line that separates two adjoining domains and one of its properties (the points on the border lie at the same distance from the two wells). In this case also the refinement of previous rough representations can be made, where the points of borders verify the equidistance between wells.

Question e

"Do the domains of wells 1 and 5 adjoin? Or: try to find a point which has equal distances to wells 1 and 5 and has larger distances to all the other wells."

Students' answer here depends on their answer to question *c*. For example, if their answer to question *c* had been representation *a* or *c*, then we would expect them to respond to question *e* by stating that the domains of wells 1 and 5 adjoin and in locating a point that is equidistant from the two wells they would indicate a point on the border, and that this would be located further away from all the other wells. This process of representation of point *X* also requires a process of idealization of the sketch made. Students have to understand that, even if the border that separates wells 1 and 5 had fallen outside of their drawing, at exactly the same distance from the two wells, ideally this is so.

However, if their answer is representation b, then we would expect them to respond that they do not adjoin and that to find a point that is equidistant from the two wells, they would use any of the procedures they know for finding a point that is equidistant from two other points (for example, by drawing a point on the perpendicular bisector, or by trial and error using a ruler and a compass). Given that these points lie outside the figure (as they have to be located further away from all the other wells), they would have to undertake a process of idealization that would lead them to understand that the area that they have to divide is the whole of the plane and it is to be hoped that they would make a new ostensive representation along the following lines:

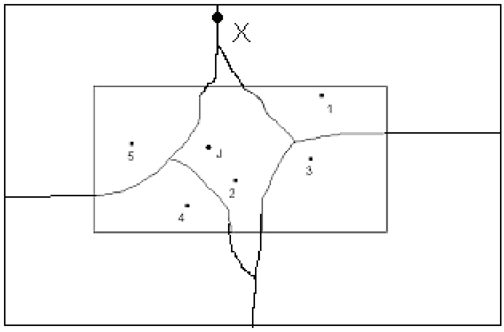


Figure 12.

An alternative answer is suggested in the solutions proposed by the authors. It is expected that students can find with a certain degree of precision a point "*X* which has equal distances to wells 1 and 5 and has larger distances to all the other wells" and that this point might not be on the border of the division sketched by students (since this is no more than a sketch). In this case a semiotic conflict would occur that could only be resolved by resorting to a process of idealization of the division sketched which would allow students to suppose that *X* would be a point on the border if the partition had been carried out accurately.

Question f
"In reality the desert is much larger than shown on this map. If there are no other wells throughout the desert than the five on this map, do the domains of wells 3 and 4 adjoin?"

Students have to undertake the process of "communication" of the task (we assume that hypothetical students can understand the wording of the problem). To respond to this question students are expected only to undertake a process of "enunciating" a statement or, more specifically, they are expected to respond "Yes, very far to the southeast." But in order to give this response, we believe that students should have undertaken a process of idealization (as the question suggests) that leads them to extend the plane; and a process of ostensive or non-ostensive representation, to indicate the border that is not shown in the sketch.

In our opinion, students who answered question *e* based on representation *b* would almost certainly have reached the answer to this question on their own.

Question g
"The edge between the domains of wells 2 and 3 crosses the line segment between wells 2 and 3 exactly in the middle. Does something similar apply to the other edges?"

Students have to undertake the process of "understanding" the task (we assume that hypothetical students have the ability to do so). To respond to this question students is expected to undertake a process of "enunciating" a statement – specifically they are expected to respond "No," and a process of argumentation, giving a counterexample such as "This is not the case for wells 3 and 4." In order to give this answer, we believe that students have to inspect visually the border that separates two wells and make a non ostensive (or equally ostensive) representation of the midpoint for the extremes of the two wells in deciding whether the border passes through the midpoint. If it does, they must continue until they find a counterexample. To do all this, they have to put into operation, both explicitly and implicitly, certain concepts (midpoint of a segment and the intersection of curves) and procedures (visual estimation of the midpoint and point of intersection).

Question h
"What kind of lines are the edges? Straight? Curved?"

Students have to undertake the process of "understanding" the task (we assume that hypothetical students have the ability to do so). To respond to this question students are expected to undertake a process of "enunciating" a conjecture (which might be either "straight" or "curved"). The authors suggest that teachers tell students that the correct conjecture is "straight" and that this will be justified at a later date. It is not entirely clear why this question is included when the reasons for the validity of the conjecture are omitted.

4. CONCLUSION

In this paper we have shown how the use of the "epistemic configuration of mathematical objects" construct, together with the processes considered in the onto-semiotic approach, allows us to undertake a better analysis of mathematical tasks and practices, one of the skills needed to analyze our own professional practice. The "epistemic configuration" tool proves useful for the static description of the structure (organization, configuration, anatomy, etc.) of a mathematical text, while the processes are tools that enable us to explore more thoroughly the operation (dynamics, physiology, etc.) of the epistemic configuration activated in the realization of the mathematical practice. Therefore, we show how the onto-semiotic approach can help us analyze mathematical texts and thus help us avoid stumbling blocks and understand students' conceptual problems.

As Hiebert, Morris & Glass (2003) affirm, a persistent problem in mathematics education is how to design educational programs that influence the nature and quality of teachers' practices. Tools for analyzing the educational practices, such as those proposed here, are necessary for designing these programs, as they provide a structured opportunity for reflection.

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LEARNING TO COMMUNICATE IN MATHEMATICS AND WITH MATHEMATICS TEACHERS IN A COMMUNITY OF LEARNERS (COL)

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ABSTRACT
This contribution deals with professional development of teachers in training (TTs) in a Community of Learners (CoL). TTs consult with a teaching methodologist (the trainer) to select an authentic assignment for their students. Communication between teachers and students and between students themselves is the key issue of this assignment. TTs map out communication flows by making use of action research, teacher-as-researcher methods, and subsequently discuss the progress of research, the results and conclusions together with a teaching methodologist in a CoL. It was jointly decided that the professional development of TTs should be concentrated on the effects of metacognitive instruction. TTs discovered that metacognitive instruction is decisive to achieve students' effective communication.

PROBLEM DEFINITION AND RESEARCH QUESTION

The problem definition is how TTs develop professionally in a CoL by learning to ask metacognitive questions to support their students' work on authentic assignments.

THEORETICAL FRAMEWORK

The stimulation of learning processes between students is chosen as a central theme of the professional development of TTs. This contribution discusses a CoL as a learning environment for TTs as a tool to reach their goals.

The educational concept of a CoL is founded on exploratory learning; scientists learning from each other. Brown and Campione (1996) elaborated on this by carrying out research in small research teams, sharing results with fellow students from other research teams, and applying the acquired knowledge to new (consequential) tasks in which the findings from each of the separate research teams were integrated. Students participating in a CoL are therefore researchers who are expected to (1) consider how teachers use role models (as an expert, model and coach) to demonstrate something; (2) listen carefully to each other; (3) be able to present results; and (4) be able to carry out subsequent steps in their research (Crawford, 2000).

The TT situation is an educational learning environment in which TTs form a CoL together with a mathematics teaching methodologist. TTs do not carry out research comparable to Brown and Campione, but conduct action research and apply the teacher-as-researcher method under the supervision of a teaching methodologist. TTs are referred to the literature by the teaching methodologist. This person is responsible for the supply of literature and its translation into the classroom setting. Under the supervision of a teaching methodologist, TTs then employ their newly acquired knowledge to search for and read subsequent articles. The aim is to jointly gain knowledge concerning the learning and teaching of mathematics in order to be able to adequately integrate this knowledge in the practical school situation and to publish the findings as researchers, teacher-as-researchers. In line with the Crawford characteristic, this context first and foremost involves a teacher, a teaching methodologist, who performs a model role (expert, model and coach). Teaching methodologists are *experts*. They mastered social scientific research methods such as action research and teacher-as-researchers. As *models*, teaching methodologists show how metacognitive questions should be posed, and how to stimulate learning processes between students. They demonstrate how to integrate educational theories, how to establish links with existing know-how and how to translate this into teaching practice. They show how to convey heuristic strategies such as problem analysis, approach, effect and subsequent reflection. They give concrete examples and stimulate TTs to think up new examples of their own. As *coaches*, teaching methodologists concentrate on the progress of students' learning processes, with questions like: What are you doing? Why have you chosen this approach? Is this approach successful? (Schoenfeld, 1985). Teaching methodologists activate reflection in the group by posing stimulating questions.

Secondly, TTs share knowledge, which means that they must be able to listen to and learn not only from teaching methodologists but also from each other. Thirdly, in the cognitive sense, this form of approach requires TTS to be capable of reading scientific articles and translating them into teaching situations. Finally, TTs must take concrete subsequent steps based on the model role of trainers, the mutual reflections and the scientific educational input.

This input is based on the research findings of Kramarski, Mevarech and Arami (2002) on metacognitive instruction in solving mathematical, authentic assignments. A characteristic feature when solving metacognitive assignments is that the approach is not immediately obvious (Darling-Hammond, 1992). In authentic assignments, students are confronted with excess information so that they have to make choices in order to get started (Cobb, 1994). The solution cannot be found through application of an algorithm or a standard solution (Prawat, 1998). It is difficult to establish links with problems already solved. Students find authentic assignments difficult (Kramarski, Mevarech & Libermann, 2001; Verschaffel, Greer & de Corte, 2000). According to Verschaffel et al., weaker students in particular have difficulty with abstraction into sub-problems, as they are unable to separate the information into relevant and irrelevant. They therefore soon give up, also due to algorithms, which do not offer a solution (Anderson, 1990). These problems go beyond the skills of solving mathematical problems. They concern the skills of solving problems together with others, for example Cardelle-Elawar (1995). Based on these findings, teaching methodology research aims at developing instruction methods to support TTs in the activation of students' metacognitive learning processes (Lester, Garofalo & Kroll, 1989; Mayer, 1987; Schoenfeld, 1987). Basic elements in the development of instruction methods for students are metacognitive questions put to small groups of students, such as: (1) conceptual questions: What is it about? What is the question? What is the meaning of a mathematical concept? (2) relational questions: Does the question resemble…? Does the question differ from a problem already solved? Why? (3) strategic questions: What is the solution strategy? Why this strategy? How does this strategy work? and (4) reflective questions: What have I done? Was it purposeful?

These metacognitive questions, which students ask each other and answer jointly can be traced back to Polya's theories (1957). According to Polya, teachers gain insight into the way in which problems can be solved but also into how students can be supported, by: analyzing various solution methods, discussing them with others, and reflecting on their effect. The hypothesis is that metacognitive instruction has positive effects on students' learning results, rather than merely on the mathematical assignments (Cardelle-Elawar, 1995; Mevarech & Kramarski, 1997).

This chapter discusses metacognitive instruction for TTs in a CoL in order to teach small groups of students to ask metacognitive questions in order to solve authentic problems. The underlying thought is that students learn to recognize mathematical concepts, and to use the correct mathematical techniques in order to solve problems (Verhoef & Broekman, 2005). This is in agreement with the goal of mathematical education to have students learn to abstract by consolidating their newly-found knowledge and skills in the existing knowledge structure (Hershkowitz, Schwarz & Dreyfus, 2001; Managhan & Otzmantar, 2004).

RESEARCH METHOD

The research method into the professional development of TTs in order to support students solving authentic assignments by asking metacognitive questions can be typified as action research of teacher-as-researchers. This type of research assumes practical learning to be the basis for theory formation (Wang, Haertel & Walberg, 1993). The criticism on this approach confirmed the differences between teaching methodology research and educational research (Kerdeman & Philips, 1993; Kliebard, 1993). Wang (2003) countered the criticism with convincing, evidence-based practical examples. This approach is certainly applicable in the case of professional development of TTs. Due to the fact that this research is practically applicable in education the action research approach is chosen (Ponte, Beijaard & Wubbels, 2004). In concrete terms, this means that TTs use social science research methods in their own teaching practice. In a CoL, they prepare their research in terms of theory formation, preparation, execution and reflection, resulting in increased personal development under the leadership of a teaching methodologist (Wubbels, Korthagen & Broekman, 1997).

Material

The material comprised the theory of metacognitive instruction. The material used by TTs in their own teaching practice comprised an authentic assignment of "Plusses and minuses" for students (aged 15). The description was: "Write down 15 plusses and minuses in a row, but make sure that there are no more than two of the same symbols after each other. ++--++ is allowed, but +---++ is not allowed for example. How many different rows can you make?" The assignment was not part of the normal teaching material but was in keeping with the subject being treated.

Participants

Three TTs (A and B male, and C female) carried out action research, teacher-as-researcher, in three different classes at three different, comparable secondary schools. Students are used to work independently in groups (3 or 4 students each).
Class A comprised 25 fourth year pre-university science and math stream students.
Class B comprised 17 fourth year pre-university science and math stream students.

Class C comprised 28 fourth year pre-higher-education students, in the Culture and Society profile, who had difficulty with mathematics.

DATA COLLECTION AND RESEARCH INSTRUMENTS

The research instruments comprised logbook reports by A, B and C prior to, during and after the research activities.

The data collection of A comprised a logbook report on:

(i) explanation on the board, writing out the possibilities for n=1 (2 possibilities), n=2 (4 possibilities), n=3 (6 possibilities), n=4 (10 possibilities), n=5 (16 possibilities);
(ii) puzzling by groups of students to find rows with 15 plusses and minuses in a row;
(iii) a number of students discovering regularity in the row of figures 2, 4, 6, 10, 16, namely that the first two figures give the third figure and that the third and fourth figures give the fifth figure. It then took very little effort for them to find the number of possibilities for a row comprising 15 plusses and minuses. Two students arrived at the answer 1974; they verified this solution and their method and had therefore solved the problem. There was no further puzzling;
(iv) analysis of the problem on the board as soon as the students had solved the problem. Rows of figures were used to show that the formula discovered was indeed the correct one. The students were familiar with this subject, as it was the subject of the last chapter. Two different formulas were presented, namely one for those rows in which the final two symbols are the same O_n, and one for those rows in which the final two symbols are different P_n:

- $P_n = P_{n-1} + O_{n-1}$
- $O_n = P_{n-1}$

The total formula was simple to deduct from these formulas, namely that the row of figures is given by the formula $U_n = U_{n-1} + U_{n-2}$.

The data collection of B comprised a logbook report on:

(i) the explanation on the board on the basis of a photocopy of attachment 1. The problem was soon understood. Students set to work in groups of four. They were promised a hint after 10 minutes, if necessary. They were also tempted with the promise of a Snickers bar if they found the right answer. After five minutes' puzzling, all the groups began to call out solutions. When asked, they admitted that all the solutions were more or less a guess and that a formula, which was associated with the problem, had been worked out, such as for example $2.^{15}$ None of the groups actually thought the problem through. The solutions called out were more or less guesswork. The class became quiet after about 10 minutes. A number of possibilities were written on the board (by A): a row with a length of 1 (n=1): 2 possibilities, and then n=2 (4 possibilities) and n=3 (6 possibilities). Two groups understood the approach and discovered a number of possibilities for n=4 (10 possibilities). The regularity in the rows 2, 4, 6 and 10 was then recognized and a group produced the right answer after two minutes, i.e. 1974. Everyone stopped puzzling and analyzing;
(ii) the proof of correctness of the result by means of the recursive rows approach. The students were not really interested (attachment 2). They were then given a photocopy with the presented solution and also a solution according to combinatorics.

The data collection of C comprised a logbook report on:
(i) the group size, sticking to the book because they were due for a test the following week. C adapted the task slightly. The total task remained the same but a stencil was provided with some sub-questions: What is the subject of the task? Explain the task in your own words; What other tasks resemble this one? How did you solve the tasks in the previous question? etc. In the end, half of them completed the tasks.
(ii) the overly difficult task, despite the sub-questions. They found it too difficult to formulate the question. It was very difficult for them to recognize the difference between a task from the book (in general that means that there was too large cognitive distance between student knowledge and the level of the question. ZPD approach suggests diminishing that distance in the process of questioning) and this task, and the students were deeply focused on finding a ready-made solution (which they could not because the task was a bit too difficult for them resulting in their disappointment). They tried to apply the theories they had recently learned to this task but that too proved to be very difficult.
(iii) the students who completed the task were those more willing and sometimes also somewhat better at mathematics than the rest. In groups which consulted each other.
(iv) the sub-tasks referred to the approach taken by students, they were not accustomed to this in their textbook and therefore found it "strange."

DATA PROCESSING AND ANALYSIS

Data processing and analysis of A:

A had probably already given too much information for the fourth year pre-university science and math students, by writing out the possibilities of n=1 to n=5 on the board. They soon found a general formula without having to work out the problem for too long. Moreover, due to A's approach a small number of students quickly found the answer, after which the remaining students soon adopted that solution. It might have been more useful to have students search for the answer in small groups rather than individually. The students now started working alone, in pairs and occasionally in groups of four, but once they had found the solution, it was called out loud in the class. Only a small number of students actually had the opportunity to understand the solution. Shared knowledge became general knowledge, only for small number of students.

Data processing and analysis of B:

These students showed little willingness to make an effort and think analytically about a task which was not part of the curriculum. Communication was spontaneous in the groups. The approach taken by one group had too much influence on the other groups. It might possibly have been more useful if systematic problem solving was part of the educational program, including instructions on how students can tackle a problem. The expectation now was that students would find it challenging and fun to work at solving such problems. That was an illusion in the case of this group. The students were accustomed to undertaking tasks as part of the curriculum. By far the largest group considered this adequate and had little need for anything extra.

Data processing and analysis of C:

C expected that this task would be too difficult for the students in this group (and that they would not be interested in mathematics and solving puzzles to the same extent as the fourth year pre-university groups). C adapted the task slightly. As expected, many students were not interested in the task. C had indicated that she needed the results for a research project at the university, and the students therefore filled it in for her. In the end, half had actually completed the required tasks. The students found it "strange" to answer sub-questions and especially to have to think about what they were doing, because their standard textbook did not do so. They looked for quick and easy answers, and once that proved difficult they soon gave up. They also started calculating the first sub-questions (attachment 3, part c) instead of the question, which really needed the solution (attachment 3, part f). That was not yet achievable: to first describe the approach and only then start calculating. A number of students worked on the task in a group. That gave very interesting results, they prompted each other and consulted on a possible solution, nice to see, nice to hear, and useful for the students themselves as well. Such groups had questions to ask and actually wanted to understand the task. The students who did not work with others but rather completed the task individually had fewer questions and were done faster.

RESULTS

The situation of A*:*

Too much pre-chewed information. Once the original task had been structured for students beforehand and any uncertainties removed, it was no longer authentic. The solution was soon passed around. The students did not start out in groups, so there was no optimum communication between the students themselves.

The situation of B:

Too much information was "given away" beforehand. The original task was no longer authentic, the general solution too obvious. The division into groups had a negative effect on the students' learning processes. Communication was spontaneous within the groups. Students were not voluntarily willing to look deeply into the context.

The situation of C:

This task was too difficult for this group. C hoped to make it simpler through the use of sub-tasks but this might have made it even more difficult. The approach of first solving sub-questions and only then writing down the solution was tricky. The students were not yet too systematic in their approach and the text book (in use *Modern Mathematics*) did not offer any support for that either. As a teacher, one is obviously trying to stimulate students to work that way, but students generally did not appreciate the approach, and instead began to calculate and to write down solutions right away. Once they started working in a particular direction, the students could not be diverted, unless they (despite being split into groups) cooperated with others (or happened to find it interesting). They were also not accustomed to reflection, or verification of their solutions (Why should I? I already have an answer, don't I? It was just difficult; etc., see also attachment 3, part g).

CONCLUSIONS

TTs commented on each other's classroom experience in training. They discovered similarities and differences. On the one hand, students have problems with (boring) authentic tasks; the lower the level, the greater the difficulty they have. The goal of the task needs to be meaningful for them. On the other hand, if group work is an accepted working method for students for this type of questions, the effect of mutual communication is more positive among lower level students than among higher level students. Communication is spontaneous and is effective as long as it is structured. The TTs learnt to focus on metacognitive instruction as an adequate principle to activate effective communication between math students. They mentioned a focus on general instruction and prevention of personal help. TTs wanted to be students' amicable coach, to stimulate students' mutual communication. They established to remark metacognitive instruction and to avoid individual students' support.

DISCUSSION

As indicated by critics, this type of research is methodically weak, crucial conclusions are insufficiently precise, it produces contrary results, is reported in incomprehensible jargon, does not lead to improved teaching results and must all take place much more thoroughly (Slavin, 2000). While good practices can certainly be identified and qualitatively described, (i) the generalization questions (will it work for my subject/with these students/in our context/with a different teacher?); (ii) the reproducibility question (will it happen again tomorrow?); (iii) and the explanation question (what is the underlying causal relationship between "treatment" and results?) are seldom adequately answered (Van Keulen, 2006). Educational research in the form of action research is difficult but is certainly recommended for professionalization of teachers. Action research activities emphasize the interaction of knowledge and professionalization (Ax, Ponte & Brouwer, 2007). Implementation of curriculum innovation requires a close relationship between design, professionalization of teachers and participation in a CoL (Pieters & De Vries, 2007). In such a Col, a synthesis can be formed between educational theory and theory for practical action in teaching; furthermore, this practical action might also be improved, although that is not shown in the described situations.

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APPENDIX 1: PLUSSES AND MINUSES

"Put fifteen plusses and minuses in a row, but make sure that there are no more than two signs of the same kind next to each other. For instance ++--++ or +-++-- is allowed, but ++---++ is not. How many different sequences can one make?" Divide into groups of four to five students and ask yourself the questions. Suggestion: First analyze the problem as if it where a row of one, two, three or four plusses and minuses.

APPENDIX 2: TWO DIFFERENT SOLUTIONS TO THE PROBLEM

A mathematical solution of this task is that the next item of the sequence can be found by adding the previous two items. The first item is equal to two, the second is equal to four. In mathematical language this leads to: $U_n = U_{n-1} + U_{n-2}$ with $U_0 = 2$ and $U_1 = 4$. Using this, one easily deduces that the solution to our problem should be 1974. In order to come to this solution, it is useful to split the sequence in two separate sequences: one sequence with the last two signs similar and one sequence with the last two signs unequal. By taking this extra step and simplifying the required abstract mathematization, students easier understand the way in which the general formula is derived. There is also another way to come to the solution. A row of plusses and minuses can be seen as a row with two types of elements. A single element (+ or -) and a double element (++ or --). Because plusses and minuses have to alter, we only have to look at the number of possibilities we can order single and double elements. Whether an element is + or -, depends on its position. The number of single and double element is not fixed. Therefore, the following sequences are possible: 15 elements containing 15 single-elements and 0 double-elements, 14 elements containing 13 single-elements and 1 double-element, 13 elements containing 11 single-elements and 2 double-elements, … 8 elements containing 1 single-element and 7 double elements.

The sum of all the possible sequences is equal to

$$\binom{15}{0}+\binom{14}{1}+\binom{13}{2}+\binom{12}{3}+\binom{11}{4}+\binom{10}{5}+\binom{9}{6}+\binom{8}{7}=987$$

The total number of different sequences is equal to: *2 * 987 = 1974.*

APPENDIX 3: ANSWERS TO THE ADDED SUBDIVISIONS

a: What is the subject of the problem?
Answers:
Putting plusses and minuses in a row, trees, probabilities; calculating probabilities; diagrams

b: Describe the problem in your own words.
Answers:
How many sequences are there? One has to make sequences and see where one needs to put the plusses and minuses; I have to put plusses and minuses and a row, without putting three of the same sign next to each other; How many different sequences can we make if we put fifteen plusses and minuses in a row and no more than two signs of the same kind next to each other; We have 15 places to put a plus or minus sign

c: Do you know other problems that look like this problem? How did you solve these?
Answers:
Draw a grate and count; calculate probabilities with the help of trees or using the function 'NCR' on the calculator; a problem with license numbers, but that was different because these were in some order; A problem with red and white balls in a vase; 15 ncr 2

d: How do these problems differ from the problem with plusses and minuses?
Answers:
It is difficult to see; I don't see any differences; In this problem, no more then two signs are allowed to be next to each other; We deal with plusses and minuses now; The way I solved these problems does not work for this problem; I have to do the subdivisions first; I can't solve this problem immediately.

e: How are you going to answer this question and why?
Answers:
I will use a tree, but I think that it will be a lot of work; I thought I had to use the function 'NCR' on the calculator, but it doesn't work. A grate also doesn't work. Do we have to write down all the possibilities?; Write everything out, because I can't use 'NCR'. I do not like that!; I'm going to use a grate and count the number of possibilities; I will use a tree. It will cost a lot of time, but the answer is right; I'm going to use 'NCR'

f: Calculate the solution of the problem.
Answers:
15 ncr 2 = 105 (I have seen some unfinished trees, but I haven't seen any other solution)

g: Look at your answers from part E and G. Do you think you are on the right track? Do you still miss some things and do you think you will have to do it some other way?"
Answers:
I don't think I did it the right way, but I don't know what else to do; I think you have to write down all the probabilities; I think I'm at the right track, but it is too difficult for me to solve; You can't calculate this with a formula, because there isn't one. And a tree is too large to calculate by hand; Yes, it is very logical; It is easy to understand, but it does cost a lot of time to calculate everything; I think that there are better ways, but I don't know them; It is possible but hard.

AN EARLY ALGEBRA GLOSSARY AND ITS ROLE IN TEACHER EDUCATION[1]

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ABSTRACT
The paper deals with issues related to early algebra in the general framework of social and educational needs, as well as of competences indicated by the PISA test and the new role of the teacher. The main focus is on the theoretical framework of our ArAl Project – an integrated project for teaching research and curricular and methodological innovation in the classroom – through an analysis of its glossary's structure. An exploration of the linked universes of arithmetic and algebra and a method to carry out this exploration are proposed to teachers. The aim is to help teachers merge theory and practice, guiding them, through a reflection on their own knowledge, towards a conception of mathematics as a language, on the way of an approach to arithmetic thinking in an algebraic view. Finally, the paper proposes the use of the glossary in the teaching activity, as a cultural support for students as well as for teachers.

1. INTRODUCTION

Socio-constructive teaching, especially at compulsory school level, seems to be a most appropriate model to educate students to deal with the complex society we live in. Nowadays the labor market is highly competitive, continuously changing and is ruled by multifaceted systems of relationships: individuals are supposed to have diversified and flexible competences to both face and manage changes. Social educational needs are reflected in the international test PISA (OCSE 2002, 2003) which aims to test students at the end of compulsory school, regarding those skills which are viewed as essential for them to be active and conscious citizens in society. This test indirectly promotes a mathematical-scientific education that goes beyond the simple transmission of knowledge to be immediately applied, and rather aims at educating students to deal with complex situations. In the case of mathematics, wide space is given to problem solving, mathematization and interpretation of phenomena that involve different registers of representation (Duval, 2006). If we look at skills evaluated by the PISA test, some, like *argumentation, representation, communication* – intertwined with methodological aspects of socio-constructive teaching – are not generally acknowledged by teachers as specific to mathematics. Other skills, like *thought and reasoning, posing and solving problems,* are viewed as typically mathematical but 'high' and difficult to be reached by the medium-bottom set of students. The PISA test stresses *mastering of cognitive processes* more than understanding and application of concepts. Moreover, competences are assessed with reference to the level of complexity which causes their enactment. Particularly relevant and meaningful are those classified in the category of reflection that

[1] Due to the themes dealt with this contribution links to Malara and Malara & Navarra's contributions in this volume.

include creative thinking, intuition, generalization, enacting strategies and carrying out complex reasoning, besides reflection as usually viewed.

In this framework, the role of teachers becomes more complex and multifaceted and they become *decision makers* (Shulman, 1985: Cobb, 1988; Carpenter, 1988; Mason, 1994; Simon, 1995; and Malara & Zan, 2002). In everyday teaching, they face a number of different situations which force them to make decisions repeatedly. These decisions concern not only the schedule of classroom-based work and the identification of suitable tasks: they also refer to the choice of communicative strategies that should be used in class interactions (Bartolini Bussi 1998; Anghileri 2006), and involve both the identification of problems emerging from the class and their solution (Cooney & Krainer 1996, Jaworski 1998, Schoenfeld 1999).

This model of teachers requires competences that can be acquired with efforts and draw on various domains of knowledge: mathematical knowledge, often poor (Ball et al., 2001); general socio-psycho-pedagogical knowledge; mathematics education knowledge; specific contents, which can be referred to Shulman's pedagogical content knowledge (1986); knowledge of mathematics teaching and learning models.

Research has pointed to the strict relationship between quality of teachers' knowledge and quality of learning and highlighted how educational projects aimed at teachers, which make them access high-quality research literature, relevant to teaching and with an accessible language, produce deep changes in their conceptual views and strongly impact on their practice (Malara & Zan 2002 and related references).

These studies show optimal ways of organizing and developing teacher education and offer precious directions on how the latter should be implemented on a large scale.

In our country, this requires a deep change in the current formative and training courses, and dramatically stresses lack of solid institutional forms of in-service teachers' long term professional development.

One of the main problems teachers encounter in their professional life, in our country, is the progressive deterioration of their conceptions, as they get far from the years of their education. Real life – sometimes hard, often frustrating – gradually prevails over cultural and methodological references (assuming that teachers had the opportunity to have some, and this is not to be generally taken for granted). This deterioration has a number of reasons and each contributes to *impoverishment of identity*. In the case of mathematics teachers – but the same might probably be said for other disciplines – impoverishment is shown by a progressive disappointment about their own teaching instruments, regarded as incapable of having a positive influence on their students' construction of knowledge. Often results – either positive or negative – seem to emerge 'by nature,' *not depending on the teacher*.

Identity can be fulfilled only through the awareness of having some *points of reference*. In the case of teaching – due to its features of being linked to social changes and, at the same time, to the continuity or discontinuity of these transformations – these points of reference should be extremely *mobile*, in the sense that teachers should become sensitive enough to progress in these changes, always making a critical reflection on their own interpretation instruments.

Teachers often acquire these instruments – we might say: they construct their own epistemology – during their studies, from infant school to graduation or post-graduate certificate. After this, they immerse in real life, slowly loosing contact with their theoretical studies, and regard the latter as a foundation set *once and for all*, frozen in stereotypes and impoverished by lack of renewal.

Negative results of tests like PISA dramatically stress the outcome of these attitudes, and initiatives, like the PDTR project, attempt to build up strategies to fight this, joining the efforts of participant countries.

This paper was devised as an answer to these issues and remains within the framework of the studies carried out by our group for the renewal and re-qualification of teaching of arithmetic in a pre-algebraic perspective and for an anticipated approach to algebra viewed as a tool for thinking. It aims to provide a contribution to mathematics teachers – almost a challenge – towards *enrichment in their identity*. *An exploration*, or better a *re-examination*, of wide conceptual universes, such as the arithmetic and algebraic ones, and a *method* to carry out the exploration will be proposed. We will use the tools elaborated within the ArAl project (Malara & Navarra, 2003; Navarra & Giacomin, 2004-2006; Malara et al., 2004, Fiorini et al., 2006).

2. THE ARAL PROJECT AND EARLY ALGEBRA

ArAl is a project aiming at *didactical innovation* in mathematics, designed for students aged 5-14, and framed within the early algebra theoretical framework. Basic aim of the project is to design and implement teaching sequences in arithmetic, in a *view that favors an anticipated approach to algebraic thinking*, aiming at a progressive construction *in parallel* with arithmetic and *not successively* to it. These sequences are meant to contribute to a reduction of difficulties – when they first appear in the arithmetic field – many 15-year-old students encounter in the study of algebra when they enter secondary school and that often turn into absolute obstacles. At the same time, these sequences aim to justify to students the role played by algebraic language *in modeling problem situations* as well as in the *production of thinking*.

Our hypothesis is that, starting from early years in primary school, the teaching of arithmetic should be implemented with the aim of letting students learn mathematics as a new *language* with modalities that resemble those used in the learning of a natural language. For this reason, we introduced the concept of *algebraic babbling*. Socio-constructive educational modalities underpin the development of this hypothesis: teachers do not transmit knowledge to be learned but rather devolve to students the construction of knowledge, during a *collective* interaction that starts from the exploration of appropriate problem situations.

3. THE TEACHER

Mathematics teachers are *the key element* to the enactment of these changes of perspective. The challenge is to lead them to *revise* the roles they *are already playing*, towards a critical reading of knowledge, beliefs and maybe stereotypes. Therefore, participating in the ArAl project is for teachers an important moment for reflecting upon central issues, like: *Which* arithmetic am I teaching? *Which* algebra? *When does 'algebra' start?* These questions concern both primary and lower secondary teachers, who, in most cases, do not have a background in mathematics (neither regarding contents, nor the teaching/educational aspects), and higher secondary teachers, often lacking in terms of teaching/educational aspects.

In the theoretical framework and instruments of the ArAl project, teachers find the necessary support for revising arithmetic and algebra, enacting, in some respect, an actual "Copernican revolution" in their mathematical culture. They actually have to come to deal with a wide, complex and somewhat maze-like set of concepts, which confuses them and makes it difficult to embed the early algebra perspective in their everyday teaching. The complex nature of this situation requires a conceptual support that may enable teachers to cast light upon their formative route. A glossary provides this support, helping teachers merge theory and practice and in particular; face the connections between mathematics and linguistics that characterize the ArAl project's theoretical framework; which leads them to a *conception of mathematics as a language*, in which a convincing *control of meanings* can be developed and transmitted to students.

4. EXPLORING EARLY ALGEBRA AS A MACHINE

Early algebra is a polycentric universe of themes. This can confuse teachers and make their approach to these themes difficult. It is therefore necessary to elaborate a framework to help teachers gradually achieve an organized vision of early algebra, through a reflection upon their own knowledge. Pre-defined approaches do not exist. There are a plurality of possible approaches that require reflection and method.

We thus analyze early algebra through the metaphor of a "machine," the functioning of which we want to find out. Recomposing the machine's devices is an individual adventure, and depends on how teachers decide to explore it. These relate to school level, features of the particular class, teachers' expertise, studies, attitudes and curiosity that lead them to approach some particular themes before others. Therefore, we are talking about a method that each teacher might apply, depending on both their individual and environment-related needs.

5. THE GLOSSARY

The starting point for observing our machine's devices is therefore in the glossary, currently made of 92 key terms. The description of each key term leads to other key terms. For example, the term *argumentation* leads to: 'collective, process/product, representation, relationship, semantics/syntax' (Figure 1):

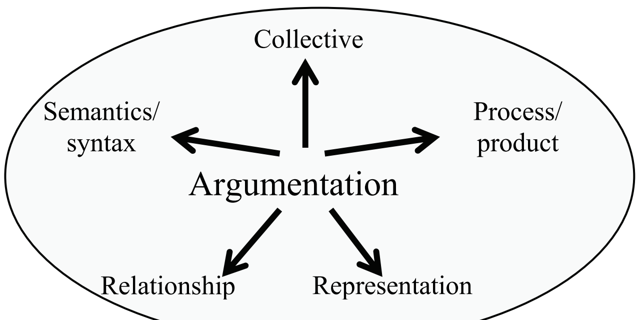
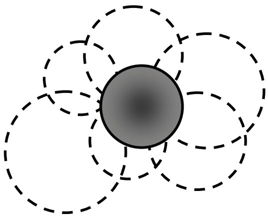


Figure 1.

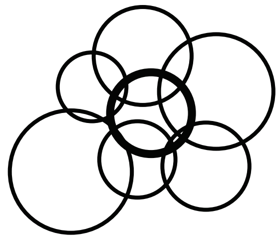
Let us call this representation a *net* of the term 'argumentation.'

Assuming the glossary as *matrix of nets*, we define a net as "the complex of references that connect a term to other terms of the glossary." Every net can be considered in two ways:

(a) through its *inner* structure
(a universe in itself)



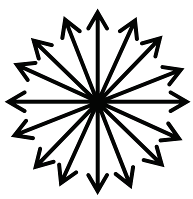
(b) how it links to *other* nets.

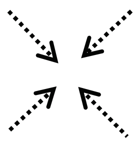


These considerations lead us to reflect on *the structure* of the glossary. Every term is characterized by two components: (1) the numerousness of its net; and (2) the numerousness of its occurrences.

Two examples:

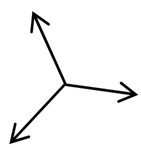
The definition of the term ‘didactical mediator’ contains a high number of glossary-terms (15) but only 4 occurrences in other nets:

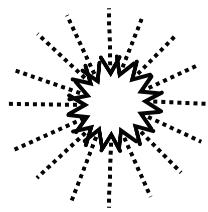




The wealth of “didactical mediator”
is due
to the numerousness of its net

On the contrary, the net of ‘mathematical phrase’ is poor in glossary-terms (only 3) but it appears in 17 other nets, and therefore its significance is of great transversal importance:





The wealth of “mathematical phrase”
is due
to the numerousness of its occurrences.

This first classification allows us to understand, for instance, that ‘didactical mediator’ works better than ‘mathematical phrase’ as a possible starting point for an initial approach to early algebra because, due to the numerousness of its net, it locates teachers within a double process of conceptual *deepening* and *extension.* Deepening of the term occurs through an illustration of the relationships that link the 15 key-terms in its definition; extension occurs because each one of the 15 key-terms is a potential stimulus to read the related definition. This definition, in turn, contains a new net, a possible basis for further exploration. Starting from ‘didactical mediator,’ teachers can thus construct their *own personal organization of the early algebra knowledge*, going through successive choices of key-terms and moving from definition to definition. At the same time, ‘didactical mediator’ is not often quoted in other nets and it is thus difficult to be found. We might say it is a mountain refuge “reached by few trails, but starting point for many excursions.”

‘Mathematical phrase’ on the contrary, leads to few nets (three only), but can be found in 17 of them. Therefore it suggests, through the second word, the *transversal* importance of the *linguistic* aspect in the conceptual structure of ArAl Project. It is a refuge “where many trails converge and from which few, but potentially panoramic, footpaths leave.”

In order to move around in this complex universe, it is appropriate to identify a *reading key*.

6. A READING KEY TO THE GLOSSARY

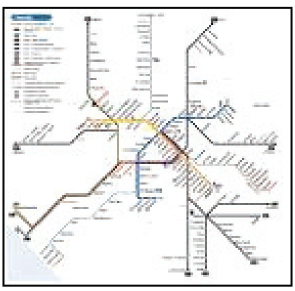
Our view is to identify a reading key with which teachers might get to know whatever they will be able to do in that particular moment of their journey, and through which they might learn to move around within the *local/global* pair along two directions: (1) inside a single *local* (a key-word); and (2) in the map of the possible connections among the locals which, rather than being approached through predefined ways, can be approached through a method that allows them to be explored.

As we said earlier, the exploration leading to a re-composition of the machine’s devices is an individual adventure and depends on how teachers *autonomously* decide to interact with the themes of early algebra. The objective is for them to gradually get to an increasingly organized and complex global view of the universe they approach. This should happen through pathways that might be heterogeneous and fractioned in time, as well as through a reflection which might also provoke a rupture with the teacher’s pre-existing knowledge and beliefs.

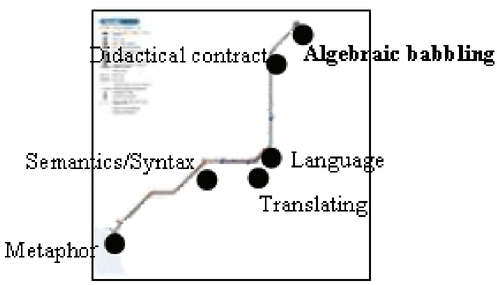
Our proposal links back to ancient problems related to the organization of knowledge, which mainly emerge when the studied topics and disciplines are wide and with hazy boundaries. These problems are linked to the need of searching for mutual influences, similarities and oppositions, inner logic of spheres of knowledge which often went through a different historical development and refer to different epistemological statutes. Applying this to the case of early algebra in the Aral Project’s perspective means exploring the *connections between mathematics and linguistics* that may favor a conception of *mathematics as a language* in which a convincing control of meanings can be developed by teachers and, later, by students.

Our main objective is, therefore, not to provide models of behavior, but rather to lead teachers to a concrete ground on which they might reflect on the “grammar” needed to move within the conception of early algebra. In this context the glossary represents a *textual map* aiming to:

(i) favor a *global* vision of the teaching activities’ theoretical framework:



(ii) define *local* situations made of pathways (the nets) linking single terms to one another:



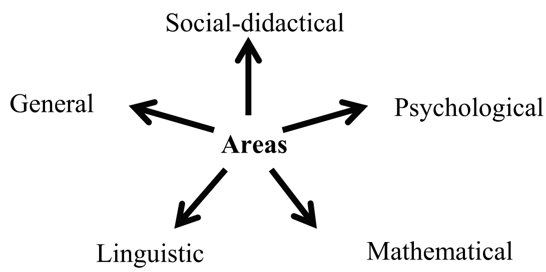
Each of the several possible routes of the glossary allows for a different reconstruction of the *sense* of the theoretical framework and of the links between reference areas (general, social, psychological, linguistic and mathematical) we will analyze in the next paragraph.

We remark that the main objective is to favor teachers' search for a necessary synthesis between theory and teaching practice. In the case of the route in the example, the synthesis of the net's terms may be:

> 'Algebraic babbling' is a 'metaphor' which puts side by side learning modalities for 'natural language' and those for 'algebraic language.' Through a suitable 'didactical contract,' which tolerates 'syntactically' promiscuous initial moments, it favors 'translations' between the two languages.'

7. AREAS IN THE GLOSSARY

Key-terms fit into five areas:



Some examples from the five areas are:

> *General area*: 'Brioshi, didactical mediator, opaque/transparent (with respect to meaning), process/product, relational thought, representing/solving.'
> *Linguistic area*: 'algebraic babbling, argumentation, canonical/non canonical form, language, letter, metaphor, paraphrasing, semantics/syntax, translating.'
> *Mathematical area*: 'additive/multiplicative form, equals sign, formal coding, mathematical phrase, pseudo equation, regularity, relation, structure, unknown.'
> *Social-didactical area:* 'collective (exchange of views, etc.), didactical contract, discussion, negotiating, sharing, social mediation.'
> *Psychological area*: 'perception, affective/emotional interference.'

General, psychological and social-didactical areas provide a *methodological support* to linguistic issues, which, in turn, represent a *conceptual junction* towards understanding of the mathematical area (Figure 2):

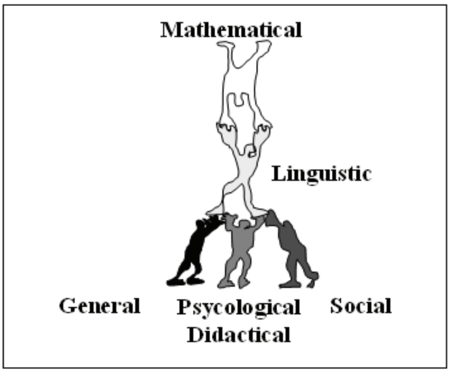


Figure 2.

In fact, since ArAl is a project concerning mathematics education, it is certainly true that the four 'bearing' components have a fundamental importance.

An early approach to arithmetic in an algebraic view is founded on a solid basis made of social and psychological assumptions as well as of a series of general basic concepts, which, in turn, sustain a strong *linguistic* component.

Teachers have to learn to promote and manage these supports, becoming aware that: (i) knowledge can be constructed through promotion of social processes that favor both *exchanges of ideas* and *verbalization* in the classroom; (ii) identifying suitable *didactical mediators* (for instance, the virtual Japanese student Brioshi who, not knowing languages other than his, 'forces' Italian students to use a correct mathematical language when they exchange messages with him) is essential to a stable acquisition of *meanings*; (iii) it is necessary to promote activities that enhance *metacognitive* and *metalinguistic* aspects.

Becoming aware of this is a fundamental prerequisite for teachers to organize and manage more specific mathematical activities.

Should these basic aspects be overlooked or developed only superficially and in a fragmentary way, the project's cultural character would be impoverished and potential didactical results invalidated.

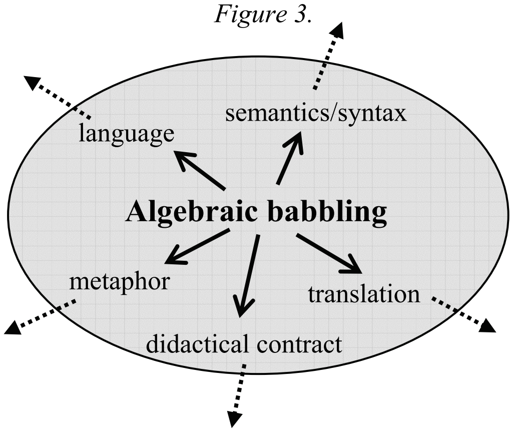
8. THE MATRIX OF NETS AND OCCURRENCES

Some terms in the glossary are particularly important. To illustrate this point we make use of a matrix (see next page) which, due to space reasons, only refers to a selected part of the glossary terms (about one third).[2] Nets of terms can be read in the rows, whereas columns contain occurrences, i.e. how many times terms are quoted in the nets. A net is thus a space that can be analyzed fully as well as in some particular areas.

Let us compare three nets: that of the 'algebraic babbling' construct, and the pair made of the nets of 'argumentation' and 'collective.'

[2] Inconveniences caused by this selection are that some nets are more numerous than they appear in the matrix. Numbers in the first row and in the last column indicate the real numerousness of the respective occurrences and nets, but in some cases the number of correspondent crosses is smaller, because many connections are excluded from the matrix. We believe that despite this limitation the matrix is still meaningful as a support to what is discussed in the text.

Figure 3.



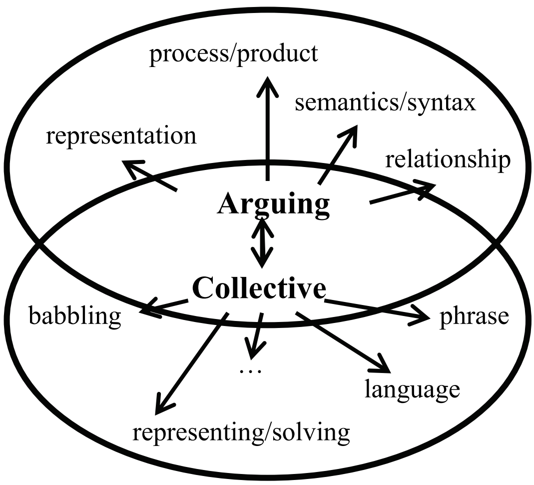
As concerns 'algebraic babbling' (Fig. 3), the five terms in its net ('semantics/syntax, translation, didactical contract, metaphor, language') lead to as many nets, thus widening its conceptual horizon. In the matrix, the five connections are highlighted in the row of 'algebraic babbling' with lightly marked crosses.

MATRIX OF NETS AND OCCURRENCES OF SOME TERMS IN THE ArAl GLOSSARY	Additive / multiplicative	Argumentation, argumentation	Algebraic babbling	Brioshi	Canonical / non canonical	Formal coding	Collective (discussion)	Didactical contract	Thrill from symbols	Mathematical phrase	Unknown	**Letter**	Language	Didactical mediator	Metaphor	Negotiation	Opaque/transparent	Paraphrase	Principle of economy	Procedure	Procedural	Process/product	Protocol	Pseudo equation	Representing/solving	representation	Relational (thinking)	Relationship	Semantics/syntax	Structure, structural	Translating/translation	Equals (sign)	Verbalizing, verbalization	Numerousness of nets
Numerousness of occurrences	16	3	4	9	5	2	8	4	3	17	3	10	20	4	3	1	5	2	1	6	4	12	4	1	14	8	5	10	11	5	12	2	1	
Additive / multiplicative (form)	■									X											X	X				X			X					**5**
Argumentation, argumentation		■					**X**															X				X		X	X					**5**
Algebraic babbling			■					X					X		X														X		X			**5**
Brioshi	X			■	X				X			**X**	**X**	**X**											**X**	X					**X**			**8**
Canonical / non canonical (form)					■												**X**					X				X								**6**
Formal coding						■				X	X	X	X					X								X		X			X			**8**
Collective (discussion)		**X**	X				■			X			**X**										X		X				X		X		X	**10**
Didactical contract								■																										**0**
Thrill from symbols			X		X		X	X	■	X		X								X					X				X					**10**
Mathematical phrase										■			X													X			**X**					**3**
Unknown			X	X			X		X	X	■	X	X	X			X			X			X		X	X			X					**15**
Letter			X	**X**			X	X		X		■	X	**X**		X								X		X			X					**12**
Language				**X**			**X**						■	X														**X**						**4**
Didactical mediator	X			**X**	X				X			**X**		■	X		X									X	X		X					**15**
Metaphor															■		X					X												**2**
Negotiation		X					X									■										X								**3**
Opaque / transparent (meaning)				X	**X**					X			X				■					**X**			X	X								**7**
Paraphrase				X						X			X					■								X		X		X	X			**8**
Principle of economy												X							■			X			X					**X**				**4**
Procedure						X				X			X							■											X			**4**
Procedural																				X	■						**X**							**2**
Process / product										X			X				**X**					■			**X**			X		X	X			**7**
Protocol		X						X												X		X	■											**4**
Pseudo equation							X																	■	X	X					X			**5**
Representing / solving				**X**						X			X								X	**X**			■		**X**	X			X			**8**

Representation																									■		**X**							1
Relational (thinking)	X				X					X	X	X	X								**X**				**X**	X	■	X				**X**		11
Relationship	X												**X**													**X**		■						6
Semantics / syntax										**X**			X																■		**X**			3
Structure, structural																			**X**		X						X	X		■				4
Translating				**X**									X													X			**X**		■			4
Equals (sign)										X															X		**X**					■		3
Verbalization						X				X			X							X		X									X		■	6

The situation for terms ‘argumentation’ and ‘collective’ is more complex: in the net of ‘argumentation’ we find the term ‘collective’ and in that of ‘collective’ we read the term ‘argumentation.’ We might say that there is a sort of one-to-one correspondence between these two terms. Cross-references in both directions, as in this case, are particularly interesting: they stimulate a process of in-depth analysis/extension which might become a fruitful instrument to deal with phenomena of circulation around a certain topic. In the matrix, pairs of terms that have this special feature are those which rows and columns meet in cells with a cross marked in bold.

Figure 4.



The exchange of cross-references is the *strongest possible connection between two nets*. In the case of the pair ‘argumentation-collective’ the importance of communication, and consequently of language (‘argumentation’) in the social construction of knowledge (‘collective event’) is once again emphasized. Examples of crossing nets often refer to Brioshi (‘Brioshi-Letter,’ ‘Brioshi-Language,’ ‘Brioshi-Translating’), and this is easily understandable, since the virtual student’s role is that of a linguistic mediator. Other pairs show a less clear connection, and this makes the search more interesting: for example, in ‘canonical/non canonical form-opaque/transparent’ pair the connection becomes clear when we find reference to the *dual* term ‘process/product’ and the consequent associations ‘non canonical form – process – transparent’ and ‘canonical form – product – opaque.’

Concluding, some terms seem to be *denser* with relationships than others. We find here two alpine refuges “each being a starting basis for goals that include the other refuge among the possible excursions.”

9. KEY-TERMS IN THE GLOSSARY

The matrix, besides providing interesting elements for a general outlook on the world designed by the glossary, also suggests another investigation.

Let us select the terms having nets and occurrences equal or greater than 10 (related numbers are highlighted in dark cells, in the first row and in the last column on the right side). Then let us put these terms into a table, grouping them on the basis of the area they belong to (see next page; colors of cells, from black to white are the same as those used for men in Fig. 2).

There are 16 terms, all leading to interesting readings.

Issues	**Nets with more than 10 cross-references**	**Terms with more than 10 occurrences**
General	Mediator 'Thrill from symbols'	Process/Product Representing/Solving Representation
Social/didactical	Collective	
Linguistic	**Letter**	**Letter** Language Semantics/Syntax Mathematical Phrase Translating
Mathematical	relational unknown	Additive-multiplicative Relationship

First remark: these 16 – either terms or pair of terms – are the only ones that appear in *all* the nine ArAl Units also published in English. This makes them highly representative of the project's theoretical framework.

There are 6 nets with more than 10 cross-references, as shown by the matrix's last column. They play a key role in the project's theory. We illustrate the latter by underlining the terms included in the table.

Nets column: In an environment characterized by a strong intertwining of mathematics and language, identifying mediators as bridges between the two disciplines is a fundamentally important *general* need. This is also the case for social aspects concerning teaching educational processes enacted with the aim of achieving a collective – and thus shared – construction of knowledge. The term 'letter' might be seen as *expected* (we will get back to this later) and inevitable difficulties in its use emerge ('thrill from symbols, unknown'). The sixth term, 'relational,' is also a paradigm of the project's main aim, that is, to favor the development of relational thinking, thus overcoming a view of the *local*, as well as a search for outlooks on *modeling* and *generalization*.

Moving to occurrences, we notice that 'letter' appears again: it is the *only* term with nets and number of occurrences greater than 10. Together with its paraphrases, (indicator, initial, place card), it is the main 'road junction' in the glossary. The peculiarities of 'letter' we are reconstructing, confirm that we are at the boundaries between language and mathematics and strongly point out the ArAl project's theoretical assumptions. We are led to pose three basic questions: (1) What does the letter *represent* when it is used in mathematics? (2) How can a “model of teaching letters” be developed? Can the letter as a mathematical object be intuited and its meaning be acquired through exercise, as it happens in traditional teaching, or rather is it necessary to use *mediators* that favor a gradual understanding through slow successive steps, as maintained by the ArAl project? (3) A correct use of letters, in particular within a formalized language, is based on a system of *rules*. In order to understand their importance and necessity is exercise enough, or rather are other strategies needed? As explained earlier, in the ArAl project, the mediator Brioshi is used to this purpose.

Therefore ArAl’s point of view is: the approach to letters is neither simple nor intuitive, their meanings become clearer and more solid through exercise, but only on the surface. They must necessarily be constructed through suitable mediators that favor students’ gradual awareness of the system of rules underlying the construction of representations in a mathematical language.

‘Letter’ is the main junction for ‘conceptual sorting.’ The table shows how the relationship/relational area and three pairs: ‘process/product,’ ‘representing/solving,’ ‘semantics/syntax,’ are junctions for *conceptual exchanges*.

Unknown and didactical mediator are the terms with the most numerous nets (15); language’ is the one with most occurrences (20); Brioshi (net with 8 terms, 9 occurrences) confirms its central role.

The construction of a deep understanding of early algebra and of its implications for practice mainly focuses on control of these terms.

The following is an attempt to use the terms in the table to compose a synthetic definition that gathers them in a sort of temporary manifesto:

Foreword

The early algebra theoretical framework supports the hypothesis that students’ weak control of the meanings of algebra, originates from how arithmetic knowledge is constructed, starting from primary school.

Algebra should be taught as a new ‘language’ one appropriates– through a number of shared ‘social’ practices (‘collective discussion, verbalization, argumentation’) – with modalities that resemble those of natural language learning: one starts from meanings (‘semantic aspects’) and gradually locates them into their ‘syntactic’ structure (‘algebraic babbling’).

Fundamental elements to this purpose are ‘metaphors,’ didactical ‘mediators’ to the acquisition of meanings during the conceptual progression towards generalization and modeling.

In this view, a natural language is the most important mediator in students’ experience and their main instrument of ‘representation,’ enabling them to illustrate the system of ‘relationships’ (initially ‘additive’ and ‘multiplicative’ ones) linking elements of a problem situation. This causes a shift of attention from ‘product’ to ‘process,’ and induces a ‘translation’ into a ‘mathematical phrase.’ In this way, attention is shifted from the *arithmetical* objective of ‘solving’ to the *algebraic* one of ‘representing.’ At the same time, mediators favor the acquisition of the use of ‘letters,’ initially viewed in their most approachable meaning of ‘unknown.’

10. THE GLOSSARY AND TRAINEE TEACHERS

The glossary provides a support on which teachers can plan and manage the didactical transposition of the subject’s contents, particularly those concerning arithmetic and algebra. The glossary is a leading thread for teachers to acquire this skill: this occurs throughout all phases of the training program for teachers – either in-service ones like those involved in the PDTR project, or those attending the specialization courses in mathematics education held at Modena University by Malara and her staff (Iaderosa, Gherpelli, Nasi, and Navarra): theoretical lectures, laboratory-based activities and training activities carried out in the mentors' classes. Teachers are supposed to complete specific assignments aimed at testing their degree of awareness in relating the management of teaching activities to the reference theoretical framework.

We report here an example of a concluding reflection, written by a trainee teacher for an exam. The text deals with a classroom-based episode, taken from one of the many experimental ArAl teaching activities, audio recorded and proposed in *six enchained scenes*, sequentially presented with twenty minutes intervals in between. The trainee teacher is supposed to analyze each scene and complete the related tasks; after the first scene's analysis he moves to the second one and so forth up to the last scene.

The structure of this didactical pathway seems meaningful as it stresses:

1. 'socially shared construction of knowledge,' enacted through a constant request of 'verbalization' and 'argumentation' by students and a 'didactical contract' characterized by the task "first represent, then solve;"
2. use of the symbolic representation register and of its supporting laws (the importance of 'language' and of the process of 'translation' from natural language to symbolic language and vice versa are particularly stressed);
3. use and analysis of 'protocols' and diaries, which enable teachers to draw information about their students' mental processes (internal representations) and, consequently, about their knowledge construction. While verbalization and argumentation help students acquire higher 'metacognitive' and 'metalinguistic' knowledge, protocols' analysis help teachers have a clear idea of the process through which students have constructed their knowledge and therefore make suitable didactical changes, when necessary;
4. risks of 'semantic persistence' in the use of letters.

As concerns representation methods, the structure of the assignment:

5. enables teachers to deal with both 'opacification' and 'transparency' of mathematical phrases and their aim ('relational or procedural');
6. puts the accent on the search for 'relationships' as well as on their 'representations.'

11. CONCLUDING HYPOTHESIS: THE GLOSSARY AND STUDENTS

The glossary, as we said earlier, was mainly envisaged for teachers. It was a final sentence in the assignment of a trainee teacher during one of our postgraduate courses that struck us and led us to formulate some hypotheses on a possible widening of the glossary function as an *instrument of cultural support to students*. The sentence was:

"I think that, for a teacher, experiencing is the most formative aspect*, especially if each one tries to 'show' his/her teaching style continuously*."

"Showing one's teaching style continuously." Possibly beyond the author's intentions, there is a strong implication of this statement: mathematics might be taught, making students aware of how the *fundamental basis of a meaningful* construction of mathematical knowledge is actually drawing on aspects that are apparently *outside* mathematics itself. Some of these aspects are: competence on the use of languages, starting from a natural language; being able to translate from one language to another; the importance of syntactic and semantic aspects of a language; the difference between representing and solving a problem situation; learning how to distinguish process and product.

This view led us to formulate a hypothesis on the widening of the glossary's function: we might think of a didactical contract that requires a *constant explicit statement of the deep motivations that guide teacher in their methodological and* *content-related choices*. In this way, students would view themselves as sharing the construction of knowledge and the glossary would become *a permanent background to teaching and learning*. Through the glossary students would be led to reflect upon the importance of sharing – with both peers and teachers – the *sense* of key terms, such as 'canonical form /non canonical form, letter, metalinguistic-metacognitive, opaque/transparent, principle of economy, argumentation, etc.'

An inevitable premise to this is that teachers come to be the prime movers of this sharing process, and therefore become conscious and convinced actors in the management of the glossary.

The concluding slogan might be: *to educate metacognitive students, we need to form metacognitive teachers.*

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EXPLOITING CHILDREN' NATURAL RESOURCES TO BUILD THE MULTIPLICATIVE STRUCTURE

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ABSTRACT
"Nothing is more practical than a good theory" (Skemp). In this paper we refer to a model of cognitive dynamics originally derived from wide empirical evidence and now also supported by some neurophenomenological research. We will show how this particular framework constitutes a powerful theoretical tool to plan didactical situations and to interpret students' cognitive behaviors. To show this in detail, we will present and analyze two excerpts of a long term class activity concerning the construction of an algebraic point of view on multiplication in children aged 6-7.

INTRODUCTION

Research in mathematical education and teacher training even with different assumptions and goals shares the need of planning and analyzing teaching practices and didactic tools through a specific theoretical background. Nevertheless, the role of theoretical framework is different: in the research field a theoretical framework is chosen as one of the possible frameworks to compare and work with, while teachers use a theoretical framework inasmuch it reflects their vision of the subject and the educational process. This changing prospective is more complex for teacher-researchers: walking on the boundary and choosing priorities between research and educational responsibility could be a tricky situation.

Starting from these preliminary remarks, in this paper we sum up a theoretical framework developed in more than thirty years of phenomenological research in a wide range of investigations in teaching-learning contexts, from kindergarten to university, in mathematics and science. The crucial feature of the framework is a model of cognitive dynamics, centered on the construct of *resonance* (Iannece & Tortora, 2008). Here we will show how this theoretical framework is a powerful and flexible tool for teachers in deep understanding their students' cognitive behaviors and in planning didactic actions. We will analyze through resonance lenses two excerpts of a long-time learning path on an algebraic approach to arithmetic with students, aged 6-8. Finally, we will stress the central role that some semiotic mediators play in teachers' behavior when they assume the role of "resonance mediator" (Guidoni et al., 2005b).

THEORETICAL FRAMEWORK

Building a model of human knowledge means hypothesizing about how people think and learn. In our opinion, a cognitive model is "living" when it allows the didactic mediation to become more effective, more useful for people (children or adults) who need to understand and learn.

The cognitive model of resonance (Iannece & Tortora, 2008) appears as an effective tool to plan and to interpret didactic situations. Our cognitive model unexpectedly is corroborated by some recent results of neural-biological research and shares some features with some neural-cognitive models. Considering the overall independence of the two research domains, this coincidence of results could be one of the main points to be further investigated by researchers.

In order to clarify our discourse, we need to briefly sketch the resonance model. The word *resonance* as we use it in its wider meaning, refers to the interaction between an individual and the "reality:" everybody, while acting in the world, produces a multiplicity of pre-representations (in the sense of Changeux, 2000) and hypotheses that they compare with the reality as they can perceive it through their senses. Each pre-representation is kept or thrown away according to the positive or negative feedback received during this comparison. In other words, all individuals look for a resonance between the cognitive simulation that they produce and the perceptual input. Through this "trial-and-error process" each human being builds up mental models needed to explain the environment and act in it.

This process is ruled by a "fit/no-fit" inner sense, that is, one of the basic features of every animal. This "resonance sense" is "automatically" adjusted and refined through our daily experience, as a biological process. At the same time it is a crucial element to be considered during any mediation between individuals and cultural models.

Teachers could use the resonance model to drive their didactic actions and to help students become aware, as much as possible, of their resonance sense and of the way it works. Teachers and students should learn to recognize whether a resonance dynamic has been activated, or not. That means to give a deeper meaning to the usual question posed by teachers to their student: "do you understand?" which does not mean "are you able to repeat what I've just said?"

Even cultural tools, along their history, pass through a "natural selection process," ruled by the sense of resonance of people who build, modify and use them (besides a multiplicity of social, political, environmental... variables):

> Our ability to make sense of the world through mathematics is due to the internalization of representations in the human mind and brain in the course of evolution. These representations are "unreasonably effective" in understanding our environment because they have been selected precisely for their representational adequacy. Indeed, two consecutive levels of evolution and selection explain the amazing adequacy of our present mathematics: first, the biological evolution of elementary representational abilities, and second, the cultural evolution of higher-level mathematics. In the course of biological evolution, selection has shaped our brain representations to ensure that they are adapted to the external world. I have argued that arithmetic is such an adaptation. At our scale, the world is mostly made up of separable objects that combine into sets according to the laws of arithmetic. Representing these combinatorial operations is useful to many organisms. Pressures of selection therefore lead to the emergence of an internal system for elementary arithmetic in the brain of many animal species, including humans" (Dehaene, 2001, 16).

According to this vision, the goal of didactic mediation should be to create conditions that allow resonance between an individual way of understanding and a cultural systematization of discipline. The main aim is to make disciplinary structures resonant with students' minds (and bodies): the resonance experience is crucial if we want cultural models to be really perceived as useful tools.

In order to reach this goal, teachers can use two subordinate tools. Firstly, teachers have to know students' ways of thinking. Recent research in cognitive science shows that individual cognitive structures are embodied in a natural language (Lakoff &

Núñez, 2000), and also in gestures and body movements (McNeill, 1992; Goldin-Meadow, 2003). In this way the first tool is the analysis of children's *discourses*[1] in problem solving situations.

Moreover, it is necessary to mediate resonance between children's thoughts and the discipline. The main tool that has to be used in order to achieve this goal is the semiotic mediation. In Vygotsky's theory the tools (like representations, symbols, etc.) play the fundamental role of semiotic mediators. The semiotic mediators are crucial to trigger the internalization/interiorization: "*the inner reconstruction of an external process*" (Vygotsky, 1987, 86, our translation). So the other useful tool for teachers is the choice of the representation that can effectively, time after time, play the role of the sensitive to resonance semiotic mediator. Therefore, a careful planning of teaching work is fundamental in order to choose the situation sensitive to the resonance of cultural tools."

TEACHER'S PRELIMINARY CHOICES AND KEY GOALS

We will show two meaningful excerpts of class activities in order to point out the power and the flexibility of the resonance framework. Firstly, in this section we have to clarify the general context where these excerpts are extracted from. Both activities are part of a path built and experimented in a long time action-research focused on the acquisition of arithmetic within a general algebraic vision (Mellone, 2007). The overall preliminary choices, made by teachers for the whole path, and directly derived from the resonance framework, can be summarized as follows:

(a) To work not just with natural number but with *small* numbers: the goal is to give priority to a deep understanding of structures rather than to numerical drilling – an achievement that can be less "fearfully" postponed, when children already move confidently within structures.

(b) To build the activities on selected action schemes. During the entire path action schemes proposed to children were chosen in order to create resonance between individual perception and cultural tools.

(c) Formal structures gradually emerge from the comparison among different contexts of factual experience, and among different events within the same context. Thanks to these comparisons, children can discover "invariants" that originate the construction and the application of mathematical models.

(d) Children work as a group: firstly, this offers children a possibility of learning by looking and questioning themselves about their peers' behavior, discussing and helping each other;[2] on the other hand, working in a group makes it easier to experience the value of inter-subjective agreement, which is at the basis of any scientific practice.

(e) Each child has to product individual representations in order to compare them with the other children. In this way the didactical mediation drives the group of children to gradually build more effective "semiotic tools" to manage the proposed experience contexts.

About point b) we want to stress that in all primary schools, teachers usually work with perceptual-motor activities but the reasons of their effectiveness is often

[1] By *discourse* we mean a set of words, gestures and body movements used by each individual to express themselves.

[2] Recently a group of neurologists (Gallese & Lakoff, 2005) discovered the so-called "mirror neurons" that offer another theoretical support of biological nature to the hypothesis of the effectiveness of peers' collaboration.

unclear: in this path the theoretical framework allows teachers to deeply feel the value of the oriented use of this kind of activities.

The semiotic value of Cartesian representation is well known at least for discrete quantities (formalized by natural numbers) (see, for example, Bernardi et al., 1991). In fact, in this kind of representations, multiplication and division simultaneously appear as two different sides of the same coin. Moreover, it is a good tool to explore the properties of multiplication, as we will show in the sequel. Then the use of the Cartesian representation scheme as a didactic mediation seems useful to allow children to directly approach the multiplicative structure in an algebraic way.

Moreover, this kind of representations is effective to manage a single experience and is also useful to discover analogies among different contexts: two games may look very different but they can be "set up" by the same graphic scheme. In this way, the sign drives a structural internal reconstruction of the action (using Vygotsky's words, the "internalization process").

In our learning path, owing to the hypothesis a), we refer to a slightly different Cartesian representation like the one in Fig. 1, that simultaneously shows the structure for discrete quantities (formalized by natural numbers), for continuous quantities with a discrete unit of measure (formalized by positive rational numbers), and for continuous quantities *tout court* (Guidoni, 2007).

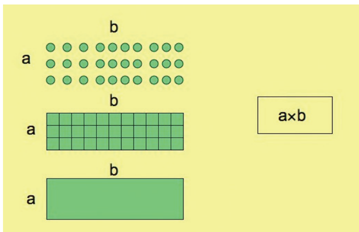

Figure 1.

A very long series of observations and reflections on young children's behaviors have convinced us that they almost never spontaneously turn to a scheme like that in Fig. 1, in order to represent their action experiences. Therefore, the schema can be used as an effective semiotic mediator only if a lot of preliminary work is carefully done by teachers who really want to take into account their natural cognitive structures.

The next sections will show two excerpts in order to highlight the role of the theoretical framework in our didactic action toward this goal.

1. The Mathematical Beady Curtain

"The Mathematical Beady Curtain Game" that we are going to describe is a teacher's proposal to facilitate a soft passage from the additive structure to the multiplicative one.

The teacher divided the class into groups of three children and gave each group a large heap of objects to count (for instance 80 shells or 75 pens). The idea is that the necessity of counting so many objects would induce the creation of small equal heaps easier to be counted. In other words, a heap of objects can play the role of *false unity*, that is, a semiotic mediator between the real constraints of the task – to count a big number of objects – and the limits of the individual cognitive structures – the ability to manage small numbers (Dehaene, 2001).

The first reactions of the children actually confirmed her forecasts; they spontaneously made small heaps of objects and counted the heaps.

Giuseppe: We have counted twelve heaps, so we have twelve pens.
T.: Are you sure?
Giuseppe: Yes.

Giuseppe's answer made the teacher aware of children's difficulties in the use of the false unity. Therefore, the teacher adopted a new tactic: she gave each group a cotton string and some holed pieces of pasta and invited them to thread one piece of pasta for each heap onto the string: her aim was to give the children a semiotic tool to manage the false unity and to reflect upon it. In fact, in her opinion, the false unity is a powerful cognitive link between the counting/adding structures and the multiplicative one.[3] The teacher's hypothesis is that a task of this kind, focusing on the rhythmic repetition of gestures more than on the objects themselves, allows children to recover the coordination between "taking and putting aside" actions and the signs representing them: the basic components of the primary "counting" structure.

Next the teacher asked children to put at the bottom of each string a sticker that specifies the "value" of each pasta piece ("one macaroni stands for one pen;" "one macaroni stands for five shells," and so on). The students had trouble writing the labeling stickers: at first, the children were confused by the double counting involved in describing "how many groups" and "groups of how many" (it was easier to state that "one macaroni stands for one group" than to accept that "three pens are represented by one macaroni").

When all the strings, one for each group, were completed and labeled and all the heaps were counted and recorded by the children, the teacher suspended the strings next to each other and realized the Mathematical Curtain, similar to a common beady-curtain keeping flies away.

The Curtain was ready to be an object of discussion: so, the teacher asked the children for the number of their groups of objects. At the beginning there were more problems than the ones foreseen. During the opening discussion, when everything seemed to be clear, the children unexpectedly came back to the "one macaroni – one object" action scheme, whatever the original agreement was.

Luisa: In our string there are thirteen pasta pieces so we have thirteen shells.
T.: But what do you read on the sticker?
Luisa: One macaroni stands for five shells – so I was wrong, we have many shells.
T.: Can you tell me how many shells have you?
Luisa: I don't know exactly, we have to do thirteen times five.
Giuseppe [member of a group different from Luisa's one]: In our string we have twelve macaronies, so we have to do twelve times five.
Luisa: Look at your sticker! You have to read on your sticker, they are different from each other!

The original problems arose again. The resonance framework helped the teacher to interpret these difficulties and overcome her own frustration: the structures coming from experience are stronger than the abstract ones, mostly at this age, this is the reason why children encountered no difficulties in the step from experience to abstraction, while had so many problems in the inverse step. In fact, they had no problems in constituting the heaps, the false units, but they had trouble when considering the different units composing each false unit. In order to overcome this difficulty, it was crucial for those children to play back-and-forth along the cognitive paths many and

[3] In this way "counting by one" is clearly seen as a particular case (the prototype!) of multiplication.

many times using the Curtain**.** In other words, the curtain game, developing their perceptual memory, allowed an interiorized experience of reversibility.[4]

Along the whole path, it was crucial to observe of children's gestures and the comparison between their gestures and words. In this way the teacher discovered that children need time and practice to interiorize a complex scheme like this, despite the fact that at the beginning they apparently understood the game rules.

Then she gave them another task: to arrange the objects on their desk making evident, even at a quick glance, "how many groups of how many things" they had on their desk. The children's representations were various and fanciful and were selected on the base of their effectiveness. Finally, the teacher introduced a representation like in Fig. 2.

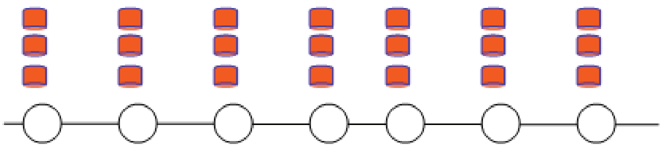

Figure 2.

Students immediately accepted the teacher's proposal:

Giulia: So we only need to count the lines!

The spatial disposition appears so strong that they can also forget the string of the beady curtain because the number of macaronies on each string is nothing more than the columns of the Cartesian representation, while the number of objects for each macaroni is the number of lines.

Even though children seemed to welcome the Cartesian representation, as proven by Giulia's words, in the following activities they did not often use it and the teacher had to recall it.

2. The Sheep's Trips

During the next year, the teacher decided to work again with the Cartesian representation in order to promote reflections on multiplication properties. It was re-introduced through several activities. Here, we present a following activity, in order to analyze the teacher's use of the theoretical tool of resonance in choosing each time a semiotic mediator more suitable to link children's thinking to the disciplinary goal.

The activity started with the following problem:

> A shepherd has to ferry his flock of sheep across a river, where pastures are more tender and green. The boatman says that his boat can carry only three sheep at a time and each trip costs one Euro. The poor shepherd has just four Euros and he does not want to get into debts. So he decides to take across the river only the number of sheep that he can pay for. How many sheep can he take? After some days the shepherd has made a little money, so he goes back to the boatman to let his sheep return to the fold. This time the boatman has a bigger boat, that can take four sheep at once, and the trip still costs one Euro. How much does the shepherd have to pay?

As usual, the problem involved small number and required both multiplication and division. Furthermore this problem, being bodily representable (or imaginable),

[4] By *reversibility* we generally refer to a property of "cognitive operations" before and rather than to a property of the mathematical ones.

allows children to play the important game of continually going in and out from mathematics on the basis of their cognitive needs.

Children immediately try to answer the first question.

Cristiana: It's easy, the shepherd can bring only twelve sheep.
Marcello: Yes, it's enough to do three times four.
T.: Are you sure? Can you explain your reasoning?

Luca, trying to answer, traced with his hands a sort of Cartesian representation: he put four imaginary objects along a vertical line and then three more imaginary objects for each line in the space in front of him, "It's enough to do: one, two, three and four; and one, two, three; one, two, three; one, two, three and the last one, two, three"

In Luca's gestures, the teacher read the association of the Cartesian representation to the word "multiplication," but the class did not share this association. She decided to go back to the bodily approach and asked the children to put the story on the stage: with a chalk they drew "the river" on the floor of the gymnasium, choose a shepherd and a boatman, while all the other were the sheep. As usual, after the body experience, the teacher asked for a representation of the story. Most of the children proposed representations that preserved the story structure, disposing sheep in groups of three.

Giuseppe: This way I can remember that I carried them in groups of three!
T.: But can you say how many trips?
Giuseppe: Yes, you can count the groups!

The teacher decided to drive children's attention to Luca's foregoing gesture and proposed to arrange the children-sheep along lines and columns.

Figure 3.

At this point Stefano, in front of the "human" Cartesian disposition like in Fig. 3, put four Euros on the floor, at the beginning of each column.

Stefano: Each Euro for each column.
T.: Can you see how many sheep for one column?
Stefano: No, you can't! You have to move to the other side.
T.: So you have two sides! And what do these sides mean?
Stefano: The columns are the Euros or the trips, while the lines are the sheep for each trip.

The teacher read the latest part of the problem in order to promote reflections about the commutative propriety.

Luca: It's the same! He has to pay three Euros.
Giorgio: Actually, it's the contrary! The first time, there were four trips and the places for sheep in the boat were three; now, they make three trips with four sheep at a time coming into the boat.
Ludovica: I agree, for the shepherd's pocket it is different because the first time he spent 4 Euros, and the second time he spent 3 Euros.
Luca: Yes, but since four times three is twelve sheep, it is obvious that twelve divided by four is three!

Giorgio: Yes, but the action is different! Let's do it!
[They bodily represented the new situation.]
Giorgio: Look, Luca! It's different now: there are three columns and four lines.
Luca: Yes, but in math it's the same!
Elena: I'm thinking that there is a way to see at the same time the columns and the lines: I can stay on the diagonal!
Gaia: We should have an helicopter to look from the sky!

Gaia's intervention showed the teacher that it was time to introduce a new and more easily-handled mediator: the egg box. Our goal was exactly to induce children to discover the properties of multiplication. The recognition of the commutative and distributive properties needs cognitive skills of second order: to reflect on operations in their formal aspect and their syntactic invariances, it is necessary to use an explicit metacognitive thinking that finds its origin in concrete operation experiences managed by a well-mastered language.

Figure 4.

By these new tools representing the trips and the number of sheep per trip, the children went on creating new situations with bigger boats (6 or 12 sheep at a time … but what happens if we have a boat for 5 sheep?). The spatial metaphor of the Cartesian representation and the possibility to turn the tool (Fig. 4) led children to discover the commutative property.

Gaia: To take twelve sheep, we can make six trips with a boat for two, but also two trips with a boat for six sheep!

But perceptive thinking always lies in ambush. For Giorgio it was very important to again draw attention to the concrete meaning of what we are doing:

Actually, it's still the contrary! In the first way, the trips are six and the places for sheep in the boat are two; while in the second way, they make two trips with six sheep at a time coming into the boat."

Referring to her theoretical framework, the teacher was able to go beyond a possible interpretation of children's behavior as a failure of her didactical action and instead to exploit and encourage this cognitive game that allowed children to make a deep experience of abstraction processes and metacognitive thinking.

Discovering the distributive property required a stronger mediation and, in this sense, it was an even better example of the crucial role of a theoretical tool in the hands of teachers.

She invented a story of sheep and boat in which the product 11×5 was involved; every egg box allowed products until 6×6, so just one box was not enough for representing the new situation. She suggested children should join two boxes and let them discuss.

Lorenza: I see… you can think about six trips for five sheep and then five trips for five sheep, maybe the boatman was tired after the first six trips and he needed to have a nap.

Alessandro: Yes, but the two boxes together are the same as a bigger box… you may directly think about eleven trips for five sheep… it's the same!

SOME CONCLUDING REMARKS

To briefly conclude, we want to recall that one of the main aims of the PDTR project was "the systematic transformation of mathematics education towards a system which, while respecting the standards and contents of the national curriculum, should be more engaging and responsive to students' intellectual needs, promoting independence and creativity of thought, and realizing fully the intellectual capital and potential of every student and teacher."

To put this in practice is a vary hard task, so what we have tried to show is how the ability of teachers in managing a model of cognitive dynamics rich enough can support this transformation. In particular, we have shown that teachers' ability to consciously rely on a suitable theoretical reference frame appears in several aspects: in the choice of global goals in mathematics education, in designing didactic strategies, in the interpretation of learners' cognitive behaviors, in the assessment, in re-design new interventions, and so on.

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A MODEL FOR VISUALIZING STUDENTS' PARTICIPATION IN MATHEMATICAL DISCUSSIONS

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ABSTRACT
This contribution draws on the analysis and reflection carried out by our group on the teaching experiments we implemented with the aim to improve both quality of teaching and teachers' professional development. In particular, we illustrate a new tool we created to support the analysis of transcripts of classroom-based discussions, so that students' participation might be controlled. During the joint analysis (with both teachers and researchers) of transcripts referred to classroom-based processes, we were able to confirm that this tool enables us to model the main parts of a discussion, visualize the flow of the discourse, highlight relational links and capture the role played by individuals. The tool also supports the evaluation of the path individually followed by students, in interaction with classmates, and provides opportunities to highlight the teachers' role and behaviors.

FOREWORD

The renewal of teacher professional training is a key condition to an improvement of the quality of mathematics teaching in our country: this renewal should not only relate to curricular aspects, but rather to the acquisition of new teaching models as well as of an everyday teaching practice, which both require a deep change in the usual way of planning and carrying out activities in the classroom. In compulsory school, where students construct their image of mathematics, constructive teaching is widely spread and "centered on the devolution to students of the exploration of paradigmatic situations, in relation with the teacher. Metaknowledge is also strongly valued, as it is strictly related to students' maturation in terms of flexibility in reasoning and elasticity in tackling and elaborating on new facts" (Malara, 2005). As a working group, we deemed fundamental to set some shared praxes to be used both in the preparatory phase of planning teaching sequences to be implemented in the classes, and in the final phase of analysis of the work carried out. We put emphasis on the lessons' transcripts, taken from the whole transcription of video recordings and we realized how appropriate it is to juxtapose the class teachers' comments and the "counter-comments" made by other members of the research group (Malara, 2008). We also realized that transcripts' analysis should be carried out from different points of view and requires the use of different reading-keys, so that the diverse variables involved in the development of a mathematical discussion can be taken into account. In this paper, we dwell on a particular reading key that we needed to use, as a group, and that led us to outline a modality of visualization of students' participation as well as to a progressive construction of a model for the latter.

The model was outlined within action-research activities in the classroom[1] supported by meetings involving teachers and university researchers on planning, analysis and reflection, with modalities that have characterized Italian research since the end of the 1970s and the beginning of the 1980s (Malara & Zan, 2002) and analogous to those characterizing Jaworski's co-partnership model (Jaworski 1998, 2003). The experimental setting up of the planned sequences goes with a detailed and critical examination of the teaching processes enacted, so that both teachers' role and students' cognitive achievements can be made explicit and compared. Many of our research studies (Malara, 1995, 2005; Malara et al., 2004) have shown the advantages of this practice, which values mutual exchanges on classroom-based processes, both between teachers and between teachers and researchers, and leads both groups to a progressive professional growth.

Our shared processes occur in different phases: (1) collective explicit statement of and exchanges on the various positions taken on the analyzed mathematical topic; (2) collective formulation of research hypotheses and planning of the work to be carried out in the classroom, fine tuning of problem situations to be posed so that the formulated hypotheses might be tested, a priori analysis of situations and expected problems, sharing of transcripts and development of related analytical strategies; (3) comparison of students' both collective and individual routes followed during the experimented sequences; shared reflection on the achieved results; analysis of classes' paths over time and reflection upon enacted changes; and (4) individual and then collective self-analysis on the role played by teachers and on ongoing evolutionary processes related to their work methodology as well as to their conceptions.

These processes led us to change, over time, our role of teachers within the group. Although we keep a solid link with the school world, we progressively tend to acquire a point of view much closer to that of educational research. In this shift towards different horizons, we developed a particular view of transcripts, which led us to consider both the issues of how to check the quality of individual contributions and of the participation of single students in the development of a collective discussion. We thus structured some tools to be used together with commented transcripts, in order to evaluate both process and students.

PROBLEMATIC NATURE OF TRANSCRIPTS

Collective discussions should produce learning. A necessary condition for this process to occur is that the individuals involved learn that there is a precise modality to get into the group dimension. All individuals involved in this process should carry out their task in this direction. Group learning initially is learning of a frame which can outline not an indeterminate time and a certain space, but rather a given time and a given space: the class, at that moment. The key element, focus of the group learning, is the set of tasks for students. The group is organized around a task or an aim. Space, time, role, task or tasks, together with behavioral norms students should respect, constitute elements of a frame which enables us to provide an access to the group dimension (Yackel, 2001). Defining the frame is a crucial moment of the a priori analysis of any teaching sequence. It outlines the features of the desired didactical contract between

[1] We refer to activities carried out for both training classes given for postgraduate courses for teachers and research and teacher training projects on a European (PDTR), Italian (Master in Science Education), and a regional ('Mathematics Together' project) level.

teachers and students. It focuses on assignments and on mainly cognitive expectations about assessment. In presenting the frame, metacognitive aspects, i.e. organizational keys of the sequence, are stressed. Students should be informed about the dimensions they are asked to work in, everybody's expectations, objectives and methodologies to be used (Yackel, 2001). This frame provides a boundary for the field in which events belonging to the group process occur. The class produces a representation that highlights the "style" or "habit" of that particular group. It is about mutually recognizing glances, gestures, assigned priorities, expected contrasts, nets made of role request, award and playing. In the end, it is about a tissue of constraints organized around the group's task. Learning processes intermingle with the structuring and showing up of these group schemes, which should be usefully highlighted as a support to transcripts.

A complete transcript of a lesson provides a full picture of what happens in the classroom; it is a detailed and faithful image of the everyday teaching practice. It reports on all the interventions that were made in class and that contributed to a collective discussion. But this is an essentially "static" image of the lesson. The double numbering of the interventions, general and specific, pointing to either the sequence of interventions made by a specific student, or to their recording time, tends to make the transcript more "dynamic," as a temporal dimension is introduced. This leads us to a "cinematic" view of the collective discussion. But this is still a limited view; gestures, use of phonetics by both teachers and students are missing. Likewise a projection of an object on a plane, a transcript reduces the expressive range of languages to the verbal register only. This is why we deem useful to perform video recordings instead of the usual audio recordings. Unfortunately, at the moment, privacy issues make it hard to use this tool in everyday class teaching.

A transcript reproduces the interactions between cognitive and relational-affective aspects of learning. They are never separated in everyday classroom work. Therefore, a transcript is not a neutral or absolute object: it is strictly linked to the involved subjects. Its "language" expresses the sum of all the active subjects' languages. It is thus a contextualized object, linked to the specific class and to all the processes enacted in everyday work. Among the various languages, we aim to look at mathematical language and follow its progressive definition and construction. On the linguistic side, the picture is still intermingled and a subtle balance between the different reading keys is needed for an analysis. The cognitive key generally prevails over the others. For teachers it is equally important to have a control over the class and the global development of the discussion. In this view, it is important that relational processes in the development of a lesson are made explicit, both for valuing individuals and for assessing them. Sometimes, in the development of mathematical discourse within a discussion, teachers might tend to focus on the main "actors," neglecting the rest of the class and not considering the quantitative ratio between actors and remaining students. We often wonder what brings about a certain behavior by a student or rather what makes the discussion take a direction or another one. For this reason, we tackled the problem of studying a way to use transcripts together with other instruments that might highlight participative processes, by making them visible and overcoming the temporal dimension that characterizes transcripts. By identifying the causes that determine a certain effect, we would be able to use transcripts together with other tools for assessing learners' participation. The "causes" we mentioned are generally not explicit, because teachers and students speak their own classroom-language, an oral language made of silence, voice inflections, specific gestures, full of slang expressions, superficial or wrong statements, not always appropriate metaphors, allusions. Comments made by teachers are important for both a global and a fine analysis and should always be made in progress, i.e. within the planned teaching sequence: nevertheless, alone, they cannot make the lesson's transcript "dynamic," i.e. able to highlight the progressive construction of emerging mathematical concepts and/or the development of the collective discussion. Although extremely useful, including a presentation of single students involved is still not sufficient.

In particular, the control over the class group is not clear at all, if not for the involved teacher. The role played by students seems to be made very little explicit. Some students get involved very little, some stay out of the discussion and some get involved too much. But, how does the class go on? A complex issue is the identification of the important steps made in the collective discussion, as concerns the construction of individual learning. It is now an ordinary practice for us to report in our transcripts the points shared in the collective discussion and written on the blackboard: nevertheless, this leaves out the dimension of students' notebooks, which would make their individual process explicit.

We view transcripts as a complex didactical object. In order to take advantage of its richness, we should split it into its components, i.e. mathematical knowledge, languages, roles played by teachers, relational processes. Researchers usually prefer to read and analyze mathematical constructions and teachers' role; as teachers, we regard as more important to highlight how the flow of ideas can be constructed starting from individual contributions, how meanings are negotiated during discussions, also with the aim to assess both individually and collectively completed processes. Therefore, we engaged in the construction of an instrument which can make students' participation in a discussion visible and, together with transcripts, may highlight group-related schemes and favor the assessment of both discussion and students.

TOWARDS A DESCRIPTIVE MODEL OF CLASSROOM DYNAMIC PROCESSES

In our analysis of transcripts, we outlined some strategies which proved to be helpful in the identification and objectification of the flows of the various micro-discussions at the basis of the process of mathematical construction that occurs in the classroom. We might view these strategies as the basic components of a model we created and named "bees' flight." The flight of a bee appears as a subtle metaphor of the path followed by a class involved in the exploration of a mathematical problem situation: worker-bees charged with their 'pickings' provide a metaphor for individual students' achievements and the harvest in the bee-hive well represents the fact that individual achievements should be made available for a shared mathematical construction. The communicative dance used by worker-bees to indicate the direction where food can be found, is a good metaphor for the communication characterizing the class group during a mathematical discussion, with relation to the institutionalization of knowledge, to the representation processes and to the synthesis of the various moments of mathematical construction within the study of the situation. Representation processes at the blackboard are to us key points in the development of the discussion. The blackboard well represents the bee-hive as a place for both representation and synthesis of the produced knowledge.

The "bees' flight" analytical model is a dynamic one, since it allows us to objectify the nets of relations that contribute to the development of a discussion. A mathematical discussion is made of a collection of micro-discussions, in each of which only a small number of students participate. The model we mentioned enables us to map the micro-discussions onto a classroom map, which highlights the position of each of the involved subjects (including the teacher and possibly other observers). On each map some numbered arrows are laid to reproduce the ongoing interventions; the aim is to generate an image of the micro-discussion's flow. The various subjects progressively involved are then connected to one another, thus establishing relationships between the interventions reported in the transcripts. The arrow starts from the position of the subject who makes the initial intervention and goes towards the subject who gets involved immediately after. An arrow might take two opposite directions, due to the intermingling of interventions. In our "bees' flight" maps the features of each didactical action carried out in the classroom during the discussion are made explicit and expressed by linking the involved subjects (Yackel, 2000).

Creating the map of a micro-discussion's flow provides a dynamic view of the transcript; recursive structures, little or big discussion whirls, made of flow lines that schematize and smoothen the broken lines drawn on the map, can be noticed. We metaphorically name these structures "bees' flights." The set of these curves provides a discussion's flow-chart. Analysis of transcripts can be carried out on different levels; we might construct charts for single micro-discussions or charts for whole lessons, depending on students' participation. "Flights" may have diverse shapes, such as circuits, knots, spirals, crosses, L-like, or T-like. In knots the involved subjects, discussion leaders, their supporters and passive subjects emerge immediately. The group map generated by a synthetic view of the enacted learning processes and the observation of cycles in their development over time, brings about the outline of an infrastructure that keeps the process under control. As it happens in a map of the eye-movements made by the eye to "look at" an object, by tracing the "flights" followed by students to discuss a mathematical problem, we can see how the class "looked at" the problem in all its features.[2]

EXAMPLES OF BEES' FLIGHTS

In the following, I report discussions' flow-charts[3] referring to two discussions that occurred at different moments, in the eight-grade class, within a teaching experiment carried out for the ArAl units about an approach to functional relationships (Fiorini et al., 2006). Due to privacy reasons, names have been changed.

[2] This analytical model is regularly presented in methodological laboratories for both pre-service and in-service trainee teachers. In particular it is widely used in the analysis of lessons given by novice teachers in their training. It was effectively used in a number of postgraduate courses final dissertations: it was also successfully used in the validation of teaching sequences belonging to the ArAl unit about functional relationships (Fiorini et al., 2006).

[3] Legend of flow-charts: Red rectangle: Center of discussion; Green rectangle: Supporter; Yellow rectangle: Start; I: Teacher's position; *Cattedra*: Teacher's desk;

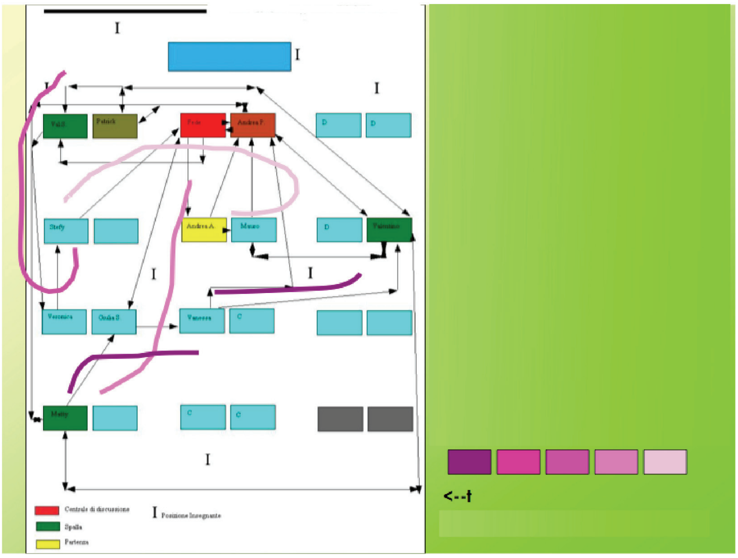

Figure 1.

This map (Figure 1) shows the emergence of various bee flights:

(1) one main chain, shaped as an ascending spiral (towards the blackboard): Andrea, Mauro, Andrea P., Federica, and Patrick (closing intervention);

(2) a secondary female chain, shaped like a descending spiral: Federica, Valentina S., Veronica, and Stefy

(3) a transversal axis: Matteo, Giulia S., and Federica

(4) a double L-like specular chain searching for links: Valentino, Vanessa, Matteo, and Giulia S.

Remarkable efforts to connect with chains are made by various students: (a) Valentina S. would like to connect with the ascending spiral, but Patrick closes it; Federica tries to connect her but she does not understand and does not get connected and prefers to create her own spiral, which nevertheless turns out to be closed in itself; (b) Vanessa tries to get connected to the ascending spiral but she does not manage to do so and turns to the chain of connections, after attempting with the transversal axis too. Unfortunately, she fails, because her classmates, particularly Giulia S. and Valentino, besides Andrea P., do not react and cut her off. These students have tried to take part in the bee flight; the group and the context, actually blocked them.

The following map (Figure 2) is enriched with signs "+" and "-" that we introduced because we realized that it is helpful to associate each intervention with a symbol to indicate its social 'value' (positive, +, or negative, -) whether the intervention:
+: expressed either individuals' position or their membership to the group they want to belong to, expressed the positive role they tend to play within the class;
-: expressed the isolation of the subject, the negative role they tend to play in the class (or rather the fact that they do not play any definite role in that context).

The concept of value enables us to observe the capacity of organizing links: low and high value characterize different groups and different group situations.

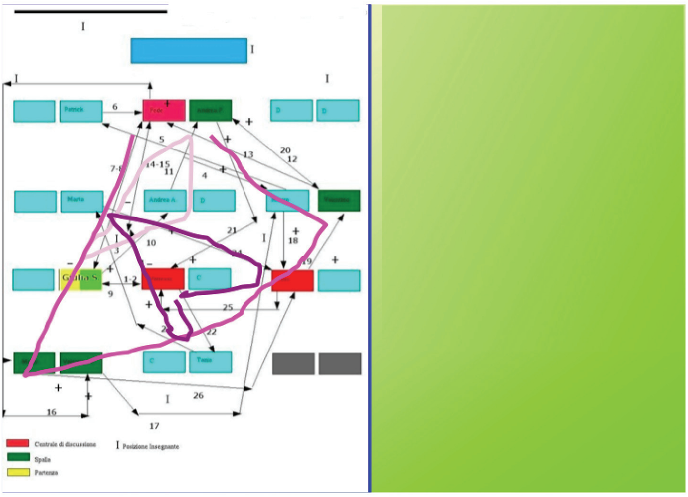

Figure 2.

Reading this map (Figure 2), we notice lack of closure of a triangular structure (Vanessa passes to Tania who passes over to Marta. Instead of closing on Vanessa she passes over to Elisa. Vanessa does not react to Marta). The number of students involved is essentially the same as in the previously analyzed lesson, but different students get engaged in discussion here. There is a polycentric and quite intertwined structure. This opening discussion is much more varied than the previous one.

The following Figure 3 is another example of mapping, referring to a completely different context (a different class, different teacher, different school, and a different problem situation).

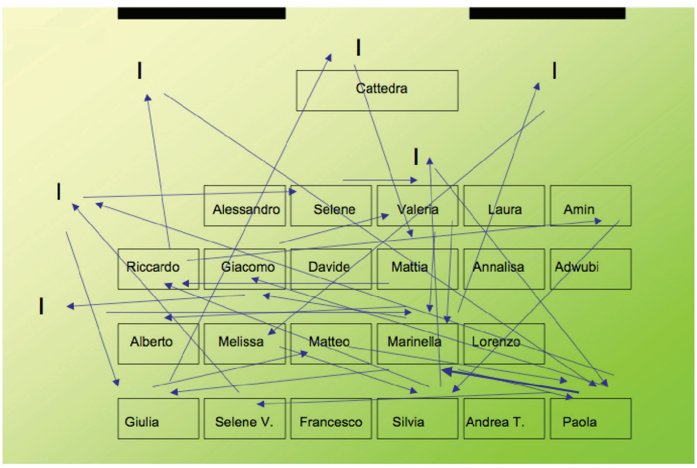

Figure 3.

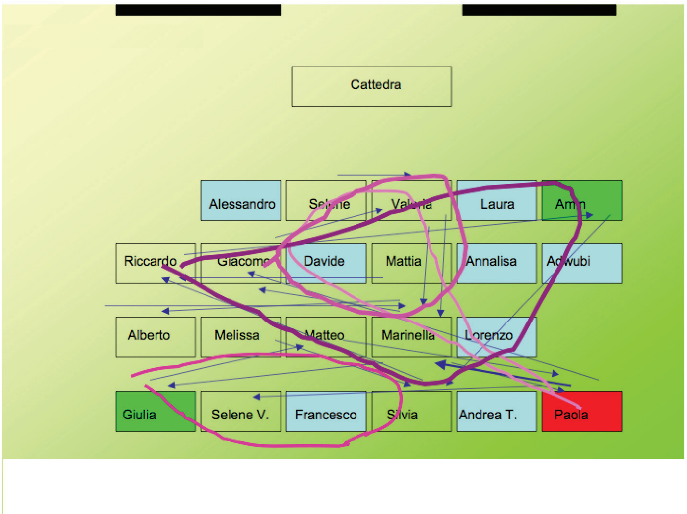

Figure 4.

In this case (Figure 4), the preferred structures for the 'shift' of 'voices' in the discussion process relate to the role of the teacher. The flights seem to be affected by the teacher's "mobility."

The work of various trainee teachers provided evidence that this model is really effective when it is used in training activities for prospective teachers. Carrying out a mathematical discussion may turn into a dialogue between the teacher and individual students, without mutual exchanges: in this case, the map would be radially shaped and the group scheme would collapse. An example is provided by the map below (Figure 5), (Pisi, 2006) where this collapse is clear: the trainee teacher had to face the need for a self-analysis of his own way to relate with students.

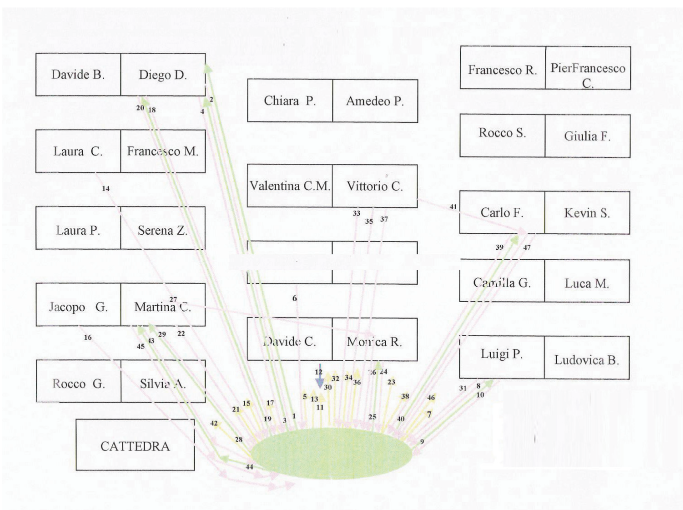

Figure 5.

During constructive work, each class, consistently with their own internal processes of interrelation and exchanges, acquire their own working style or habit for discussions, characterized by a certain flight rhythm, made explicit in a certain type of map. The use of these graphs, together with transcripts, enriches the a posteriori analysis and provides both teachers and researchers with new reading keys and further opportunities for reflection. Teachers, in particular, are given an opportunity to look at themselves and, in the hard work needed for transcribing and carrying out discussions, to improve their work methodologies, to evaluate their students' and their own paths. The model we proposed proved to be exportable to the context of pre-service teacher training, showing a wider value than we initially expected.

The instrument presented here may turn out to be useful to teachers-researchers, as it might enable them to analyze in-depth their own teaching sequences in the class, making links between cognitive and non-cognitive aspects; it might help them relate the construction of concepts to the evolution of a collective discussion; it might determine an immediately visible infrastructure for how ideas have been constructed and represented with relation to the teacher's action; finally, it may enable them to have a critical approach towards their own actions and globally revise everyday teaching methodologies. This instrument was born in the field, from my own work as teacher-researcher and rapidly spread at local level, as a useful key to open the doors of the complexity underlying processes of both management and control of collective discussions.

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THE USE OF "GRAPHICS FOR INTERACTIONS" IN SOLVING MATHEMATICS PROBLEMS WITH MULTICULTURAL STUDENTS

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ABSTRACT
The Pisa Assessment has highlighted the difficulties experienced by students in learning of mathematics in Spain. The present study focuses on the use of graphics, which allows teachers to visualize and to think about the interactions that take place during learning math in new immigrant students' classroom. The graphs are displayed as a good instrument for analyzing communicative networks on the interactions that are carried out during the resolution of mathematics tasks and at the same time, this instrument enables teacher-researchers to think about the basic teaching-research.

INTRODUCTION

When students arrive to our country, they are assigned to a school of the town they live in and since they ignore the official language (the Catalan language) these students are integrated in special classrooms, in order to introduce them to the knowledge of the language and the culture of our country. In general, teachers of these classrooms are teachers with a significant experience who receive basic training in linguistics, but they do not receive any special training in mathematics. This it is the case of the classroom teacher where the investigation is carried out.

The presented experience takes place in (1) a classroom with multicultural students; (2) with the classroom teacher who has had a lot of experience and knowledge of the language of our country; and (3) with a teacher-researcher as a collaborator (expert in mathematical knowledge).

In the recent years, Spain has become one of the countries receiving immigrants from different countries: North Africa, Latin America, and from countries of the former Soviet Union. This has made the autonomic government arbitrate several measures for the integration of these students in schools. When they arrive in our country, they are assigned to a school in the town where they live. They do not known the official language (Catalan) and for this reason, these students are integrated into new immigrant student classrooms in order to introduce them to the knowledge of language and culture of our country. In general, teachers in these classrooms are experienced teachers who receive training in basic aspects of linguistic, but receive no special training in mathematics. This is the case of the teacher from the group where the investigation takes place.

THEORETICAL FRAMEWORK

Since the study investigates the use of the interactions involved in conversation learning, first we try to define these interactions. We define as communicative interactions those, which are produced between two subjects orally in a conversation. In our case these interactions are between the teacher and students or students and students, and their content is about teaching and learning of mathematics. Therefore, the interactions are in the middle of the communicative process. So, we believe that in a systemic model, they can be seen as an "interactive relationship[s] and a whole dynamic" (Marc & Picard, 1992). Morin (1999) provides us with a very general notion of communicative interactions saying that communicative interactions are reciprocal actions that change the behavior or the nature of the elements in the presence of influence." It is clear that the interactions produced in a pedagogical conversation are oriented, from the beginning, to specialized interactions as the ones noted by Vion (1992). The difference between a didactic dialogue (Amigues, 1996) and other kind of dialogues is that the first tends to create a cultural relationship referring to a known object.

We share with Goffman the idea that an interaction is defined as a reciprocal influence carried by participants on the respective actions while they are in the physical presence of each other (quoted by Vion, 1992).

It is important to point out the structure and functionality of the interactions. Kebrat-Orecchioni (1990) presents a hierarchical model in which different components establish inclusion, subordination and functionality relationships. Every interaction can be broken down into sequences, that is to say, exchange blocks linked by a sharp degree of semantics and or pragmatic coherence.

This investigation asks: (1) how can we improve communication between teachers and students? (2) how can we improve communication among students themselves? (3) how are we able to assess whether the use of educational materials helps understanding?

METHODOLOGY

One of the main points of a multicultural classroom is the use of language as a learning tool, in our case, of mathematical language. That is why we want to analyze conversational networks that take place around the resolution of mathematical tasks, in the classroom with multicultural students. This article presents an analysis of the resolution of a mathematical problem as well as our reflections on it.

Given the communication difficulties that take place in a multicultural classroom, our starting point is a hypothesis that the use of teaching materials will improve communication (understanding) and the resolution of mathematical tasks.

That is why the goals that this research proposes are: (1) to determine the degree of communication established between teachers and students and among students in activities with and without use of mathematical content materials; (2) designing activities involving the use mathematical materials.

Two teachers are involved in this experience: one as a classroom's teacher and another as a teacher-researcher. The researcher is involved in this experience by collaborating in the design of learning tasks and contributing with her experiences in teaching mathematics as a person skilled in teaching of mathematics. The motivation for the participation of the teacher-researcher in this experiment is to learn how to manage a multicultural classroom and at the same time to contribute her experience to teaching of mathematics.

The methodology of this case study was designed on the basis of the design-based research (DBR). This choice responded to the characteristics of the study.

The students involved in this study were students from a new multicultural classroom in a public school aged 3-12. The school has a rate of 20% of students from other cultures.

The group study was formed by the following students: (1) Three students aged 11-12 from Morocco with a one-year stay in school. They attended the fifth grade of primary; (ii) one student aged 11-12 from China with a year of permanency in school who studied in the fifth grade of primary; and (iii) two students aged 10-11, from Morocco. They stayed one year in school.

The instrumental techniques used in the investigation included the following: (1) observations of the participant-observer Gonzalez and Latorre (1987); (2) class journal where the experiences during the sessions were recorded by the class teacher and researcher; (3) e-mails between the two teachers; and (4) five audio recorded class sessions.

EXPERIMENTAL PART

The main aspect of the study was the role played by the mathematical content in new immigrant student classes. That is why it was decided to adapt for students both arithmetic and geometric tasks.

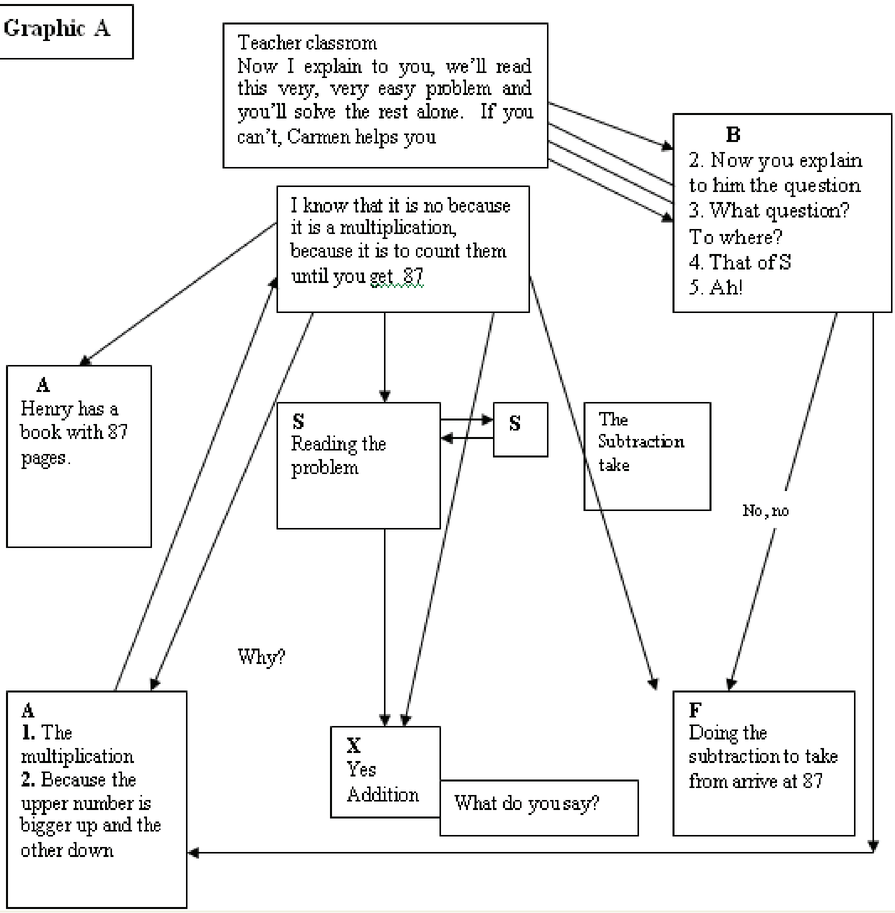
The arithmetic activities focused on a game called "Fermez la boite" which took three sessions. It was decided to use the game due to the fact that it was one activity that could motivate students to participate in dialogues on math tasks. The game had chips and dice and students had to operate with the number obtained to get the quantities of chips and win.

In geometry we also choose the game format. The game took place in two teams. The game consisted of composing geometric shapes using the polydron. In order to get it, one had to compose a geometric shape and verbally describe to the opponents the figure that the other team had to compose without seeing it. This game also sought the communicative participation.

The type of analysis used was a communication networks analysis with the help of graphics. To this end (1) the sessions of class were transcribed; (2) hypertext schemes were made from communicative units of the speakers participating in the dialogue, in order to see how many interactions took place and what the degree of their quality was; and (3) schemes were reduced to identify the meaning knots.

The situation that we present in the graph A shows an activity made by the classroom teacher without using didactic material. The teacher proposes the resolution of a simple problem. The problem says: “Enrique has a book that has 87 pages. If he has already read 23 pages, how many pages are left? How many pages does he have to read?” We can see that it is a problem of the second grade of primary level.

Graphic A

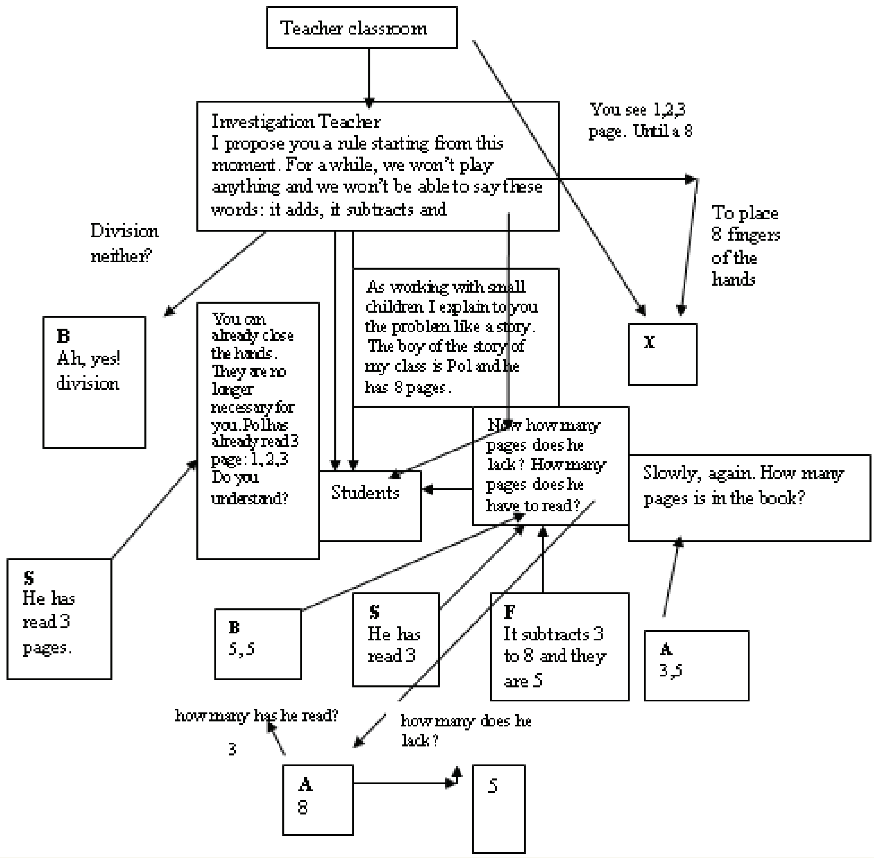


In this figure one can observe that the classroom teacher starts solving the problem talking to the whole group, but at this moment all questions are always unidirectional, in other words, the questions are directed at one student (A, B, S, etc.). The kind of questions that the teacher poses are very concrete, awaiting monosyllabic answers (if not a number, etc.), but they do not allow discussions at group level, that is between students and the teacher. In this figure, we also observe the absence of conversational knots and the presence of students without any participation.

Since students are not clear which operation they must carry out to solve the problem, the session of the resolution of the previous problem continues. The classroom teacher introduces the expert teacher in mathematics to see if she succeeds in facilitating the resolution of the problem by students.

In the graphic B we see the representation of the interactions that take place with the teacher and researcher, and we can observe that most of the time the teacher talks to the class in general, giving clear slogans that help in the resolution of the problem, reducing the number of pages to make it clearer to the students who achieve their solution; most of the students participate in the discussion.

Graphic B



If we compare both graphics we can clearly see two forms of teachers' behavior in this investigation, giving an example how to plan math activities in a group with great communicative difficulties. In figure A the teacher maintains unidirectional dialogue, while in the figure B the teacher manages to involve the whole group as the main interlocutor.

DISCUSSION

Three different arithmetic games and one geometric were analyzed with similar graphics. The educational games play an important role and one can observe improvement of the communication but not in an outstanding way. The use of the graphics has allowed the two teachers involved in the experience to understand how to bring the mathematical tasks into communicative play in didactic learning atmosphere. These teachers realize that teaching materials are interesting and provocative for the activity, but what really matters is the type of communicative interactions that the teacher develops with them. Consequently, the mathematical activities designed with teaching materials for immigrant students must take into account the role of such interactions.

The study suggests some recommendations for the teacher training about the design of activities which encourage dialogue between students and the teacher. For this reason, we recommend using graphics as a good tool for research teachers who want to study all the classroom communication.

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THE PRINCIPLE OF MATHEMATICAL INDUCTION: AN EXPERIMENTAL APPROACH TO IMPROVING AWARENESS OF ITS MEANING

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ABSTRACT
This work is based on our conviction that it is possible to minimize the difficulties that students face in learning the principle of mathematical induction (PMI) by clarifying its logical aspects. By referring to research results that outline the most problematic aspects in the teaching of PMI, and to research that suggests possible ways of overcoming such difficulties, we designed a method for fostering students' understanding of the principle. Together with two teachers involved in the PDTR Project, we planned a path for the introduction of the PMI as a proving tool, and implemented it in two secondary school classes in a school for future primary school teachers. In this paper we will describe the main phases of this path, highlighting in particular the student-teacher discussion in the central lesson, and the teachers' observations on key aspects of the class activity in the development of the path.

1. INTRODUCTION

In her paper "Proofs that Prove and Proofs that Explain" (1989), Hanna defines a clear 'boundary marker' between non-explanatory and explanatory proofs. As an example of non-explanatory proof, she proposes the proof by mathematical induction that the sum of the first *n* natural numbers is equal to $n(n+1)/2$. The author observes that this proof guarantees the truthfulness of the statement, but it does not explain why this equality is valid and it does not clarify the relationship between the sum to the left of the equality and the expression $n(n+1)/2$. Given that schools aim to foster students' comprehension, Hanna suggests that the mathematics curriculum should place greater emphasis on proofs that *explain* than on proofs that *only prove* (i.e. those that use mathematical induction).

We agree that there is always a need to direct efforts to those activities which favor students' deep comprehension of mathematical concepts, rather than limiting ourselves to the mere transmission of formal notions. Nevertheless, we believe that our proposal to use the principle of mathematical induction as a proving tool in upper secondary school is not necessary in conflict with this approach. Using the PMI to prove a statement obviously does not "explain" how the statement was constructed. Therefore, activities aimed at the formulation of conjectures are essential. All the same, we propose that it is possible to make students "grasp" the meaning of the use of this principle, to the point that they can consider a proof by mathematical induction convincing. In this sense a proof by mathematical induction could also 'explain.'

For this to occur, it is necessary to revise the approach traditionally used in Italian schools which devotes little time to the teaching of a solid understanding of the principle. Most traditional text books, for example do not cover the PMI in depth and require students to apply it "blindly" in proving equalities. Students learn to reproduce these exercises mechanically but do not develop a true understanding of the PMI.

We propose that it is both important and possible to promote an understanding of the PMI, along with its applications, by using non traditional methods. We are convinced that it is possible to minimize students' difficulties by clarifying its logical aspects. Based on previous research and theory, we designed a method of fostering students' understanding of the principle as a proving tool.

2. THEORETICAL FRAMEWORK

Previous research has highlighted that the reason why students encounter difficulties in learning the PMI is due to certain misconceptions they have about it. For example, Ron and Dreyfus (2004) argue that three aspects of knowledge are required to foster a meaningful understanding of a proof by mathematical induction (MI): (1) understanding the structure of proofs by MI; (2) understanding the induction basis; and (3) understanding the induction step. Based on our experience in teaching the PMI, we believe that the third aspect, the induction step, is the most important in fostering understanding. Ernest (1984) observes that a typical misconception among students is the idea that in MI "you assume what you have to prove and then prove it" (181). Fishbein and Engel (1989) also stress that many students are "inclined to consider the absolute truth value of the inductive hypothesis in the realm of the induction step" (276). Both Ernest (1984) and Fishbein and Engel (1989) argue that the source of this misconception is in students' lack of understanding of the meaning of proofs of implication statements. They suggest that a proper approach to teaching the PMI must include logical implication and its methods of proofs. We (Malara, 2002) agree with Avital and Libeskind (1978) who suggest that a way to overcome students' bewilderment before the "jump" from induction basis to induction step is to approach MI by means of "naïve induction," which consists of showing the passage from k to k+1 for particular values of k "not by simple computation but by finding a structure of transition which is the same for the passage from each value of k to the next" (431).

Another conceptual difficulty experienced by students that is highlighted by research is that many students look at the PMI as something which is neither self-evident nor a generalization of previous experience. Ernest (1984) suggests that a way to overcome this problem is to refer to the well ordering of natural numbers, that is, if a number has a property and "if it is passed along the ordered sequence from any natural number to its successors, then the property will hold for all numbers, since they all occur in the sequence" (183). Harel (2001) also refers to this way of introducing the PMI, calling it "quasi-induction," but he observes that there is a conceptual gap between the PMI and quasi-induction (namely quasi-induction has to do with steps of local inference, while PMI has to do with steps of global inference) which students are not always able to grasp.

In addition, Ron and Dreyfus (2004) highlight the usefulness of using analogies with students when teaching the PMI for two reasons: (1) analogies illustrate the relationship between the method of induction and the ordering of natural numbers and (2) they are tools for fostering understanding of the use of MI in proofs.

3. RESEARCH HYPOTHESIS AND PURPOSES

We propose that the effective teaching of PMI requires a combination of the approaches described above. In particular, we propose that the essential steps in a constructive path toward PMI should include: (1) a thorough analysis of the concept of logical implication; (2) an introduction to PMI through the naïve approach, drawing parallels between PMI and the ordering of natural numbers, and the use of reference metaphors; and (3) the use of examples of fallacious induction to stress the importance of the inductive basis. We hypothesize that a path that includes each of these aspects could lead to real understanding of the meaning of the principle and its more conscientious use in proofs.

In this paper we present this path, describing in detail its central lesson, which is devoted to the proper introduction of the principle of mathematical induction. In particular, we refer to a discussion in class guided by a teacher involved in the PDTR Project.

4. THE CLASSES

The students involved in the experiment belong to two secondary school classes (grade 12) in a school for future primary school teachers (the first class with a linguistic orientation, the second with a social-sciences orientation). Since mathematics does not play an important role in the educational curriculum of this school, these students are usually not very interested in studying mathematics and most of them do not have particularly strong aptitude for the subject. Although arithmetic and elementary number theory are central topics in the mathematical curriculum for these two classes, the PMI is not. Moreover, these students display difficulties in facing non-traditional topics. For these reasons, introducing a topic like PMI to students of this kind, and in this particular context, constitutes a challenge for us and represents a 'litmus test' for the effectiveness of our method.

5. THE PATH AND THE METHODOLOGY OF WORK WITH STUDENTS

The path we propose can be divided into six main phases: (1) an initial diagnostic test; (2) activities which lead students from conditional propositions in ordinary language to logical implications; (3) numerical explorations of situations aimed at producing conjectures to be proved in a subsequent phase; (4) an introduction to the method of proofs by MI and to the statement of the principle; (5) analysis of the statement of PMI and production of proofs; (6) a final test (given 3 weeks after the last lesson). In the following we will briefly describe the six phases, focusing in particular to the "central" lesson of the path (the one which we label 'phase 4').

5.1 Phase 1: An initial diagnostic test

Some aspects of the work we did during the first year[1] of experimentation can be considered essential requirements for the understanding of the method of proof by MI: (1) abilities in correctly interpreting algebraic expressions; (2) abilities in performing transformations in order to complete a proof; (3) abilities in recognizing the difference between formulating a conjecture, verifying it, "proving it" or refuting it. For this reason, we decided to start this second year of the project by analyzing, through an individual test, what students really assimilated from last year activities, in particular their vision about the concept of proof, and what kind of competencies students really had. The test was subdivided in three main parts: (1) activities which require translations from verbal to algebraic language and the interpretation of algebraic expressions; (2) analysis of statements and analysis of given proofs; (3) proofs of given theorems. The test was followed by a "crossed" analysis of students' protocols and a discussion by the mentor and teachers (who were asked to highlight, in particular, the areas in which students' results did not come up to teachers' expectations). Subsequently, teachers discussed the main aspects highlighted by our analysis with their students.

[1] The first year of work with these two classes was devoted to the implementation of a path on proof in elementary number theory.

5.2 Phase 2: From conditional propositions in the ordinary language to logical implication

As observed above, it is fundamental to help students overcome a common logical misconception associated with the use of the principle of mathematical induction as a proving tool ("in MI you assume what you have to prove and then prove it"). In order to achieve this objective, we believe it is essential to foster a real understanding of the logical structure of the principle. This understanding can be achieved by students only if they are aware that proving the implication $P(k) \Rightarrow P(k+1)$ is independent of the truthfulness of the two proposition-components. Since the students of the two classes have never studied logic, we decided to foster this awareness by introducing some activities which would gradually lead them to the construction of the table of logical implication. These activities were guided by the teacher through a class discussion. Because of space constraints, we cannot dwell on the presentation of these activities.

5.3 Phase 3: Numerical explorations

Since students' motivation in proving a statement is greater if the statement is posed by student themselves through numerical explorations, we chose to propose to students the formulation of some simple conjectures. Following Avital and Libeskind's suggestion (1978) to choose examples of naive induction simple enough so that technique does not overshadow the idea of transition from one case to the next, we selected only conjectures involving sums of natural numbers. In Italian schools students are not used to carry out exploratory activities on numerical regularities. Because of students' widespread difficulties in exploring examples and in constructing and managing algebraic expression, we prepared a guided path for them and asked them to work in groups. To foster the involvement of all students, and to avoid that only those students considered clever by their classmates would participate, we decided to create homogeneous groups.

5.4 Phase 4: Introduction to the principle of mathematical induction

The approach we use in this phase of the path is inspired by the suggestion taken from Avital and Libeskind (1978) and Malara (2000). Because of the difficulties our students have in working through numerical explorations, we decided to propose this topic through a discussion led by teachers. According to us, this is a key-lesson in the development of our path. This is why we will dwell on this central lesson in a following section, devoted to the description of its main phases.

5.5 Phase 5: Analysis of the statement of the PMI and production of proofs

This phase can be divided in two essential phases: (1) initial class discussion; and (2) working group activities. The class discussion was aimed at stressing some fundamental aspects about the structure of a proof by MI (see Ron & Dreyfus, 2004). In particular, the teacher presented examples of fallacious induction, stressing the importance of the inductive basis in the proof. He also made students clarify that the proofs of the "particular implications" $P(2) \Rightarrow P(3)$, $P(3) \Rightarrow P(4)$, …, proposed to them during phase 4, are not part of the proof by MI: they only helped students in extrapolating the strategy to be followed in the proof of the implication $P(n) \Rightarrow P(n+1)$.
After the discussion, students worked in small groups to construct proofs of new statements using MI.

5.6 Phase 6: Final Test

Taking the questionnaire proposed by Fishbein and Engel (1989) as a starting point we constructed a test aimed at verifying: 1) if students really had understood the meaning of the inductive step and the importance of the inductive basis as an integral part of the proofs by MI; 2) if students were able to reconstruct proofs already faced during the previous lessons; 3) if students were able to single out the key-passages in performing proofs by MI concerning new conjectures.

6. THE CENTRAL LESSON OF THE PATH: TEACHER'S BEHAVIOR AND STUDENTS' REACTIONS

This section is devoted to the description of the essential moments of the lesson through which one of the teachers involved in the project introduced his class to the use of the PMI as a proving tool. We also propose some excerpts taken from significant moments in the class discussion guided by the teacher. Since our main aim is to verify the effectiveness of our approach to foster comprehension of the use of the principle, we analyze these excerpts from the point of view of the significance of teacher's interventions and students' responses.

In addition to our analysis, we add some reflections offered by the teacher after having read the transcript of the lesson and having analyzed the role he played in it, highlighting in particular what impressed him most. To foster the teacher's reflections, we suggested that he consider the following questions: 1) What were the "best" parts of the lesson do you think? 2) What were the worst parts? 3) What kind of difficulties did you face, and what kind of choices did you make (or should you have made) to overcome them? 4) How interactive were you with your students?

The lesson could be divided in two main parts. The *first part* is devoted to encouraging the students to introduce new specific symbols. The statement we chose to observe is the one about the sum of the first n even numbers: $2+4+\ldots+2n=n(n+1)$. We chose it because of the simple outline required by its proof. Initially, the teacher leaves students to talk about the difficulties they could face in trying to prove this statement and makes them observe that the tools they posses make it difficult for them to approach the proof of the statement directly. In this way students feel the need for a new method of proof.

Subsequently the teacher leads students to: (1) look at the statement in general terms, as a function of the number n, and understand that it is possible to represent it through the expression $P(n)$ (in order to favor this development the teacher asks the students to make the meaning of particular cases explicit, with questions such as 'what does P(2) mean? And P(3)?...); (2) reflect on the fact that the sum to the left of the equality depends on n and could be represented with the symbol S_n; (3) recognize that every sum S_n is obtained from the previous sum by adding the n^{th} term of the sequence of the even numbers; and (4) observe that the proof of the statement represented by the proposition $P(n)$ is required to prove that $P(n)$ is valid for every value of n.

The teacher considered the worst part of the lesson to be to be when he introduced new symbols because it involved him being "intrusive." The teacher observed that it was difficult for him to "stay in the background" and "resist the temptation" to methodically comment upon students' assertions in an attempt to direct their learning.

The *second part* of the lesson is devoted to a gradual "construction" of the statement of PMI. The teacher refers to observations made during the initial discussion and constructs a table which makes explicit some common aspects of the sequences involved in the formulation of the proposition:

n	S_n	$P(n)$
1		
$\downarrow +1$		
2	$S_2=6=2+2\cdot 2$	P(2): $S_2=2(2+1)$
$\downarrow +1$	$\downarrow +2\cdot 3$	
3	$S_3=2+4+2\cdot 3=S_2+6$	P(3): $S_3=3(3+1)$
$\downarrow +1$	$\downarrow +2\cdot 4$	
4	$S_4=S_3+2\cdot 4$	P(4): $S_4=4(4+1)$
…	…	…

The teacher helps the students observe that the structure of natural numbers is such that every number n could be obtained from the previous number *(n-1)* by adding 1. Also the sums S_n are related to each other by an analogous "recursion:" every sum is obtained from the previous sum by adding the n^{th} even number $2n$. Therefore, the terms of the succession of natural numbers and the terms of the successions of the sum S_n have the common property of depending on the terms which precede them. At this point it should be easy for students to see that there could be a connection between the propositions P(2), P(3), … and that every proposition can be derive recursively from its prior. Since our purpose is to prove these propositions, the new idea is to refer to P(2) to prove P(3), to refer to P(3) to prove P(4), etc.

Starting from this intuition, the teacher discusses the possibility of proving P(3) by starting from P(2), and constructs this proof together with the students.
Here we detail an outline of the proof of the implication P(2)□P(3) that the teacher showed to his students:

Can I deduce P(3) from P(2)?
hp) P(2): $2+4=2\cdot 3$ th) P(3): $2+4+6=3\cdot 4$
$2+4+6=2\cdot(2+1)+2\cdot 3 =2\cdot 3+2\cdot 3= 3\cdot(2+2)= 3\cdot(3+1)$
So, if P(2) is true, P(3) is also true!
[In particular the teacher highlights the importance of referring to the hypothesis in the passage which leads from *2+4+6* to *2·(2+1)+2·3*))]

Subsequently, the teacher asks students to try to construct the proofs of the implications $P(3)\rightarrow P(4)$ and $P(4)\rightarrow P(5)$, and to identify the common strategy which underlies them. At this point, the teacher encourages the class to explain the sense of the activities they are involved in. The following excerpt is taken from this particular moment in the discussion.

T: Let us observe something: we insisted in proving these implications. But I also could make a more trivial choice: Is P(2) true? I simply make some calculations and I can verify it. Is P(3) true? I make some calculations. I can verify and numerically check if everything is balanced. Why do I have to insist in this direction? What do you think?
G: To generalize it?
T: Yes, but, more that a generalization, I need another …
G: To find a mechanism, which is always valid.
R: Without doing all the examples …till 50 …
T: It looks like a good idea to create a mechanism which, once it is engaged, will never break off. What was the limit of numerical verifications? P(3), P(4), P(5) can be verified easily… but, after a while ...
R: If numbers are greater than 10 …
T: I should always check. There is no moment in which I can say "it is the same more or less." [Students nod.]
T: This idea, on the other hand, generates this recursive mechanism, so we can say that … if I wanted to prove the truthfulness of P(18), It would be enough to have proven …
S: P(17)

T: … the truthfulness of P(17), because I am sure that the implication "works." Therefore, if P(17) is true and the implication is true, P(18) must also be true. This observation refers to a competence related to previous topics. Do you remember the truth table of logical implication?
C: …but … can I say that it is sufficient to prove P(2) if I have to prove P(18)?
T: go on with your reasoning…
C: … because if P(2) is true, then P(3) is true. If P(3) is true, then P(4) is true … and we can 'reach' P(18).
T: Is this observation convincing? [he is talking to the other students]
[they do not answer]
T: You [he addresses C] speeded up with respect to what I was going to say, but it is perfect. If you [he addresses the other students] thought something similar and you agree with that, it is ok, but I want to repeat what I was going to make you observe. If I need to verify P(17) in order to prove P(18), because I know the strategy to prove the implication, then I can shift the problem to P(17). But this problem is not going to be shifted endlessly: at a certain point there will be a first value, on which this observation can be done.

We chose to insert this transcript because we think it provides evidence that our method can foster the development of a deeper level of awareness about the meaning of the process related to the concept of mathematical induction. The whole class, in fact, display an understanding that numerical verifications are not proofs of the statement. Moreover, students display an understanding that the proofs of particular implications are propaedeutic to the "generalization of the proofs themselves." Besides, C's intervention even amazes the teacher, because the student anticipates further observations about the connection between the verification of the inductive basis, the proof of the general implication and the "transfer of the truthfulness of the propositions related to every natural number" along the whole sequence of natural numbers. In his analysis of this excerpt, the teacher observes that interaction is a complicated practice because it always brings with it a risk of running into unforeseen events, but he recognizes that, although unexpected students' responses and intuitions can often wrongfoot him in the classroom, they bring added knowledge and can help to improve the teacher-student relationship. This passage offers evidence of this fact.

During the following phase the teacher is referred to the meaningful metaphor "the falling of the domino" which describes this process. In this phase the students also display having assimilated the essence of the previous observations. In fact, they are able to 'catch' the connection between the falling of the dominoes and the statement's proving process.

T: C was saying: "if I prove that P(2) is true, and that the implication is true…" ….we still have to consider this second aspect …! However, if I prove that the starting point is true and the strategy I use (to prove the particular implications) 'works,' what happens is something that can be represented with these dominoes [he indicates the dominoes that he has previously put on his desk].
T: What kind of meaning could these dominos have?
G: …they are all connected.
C: If one of them stands, then all the other dominos stand … if one of them falls, then all the other dominos fall.
T: If one of them falls, is it certain that all the other dominos will fall?
S1: No! It depends on how you push it.
S2: It depends … if they are in a line … if they are connected...
S3: for example … if one is over there … nothing falls [she refers to the distance between the dominos]
G: if the first falls, the others will also fall.
T: You can observe something. When does the falling of the first domino not generate the falling of the following dominos?
C: When it does not touch the second domino.

Having read the above transcript, the teacher, who initially had not believed that this metaphor would be effective in promoting better comprehension of the meaning of the principle, considers the use of the metaphor of the "falling of the domino" as one of the best parts of the lesson. It was the first time he used a metaphorical model as a teaching instrument and he discovered that metaphors are effective tools, that they can capture students' interest, and help teachers convey the meaning of key-concepts in the lesson.

At this point, most students have a clear vision of the proving schema applied in the proofs of the "particular implications" and of the proving process. The teacher is now able to guide them to observe that this schema can be followed every time it is necessary to prove a proposition $P(n+1)$ starting from the previous proposition $P(n)$.

During this discussion, students learn that the complete proof of the statement is based on an infinite set of implications such as the ones proved. In this chain of implications each overlaps the following because every $P(n)$ is first the thesis of an implication and then the hypothesis of the one that follows. A crucial passage in the discussion is the one in which the teacher leads students to observe that this chain can be 'summarized' in "$P(n) \rightarrow P(n+1) \ \ \forall n$" and in which he then constructs with them the prove of this general implication, as a generalization of the previously constructed proofs.

It is fundamental that students be reminded that an implication could also be valid where the two components themselves are not valid, in order to make them realize that proving $P(n) \rightarrow P(n+1)$ does not guarantee the validity of $P(x)$ for every x. In this way students are gradually led to the realization that proving "$P(n) \rightarrow P(n+1) \ \ \forall n$" means proving that "$P(n)$ is valid $\forall n$," ONLY IF the first proposition of the chain, P(2), is valid.

7. CONCLUSIONS

One of the main purposes of this work was to test the effects of our approach on students who had never studied the PMI and to highlight the role played by the teacher in guiding students during these activities. Our observations of class activities and our analysis of student protocols and behaviors during class discussions allow us to offer some conclusions on these aspects.

The detailed description of the central lesson in our approach highlights how meticulous this path was planned. The activities described in this paper are, in fact, the result of many meetings, aimed at making teachers aware of the problems related to the teaching and learning of the PMI.

In particular, we think it is fundamental to stress the role played by the teacher in the management of the whole path.

Students display having "grasped" the essential aspects of the proposed activities and respond to the teacher's questions in an appropriate way, sometimes surprising him with the perspicacity of their considerations. Their contributions suggest that this method could represent an effective approach to the introduction of the use of PMI as a proving tool.

Moreover, these observations testify the validity of our research hypothesis regarding which aspects should be considered fundamental for an effective introduction to the use of PMI as a "proving tool:" (1) an in-depth introduction to the concept of logical implication; (2) a naïve approach to the introduction of PMI, with an emphasis on the parallels between the PMI and the ordering of natural numbers; (3) the use of the metaphor of the "falling of the domino" as a tool to better clarify the meaning of the principle; and (4) the presentation of examples of fallacious induction to stress the importance of the inductive basis.

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PART 4
CASE STUDIES OF TEACHING-RESEARCH

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TEACHING ISOMETRIES IN GRADE 7 (DEVELOPMENTAL TEACHING EXPERIMENT)

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ABSTRACT
A teaching experiment is described where the teaching of isometries was designed with the help of Van Hiele, Brunner and Tall theories of learning. The experiment was carried out in the mathematically oriented class of the school. The central aim of the teaching-research was to create a strong concept image of the isometries concept and to facilitate the transition to the concept definition. Escher tilings were used as the medium to study the properties of isometries. Students have formulated their "own" definitions, revised them and as a result they were able to master them significantly better. Moreover, their learning was much more enjoyable.

INTRODUCTION

Teaching isometries is one of the most difficult but most beautiful fields of mathematics teaching. In the case of most students, topics studied at mathematics lessons are not linked to their real-life experiences. Moreover, their knowledge is not even linked to other parts of geometry. They have difficulties memorizing definitions because they are too complex. As constructions are complicated and because of lack of time, they see few examples how isometries "work." So only a few of them the best ones get to the point where they are able to identify congruent or similar shapes and transformations between them.

In Hungary students encounter isometries in the form of axial reflection in sixth grade (aged 12). They study "central reflection" (rotation by 180°) in grade 7 and 8, then rotation and translation in grade 9. Students in special mathematics classes study all isometries in grade 7 (in their first year of secondary school). Later students only meet them in problems, where they (would) need to apply their knowledge. One of the aims of teaching isometries is using this knowledge for solving problems (construction problems, complex geometrical problems including proof problems, problems for maximum and minimum values). The first trouble is the density of material, as students have to process a big quantity of information without revision and settling enhanced by a circular syllabus. Secondly, many students lack the routine of constructions, so they construct transformations quite imprecisely. Thirdly, the preciseness of wording definitions is also new to most of them.

To help overcome these difficulties we designed a developing experiment comprising 25 lessons. At first, students got familiar with isometries on Escher's tilings. Before making constructions, we worked a lot with tracing-paper. And in the spirit of mathematics teaching based on exploration, students constructed definitions of isometries and "discovered" their properties on their own.

With the teaching of this topic we can develop following PISA competencies: mathematical thinking, problem handling, reasoning, representations, communication, aids and tool.

THE MATHEMATICAL BACKGROUND

There are two different approaches in Hungarian mathematics teaching relating to teaching of congruent transformations.

The first approach is based on Hilbert axioms. A transformation can be defined by a rule which attaches unique point to corresponding point of a plane (Point by point or analytic definition). Example: Axial reflection: There is given a line *t* on the axis. To get the corresponding image of a point P, we shall draw a line through it which is perpendicular to the axis. This perpendicular line intersects the axis in a point T. The image of P is P' is lying on the perpendicular line, and has equal distance from T as P has on the other side of it. PT =P'T. If P is lying on the axis than P'=P

This kind of definition directly provides an instruction, how to construct the image of any point by ruler and compass. To construct the image of a shape with infinite number of points, we need to know the properties of the transformation, e.g.: the image of a straight line is a straight line; the image of a circle is a circle with same radius. So we need to introduce the main properties of the transformation, but unfortunately in this approach the isometry properties of a transformation are not direct consequences of the definition. While teaching teachers are moving a shape on a prescribed route when illustrating the transformation and read the properties from this concrete example.

The second approach is based on the motion axioms (HAJÓS). The basic notion here is the *motion.* To define the motion for children we give two "flags" which have the same shape and the same size but one is white and the other is black. We need tracing paper too. To get the image of a certain shape we just have to trace the shape together with the white flag, then lift the paper and move the white flag on the black one. If you trace after that the original shape again, then you can see on the tracing paper the shape together with the transformed image of it. In this approach the route of the motion is not defined, only the beginning and the end position of the moving plane are marked. Example: Axial reflection: There is given a straight line *t* in a plane – the axis. We rotate one of the half planes around the axis *t* by 180 degrees. So this half plane goes into the other half plane. The points of the axis remain as fixed points.

Advantages of this approach: This action "tells" us how to construct the image of any shape with the help of tracing paper. It makes immediately clear that the transformation conserves the distance, straight lines, the measure of angles and the "form and measurements of the shape."

After such introduction we give the point by point definition on the basis of the special properties, because we need a direct prescription how to construct the image of a shape. Main characteristic of geometrical transformations include: (i) existence of inverse transformation; (ii) fix elements; (iii) invariant figures; (iv) preserving the distance; (v) preserving the angle; (vi) preserving the straight lines; (vii) preserving the circles; (viii) preserving parallel straight lines; and (ix) preserving circulation.

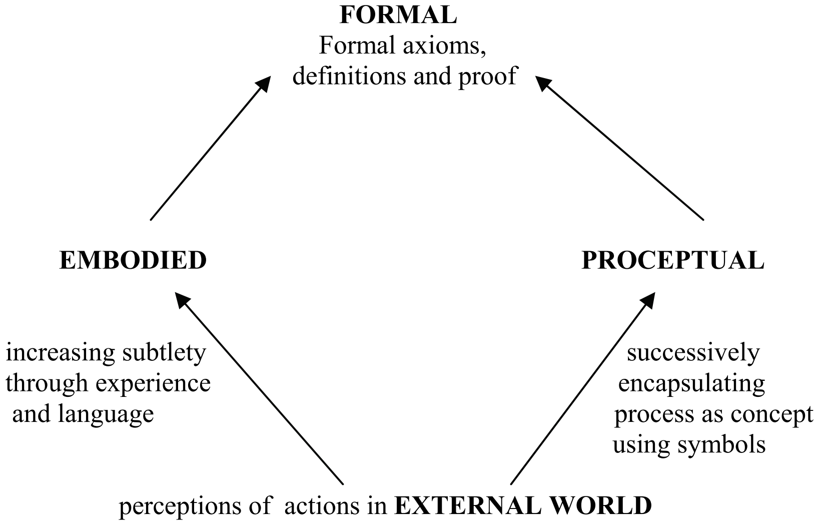
THEORETICAL CONSIDERATIONS

I am a product of Hungarian mathematics education. In Hungary the science mathematics plays a dominant role in teaching, the psychology and pedagogy aspects of the students are neglected. This is the reason why I give a detailed theoretical basis.

These are the most important factors which convinced me about the necessity of the change of my teaching style.

At the end of this part of my article I cite some world famous Hungarian mathematics didacticians, because these quotations helped me a lot and we Hungarian people evaluate very high our famous scientists.

Mathematical growth from child to mathematician



> *The embodied object-based theory* is based on perceptions of objects, analyzing their properties, describing them with language and seeking relationships. This can occur in a variety of contexts, from studying geometrical figures, or looking at visual representations of data in the form of a graph or a diagram. The increasing subtlety of experience and use of language enables us to describe perceptions more precisely, and to move from *description* of our perceptions to *definitions* that prescribe our ideas in a more precise sense. Perhaps the most sophisticated development of an embodied object-based theory occurs in geometry, where increasing sophistication leads naturally to the development of Euclidean proof. However, *throughout our mathematical growth, the underlying embodiment of ideas remains a powerful source of human being*. (Tall, 2004)

Tall emphasizes the different aspects of *mathematical thinking*: (1) *Perception* focuses on objects and their properties; (2) *Action* can be built into action-schemas that are streamlined to carry out routine processes. Then these actions can by symbolized and conceived as mental cognitive units that can become the focus of attention of our mathematical thought; (3) *Reflection* on perceptions and actions not only lead to more sophisticated kinds of mathematical objects and symbols, it can also lead to formal mathematical thought based on axioms, definitions and formal proof (Tall, 2004).

The ideas of Tall can be enhanced by the recent discoveries about the nature of mind (Lakoff & Nunez, 2000): (a) *The embodiment of mind.* The detailed nature of our bodies, our brains, and our everyday functioning in the world structures human concepts and human reason. This includes mathematical concepts and mathematical reason; (b) *The cognitive unconscious.* Most thought is unconscious – not repressed in the Freudian sense but simply inaccessible to direct conscious introspection. We can not look directly at our conceptual systems and at our low-level thought process. This includes most mathematical thought; (c) *Metaphorical thought.* For the most part, human beings conceptualize abstract concepts in concrete terms, using ideas and modes of reasoning grounded in the sensory-motor system. The mechanism by which the abstract is comprehended in terms of the concrete is called *conceptual metaphor.* Mathematical thought also makes use of conceptual metaphor, as when we conceptualize numbers as point on a line.

The Van Hiele levels of geometric thoughts

(1) *Holistic level.* Children recognize shapes globally as a whole and not as their characteristics or parts. They learn the names of geometrical figures, can identify them, they can copy the figures. On this level the outside presence is decisive.

(2) *Analytic level.* Students determine the characteristics of figures with help of observation, measurement, paper folding, tessellations also with help of concrete activities with physical objects. With help of the characteristic they start to build classes of figures. At this level they see the figures as the holder of their characteristic. Students do not understand yet the relation between the characteristic of a figure and the relation between different figures. Also they do not see yet the necessity of the definitions.

(3) *Informal deduction.* Students understand the relation between the characteristic of a figure and the relations between different figures (definitions, hierarchies of concepts). They can follow informal (concrete, using the content) arguments, they can construct such arguments by themselves. They can not understand the importance of deduction, the role of axioms. They use very often informal, empirical results with deductive technique.

(4) *Deductive level (formal deduction).* At this level, the mathematical structure of geometry completely emerges for students. Proof is viewed as the final authority in deciding the truth of a conjecture. The roles of the components in a mathematical discourse are understood (undefined terms, axioms, a system of logic, theorems and proof). Students thus are able to reason mathematically within a particular mathematical system, although perhaps not realizing that different axioms would produce a different system, and hence different theorem.

(5) *Rigorous level.* At this level students appreciate the investigation of various systems of axioms and logical systems and also the ability to reason in the most rigorous system way within various systems. This is a university level.

Some characteristics of van Hiele model

The levels build a sequence, they follow each other in the order we presented above. The movement from one level to the next higher one depends on the content and method of teaching rather than on biological reification. Each level has its own language and symbolic structure. If the language of teachers differs from the language of students, misunderstandings will happen.

In our developmental research we used the second and third level since in grade 7 it is only realistic approach.

Representations

Bruner introduced the notion of representation in the cognitive psychology:

> Any domain of knowledge (or any problem within that domain of knowledge) can be represented in three ways: by a set of *actions* appropriate for achieving a certain result (*enactive representation)*; by a set of *summary image or graphics* that stand for a concept without defining it fully (*iconic representation)*; and by a set of symbolic or logical propositions drawn from a symbolic system that is governed by rules or laws for forming and transforming propositions (*symbolic representation)* (Bruner, 1966)

New view of concrete and visual representations. What counts primarily are *mental images* of concepts. Visual representations can support these to some extent. Mental images are not just copies of external representations, but they are formed by the constructive activity of the individual. These constructions are idiosyncratic, that is, they are determined by the experiences and personal knowledge of the individual.

Concrete and visual representations are no "speaking pictures," they do not fulfill the expected function as carriers of mental images per se. Concrete and visual representations are neither only aids for the so-called "slow learners" nor is their use restricted to the early steps of the learning process. They are important for all children and they are useful for the whole duration of the learning process. Concrete and visual representations are not automatically the better, the more specifically they represent the intended concept. "Perfect" representations can be counterproductive. In order to fulfill their function, good representations must involve a certain vagueness (Wittmann, 1998).

In our experiment the use of concrete, visual representations was in the center.

Concept image

Concept image is something non-verbal associated in our mind with the concept name. It can be a visual representation of the concept in the case concept has visual representation; it also can be a collection of impressions or experiences.

The visual representations (pictures, tables, and graphs), concrete examples, mental pictures, impressions and experiences associated with the concept name are referred to as concept image, which can be translated into verbal forms. But it is important to remember that these verbal forms were not the first thing evoked in our memory. These came into being only at a later stage.

The main aim in our experiment was to build a rich concept image for congruent transformations with help of manipulations with concrete objects (ruler and compass too).

Famous mathematics didacticians about the use of concrete, visual representations

Dienes:

> Children think rather constructively than analytically. Children see first the whole, they construct first and only later analyze. I have found a contradiction: the whole mathematics teaching is based on analysis. The basic idea of the "new math" is that the mathematics is a language, and if children learn the structure of this language, than they understand the meaning also and can apply it also. It similar to the situation when a coach would draw a horse, because first comes the abstraction and then the concrete. In the reality it is the opposite; we are going from concrete to abstract.
> My system is based on the idea that the children need to know, to discover the complicated mathematical concepts, structures first on the base of concrete experiences, in the frame of real games, during perceptual actions. This means, that we need not only to learn the mathematics in a narrow sense, but it means a spiritual attitude also, we need to learn on the best mode*: to learn based on our own experiences*.
> I can not learn to ride on a bicycle knowing only its structure, but if I can ride, then I can learn easier its structure too. (Győrí, 1973)

Varga:

> We may abstract only from concrete, to abstract effectively we need to know a lot of concrete things. Mathematics is very abstract, it is its power, in this sense it condenses the common essence of many concrete phenomena. To this very abstract we can lead the children most effectively with help of a very concrete start, so we supply them with a sufficient number of challenging concrete experiences. For young children it means perceptual-motional experiences. (Klein, 1980)

Pólya:

> It is not allowed to miss anything, if it has a chance to bring mathematics closer to children. Mathematics is a very abstract science, just for this reason we need to teach it in a very concrete way. (Pólya, 1977)

To summarize our view we share the opinion of Eisenberg: "Whenever and wherever possible, both visual and analytical modes of representation must be used in the mathematics classroom." (Eisenberg, 1994)

TEACHING EXPERIMENT

Research questions

(1) Does the use of Escher tilings make the identification of congruent shapes and the isometries between them easier for them later on? (2) Does the use of tilings and tracing paper help students carry out constructions accurately? (3) To what extent does the use of tracing paper enhance students to view the movement of the whole plane in course of isometries? (4) Is it easier for students to recall definitions constructed on their own? Does this process develop their standards of speaking and writing precisely about Mathematics?

Research methodology

The experiment was carried out in class 7a at Fazekas Mihály Secondary School in Debrecen, Hungary. The class is specialized in mathematics: students study mathematics in separated groups and have 6 lessons a week. The first cycle was implemented in spring 2006, with the better 16 students of two groups. The second cycle was carried out in spring 2007 when both groups studied isometries with the method in question (17 students in the better, and 14 in the other group).

The participating students are interested in mathematics, their mathematical knowledge and problem solving competency is better than the average for their age. They like to work individually and they like to compete with each other. But they like to work in smaller groups too (2-3 students in one group). For this reason these two forms of work were characteristic in our experiment.

The learning trajectory

Lesson 1:	Pre test
Lessons 2-3:	Motions on Escher's tiling
Lesson 4:	Motions with help of "pfiles"
Lessons 5-10:	Reflection on the straight line Axial reflection on Escher tiling with help of tracing paper Formulation of the definition Simple construction problems Characteristics of axial reflection Complex construction problems Axial symmetric figures
Lessons 11-15:	Reflection on a point (the structure of these lessons was similar to the axial reflection
Lessons 16-20:	Rotation around a point
Lessons 21-24:	Translation
Lessons 25-26:	Congruent transformations; congruence of figures; basic rule of congruence of triangles

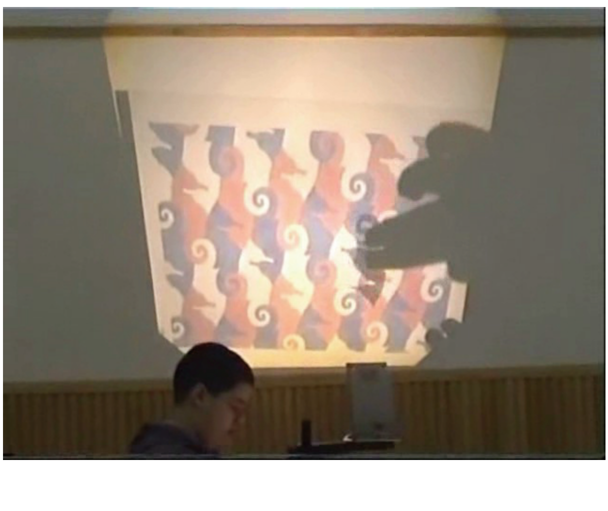
The experiment started with a pre-test, in order to gather information about students' experiences with isometries (Appendix 1). The pre-test consisted of three types of problems. In the first we intended to find out about students' experiences with isometries. Students had no difficulties with reflections. Most students demonstrated competency in problems with reflections due to primary school knowledge, others were very creative in solving them (for example folded their paper so that the two shapes would cover each other, and copy the required points). They had more difficulties with translations and rotations. In problem 6, for example, 2-3 students from each group drew a coat-rack on the wrong side of the sofa, and some asked in which direction the stage was turned to come to this position. Moving on Escher's tiles was the second type of problem. This posed great difficulties for students: they had problems with wording their thoughts. The third group of problems dealt with symmetries. We found that it was very difficult for students to explain the word symmetry, and for most of them it only meant line symmetry.

Let me present some of the answers for problem 8:

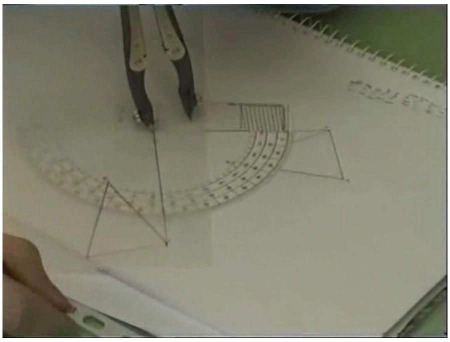
We use the word symmetry when we can divide a shape into two identical parts by a line;
When 2 sides of something are the same or are not deformed;
When a solid or a shape is mirrored, so if we put a mirror on the line of symmetry (in an appropriate angle with the figure) we could see the same picture as it covers;
If we mirror it, its image is the same as the original.

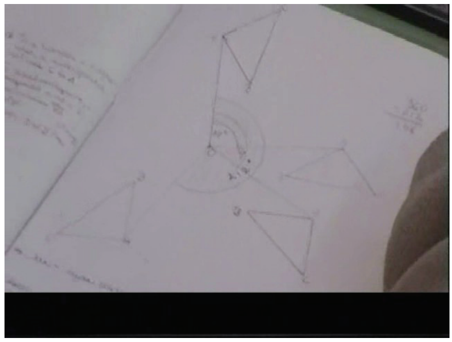
In part b) everyone could show examples from nature, my favorite answer is: "Nothing is perfectly symmetrical, that is why nature is beautiful.

In the lesson after the pre-test we moved animals on Escher's tilings. Each student received a sheet with a tiling and cut out animals. First, they had to follow simple instructions (let us rotate the highlighted lizard by 120º around its left leg), then multi-step instructions. In the end they had to find isometries where objects and their images were given. Our first aim with these exercises was to develop a common language concerning transformations. Students started to see more and more clearly what data they needed to give for each movement so that the operation would be unambiguous. Secondly, problems of the third type prepare students to see dynamic features in static figures, and to notice if a shape can be moved to another. Then students got to know isometries in detail. We started each isometry by moving on Escher tiling.

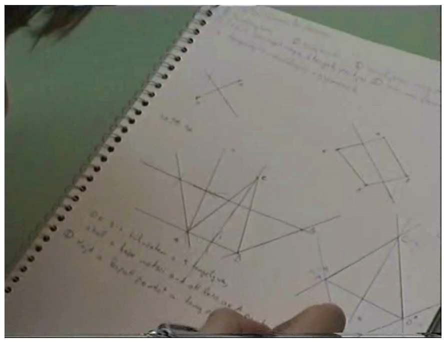


The next step was “construction” with the help of tracing paper. First, I performed isometries on an overhead projector following students’ instructions; then they solved problems individually in their exercise books. The first aim of these exercises was to overcome students’ lack of construction skills. The second aim was to show them many examples of how isometries work in order to develop the idea that isometries move all, not only highlighted, points of the plane. This was followed by constructing the isometry.





This was followed by constructing the isometry.



Having gathered experience, students tried to word the definition of the isometry on their own. Some of them read out loud what they wrote, others corrected and completed what they heard. Then they all revised their own definitions, followed by another class discussion. It is very important for students to feel that the definition they gave is not incorrect, but it is a crucial step towards the final, precise form. This feeling gives them security and a growing routine in giving definitions. We found that such definitions are easier to recall. Students forget exceptions less frequently because they know why these exceptions are part of the definition. At the same time students learn to be more precise about mathematics, develop their skills to talk accurately about mathematics in all topics, and their aesthetics of mathematics develops as well (“This is a good definition, it includes everything, but couldn’t we put it in a nicer, simpler way?”)

Giving the definition was followed by more and more complex constructions. Then we finished dealing with the isometry by examining the symmetry related to the isometry. As an example, let me present a discussion about the definition of rotation.

Alida: Rotation is when we rotate every point of the figure by a given degree around a centre.
Ádám: This is the same. There is rotation in the definition of rotation.
Miki: You explain rotation by rotation.

After students convinced Alida, we discussed that we should explain precisely how a point is rotated. Definitions were constructed. Let us see some literal translations of these definitions.

There is a centre and a point, I connect the two points – this is one arm of the angle. The other arm starts from the centre, and the size of the angle is by which we want to rotate the point. I indicate the distance of the point from the centre on the other arm, too. If the point is the centre, then it will be.

A point P is given, we rotate points around it. We rotate a point around P by the angle α as follows: we connect the two points, construct the angle α on the line, and on the new arm we copy the distance of P and the point. If the point is P, than its image is itself.

After another discussion we obtained the final definition:

A point O is given, which is the centre of rotation and an angle with a sign is also given. The image of O is itself. We get the image of a point other than O by connecting O with the point, and at O on this ray we measure the angle and on the new arm we measure the distance of the point and O.

We finished the experiment by a post-test (Appendix 2**)**, and we are planning to arrange a follow-up test later. Students solved problem 1 easily, only the rotation between pair number 5 posed difficulty to 2-3 students. Problem 2 measured the acquisition of theory, which has not posed serious difficulties either (3-4 students omitted one property of central reflection). Problem 3 is a basic construction, which showed that we had managed to overcome the lack of routine in construction. It was optional to find out the area of the intersection of the two triangles. Most students conjectured the result, and two of them were able to prove it by breaking up the shape into congruent triangles. Problem 4 was the most difficult and around 75% of both groups solved it correctly. In problem 5 all students knew the three types of symmetry (line, central and rotational symmetry). Several students had problems with empty fields of the diagram (5-6 students thought that they could draw a shape with central but no rotational symmetry). And only a few managed to draw a shape which has rotational but no axial or central symmetry.

According to my observations, this type of teaching of congruent transformations was more enjoyable for the students than the earlier one. Their “own definitions” were fixed better in their memory; they can remember these definitions some years later too thinking about the Escher tilings. The use of tracing paper was a real help for the students, but it caused more difficulties than I had imagined before. Better students wanted to leave this phase as soon as possible; they wanted to use the tools for constructions. In the second cycle of the experiment I gave the students more freedom: it was for them to decide which solution method they prefer to choose. Interesting was for me to observe that the students who were uncertain of the construction with tools, they used the tracing paper further on, sometimes for the control of their construction with tools. Officially, we have about 10-13 lessons for teaching congruent transformations but I see soon that due to my plus 10 lessons the achievement of my students developed better than during traditional teaching of the subject.

The most important experience was for me to see how students enjoy their mathematics learning when they have a chance to create mathematics as their own world. I changed my teaching style since the experiment. I try to teach other topics in this style too. For the first time in my teaching career it was clear to me during the experiment that students can remember much better an activity schema than a pure formal definition. Earlier I preferred the latter in my teaching. Of course, the definitions are important too, but they can work better with help of action schemas. I try to build in students' heads a rich concept image (action sequences, a lot of drawings, figures, and definition formulations by students themselves), so the concept will be more livable for them and they will consider the concept, topic and, hopefully, mathematics as their own.

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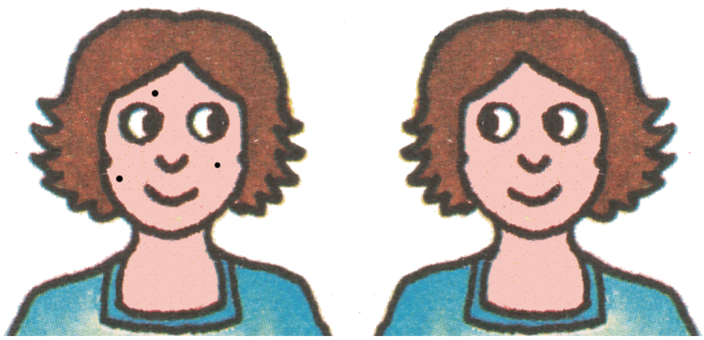
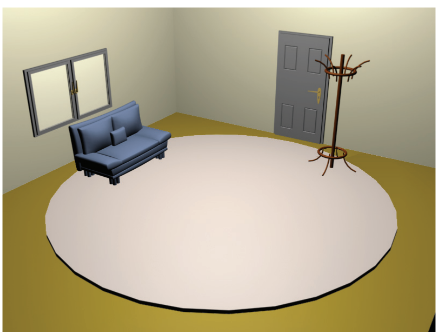
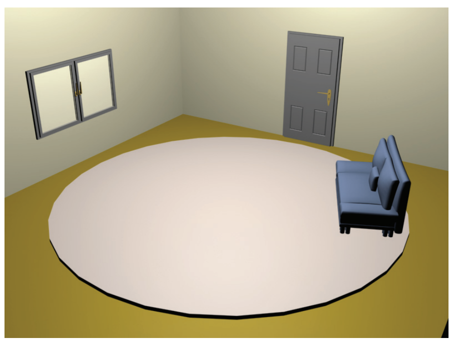
APPENDIX 1 (PRE-TEST)

1. Pete's cousins, Rachel and Rebecca are identical twins, and everyone says they're as like as two peas in a pod. Pete drew them yesterday afternoon, and noticed that this is not exactly true; they are rather each other's mirror image. Unfortunately he hasn't had time to finish the picture, so birthmarks are missing from Rebecca's face. Help him by drawing Rebecca's three birthmarks.



6. A coat rack has to be put on a revolving stage for a theatre performance as shown on the first picture. However, during rehearsals someone moved it, and meanwhile the stage was turned. Mark where we need to put the coat rack so that it goes to its original place when the stage is turned back.





8. a) In what situations do we use the word symmetry? What do we mean by it?
   b) Give some things from nature that are symmetrical.

APPENDIX 2 (POST-TEST)

1. What isometries can we use to move corresponding fish to each other? (Mark the mirror line, the centre of rotation or the vector of translation if you can.)

2. a) Give a definition of reflection.
   b) Give the properties of central reflection.

3. Construct an equilateral triangle and the centre of its excircle. Translate the triangle so that its vertex A is at the centre of the excircle. What shape is the intersection of the two triangles? (What part is the area of the intersection of the area of the original triangle?)

4. Take a point and two parallel lines. Construct an isosceles right triangle so that the given point is the vertex with the right angle, and the other two vertices are on the two lines.

5. What kind of symmetries do you know? Make a Venn-diagram and draw an appropriate shape in all of the fields.

TEACHING PROOF TO NINTH-GRADERS

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ABSTRACT
I present the proceedings and the results of a teaching experiment focusing on geometrical proof, which made use of visual aids such as tangrams. The hypotheses of the research were that by using concrete and visual representations students would obtain a deeper understanding of the area of polygons, students' visual thinking would develop, too. Finally, students would be able to combine all their formerly acquired knowledge and prove the Altitude Theorem and the Leg Theorem related to right triangles. Most of the time students worked in groups or in pairs, which was meant to help them become more successful problem-solvers. Thus, less able, less motivated students would experience success and become more motivated in studying mathematics.

INTRODUCTION

Looking back on the tradition of teaching mathematics in Hungary a serious contradiction can be found between the view of our great mathematicians and the current tendencies in teaching mathematics. For instance, Farkas Bolyai, the father of János Bolyai, in the 18th century always approached mathematical problems with illustration, starting out from the concrete then turning to more abstract aspects of the same problem (Dávid, 1979). Another excellent example could be Tamás Varga who firmly believed that

> We may abstract only from concrete, to abstract effectively we need to know a lot of concrete things. Mathematics is very abstract, it is its power, in this sense it condenses the common essence of many concrete phenomena. To this very abstract we can lead the children most effectively with the help of a very concrete start, so we supply them with a sufficient number of challenging concrete experiences. For young children it means perceptual-motional experiences (quoted in Klein 1980).

However, having a look at the Hungarian mathematics teaching tradition and the new National Curriculum (NC), we can observe that using tools is required in lower elementary school mathematics (Grade 1-4) as they are seen as "a crutch, the necessary evil. On the basis of this thinking the 'real' teaching of mathematics does not require the use of any tools, it merely builds on thinking." Though if "thinking is helped by the use of hands and an activity, it does not mean thinking on a low level but it means the accommodation to the possibility of abstraction related to a particular topic." These teachers allow students to use tools in the lesson to make abstraction happen gradually and finally reach a state when concrete tools become mental images (Szendrei, 2005). In brief, the more visual aids students use, the more likely is their mathematical comprehension.

The previous discrepancy can be detected in one particular area of mathematics, in our case in the geometrical proof. On the one hand, the role of proofs has changed enormously. It is a highly controversial issue in teaching and in learning as well. As far as the teaching of proofs is concerned, a new approach to teaching theorems and their proofs came to existence when the new Matura Exam was introduced in 2005. The new state of affairs is that teachers are obliged to teach different proving methods to mathematically-committed advanced-level students as the new final exam system makes a crucial differentiation between the requirements for intermediate and for advanced level students. As a result only those taking the advanced level GCSE in mathematics have to know proofs. Students are expected to memorize and be capable of using indirect and direct proof techniques, be aware of the logical sieve and that of full induction. Candidates are supposed to know certain theorems and their converses, e.g. Pythagorean Theorem, Leg Theorem, Theorem of Altitude, etc. However, most often they have to deduce various formulae, such as the cosine rule or the sine rule, etc. Presentation of their knowledge related to theory and exercises is another key issue as verbal and written communication is an important criterium in this exam. Whereas in the previous final exam system graduating Hungarian secondary school students were supposed to be able to reproduce the proof of only a few theorems from a given list included in the curriculum. Students had to find regularities, draw deductive conclusions, formulate conjectures, carry out one-step or few-step proofs, and differentiate between definitions and theorems (NC, 1995).

On the other hand, the Hungarian tradition is forced to change as the needs of the world have changed, too. We are heading towards a new society in which people are supposed to have various competencies; they have to be able to apply their knowledge, capabilities in any kind of situation. What is needed is not bookish, fixed mastery but rather flexible and applicable knowledge.

Bearing in mind all these conditions and requirements I toyed with the idea of trying out a new way of teaching of three related theorems in geometry in spring 2007. I decided to take the advantage of teaching mathematical terminology in a class of ninth- grade bilingual students. The reason why I chose this class and not another one was that I could insert an experiment into my plan of teaching only related to this class. On the one hand, the national curriculum does not leave much room for such intentions. On the other hand, these students devoted that year to intensive language-learning and they were supposed to learn the technical terms of mathematics in English, which offered a chance for a smaller experiment. Last but not least, the experiment itself was challenging enough for both parties as the students did not have much experience in solving unusual geometry problems or proving their findings. Moreover, I, the teacher, did not have much practice in teaching proof.

THEORETICAL FRAMEWORK

The focus of our attention is on the interrelatedness of cognitive psychology, representations and cognition. Having a look at cognitive psychology, it was Bruner who introduced the notion of representation. According to him, "Any domain of knowledge (or any problem within that domain of knowledge) can be represented in three ways: by a set of *actions* appropriate for achieving a certain result (*enactive representation)*; by a set of *summary image or graphics* that stand for a concept without defining it fully (*iconic representation)*; and by a set of symbolic or logical propositions drawn from a symbolic system that is governed by rules or laws for forming and transforming propositions (*symbolic representation)*(Bruner, 1966).

After the introduction of the notion of representation, new views of concrete and visual representations appeared. First of all, what primarily counts are mental images of concepts. Visual representations can support these to some extent. Moreover, mental images are not just copies of external representations, but they are formed by the constructive activity of the individual. These constructions are idiosyncratic, i.e. they are determined by the experiences and personal knowledge of the individual. However, concrete and visual representations are not “speaking pictures,” they do not fulfill the expected function as carriers of mental images per se. This approach leads to that principle which states that “Concrete and visual representations are neither only aids for the so called ‘slow learners,’ nor is their use restricted to the early steps of the learning process. They are important for all children and they are useful for the whole duration of the learning process” (Ambrus). Last but not least, “concrete and visual representations are not automatically the better, the more specifically they represent the intended concept. ‘Perfect’ representations can be counterproductive. In order to fulfill their function, good representations must involve a certain vagueness” (Wittmann, 1998).

The key organ where all these representations are created is the brain. It is known that our brain is an asymmetric organ, resulting in the fact that its two hemispheres are responsible for different activities, functions. These distinctions are displayed in the following table.

Dominant functions of the left hemisphere	**Dominant functions of the right hemisphere**
speech, use of language	speechless, seeing, space manipulator
sequences, digital	simultaneous, analogous
logical, analytic	synthetic, holistic
algebraic	geometrical
intellectual	intuitive, spontaneous
convergent	divergent
reasoning	creativity, imagining power
rational	irrational
abstract (thinking)	object centered (thinking)
realistic, objective	subjective, impulsive
guided	free
time sense	timeless
no sense of humor	sense of humor

Normally, in a complex problem solving activity the creative right hemisphere gives the idea of the solution and the left hemisphere works out the details, organizes the solution-steps in logical order. Therefore, in order to be effective we need to use our both hemispheres. Unfortunately, the Hungarian tradition works contrarily, while there the analytic, symbolic and logical considerations are dominant, we do not use the capacity of our right hemisphere.

When all the representations and the related ideas, notions consolidate in our brain, we fathom a given problem. Various ways exist in which cognition can occur. In the following, the narrative and the paradigmatic modes of cognition will be analyzed.

Narrative psychology refers to a viewpoint within psychology which is interested in the “storied nature” of human conduct. Human beings deal with experience by constructing stories and listening to the stories of others. Psychologists studying narrative are challenged by the notion that human activity and experience are filled with “meaning” and that stories rather than logical arguments or lawful information are the vehicle by which that meaning is communicated. This dichotomy is expressed by Bruner as the distinction between the “paradigmatic” and the “narrative” forms of thought which, he claims, are both fundamental and irreducible one to the other (Bruner, 1966). Focusing on the narrative forms of thought as Sarbin proposes that the "narrative" becomes a root metaphor for psychology to replace the mechanistic and organic metaphor which shaped so much theory and research in the discipline over the past century. His fundamental concepts in connection with cognition are summarized in the table below (Sarbin, 1986).

The mode of cognition	Narrative	Paradigmatic, theoretical
Organization	time, sequences, actions	timeless, categorical, subordinating
Textual	story, deliberate	description,
Correspondence	intended	hierarchy-relations
Idea	special, episodes	impersonal, validity
Embedded	contextual, personal, social	context free tendency

The most recent discoveries about the nature of mind focus on the *embodiment of mind*. The detailed nature of our bodies, our brains, and our everyday functioning in the world structures human concepts and human reason. This includes mathematical concepts and mathematical reason. The *cognitive unconscious* plays a significant role as most thoughts are unconscious – not repressed in the Freudian sense but simply inaccessible to direct conscious introspection. We can not look directly at our conceptual systems and at our low-level thought process. This includes most mathematical thoughts. Not only mathematical but also *metaphorical thoughts* are relevant in this theory. For the most part, human beings conceptualize abstract concepts in concrete terms, using ideas and modes of reasoning grounded in the sensory-motor system. The mechanism by which the abstract is comprehended in terms of the concrete is called conceptual metaphor. Mathematical thought also makes use of *conceptual metaphor*, as when we conceptualize numbers as point on a line (Lakoff & Nunez, 2000).

THE MATHEMATICAL ANALYSIS OF THE THEOREMS

To make the situation clear, I would like to briefly explain the theorems in question. These theorems are related to right-angled triangles. According to the Pythagorean Theorem, "In all right-angled triangles the square of the hypotenuse is equal to the sum of the squares of the legs" (legs are those sides of a right-angled triangle which enclose the right angle. The side opposite the right angle is called the hypotenuse.) The Altitude Theorem goes as follows: "In all right-angled triangles the altitude belonging to the hypotenuse is the geometric mean of the two parts of the hypotenuse into which its altitude divides it." The third theorem is the Leg Theorem: "In all right-angled triangles the leg is the geometric mean of the hypotenuse and the orthogonal projection of this leg on the hypotenuse."

THE RECENT HUNGARIAN CONTEXT

Pythagorean Theorem (Grade 8)

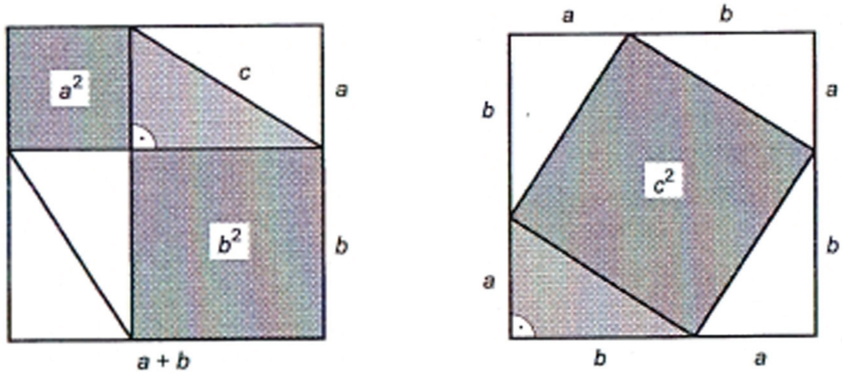

Figure 1.

Altitude Theorem (Grade 10)

Let us denote the congruent angles with the same sign. ΔATC and ΔCTB triangles are similar because their corresponding angles are congruent. It follows that the ratio of the lengths of the opposite sides are equal.

$$\frac{h}{x} = \frac{y}{h} \quad \Rightarrow \quad h^2 = xy$$

In our case two polygons are regarded equal if there is a one-to-one correspondence between their sides. Another case is when a polygon with the same area can be obtained by cutting the original polygon into different smaller ones which are reused to complete a given figure.

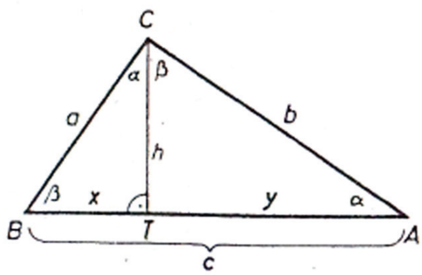

Figure 2.

Leg Theorem

Using Figure 2 we may find another similar triangle pair: ΔABC and ΔCBT. It follows that:

$$\frac{a}{c} = \frac{x}{a} \quad \Rightarrow \quad a^2 = xc$$

Comment: the proof methods of the Pythagorean Theorem and of the other two theorems are quite clearly different and there is a two-year difference between teaching them.

One possible connection among the three theorems: using the Leg Theorem for both legs we get: $a^2 = xc$ and $b^2 = yc$. By adding these two equalities we get: $a^2 + b^2 = xc + yc = c(x + y) = c^2$

Comment: in this case the Pythagorean Theorem is a consequence of two leg theorems, which means that it will be taught after teaching the Leg Theorem. For some reasons it is taught rather late because the introduction of irrational numbers takes place in grade 7-8, and e.g. the place of $\sqrt{2}$ on the datum line requires the knowledge of the Pythagorean Theorem.

A NATURAL CONNECTION AMONG THE THREE THEOREMS
Use of enactive representations: Pythagorean Theorem (Figure 5-6)
Configuration I.

We cut out two squares with sides *a* + *b* from a red-colored paper and four right-angled triangles which are congruent to the original ones in the square from a blue-colored paper. Using these figures we can create the first configuration. (The basic plane is the square and the four triangles are placed on it as seen in the figure. Every student builds this configuration and they draw this figure into their notebook. This manual process plays a vital role in building a firm ground for the understanding of the proof. The next step is to translate the four triangles into the second position (See it on the figures.) The acute angles of the right-angled triangles are denoted by symbols:α, β. On the second configuration students can see that the angles of the four-sided polygon are right angles and its sides are *c long,* this figure is also a square with side *c.* The area of the four-sided polygon is c^2. Comparing the two configurations we may see that both figures have four congruent right-angled triangles, they cover the same amount of area on the red square, so the remaining red part will have the same area in both configurations, which means that $a^2 + b^2 = c^2$.

Students need to draw the second configuration beside the first one in their notebook and the argumentation described above. I think in grade 9 mainly for weaker students as the students from a classical school it is enough to understand why the two configurations depend on the constraints of the border. Better students will obviously realize the limits of cutting and will feel the need for a more formal proof. Though, the complete formal proof, based on these figures and the use of symbols can be postponed until grade 10.

Theorem of Altitude (Figure 7-8)
Configuration II.

We cut out a right-angled triangle with legs *p+h vs. q+h* from a red-colored paper and we cut the original right-angled triangle, from a blue-colored paper, into two parts along its altitude related to its hypotenuse. We can build the first configuration with red and blue triangles. Translating the blue triangles along the hypotenuse of the red triangle, we may build the second configuration. (See on the figures.) Using the same basic idea we used above, comparing the red uncovered parts, we may get the Theorem of Altitude: $h^2 = pq$.

Comment: some students may doubt that how we know for sure that a right-angled triangle with legs *p+h* vs. *q+h* and with hypotenuse *a+b* really exists. We can give a constructive explanation to them. We can place the blue triangles along a line as seen on the first figure. If we translate them along the relevant legs, we get our right-angled triangle.

Leg Theorem (Figure 9-10)
Configuration III.

The existence of the first configuration can be explained as above. We may place the blue right-angled triangles along a line as we have seen in the previous figure. Translating these blue triangles along the relevant directions (see on the figure) we get the second configuration. From it follows that: $pc = a^2$.

Didactical Analysis and Recommendations

1. Students need to cut out all the necessary figures at home before the lessons.
2. The basic proof idea is: by covering a part of a figure with some other figures, then transforming the covered figures into another covering position on the same, original, figure, we will get uncovered parts whose areas are equal. (This phenomenon is known as the additive characteristic of area.)
3. The Pythagorean Theorem is understood by students in grade 9 because they learnt it in the previous grade. Therefore, building the first configuration will be not a hard work for them. Teaching the other two proofs should be organized in group work. The first configuration will be constructed with the leadership of the teacher. Though, in order to make students find the relevant transformation we have to emphasize the aim, namely that we want to find a relationship between the altitude and the two parts of the hypotenuse into which the altitude cuts it. The other relationship should be found among a leg, its orthogonal projection on the hypotenuse and the hypotenuse. It is the question of the experiment whether it is realistic to leave the conjecture to students or we give the right relationships in advance.
4. These types of proofs are very unusual in Hungarian mathematics education. For this reason we have planned problems, tangram problems, covering and area problems, so as to prepare the students for the main task.

A DEVELOPMENTAL TEACHING EXPERIMENT

The Research Question

Can a group of classicists get closer to such an abstract phenomenon as the proof of the Leg Theorem and the Theorem of Altitude related to right-angled triangles by using a proof method involving enactive representations?

The Research Hypotheses

The abundance of preliminary problems and the involvement of concrete and visual representations will offer enough opportunity for gaining a deeper understanding of the area of substantial polygons. Working with tangrams will enhance students' visual thinking. After establishing firm grounds, most students will be able to combine all that knowledge to successfully prove the given theorems. By making students work in pairs, they will be more successful problem-solvers. Less able, less motivated students will experience success, so they will become more motivated in learning mathematics. By solving these problems students' argumentation skills, the use of formulae and symbolic language will develop.

Methodology

The subjects of the experiment were a class of ninth-graders who studied mathematics in two groups. The groups were made up of students with different mathematical abilities and backgrounds. During the four lessons of the experiment photos, photocopies and notices of 35 ninth-grade students' works were taken, observations were carried out, interviews were conducted, video recordings were made for further analysis and reflection. Students were given a great variety of area-related problems, distinct methods were revealed by them. Moreover, a true or false questionnaire was handed out in the middle of the research to help their revision of the area of major polygons. After a proper introduction of the new proof method, the final tests were the theorems that they had to prove under some guidance. Knowing that the students are more interested in humanities than in science, they were asked to work in a new form of work, i.e. pair work. The novelty of the form of work and the challenging problems made students enthusiastic.

Learning trajectory

Stage 1: Freedom of creativity, imagination: working with sets of tangrams, either individually or in pairs; creating polygons individually from tangrams by following instructions

Stage 2: Area-related problems: working on a true or false questionnaire, giving written evidence for findings, pair work followed by class discussion

Stage 3: Learning new problem-solving methods related to the area of polygons, individual work turned into pair work, later to class discussion

Stage 4: Proof of the Pythagorean Theorem, the Leg Theorem and the Theorem of Altitude

Results

The actual experiment lasted for four-five lessons. In the first half of the initial lesson each student was given an envelope containing a set of tangrams. After identifying the pieces they were asked to create polygons, figures for their own sake. They could either work in pairs or alone. Photos were taken of these sometimes creative, sometimes conventional shapes. Two of them can be seen below. Most students enjoyed the lesson because they were not instructed how to carry out the task, so they could rely on their own imagination and creativity. There were a few students who needed some encouragement and only then they started to fly high and made gorgeous figures. A small minority suffered from the unusual task as they seemed to struggle a lot to come up with a meaningful figure. They might be short of imagination or felt insecure because of the lack of instruction. I could also imagine that they might not have much experience in such activities since their shapes showed lack of geometric intuition or imagination.

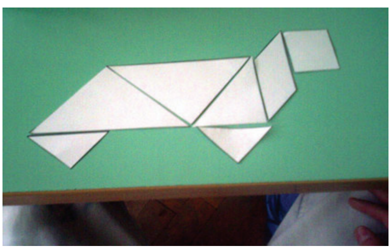

Figure 3. Animal

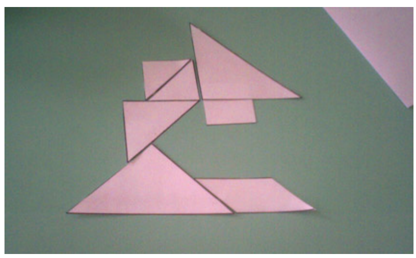

Figure 4. Microscope

In the second half of this preparatory lesson students worked alone and were given instructions to construct simple tangrams and determine the area of each tangram piece so that they could calculate the area of other polygons next time. Students took notes of their work in their notebooks. Whenever they solved a problem we discussed their solutions, the good ones and the bad ones as well. Thus, students had the opportunity to alter their conclusions if it was necessary. The initial problems seemed to be far too easy for most students, meaning that in the forthcoming experiment I will have to select the problems more critically. The consecutive problems were of levels of increasing difficulty, which required the use of the concept of similarity and congruency of polygons as well. Sometimes they were the sources of mistakes since some students did not know exactly what similarity or congruency meant. They needed some explanation and afterwards they could solve the task. Most students liked these challenging problems because they started to compete against each other.

During the first two lessons we revised students' knowledge of the area of polygons without using any formulae. It was followed by the formal revision of area formulae and then students were tested with a true or false questionnaire. Students worked in pairs, which proved to be very fruitful on the basis of their results. They did not only have to decide whether the statement was true or false but also had to write some justification for their answer. After completing the questionnaire we discussed their answers and explanations. On the basis of the interviews carried out after students' cooperation, pair work proved to be a more efficient way of learning mathematics in both groups, especially in case of a topic in which neither student was confident.

While interviewing the participants it turned out that some students had quite unusual strategies to solve these problems. Whilst I was listening to them it became quite evident that their thinking was so diverse and was deeply rooted in their previous mathematics-related experiences. The solution of the geometric problems they were given depended mainly on the way they understood geometry, i.e. they had mental images, drawings or relied on concrete drawings or used algebra to solve the problems. The most effective strategy proved to be the one that made use of intangible mental images. Those students who possessed this ability managed to achieve better results because they could carry out the alterations in their mind, thus they had a much larger perspective of the changes that the alterations evoked. It seems that if you have a creative mind, you are more likely to cope with such mathematics tasks. Another group of students, the majority, drew the given problems into their notebooks and tried to carry out the necessary alterations on the given figures. They relied mainly on algebraic calculations, that is, whenever it was possible instead of a general polygon they used one with concrete sides. They tried to calculate the answers instead of seeing them. The third group of students, the minority, decided to guess. In my opinion, these students might have problems with geometry, they do not have strategies at all or the problems were not interesting enough for them. In the followings some examples of more creative strategies are presented. The statement is taken from the true-false questionnaire.

Statement 1: If we double the sides of a square, then its area will be doubled, too.

Teacher: How do you solve such problems in general?

Dóra: I imagine figures in my mind and I try to join them together. I rotate the squares in my mind, ... I'm not calculating then, but I'm drawing. I see things in three dimensions….I usually solve problems by drawing them. …If I had to calculate a problem on a piece of paper, I would have no chance to solve it. …I like playing with things. In my mind every figure has a color, e.g. triangles are yellow, squares are tomato-red, rectangles are blue, and curves are either purple or pink.

Judit: In the first five problems I tried to think, come up with theories on the basis of what I knew…. It came to my mind that if both sides were doubled, then its area couldn't be its double. Afterwards I drew it. I imagined the square and that I would increase its sides, and then I had the answer…. I usually think in terms of 4.

During the next lesson, in the third stage of the experiment, students were given area-related problems to acquire new problem-solving methods such as leaving out superficial pieces/polygons from a given drawing, leaving out equal parts from polygons with equal area or cutting/drawing lines into a given polygon. In this part of the experiment students were exposed to math competition problems made especially for seventh and eighth graders. At the beginning the problems did not prove to be invincible, but later they were more than thought-provoking. According to the interviews, students made use of the previously acquired techniques, such as using mental images, drawing in helpful lines, rotating, translating certain parts of the given polygons into new positions. However, in the middle of the lesson most of them seemed to lack an effective idea. In order to help students overcome their difficulties they were asked to work in pairs. When they were totally lost, they tried to ask for help from others, so they started forming groups of four. After every third problem, we stopped and skilful problem-solvers presented their solution to the others on the board. Those who came up with other strategies were given the opportunity to show what they found. Later, from the interviews I learnt that although most students badly needed help in this situation, still some students decided to struggle alone because they wanted to find the solution on their own. Smart students told me that they played a lot with puzzles in their childhood or had seen such problems before, which helped them find the solution to the given problem. By solving these exercises, students acquired all the necessary background knowledge to be able to prove the Pythagorean Theorem, the Leg Theorem and the Theorem of Altitude related to right-angled triangles.

After this long preparatory phase the experiment reached its climax, namely, students had to cope with "real" proofs. The introduction to the last phase of the experiment was the revision of the Pythagorean Theorem. Most students managed to recall the theorem itself, but when they were asked to prove it, nobody volunteered to carry it out. To make that work easier, the participants were given 2 white squares with sides $a+b$ and four orange congruent right-angled triangles with legs a and b (see Figure 5). Students were shown that there were two squares within the original square with areas a^2 and b^2. They had to trace the first phase of the problem into their notebooks. Their task was to find another arrangement of the right-angled triangles within the original square which contained another square. For most students it was not most difficult to find the arrangement asked for but they had problems with the understanding of the equality of the areas of the parts in question because on the basis of the drawing it was not so inevitable for them. So, I had to make them recall what they had learnt in the previous lesson, when they had to determine the area of certain polygons and that of polygons drawn into other polygons. Eventually, when they had a look at their notes Nándi said, "Oh, yes. I should have known that. If we leave out parts of equal areas from equal polygons, then the areas of the remaining parts will be equal, too." Students were asked to draw the solution into their notebooks (see Figure 6) and I tried to make them formulate the symbolic proof but it was beyond their knowledge. So, I had to give it to them myself. Most students were satisfied with this answer, but for the sake of those who still had problems with the understanding of this proof we used a pair of scissors and cut the two squares with areas a^2 and b^2 into smaller pieces so that less able students could see that from the smaller pieces we could build the third square with area c^2.

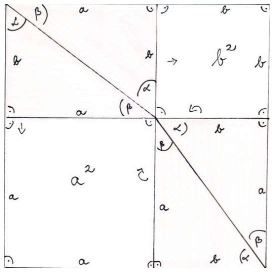

Figure 5.

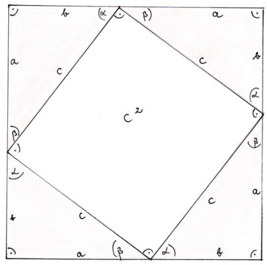

Figure 6.

Due to the comprehension hardships, I decided to make students work in mixed groups of 3-4 in the other part of the lesson because I wanted to make them experience success instead of failure. The second theorem of the lesson was that of Altitude. The students were given a drawing containing a quadrilateral which contained a square with area $a^2$and two pink right-angled triangles with legs *a* and *b* (see Figure 7). They traced the initial step into their notebooks. The task was similar to the previous proof, i.e. they had to find such an arrangement of the triangles which led to a polygon whose area was the same as that of the square. Reni shouted, "We've got it! We have simply shifted the triangle on the left to the right, the triangle on the right to the left. They changed places" (see Figure 8.) Then, I asked the students, "How can you make sure that the area of the square with area h^2 will be equal to the area of this rectangle you have just drawn?" Erika, "Well, … Nándi has said that if we take out parts with equal areas, then the areas of the remaining parts will be the same, too." Nándi, "Erika, you said it right." This time, everybody understood the justification. Again, students took notes of their works into their notebooks. They drew the initial step and the forthcoming step of the translation. When students were asked to give the symbolic proof of the theorem, to my surprise, Mór wanted to take the opportunity.

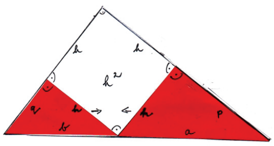

Figure 7.

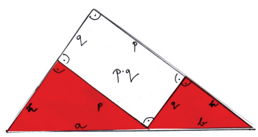

Figure 8.

The third theorem that students had to prove was the Leg Theorem. Students were given a drawing which they used for the proof of the theorem in question (see Figure 9). They took notes of the initial stage and the final stage of their proof into their notebooks. Most groups struggled with the problem because they believed that they had to find another square, like in the exercise above. They were on the edge of giving up, Diána said, "I'm stupid. I can't do this. There is no other square in this drawing. It's impossible to find it." Others asked Nándi to help them. I did not want to let them down and I did not want to let the experiment fail so I dropped a hint, namely that they might not have to look for a square but for a quadrilateral in general. That sentence made them think their ideas over again and a few minutes later every group managed to find the arrangement of the triangles. I asked Erika to come the board to draw and explain what her group realized. Erika, "First, we tried to move the left-side triangle to the right, but it did not work that way… Then, I saw that it would fit into the top of the original quadrilateral and we could see that it was good. Then, Viki said, 'Why don't we move the standing triangle to the right and we have a rectangle in the middle'. So, that's it" (see Figure 10). When students were asked to prove the theorem symbolically as well, Erika volunteered for the task. She wrote area $_{\text{rectangle}} = p\cdot(q + p) = p\cdot c$ and area$_{\text{square}} = a^2$ on the board. She added, "the areas of the remaining parts are equal, that's why $p\cdot c = a^2$, so the area$_{\text{rectangle}}$ = area$_{\text{square}}$."

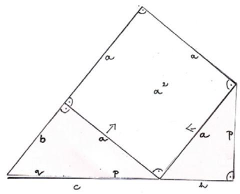

Figure 9.

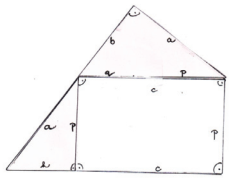

Figure 10.

CONCLUDING REMARKS

Before and during the experiment several problems occurred. The first and the most important one proved to be the choice of topic and age-group. In the current state of affairs, the Hungarian National Curriculum would prefer competence-driven teaching. Still, there is a significant problem, namely that the amount of teaching material is far too large, thus it rarely offers the opportunity for a similar experiment that I carried out in my own class. Maybe the reduction of the teaching material could give greater freedom to teachers in order to experiment in the classroom. Another possible solution could be the enrichment of teachers' methodological and technical mind.

Another major challenge was the planning of the experiment, especially as I did not have any experience in it before. Originally, I wanted to spend only three lessons on the topic, but after a discussion with my fellow colleagues who had a lot of experience in teaching proof, the plan had to be altered fundamentally. One of my colleagues was a very good critic as she called my attention to the fact that I expected too much from my students and a more thorough preparation was needed for a successful outcome. She also suggested problems which proved to be extremely useful and essential for the experiment. I am convinced that working on a project like this should be done in cooperation with other colleagues as the experience of the colleagues is accumulated and leads to a more fruitful, effective learning-teaching situation.

Another novelty was the need for reflections on the work of my students and on my own teaching in new ways. I made quite a few interviews with my students, either on their own or together with their actual pair, on particular problems within the research which could reveal the most about their way of thinking or problem solving methods, and eventually I got a different insight into their mathematics knowledge. Moreover, shy students opened up during these interviews, thus I may use this technique in my future teaching practice as well.

For me, one of the most difficult parts was to reflect on my own teaching. Traditionally, in our school all teachers have to write a self-evaluation at the end of each school year. During an experiment more frequent self-evaluation would be advisable. Sometimes, when I had a few-minutes' break after a lesson I tried to recall the students' reactions to the problems that I had given to them, or my own reactions towards problematic situations, which resulted in the consideration and realization of the possible causes of these problems. There were a few occasions when I took notes of these recollections. The video recordings of the last lesson proved to be the most important source of information in this respect. I could face the problems that I felt more explicitly and I could observe other problems that I did not take into consideration before. Now, I am convinced that I am going to use this device more often and take notes of my reflections more often.

Among the concluding remarks one important detail regarding data collection has to be highlighted. Namely that abundant data and sufficient number of interviews and/or video recordings or the help of an assistant teacher are essential for the ability to construe the happenings in the classroom, to reflect on the students' work and on our work and eventually, to finalize the findings of the experiment. Otherwise, it is immensely problematic to write an acceptable article on a given experiment.

Last but not least, I would like to say a few words about the preparation for the lessons during the experiment. At the beginning, it was extremely laborious to collect material, to plan the experiment, its stages, the tasks and the instructions. Because of the lack of printed help I relied on the Internet sources. Initially, I did not manage to find any hints about how to build up an experiment, what to expect, etc. Strangely enough, I found some relevant resources after finishing the experiment. The lack of access to the theoretical literature made the preparation phase and the realization phase challenging. Because of that, I appreciated any help that I was given, still I needed more help. So it would be a good idea that experienced teacher-researchers helped teacher-researcher trainees during their initial attempts. As far as the technical part is regarded, sometimes I found it very time-consuming to draw and cut all the necessary polygons for the next lessons. Although, it extended my preparation time with hours but the results compensated for it.

All in all, it is worth researching one's own teaching practice as well as experimenting with new ways of teaching. We not only learn a lot about our students' thinking, their knowledge of mathematics but also about our own teaching methodology and strategies. By reflecting on the actions of the participants of the learning-teaching episodes of the given lessons, mainly teachers but also learners are offered the opportunity to detect possible obstacles in their learning and/or teaching. If they are aware of their own difficulties, it can make them search for new methods, strategies, thus eventually they will be able to find more effective ways of learning and teaching. One useful tool to achieve this goal could be the conduct of the experiment.

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PROBLEM SITUATIONS TO PROMOTE PROPORTIONAL REASONING: SOLVING STRATEGIES BY STUDENTS AGED 11-14

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ABSTRACT
This paper concerns the analysis of thought strategies applied by students in the first cycle of secondary education (aged 11-14) in situations concerning proportionality as proposed within a project extending through the three-year cycle. The teaching path that was established is based on the hypothesis that proportional reasoning can be actually achieved only through non-episodic, metacognitive, by-problems teaching, which allows the exploration of possibly complex instances of proportionality in different contexts and promotes the comparison with non-proportionality instances. After a few indications on the teaching path, a few examples of the activities performed within the three-year cycle are presented, with the analysis of solutions provided by students. The paper then concludes with considerations on the usefulness of the proposed approach based on the long-term collection of impressions of the mindset and attitude of the students involved.

INTRODUCTION

The conscious achievement of proportional reasoning by students of the first cycle of secondary education (aged 11-14) is certainly one of the most daunting tasks for teachers. Though all teachers deal with the subject of “proportionality,” few of them bother to verify whether the students, at the end of the school cycle, are actually able not only to reason through the proportionality pattern but also to recognize, within a diverse range of problem situations, the ones which fit the proportionality pattern (Grugnetti, 1996). Such behavior in teachers is induced, among other things, by: 1) the approach used by many textbooks, which, to this day, simply explain “proportions” and their properties as a subject rather than promote the development of proportional reasoning, and 2) the urgent need to economize time. This old-fashioned approach still persists even though the introduction, as early as in the 1970s, of the so-called “new math” challenged the Euclidean view on proportionality (based on equality of ratios and simple “rule-of-three” problems) describing it through a relational point of view focused on the study and exploration of more and more complex situations, the analysis whereof would bring about the nature of the relation between the pairs of values of the two quantities involved. This latter (and still novel) approach, focused on the concepts of variable and function, poses a significant contribution to the modelization of reality (Freudenthal, 1978) and promotes the development of *proportional reasoning*.

The problematic nature of teaching/learning proportionality is documented by the ample literature on the subject. As early as in the 1980s diagnostic studies were made concerning the behavior and issues of students in solving proportionality problems, wherein: (i) aspects affecting performance, e.g. the type of numbers involved or the type of context, are brought into focus (Noelting 1980a, 1980b; Hart 1981, 1984; Karplus et al. 1983; Mariotti et al. 1988; Lesh et al.); (b) research reviews on this topic are made, underlining aspects which deserve further investigation (Tournaie & Pulos, 1985); and (c) wide theoretical frameworks for proportionality are established, such as the Vergnaud's conceptual fields theory (1983).

In the early 1990s the first studies which take into consideration informal reasoning methods (Behr et al., 1992; Harel & Confrey, 1994) appear, whereas in the late 1990s we have a stronger presence of qualitative studies wherein more consideration is given to spontaneous thought processes within the framework of a socio-constructive teaching (Ben Chaim et al.,1998; Christou & Philippou, 2002; De Bock et al., 2001; Hino, 2002; Pesci, 1998, 1999; Nabors, 2002) leading to studying how to write the texts for problems, which may motivate and strengthen the students' approach to proportional reasoning in the classroom environment (Anley et al., 2005)

Our project falls within this trend and is based on the assumption that in the first cycle of secondary education it is preferable to work towards achieving proportional reasoning through activities consisting of the exploration of problem situations without teaching proportions and their properties beforehand, pervasively carried out throughout the first three-year cycle of secondary education (and not in a limited period in the second year of the cycle, as it usually happens). This approach encourages in students the attainment of a mindset which, in time, will lead them: 1) to understand the proportional relation and its multiplicative nature; 2) to correctly handle proportionality situations even when not standard; 3) not to improperly extend the proportional pattern, recognizing the difference from other kinds of relation. We also believe that in following this teaching path, the building of the concept of rational number as a class of equivalent fractions (Malara & Gherpelli, 2002; Malara, 2003) may favor in students the understanding of the inherent variability to the proportionality relation and assist in studying rather complex situations.

METHODOLOGY

This study was carried out on a class of the first cycle of secondary education over three years (teacher: L. Gherpelli) with a full collaboration of both authors. The teaching path was implemented in the framework of the usual study program for the class and involved careful scheduling in order to make room for shared activities. The teaching methodology included both individual and small-group exploration of problem situations and group discussions for confrontation and considerations on strategies applied and errors noticed. The class developed an atmosphere that allowed students to take on a responsible, research-oriented attitude, which helped the participation of all students in the discussion sessions. In leading the discussions, the teacher strove to operate non-didactically (Brousseau, 1988) without commenting on the correctness of each observation and allowing the students themselves to validate them, while at the same time acting as a mediator and leader.

The analysis of students' protocols and the accurate reviewing of recorded class conversations were vital elements to this research effort. This allowed us to monitor the students' actual grasp of concepts and follow their argumentative and cognitive evolution.

DISTINCTIVE FEATURES OF THE TEACHING PATH

In developing the teaching path: (i) an effective connection was made between division problems (according to partition and capacity models) and problems introducing proportionality, based on the language construct "for each set of... a given number of...";

(ii) the "table" was used constructively and mindfully as a tool, as early as in the first year; (iii) a comparison was drawn between situations involving proportionality relations and non-proportionality relations; and (iv) a teaching route was developed which aimed at providing multiple strategies both for studying and for recognizing proportionality situations.

We also operated in different contexts, agreeing on the fact that in the development of proportional reasoning the context plays a crucial role in students' performance and that the use of a wide variety of contexts is needed in the teaching of this domain (Touraire & Pulos, 1985). The following paragraphs provide a broad description of the activities carried out in the three-year cycle.

In the *first year*, after reviewing division problems, students were confronted with situations following the pattern "for each set of... a given number of..." in the context of problems with progressively more difficult numerical data. At first, the proposed situations suggested the enactment of ingenuous strategies; later, situations were introduced which stressed ratios as the invariant elements characterizing the relation between pairs of values of given quantities.

In the *second year* problem situations leading to the comparison between ratios were introduced, and students were given explicit requests about managing proportionality situations.

In the *third year* more complex problem situations were addressed, aimed at controlling the sedimentation of proportional reasoning. The proportionality relation was also studied as a function by analyzing its graphical and algebraic aspects.

Throughout the three years, we cyclically proposed situations having strong analogies in order to have an accurate perception of students' maturation process and stress with them the point that a higher degree of knowledge facilitates the simplification of the ways, in which they perceive and operate.

CHARACTERISTICS OF PROBLEM SITUATIONS

Problem situations were developed considering several parameters, such as: context, numerical data, linguistic aspects, and the articulation of questions (which usually comprised several points). In order to convey the general idea of our work, we present here a limited selection of four problem situations, the 1st presented in the first year, the 2nd and 3rd in the second year, and the 4th in the third year (see Table 1).

A SAMPLE OF PROPOSED PROBLEM SITUATIONS

First Situation: THE NOTION POTION

- "The long section of scroll lay unrolled on the floor, held down by stacks of books so that it didn't roll back up. After studying carefully, once again, the instructions for use, Preposteror and Tyrannia skipped over to the recipe for the Notion Potion, one of the most powerful and ancient spells in the Universe." (from "The Night of Wishes" by Michael Ende)

These are but some of the indications reported in the scroll: "and finally, carefully mix together 3 spoonfuls of HOT AIR for every 120 g of IPECACUANA POWDER"

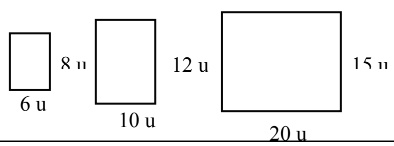
Now imagine you want to prepare the Notion Potion with 240 g of IPECACUANA POWDER:

a) How many spoonfuls of HOT AIR do you think you should use?

b) Try and explain how you would calculate the number of spoonfuls of HOT AIR you need to prepare the Notion Potion with 200 g of IPECACUANA POWDER. How many would you need?

Second Situation: COMPARING RECTANGLES

Look at the 3 rectangles shown and check whether all, two, one or none of them may be reproductions in scale of a rectangle with dimensions $3u$ x $4u$.



Third Situation: IS THIS PROPORTIONALITY?
Read the following situations and verify whether any of them are about proportionality:
a) Three workers have received 216 Euro for a job they worked on together. If they worked respectively for 4, 5, and 3 hours, how much money should each of them receive considering that the hourly wage is the same for each worker?
b) Luca and Marco are playing "Guess the Trick:" Luca says 2 and Marco answers 5, Luca says 3 and Marco says 7, Luca says 10 and Marco says 21. What's the rule used by Marco to answer Luca?
c) A supermarket has a special offer: for every 50 Euro you spend, you get a free plate. How many free plates will Elisa get for spending 150 Euro?
d) With a single length of string I first built the rectangle *ABCD*, with dimensions *AB=8u, CD=6u*; then, with the same string, I built the rectangle *EFGH*, where *EF=12u*. Calculate the length of the *FG* side of the *EFGH* rectangle.

Fourth Situation: THE SHED[2]
Anna and Giulia are painting the shed in their garden. Anna's dad wants to paint it gray. The store only has small cans of black paint and white paint.
a) Anna mixes 2 cans of white paint and 7 cans of black paint; Giulia mixes 3 cans of white paint and 9 cans of black paint. Will the two sides of the shed be of the same color? If they are, which side will be the darkest?
b) Anna adds one more can of white paint and one of black paint. Will the color of her paint become darker or lighter?
c) Anna keeps adding one can of each color. Will she ever reach Giulia's color? If she does, how many cans of each color will she have mixed?

2 a 7
3 a 8
4 a 9

3 a 9
4 a 10
5 a 11

d) After a while Anna mixes 7 cans of white paint with 12 cans of black paint. The two colors are still different. Giulia starts adding cans herself: one of white paint and one of black paint each time. Will she ever obtain Anna's color? If she does, how many cans of each color will she have mixed?
e) Do you think this is a proportionality situation? How can I check that?

Table 1.

ANALYSIS OF PROBLEM SITUATIONS

First Situation (The Notion Potion)

The first question facilitates exploration by proposing a numeric value, which is twice the initial value; the second question – which offers a numerical value not immediately recognizable as connected to the original recipe – favors the enactment of different strategies.

Second Situation (Comparing Rectangles)

This situation is based on the mental image of a *3u* × *4u* rectangle and poses the problem of relating rectangles of different dimensions. It deliberately contains a strong element of distraction: the graphical rendering of the *20u* × *15u* rectangle is different from the others, which may induce students in not establishing the correct relations (shorter side with shorter side, longer side with longer side). This situation therefore allows for testing students' flexibility in handling images mentally: the *15u* × *20u* rectangle can be put in a relation with the other rectangles, using the graphical strategy of homothety, only by visualizing it after a 90 degrees rotation.

Third Situation (Is This Proportionality?)

This situation was designed to verify the students' ability to recognize, in a set of situations given in different contexts, those that were proportionality situations. One peculiar aspect of this situation is the fact that students are not necessarily asked to solve the problems but simply to evaluate their type. Students must therefore assume a "meta-problem" view. The first problem is a classic example of compound partition, which requires calculating the unit (hourly wage per worker). The second, which is about an activity familiar to the students, aims at recognizing the type of function involved which is not proportional but affine. The third, deliberately simple, is based on known situations. The fourth, a geometrical problem, is quite complex both in language and in mental representation. The specific request to "calculate" is actually an element distracting the attention from the more general request, the meta-request, which can, however, be answered through qualitative considerations.

Fourth Situation (The Shed)

This five-stage situation is similar to other previously studied situations about mixtures, with a wider scope. It is complex on account of the dynamics of the situations it proposes and the mental processes it requires to be activated. In order to follow it correctly, students must compare several paint mixtures and recognize whether there is any equivalence in ratios. Question a) requires a comparison between the shades obtained in each case; the goal is to have students recognize the necessity to associate a given mixture with the ratio between the quantities involved, i.e. the number of cans of the two colors being mixed, and therefore compare the shades by analyzing the respective ratios. Question b) proposes the analysis of the color variation obtained by adding one black and one white paint can to Anna's first mixture; the goal here is to verify that students interiorized the proportionality relation and the ability to recognize that the quantities that are obtained are not proportional, even though the difference between the parts remains constant. Question c), which is more complex, requires a comparison between the shades obtained by Anna at each step and Giulia's initial shade; the originality of the question lies in the iterative procedure of adding another can of each color to the original first shade, leading the students to generalizing the situation. This request forces the students' attention to focus on comparing the sequence of ratios obtained at each step with the ratio characterizing the second original shade. The aim is to verify the students' ability to recognize how the ratios obtained at each step are different. Question d), similar to question c) but with less simple numerical data, can complicate the situation by shifting the iteration procedure from the first to the second shade. Question e), finally, was introduced by the teacher in order to verify whether the students had interiorized the concept of proportionality.

SAMPLES OF STRATEGIES USED BY STUDENTS

In this context we will only analyze the third and fourth problems in Table 1, which involve issues that more closely represent the value of our study.

Problem Situation: (Is This Proportionality?)

Contextualization: Problem included in a test.

Below are some of the answers for each question which underline different levels of maturity in the students.

PROTOCOLS RELATED TO THE SITUATION *IS THIS PROPORTIONALITY?*

- FIRST QUESTION

Ingrid: You obtain equivalent fractions like: 18/1 = 72/4 = 90/5 = 54/3 = 216/12. Marius: 216 : x = 12 : 4, first proportion: 216·4=864, 864:12=72 Euro. 216:x = 12:5, second proportion: 216·5=1080, 1080:12=90 Euro. 216:x=1:3, third proportion, 216·3=648, 648:12=54 Euro.

Davide A.: Yes, because there's 18 Euro for each hour.

Simone: Yes, there is proportionality (see table).

Cristina: Yes, because the more hours they work, the more they get paid.

Federica: It's not proportionality because there is no fixed ratio between the number of workers and the hours they work.

Elena: It's not proportionality because the three workers didn't work for the same number of hours, or at least there's no logical relation.*

Hours	Money
4	72
5	90
3	54

A	B
2	5
3	7
10	21

SECOND QUESTION
Ingrid: 2/5 ≠ 3/7 ≠10/21 .
Marius: The rule is ·2 +1 (see table)
Davide B.: It's about difference, so there's no ratio.
Cristina: Yes, because the ratio between the two numbers is 0.4 so the numbers are proportional.
Davide A.: It's not proportionality because it's the rule that's fixed, not the number you get to add each time, like: 10 +13 = 23, 20 + 13 = 33 – in this case it would be proportionality because you always add 13.
Elena: This is a proportion because 2·2 = 4 + 1, 3·2 = 6 + 1, 10·2 = 20 + 1, so one can say there's a relation* between these numbers.

THIRD QUESTION
Ingrid: 1/50 = 3/150.
Marius: Elisa gets three plates. 150:x=50:1, 150·1=150, 150:50=3.
Giulia Z.: Yes, 50 : 1 = 150 : A, A =150 : 50 = 3.
Elena: Yes, because there is a relation* between 150 and 50. 50 gets you a free item, 150 gets you three free items.
Davide A.: For every 50 Euro they give you a plate.
Edwige: Even though there is no comparison, it's that they give them 1 plate every 50 Euro.

FOURTH QUESTION
Giulia Z. "If there was proportionality, I'd get 8 : 6 = 12 : A , A = 9, (9 + 12)·2 = 42 but 42 is not 28.
Davide A.: This is a little difficult for me, but I think it's not a proportionality because there's no "for each" rule.
Giulia L.: Here we have proportionality somewhere because both rectangles have their perimeter in common.
Davide B. [draws two rectangles. the first with dimensions 6×8, and the second with dimensions 12×16] 12 : 6 = 2, 8·2 = 16, 6/8 = 12/A, 8·12 = 96, 96:6 = 16 = A. This is a proportionality situation. There is a ratio; it can be solved with proportions.
Marius: 8:6 = 12: x, 12·6 = 72, 72:8= 9 number x. I don't think it can be solved with proportions because this result doesn't add up: 8+6=14, 14-12=2.

Table 4.

* *Translator's note:* it is important to notice that, in these instances, the students use the word *rapporto* which in Italian stands both for "relation" or "relationship" in a general sense, and for "ratio" in the context of mathematics.

Analysis of the above protocols

First Question. There are examples of correct and articulate reasoning: Not only does Ingrid recognize the proportionality situation but she also advances arguments for it and reinterprets it in a purely numerical context as a class of equivalent fractions making a connection between the two concepts. Marius recognizes the proportionality situation and, as such, translates it by building correct proportions and thus demonstrating his firm grip on thee quantities involved. *There are examples of correct but one-way reasoning*: Davide A. synthetically recognizes the hourly wage as the proportionality factor, which is probably also underlined by the language factor "…for each…" Simone appears more closely tied to visual models and therefore the use of the table. *There are examples of reasoning to be further expanded*: Structuring a table is enough for Simone to declare that there is a proportionality relation; however, there is no actual check of the equality of ratios. *There are examples of intuitive reasoning*: Cristina gives a mostly intuitive answer unrelated to the actual calculation of the hourly wage. (Cristina's argument allowed the teacher to discuss whether the answer given was actually correct. The example of train tickets, whose prices are based on the distance traveled but also include a fixed amount, has been used). *There are examples of lacking or undeveloped reasoning*: Federica expresses the correct notion that in a proportionality situation there should be a constant ratio between two variables, but she does not recognize the actual correspondence between the quantities involved. Elena with her argument shows an undeveloped proportional reasoning.

Second question. *There are examples of correct reasoning* characterized by the recognition of the relation ($y=2x+1$) and its not being a proportionality relation. Ingrid manages the situation, as she is wont to do, analyzing and comparing ratios and their equivalence classes. Marius identifies the relation between Marco and Luca's numbers using the table to better visualize the data. In order to "classify" the relation he then uses its representation on the Cartesian plane reporting Luca's numbers on the x axis and Marco's on the y axis. *There are examples of unclear reasoning*: Davide B. seems to refer a problem situation studied with the class where two relations of the type $y = kx$ and $y–x=m$ were compared; however, his absence of any argument does not allow further understanding. *There are fatal misconceptions*: Cristina, while correctly looking for a constant ratio to recognize a proportionality situation, mistakenly identifies that ratio with its approximate decimal representation. *There are wrong or immature notions*: Davide A. has not achieved a proper proportionality pattern yet. His lack of understanding of the situation is also underlined by the fact that he does not realize that the cases he proposes ($10+\underline{13}=23$, $2\cdot10+\underline{13}=33$) do not satisfy the rule $y = 2x + 1$ which is the basic pattern of the game. Elena grasps the non-straightforward relation between Luca's question and Marco's answer ($y = 2x+1$) but mistakes it for a proportionality relation. There is confusion with the meanings of the Italian word *rapporto* in a natural language ("relation," "relationship") and in the context of mathematics ("ratio"), which, as far as we can tell, confirms the confusion between the general concept of rule and the concept of proportionality "rule."

Third Question. *There are examples of correct reasoning* – they mainly diversify according to the advanced argument: Ingrid's is straight to the point and only relies on the equality of ratios between variables, Marius' concerns building a proportion. Other arguments are based on using proportions and classifying the problem as a "rule of three" (see the protocols by Giulia, who identifies the fourth element in the proportion, and Elena). *There are examples of unclear reasoning*: in Edwige's case, the student understands the functional relation – "for every so-and-so there is one such-and-such…" – but she does not recognize in this a typical language structure for expressing proportionality situations.

Fourth Question. *There are examples of correct reasoning*: Giulia Z. formulates a hypothesis and attempts to validate it. She assumes that a proportionality situation must be characterized by equal ratios, thereby building a proportion, from which she infers the dimensions the second rectangle should have. Since that datum is incompatible with the necessary isoperimetry between the rectangles she concludes that this is not a proportionality situation. Marius shows an effective handling of the quantities involved by correctly building the proportion. One may assume that in this case Marius had already calculated mentally $14-12=2$ since after determining the length of the second side of the scaled rectangle by means of the proportion he states that it is not compatible with the length obtained through the isoperimetry between the rectangles ($9\neq2$). *There are examples of improper or immature models of proportional reasoning* even in students who correctly state there is no proportionality between the two isoperimetric rectangles considered. Davide A. clearly expresses his perplexities about the situation he is facing. In particular, being unable to manage the variables involved, he loses his grip on the situation and justifies his answer with the only tool he masters: the now familiar language construct "a given number of… for each/every….of…." Giulia L. states there is no proportionality but gives an unclear reason for it and her observation on perimeters shows unclear reasoning (isoperimetry is actually possible only in the particular case where the scale ratio is 1:1). Davide B. does not consider the information somehow implied in the text that the rectangles involved must be isoperimetric. He builds a second rectangle in a 2:1 scale from the first, by determining its dimensions. However, his work on the ratios between the corresponding sides of the two rectangles is correct, and after something very close to a cross-check between dimensions, he concludes that this is a proportionality situation.

Fourth Situation: (The Shed)
Contextualization: The problem was given in the second part of the third year.

SAMPLE PROTOCOLS RELATED TO SITUATION FOUR: THE SHED

• Ingrid: Anna 2/7, Giulia 3/9. Giulia has 1/3 of white paint, Anna has less since if she'd had 2/6 it would have been 1/3 but she has 2/7 instead, and that's a little bit less. So Giulia's paint will be lighter than Anna's.
Second Question: giving the same denominator to all fractions one can see that the amount of white is progressively higher. 2/7, 3/8, 4/9 that is 144/504,189/504, 224/504 and the paint gets lighter and lighter.
Third Question: Anna will never reach Giulia's color because as she keeps adding cans of both colors the amount of white increases continuously and passes above Giulia's: in fact, from less than 1/3 (2/7) of white she passed above 1/3 (4/9) and got to 1/2 white (5/10): 3/9=6/18=9/27=12/36 – Giulia; 4/9, 5/10, 6/11, 7/12 – Anna's additions.
Fourth Question: Giulia will never reach Anna's color because adding one can at a time for each color the difference is always 6: 3/9,4/10, 5/11, 6/12, 7/13.
Fifth Question: Yes, it's a proportionality situation because everything is based on a color that must be equal to another but with different values. It's obvious that these two friends knew nothing about proportionality because adding cans on both sides doesn't help.

• Cristina: Anna 7:2= 3.5 – for each white 3.5 blacks (7/2). Giulia 9:3=3 – for each white 3 blacks (that is 9/3). They're not the same color. Anna's color is darker.
Second Question: Anna: 3 whites + 8 blacks, 8:3=2.6– for each white 2.6 blacks. The color is whiter, then.
Third Question: No, Anna will never obtain Giulia's color, because each time she adds cans the color gets lighter and lighter so she'll never get to 3 blacks for each white.
Fourth Question: Anna 12/7, 12:7= 1.714285. Giulia: 10/4, 10:4=2.5, 11/5 11:5=2.2. No, it just gets closer but it's not the same.
Fifth Question: Yes because white is directly proportional to black. If the ratio changed, the color shade would change also.

• Martina: 2/7, 3/9, 2:7 =0.28, 3:9= 0.33 the ratio of the two mixes is not the same. The darker side will be Anna's because: N N N N N N N (Anna) N N N N N N N N N (Giulia)
Looking at the ratio, too, you can see B B B B B
that whites distributed on blacks are less in Anna's paint than in Giulia's.
Second Question: 2:7=0.285 , 3:8 = 0.375 Anna's color gets lighter and lighter.
Third/Fourth Question: Anna 2/7, 3/8….9/14……14/19, Giulia 3/9,6/18, 9/27…18/54…..24/72 Anna will never reach Giulia's color because between Anna's numerator and denominator the difference is always 5 which is not the case with Giulia.
Fifth Question: You certainly can't do that without proportions.

• Davide A.: The two sides of the house will be different because 2/7= 18/63, 3/9=21/63 and 18/63 < 21/63 – Giulia's color will be lighter.
Second Question: It will get lighter because 2/7=16/56<21/56=3/8
Third Question: Anna will never reach Giulia's color.
Fourth Question: Also here it will never reach the same color because of the "stuff" I said before, also (this is for the third question too) the difference between numerators in the initial fractions is 4 and between denominators is 3.
Fifth Question: The girls want to reach a situation of proportionality, that is, they want to scale up the amount of paint to reach the same shade, even though they don't get it. They want to find the equivalent fractions but they got the method wrong. I mean, proportionality was their goal, but they don't manage to use it.

• Davide B.: Anna 2/7 Giulia 3/9 With a single white can Anna mixes 3.5 black cans, while Giulia mixes only 3 blacks for one white, so Anna's color is darker.

Second Question: If Anna adds one can of white and one of black, the color gets lighter because 2/7 = 0.29 , 3/8=0.38, 4/9=0.44, 5/10. Third Question: She won't get it. Fourth Question: 7/12=0.58 Anna, 6/12 Giulia – with that procedure she won't get it, if she wants to get it she needs to mix 29 cans of white and 50 cans of black 29/50=0.58. Fifth Question: I'm not sure, maybe not, because you don't do proportion by adding but by multiplying the fraction by the same number.

Table 5.

The protocols show the enactment of different strategies which show how the students' activity is an actual exploration: they resort to their skills and use all their resources. Their references are in many cases equivalence classes and fractions, in other cases the decimal representation of fractions. Graphic representations sometimes contribute to reasoning. Arguments are explicit and show complex skills as well as uncertainties and difficulties. We will now analyze and comment on each of the protocols above.

Ingrid, in the first question, explores the situation by analyzing ratios as inferred from the text, while in the second question she chooses to compare the fractions by adopting the same denominator, therefore eliminating the simultaneous variation in both denominator and numerator, in order to achieve a better control on the color shade obtained. In order to answer the third question she analyzes the subsequent changes to Anna's shade, at the same time working within Giulia's classes of equivalence. Like other students, in the fourth question she is misled by the data, which are less straightforward (7/12 is more difficult, also from a perceptual standpoint, than 1/3), and by the iterative process of adding cans. In comparing the ratios obtained after subsequent additions, her attention focuses on the constant difference between denominator and numerator: Ingrid, in fact, emphasizes how Giulia's strategy, where the difference between denominator and numerator is 6, is different from Anna's where the difference is 5.

Davide A., like Ingrid, resorts to fractions with a common denominator to facilitate comparisons and the analysis of the situation. Specifically, he determines the ratio between white paint cans and black paint cans and correctly interprets the results he obtains by stating that Giulia's color will be lighter. As far as the fifth question is concerned, Davide A.'s argument suggests that the student has finally set his skills straight and attained a good degree of confidence in facing proportionality situations.

Cristina prefers referring to the number of black cans needed for each white can rather than the other way round, whereas Ingrid, for instance, analyzed the situation by distributing white cans on black cans. Cristina employs all the analysis strategies she has at hand: not only she expresses the ratios in term of decimal representation but she also articulates her answers to the first and second question using the typical language of proportionality relations "for each set of… a given quantity of…" She answers the third question correctly but without giving any reason, simply stating that Giulia's shade will never be reached by Anna. In order to answer the fourth question she goes back to her initial strategy: she analyzes the ratios using their decimal representation and underlines their variation citing two examples. Then she shows a good grasp of the situation by stating, in the fifth question, that the change of ratio leads to different color shades.

Martina explores the situation by initially determining the decimal value of ratios distributing white paint cans over black paint cans, and infers that the two shades will never be equal by numeric comparison. Initially, she has trouble interpreting the ratios, and only after resorting to iconic representation she recovers the meaning of the ratios she had determined earlier. The iconic strategy therefore becomes an interpretative key, which is vital in handling the ratios between numeric data. In particular, the graphical-iconic check is favored, at least initially, by the fact that 9/3 represent an integer: black is therefore distributed over three white cans, and starting from this analysis Martina correctly manages also Giulia's choice of colors, thereby mastering the meaning of the increasing ratio. The third question shows how Martina, while evidently having a good grasp of equivalence classes, does not see the need to make a comparison with the initial situation. Specifically, she considers Giulia's and Anna's situations at the same time. Her final comment to the comparison is interesting though, while being true, it does not justify her negative answer: the best answer would have been that 2/7 and 3/9 are not in the same equivalence class. Finally, her answer to the fifth question is brief but correct: she recognizes "proportional reasoning" as the only way to obtain the right shade

Davide B. directly skirts numerical calculations by cutting in half, in his iconic representation, a can of black paint. In the second question, however, he processes the ratio only through an approximation, which nevertheless allows him a correct analysis and comment. Often, though, the use of decimal numbers leads students to a superficial and approximate analysis of the situation. In fact, Davide B., while checking both representations of rational numbers, reaches the wrong conclusion by stating that it would be necessary to mix with 29 white cans with 50 black cans, not realizing that 0.58 is not the exact decimal representation of 7/12 but only an approximation thereof. He answers the fifth question sincerely, expressing his uncertainty, but stating that proportions are built through equivalent fractions, which in turn are obtained by "multiplying a fraction" (the student is actually thinking of multiplying both numerator and denominator by the same amount) while Anna and Giulia are proceeding by additions.

OVERVIEW

Let us consider an overview of the students' behaviors. When analyzing their work, only a small part of which has been included here, it is clear that their behavior in facing proportionality problems depends on several variables and includes a number of development phases. As reported in the related literature, one of the most frequently encountered obstacles, especially at the beginning, is the mistake of resorting to the additive model and not to the multiplicative one. It is a strategy that the students use quite often at an elementary level, and can work as a bridge to a multiplicative strategy. Resorting to such strategy, however, is not always an immediate or conscious effort; often, the solution is reached procedurally, as shown in this example: "I have to calculate the number of spoonfuls for 240 grams: 40 + 40 = 80, 80 + 40 = 120, 120 + 40 = 160, 160 + 40 = 200. So I need 5 spoonfuls."

Among those who resort to the multiplicative strategy there are some who spontaneously employ the reduction to unit; that is to say, they reach the solution to the problem via the intermediate stage of calculating the value corresponding to the unit, even when it is not specifically requested. From an operational standpoint, at first the students have difficulties in divisions and reverse processing, as well as in interpreting the text and the variables involved: the confuse dividends and divisors or resort to the less convenient unit. It has also been noticed that often analysis and comparison between tables, even with the cleverer students, proceeds by separate categories and columns wherein the students mostly look for additive relations; therefore, the choice of the numerical data involved and the order of request in a problem situation is particularly important.

Requesting comparison (see "The Shed"), while leading to more difficulties, is the best choice to let the unique characteristics of proportionality become apparent, and to allow the students to notice them. It is vital never to assume that students *consciously* resort to the resolution strategies they employ: to that end, it is imperative to require a verbal description of both the procedural choices and the answers (see Davide A., 3rd situation). The tools and resolution strategies that the students were able to employ in order to identify the proportionality relation were numerous: in fact, besides putting to use the "for each set of … a given quantity of…" language construct and the table, they resorted to comparing ratios, using proportions, graphically representing the numerical data on the Cartesian plane or, in a geometry context, checking for homothety. Especially by virtue of the cognitive grasp of equivalence classes, proportionality situations involving several ratios have become, for most students, easier to manage and analyze. From this standpoint, proportions were perceived as a particular case of equivalence of fractions with a single pair of values. (An example of this is the behavior of students when facing the 4th problem, "The Shed").

As for the path's productivity, it can be said that the early approach to proportional reasoning as we suggested it, while being onerous at first for both the teacher and the students, leads in the long term to a strengthening of language accuracy and is especially valuable for less apt students. It has been noticed that studying problems focused on the "for each set of… a given quantity of…" language construct facilitates the spontaneous use of reduction to unit; moreover, that same language construct contributes to stressing the correspondence between classes of quantities and induces the use of correct resolution strategies. At the end of this teaching path most students have developed and strengthened an approach and analysis method which has become more and more conscious, especially concerning relational connections between the variables involved.

An important factor was resorting to tables as tools to highlight the relation between quantities involved in order to identify corresponding pairs and to facilitate the use of the language of algebra, as early as in the first activities, and then to proceed with Cartesian representation and with viewing the relation as a function. One somewhat unpredicted aspect is worth noticing: some students, having difficulties with rational numbers, have often resorted to a known situation in geometry (a lesson on rectangles where the difference between sides remains constant or the ratio between sides remains constant) in order to check for proportionality in a context that was different from the reference situation. Therefore, it can be said that sometimes an experienced reference context is important for students especially if they have difficulties with learning. For example, there have been positive results in analytic geometry: in a question that required the discussion of relations such as $y = kx$ and $y = mx + n$, some of the students assumed that x and y might have been sides of rectangles, and by representing and discussing the problem graphically (overlapping rectangles on the Cartesian plane) whether the relations represented reductions or enlargements, they determined, which of the two sets of relations were proportionalities based on the graphs they obtained. That is to say, the students passed from the analysis of rectangles to the more complex analysis of the two functions.

From a methodological standpoint, collective discussion after a group effort has proven to be an efficient teaching strategy for achieving proportional reasoning even for the less apt students, due to the support of thoughts and reflections expressed by their classmates. However, this activity requires the teacher to pay special attention in observing interpersonal dynamics and to closely monitor the development of arguments in order to avoid contrasts or stalemates.

The followed path and the operational activities carried out, due to their pervasive character, built the foundation for consolidating correct conceptual models at the end of the three-year cycle, as evidenced by the multiplicity of solution strategies they resorted to in individual tests during the third year, a long time after the lesson on those specific aspects. There have been significant achievements in personal maturity, though not in the same way by all students, regardless of the notions acquired; the fact that some students with good problem-solving skills sometimes fell back to additive strategies makes us consider the fact that they are in a proximal stage of development and that the development of these skills is a long-term process that needs to be followed up.

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BEHAVIOR AND DIFFICULTIES OF STUDENTS INVOLVED IN EXPERIMENTAL ACTIVITIES AIMED AT AN INTRODUCTION TO THE STUDY OF LINEAR OPTIMIZATION PROBLEMS

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ABSTRACT
This work stems from the experimental implementation of a teaching and learning project outlined by a group of mentors together with a group of teachers (including the paper's author) and referred to as a teaching sequence aimed at favoring students' approach to linear optimization problems. The paper deals with behaviors and difficulties encountered by students, aged 12-13, tackling problems that required the identification of intervals for variables (given quantities) and the representation of these intervals through verbal, algebraic, tabular and graphical languages. The detected difficulties were classified in different typologies, depending on the required skills, ranging from mathematical to linguistic and relational ones. Our teaching experiment provides evidence that these difficulties can be overcome in the long term, by enacting a teaching methodology that favors exchanges among students, negotiation of meanings, choice of the most suitable tools to represent problem situations, search for shared solutions, reflection upon progressive achievements.

1. INTRODUCTION

In Italian schools, the traditional approach to equations and inequalities often pays little attention to formal and syntactic aspects of algorithmic solution: at the same time, increasingly higher difficulties are introduced on the basis of the equations' degree and the number of unknowns. At the end of the lower secondary school students (aged 13-14) tackle linear equations and inequalities with one unknown, whereas in the higher secondary school, higher degree equations ad inequalities are dealt with, together with systems of equations and related Cartesian representation.

This teaching practice presents various inconveniences: (i) at motivational level, many students are not sufficiently stimulated by a repetitive activity which only entails an extension of the application of algebraic rules; (ii) we miss out the chance to convey the idea that equations and inequalities might be instruments to model real life situations; and (iii) Cartesian representation is introduced too late, as a simple transposition of acquired algebraic notions: the opportunity of contributing to the construction of algebraic knowledge is thus missed out.

A different approach views algebraic expressions as an instrument to model real life situations: this approach can be enacted in the early years of students' mathematical education. Linear optimization problems turn out to be particularly interesting, as the problem situation itself suggests which constraints and conditions should be respected: in this way, students are engaged in an attempt to transcribe them in mathematical language. In the comparison of the different statements produced, students are involved in a process of negotiation of both meanings of the enacted symbolic structures and tools available to get to an algebraic representation of the situation.

In order to manipulate the algebraic written expressions enacted in the representation, students need to carry out a syntactic elaboration connected with their own interpretation of the initial real life situation. The latter can possibly be retrieved in any phase of the work, through a semantic control over the transformation process and its results.

In this case, students are aware that the written equations and inequalities they constructed are actually related to the initial situation that brought them about, and that, due to their structure, the same equations and inequalities can potentially be extended to a wider category of situations.

The Cartesian representation plays a double role in this context: on the one hand, it provides a working environment where relations constructed by students in algebraic form shape up and get a geometrical meaning. On the other side, it is an instrument that can explicitly show the involved relations, and at the same time, suggest solution strategies, interpretations and extension of the initial problem situation. In order to make the Cartesian plane play this double role, we need to design problem situations that require the Cartesian representation as a useful solution instrument. Not only so, we need to propose Cartesian representations of already modeled relations that might enable students to reconstruct the problematic features of the initial situation through an interpretation of the representation itself in a new light.

2. THEORETICAL FRAMEWORK AND AIMS OF A TEACHING EXPERIMENT

The presented approach, based on both comparison and merging of different forms of mathematical representation (algebraic, tabular, and graphical) refers to the theoretical framework underlying the *Program for International Student Assessment* (PISA), where 8 different indicators of competencies are set: (1) thought and reasoning; (2) argumentation; (3) communication; (4) modeling; (5) formulation and solution of problems; (6) representation; (7) use of symbolic and technical language of operations; and (8) use of teaching aids and instruments.

Each of the listed indicators is then specified in terms of *reproduction* (capacity of solving problems making use of acquired knowledge), *connections* (capacity of relating knowledge from different areas) and *reflection* (capacity of critically applying acquired knowledge, including a meta-cognitive attitude).

Problems proposed by PISA, highlight the importance of mathematical *modeling,* as an autonomous thinking process which comes before and supports the subsequent *representation*, and the indicators of competencies listed above come to be transversal content items. In the work presented here, they are rather applied to more homogeneous issues, relating to the representation of inequalities and the solution of linear optimization problems.

Specification of disciplinary objectives involved in the teaching experiment goes with identification of other categories of objectives, more transversal, of linguistic, metacognitive and relational type, strictly linked with curricular objectives. Correspondingly to listed objectives, we need to consider pre-requisites necessary to tackle the proposed activities as well as the long term results to be achieved, divided in different, but open and flexible, categories. The net of relationships linking these different components is represented by the map in Figure 1. Objectives are specified as follows:

(1) Curricular Objectives: they are linked to specific mathematical topics;
(2) Transversal Objectives: they are mathematical and, at the same time, able to become instruments for interpreting and representing real life, thus being helpful in the solution of problem situations;
(3) Linguistic Objectives: they are strictly related to the previous ones, and linked to the capacity of expressing oneself and arguing in natural language, mastering the different linguistic registers (verbal, symbolic or graphical);
(4) Metacognitive Objectives: they concern reflection on one's way of knowing and comparison with others;
(5) Relational Objectives: they intervene in the collective construction of knowledge and embrace listening skills, co-operation and mutual respect.

It is important to underline the double aim of the present work. On the one hand, we aim to share the attention posed on students' learning processes through a description of the different moments of the teaching experiment we implemented. On the other hand, by presenting the methodology underlying the design and implementation of this teaching experiment, we aim to give teachers the opportunity to reflect upon a possible attempt to carry out an original teaching experiment, both for content and for modalities that differ from traditional ones. In this work, we made a clear effort to get distant from the frontal lesson model, and open the way to a constructive type of teaching: teachers involved in this effort should be aware that they need to be open to experience new modalities and question themselves.

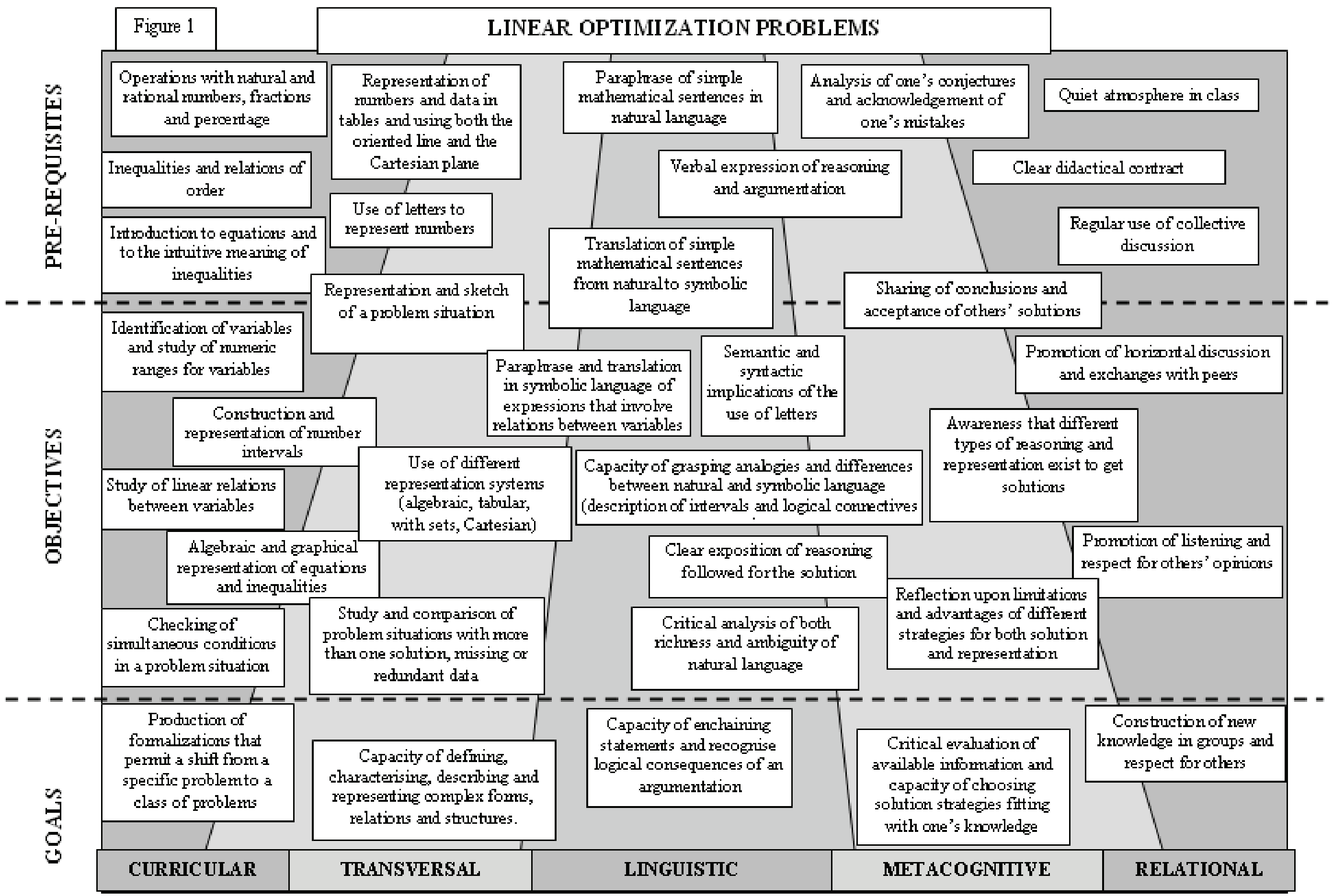
Figure 1
LINEAR OPTIMIZATION PROBLEMS
PRE-REQUISITES
OBJECTIVES
GOALS
Operations with natural and rational numbers, fractions and percentage
Inequalities and relations of order
Introduction to equations and to the intuitive meaning of inequalities
Representation of numbers and data in tables and using both the oriented line and the Cartesian plane
Use of letters to represent numbers
Representation and sketch of a problem situation
Paraphrase of simple mathematical sentences in natural language
Verbal expression of reasoning and argumentation
Translation of simple mathematical sentences from natural to symbolic language
Analysis of one's conjectures and acknowledgement of one's mistakes
Quiet atmosphere in class
Clear didactical contract
Regular use of collective discussion
Sharing of conclusions and acceptance of others' solutions
Identification of variables and study of numeric ranges for variables
Construction and representation of number intervals
Study of linear relations between variables
Algebraic and graphical representation of equations and inequalities
Checking of simultaneous conditions in a problem situation
Use of different representation systems (algebraic, tabular, with sets, Cartesian)
Study and comparison of problem situations with more than one solution, missing or redundant data
Paraphrase and translation in symbolic language of expressions that involve relations between variables
Semantic and syntactic implications of the use of letters
Capacity of grasping analogies and differences between natural and symbolic language (description of intervals and logical connectives
Clear exposition of reasoning followed for the solution
Critical analysis of both richness and ambiguity of natural language
Promotion of horizontal discussion and exchanges with peers
Awareness that different types of reasoning and representation exist to get solutions
Promotion of listening and respect for others' opinions
Reflection upon limitations and advantages of different strategies for both solution and representation
Production of formalizations that permit a shift from a specific problem to a class of problems
Capacity of defining, characterising, describing and representing complex forms, relations and structures.
Capacity of enchaining statements and recognise logical consequences of an argumentation
Critical evaluation of available information and capacity of choosing solution strategies fitting with one's knowledge
Construction of new knowledge in groups and respect for others
CURRICULAR
TRANSVERSAL
LINGUISTIC
METACOGNITIVE
RELATIONAL

3. THE TEACHING SEQUENCE

The paper presents the results obtained within a teaching project designed by a group of mathematics teachers based in Modena (Roberta Fiorini, Sandra Marchi, Romano Nasi and Paola Stefani), involved in both design and experimental setting of innovative teaching sequences relating to the development of pre-algebraic thinking (Fiorini et al., 2006) in the context of the wider ArAl project (Malara & Navarra, 2003).

The teaching sequences constructed by these teachers are structured through different phases, aimed at both the development of techniques for the representation of problem situations in particular those designed for an approach to optimization and choice problems, and the identification of related solution strategies (Pelillo, 2008).

The activities described here refer to the teaching experiment carried out by the author, together with Romano Nasi, the assigned PDTR mentor, in a seventh-grade class (students aged 12-13). In this experiment, which required 12 hours of classroom work during 6 weeks, 4 problem situations, taken from the global teaching sequence previously designed, were chosen and tackled, as reported in the table below.

The chosen problems entail the identification of inequalities and aim at the development of different techniques for representation, ranging from the use of arithmetical and algebraic symbols, to the graphical representation in the Cartesian plane, going through the organization of numeric data in tables. The problems concern contexts students are familiar with, and privilege situations related to home, work and spare time. The requested answer is never a numeric one, as it concerns the different ways in which the various quantities can be represented, as well as the co-ordination of the different representation registers (Duval, 2006).

4. METHODOLOGY

In order to provide a global view of the methodology used in this work, a distinction ought to be drawn between methodology of the work in class and research methodology.

4.1. Methodology of the work in class

Each problem was posed by giving students three worksheets, the first with the text, the second for an individual elaboration and the third with the space to write down shared conclusions reached by the class. The worksheets' structure is reflected in the time scheduling of the teaching interventions:

(1) Phase of individual reflection: students read the text and try to answer the questions individually. In each question students are asked to "represent the situation in mathematical language" and not to solve the problem.

(2) Phase of collective discussion: students propose their own solution strategies; compare them and explicitly state their doubts or the difficulties they encountered, then they are asked to provide arguments supporting their choices so that an agreement on a strategy might be reached.

PROPOSED PROBLEM SITUATIONS:

PROBLEM 1. NOT MUCH MORE THAN THAT

-On a shelf we put some flour and some sugar, as long as we get to 10 kg. How much sugar and how much flour can I put?

Would 2 kg of flour and 5 kg of sugar be ok?

Yes, together they are 7 kg, less than 10.

Would 7 kg of flour and 7 of sugar be ok?

No, because the sum is greater than 10.

Would 6 kg of sugar and 4 of flour be ok?

No, because we must not get to 10 kg.

Let's represent these cases in a Cartesian graph.

a) - Would 7 kg of sugar and 2 of flour be ok?
   - Would 6 kg of flour and 10 of sugar be ok?
   - Would 9 kg of flour be ok?
   - What about 9 kg of sugar?

b) Suggest other quantities of sugar and flour and decide whether they fit with the request not to get to 10 kg.

c) Now that you inserted many points in the plane, can you say in what position are the points which refer to acceptable quantities of flour and sugar?
   And where are the points to throw away?
   Did you identify the two regions which indicate acceptable points and non acceptable points? How are they separated?

PROBLEM 2. THE ILLUSTRATED BOOK

A book for children features 25 fairy tales and it is richly illustrated; each fairy tale is illustrated with a number of images ranging from 8 to 15.

a) What are the minimum and maximum values for the illustrations included in the book?

The book illustrator received from the publisher 30 Euros for each illustration.

b) What is the range of values for the illustrator's pay?

The second edition of the fairy tales book will prospectively have 300 illustrations altogether; the publisher enters into a new contract with the illustrator, agreeing on a variable pay, depending on the complexity of the illustrations, ranging from 20 to 45 Euros.

c) How much might the illustrator get?

Represent the situations using the language of mathematics in the form you see as more appropriate.

PROBLEM 3. BIG FEED WITH CREAM PUFFS

Some cream and chocolate puffs are displayed in a confectioner's window.

The former cost 0.60 Euros, the latter 0.90 Euros.

Pietro buys 8 puffs, we don't know which type though.

Marco, instead, buys 8 to 12 puffs, all of the cheapest type.

Represent the situation using the language of mathematics in the form you see as more appropriate.

In your opinion, which boy might have spent more money?

If only 3 out of the puffs bought by Pietro are filled with chocolate, how many puffs can Marco buy at most, in order to spend less money than his friend?

If Marco buys 10 puffs and Pietro spends more money, how many chocolate puffs can Pietro buy?

PROBLEM 4. THE CAKE

Marco's mum wants to bake a cake with a new recipe. She needs butter and margarine in quantities that may vary within certain limits.

She has small 25 grams butter and margarine packets. Butter ones are not more than 8 and margarine ones are not more than 7.

For the cake she needs 5 to 10 small packets altogether, and she'd better use more butter than margarine.

Represent the situation using mathematical language and ask yourself how many butter packets and how many margarine packets that mum can choose for her cake.

(3) Phase of sharing conclusions: shared choices, sometimes resulting from a negotiation on symbols and possible alternative choices, sometimes coming from the acceptance of a solution strategy as the most suitable to a situation, are transcribed on the last worksheet and become common heritage of the class.

With reference to the table, it is important to underline that this type of whole class process highlights students' individual skills that go beyond the achievement of curricular objectives relating to mathematics. Students need to be able to use their own linguistic instruments to interpret the text as well as to argue about their own solution strategies and communicate them to their peers. Attention paid to the phases of explicit statement, translation, argumentation, negotiation, accounts for the importance attached to the verbal level, often neglected in mathematical activities. Moreover, communication with peers cannot be constructive unless relational conditions are satisfied in the classroom: these conditions entail students' capacities to listen, their will to question their own statements and possibly modify their own opinions and beliefs.

4.2. Research methodology

The teaching action was the result of a collaborative work in which a fundamental role was played by the exchanges between the group of teachers who designed the project and a restricted group of teachers who chose some of the problems for a teaching experiment in their classes.

The teaching sequence was studied by the latter group of teachers before actually implementing the activity in the classroom, in order to predict, through an a priori analysis of the chosen problem situations, which behaviors they might expect from students.

During the activities, the mentor teacher was in the classroom together with a third teacher who played the role of external observer. They contributed to a constructive and critical exchange with the author (the class teacher) on the implemented activity, thus enabling him to an ongoing refinement of his intervention.

Teaching sessions were video recorded and whole class discussions were transcribed: this provided an objective tool for an a posteriori analysis of students' behaviors, for testing them on the achievement of both mathematical and argumentative objectives, for a control over students' participation and relational processes. Moreover, the analysis of discussions' protocols enabled us to detect some difficulties encountered by students when they were exchanging ideas and constructing meanings and, at the same time, to give a tentative interpretation of the roots of these difficulties, using a hopefully more efficient tool than the reading of written individual productions.

5. STUDENTS' BEHAVIORS AND DIFFICULTIES

Students' behaviors in this type of activity brought to the surface the existence, and often the overcoming, of difficulties related to syntactic elaboration in the use of formal algebra, and most of all, difficulties to coordinate the different representation registers and others linked to translation from verbal language (in which the problem situation is presented) to algebraic language; finally logical-argumentative obstacles or obstacles related to communication in the classroom were detected.

By highlighting the different levels of difficulties encountered by students we aim to provide an outlook on the complexity of the activity and propose some points for reflection upon typical behaviors shown by students. Therefore, we will distinguish: (i) difficulties with interpretation and translation linked to linguistic aspects of the text; (ii) difficulties with naming variables; (iii) difficulties with the construction of range intervals for variables; (iv) difficulties with graphical representation; (v) difficulties with argumentation supporting solution strategies; (vi) difficulties with communication; and (vii) relational difficulties.

5.1. Difficulties in interpretation and translation, related to the text's linguistic aspects

Students encountered difficulties with the interpretation of texts written in natural language in tackling all problems; this type of difficulty depends on the fact that, unlike standard problems, where it is enough to extract numbers to be inserted in set algorithms to get a unique answer, in the situations we presented it was necessary to discuss key-sentences of the text collectively, because there were conflicting interpretations.

In Problems 1 and 4 expressions like "some flour and some sugar" or "butter *and* margarine" where the logical value of the connective "and" prevents students from accepting solutions that make one of the two variables null.

In the problems we find locutions that induce students to construct number intervals: "as long as you don't get to …" (Problem 1), "between 8 and 15 illustrations," "between which minimum and maximum value …" (Problem 2), "*between* 8 and 12 puffs" (Problem 3), "between 5 and 10 mini-packets," "no more than 8 and no more than 7" (Problem 4). Translating these expressions into algebraic expressions with "smaller than" and "greater than" symbols (either inclusive or exclusive, depending on the situation) is a helpful moment of exchange for students.

Students themselves identified potentially ambiguous expressions in the texts, which favored the debate and allowed them to make conscious choices.

In Problem 2, for instance, it is said that the book has 25 fairy tales and each fairy tale has between 8 and 15 illustrations, but it is not clear whether each fairy tale should have the same number of illustrations as the others. Students discussed two possibilities and got to the conclusion that, in each of the two cases the interval of possible illustrations is defined by the same extreme values, but the numerousness of the possible solutions changes as the interpretation varies. In order to avoid too complex mathematical interpretations, they decided to consider an equal number of illustrations in each fairy tale.

In Problem 3, the statement "Pietro buys 8 puffs, we don't know which type though" led the class to a discussion about the possibility that all possible combinations of cream and chocolate puffs were acceptable, including the null values for one of the two variables.

Finally, in Problem 4, the expression "quantity that, within certain limits, can vary randomly" together with the subsequent "*you'd better* put more butter than margarine" raised a doubt in some students, about the possibility that not all the conditions posed by the problem were equally binding and that subjective interpretations were allowed, on the contrary. After that they decided to ignore the first statement and substitute the second for "it's necessary to put more butter than margarine," in order to cancel any ambiguity and get to a shared conclusion.

An interesting episode about the problem's text reading occurs with relation to the sentence "with pay that varies depending on the complexity of illustrations" in Problem 2:

Regina: But this time you cannot say that the pay is between 6,000 and 13,500, it's either 6,000 or 13,500… you can't represented like you did before because the total is not between ... is not between 6,000 and 13,500 it's either 6,000 or 13,500.

Marisa: The question asks: how much can the illustrator get …It's among the values we have, not maximum or minimum …

Francesco: I'm not convinced that he gets either 6,000 or 13,500, because it's not that he makes all drawings well or all drawings little detailed ... there will be some with more details and some with less .. I'm not convinced that he can get either 6,000 euros or 13,500...

Mentor: So, in your opinion the pay is between those two numbers …

Francesco: Yes, because it's not true that the illustrator makes all detailed drawings or all not detailed drawings, he will make some detailed drawings and some less detailed ones... there's an infinity of possibilities …for instance a little detailed drawing is 20, a more detailed one might be 30, and a very detailed one is 45….

Alex: But it's saying that it's either 20 or 45.

Luca C.: No, it's saying "from 20 to 45"!

In this case, the whole class is involved in the discussion on the meaning of variability of the pay. A long negotiation leads to the acceptance of extreme values given by the text as well as intermediate ones: this implies an extension of the number field from natural numbers to decimal numbers, with relation to payments expressed in Euro cents.

5.2. Difficulties with naming variables

Once the variables have been identified, one of the most frequent difficulties in this kind of activity is a representation of those variables with letters, i.e. the *naming* of variables.

Abbreviations (for instance "ill.=illustrations") or the use of letters as label are typical of "algebraic babbling" (Malara & Navarra, 2003) in very young students. In the study of Problem 2, for example, Marisa stumbled in both obstacles; to calculate the number of illustrations in the book, she multiplied the number of stories (25) by the minimum number of illustration per story (8) but she wrote "25 x 8 = 200 L", where L stands for the number of illustrations in the book (which is in this case 200), then, correcting herself, she wrote "L = ill x 25".

Another recurrent phenomenon is an attempt to choose the symbol used as the unit of measurement of the quantity represented, as the letter that identifies the variable. In Problem 2, the illustrator's pay was labeled with the symbol €, and, when the teacher invited students to use a different letter, an interesting dialogue between Alex and Cristina arose:

Teacher: So, which letter can we use in this case?

Alex: one…. I don't know, anyone.

Teacher: Except?

Alex: Except for big *i*, little *i*, *c*…the ones we used earlier, substantially

Teacher: Label this variable.

Alex: This D [points to 20] and this B [points to 45]

Cristina: I did not get the criterion he used to choose letters

Alex: Random!

Cristina: But I would choose a criterion…

[Marisa and others agree with her.]

At that moment, the class shared Cristina's need to choose letters through a criterion, whereas Alex, who was aware that the choice of the letter is an arbitrary one, made the mistake of attaching different letters to different values of the same variable.

Episodes like this highlight the need to carry out a whole class discussion to clarify these complex aspects of algebraic formalization. The spontaneous choice of using the initial letter of the variable as symbol for the variable itself makes it interesting to propose situations in which the two variables have the same initial letter (as we purposefully did in the case of "number of cream puffs" and "number of chocolate puffs," Problem 3) to help students get free from the problem's semantics in *naming* operations.

In the problems we proposed to students, these operations were a relevant obstacle, also due to the simultaneous presence of variables related to homogeneous quantities, as it happens in Problem 2, where quantities like "number of illustrations per story" and "total number of illustrations," or rather "payment for each illustration" and "total payment" coexist and vary in different ways in the different phases of the same problem.

An additional difficulty can be found in Problem 3, where the same variable ("number of cream puffs") undergoes different conditions depending on the subject of the situation (Marco or Pietro). In this case, students used the same letter, *a,* to define this quantity, comparing Marco's and Pietro's situations on the basis of numbers, without constructing algebraic relations. If this attempt emerged, we should face the issue of indexing variables; for instance, if we had to state that Pietro's expenditure must exceed Marco's, we should have written:

$0.60\ a_P + 0.90\ b > 0.60\ a_M$.

This example highlights one of the substantial differences between natural language, equipped with an index or a subscript (words that need additional information about the space-time context they are inserted in, to be understood) and algebraic language, where the same variable, in different contexts, must be represented with different symbols (Ferrari, 2004).

5.3. Difficulties with the construction of range intervals for variables

The proposed problem situations required a transposition of conditions imposed over the different variables into mathematical language. The process of algebraic formalization of these relations was gradual and extremely fruitful, starting from the first spontaneous representations to get to a shared and standard criterion for representation.

In Problem 1 the sum of the two variables cannot get to 10. The introduction of the "<" symbol was the first achievement in the representation of the predicate "to be smaller;" among the initial proposals, we actually find those with the use of the sign "minus" for "smaller than" and those constructed as truth tables on the single pairs of numbers examined in the cases where they either satisfied or contradicted the relation:

Alex's proposal: x + y = (- 10)

Marisa's proposal:

2 + 5	7	V
7 + 7	14	X
6 + 4	10	X

Luca's proposal:

F	Z	OPER	T	F
2	5	+	7	
7	7	+		14
4	6	+		10

Francesco's proposal:

V x + y = (-10)
X x + y = (+10)

Discussion brought to the surface the idea, later shared by the whole class, that the symbol "<" might be introduced, modifying the first proposal but keeping the equal symbol, which was clearly viewed as necessary in the expression of any relation: $x + y = < 10$.

In the next phases, due to a comparison with the representation on the line, it was possible to construct number intervals in algebraic form (Figure 2, Video 1).[1]

The constant parallelism between these two representations proved to be useful, also when it came to the study of multiplicative relations among the three members of the expression.

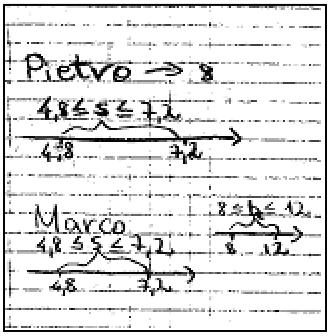

Figure 2: Representation of ranges of expenses on the oriented line by Rexhina, with relation to Problem 3.

In Problem 2, for instance, chains of inequalities like those listed below are constructed: writing natural numbers in non canonical form helps students understand the role played by the multiplicative factor in the transition from one inequality to the next one.

$$8 \leq i \leq 15$$
$$25 \times 8 = 200 \leq 25 \times i \leq 25 \times 15 = 375$$
$$25 \times 8 \times 30 = 6000 \leq 25 \times i \times 30 \leq 25 \times 15 \times 30 = 11.250$$

In this phase, it is important to compare algebraic intervals to segments constructed on the line, in order to highlight the different role played by positive multiplicative factors, respectively higher or smaller than one, in the enlargement or restriction of the resulting interval.

We did not have the chance to deal with the case in which negative factors are introduced, with the consequent inversion of the number interval and a change in the sign of the inequality.

Although students already knew how to use both “greater than” and “smaller than” symbols, some encountered difficulties in the interpretation of these symbols in complex algebraic expressions.

In the study of Problem 4, for instance, Cristina read the expression $5 \leq B + M \leq 10$ as if they were two separate expressions where the “+” sign played the role of conjunction ($5 \leq B$ and $M \leq 10$), whereas Francesco kept rewriting the same relation in the form $5 \geq B + M \leq 10$. This difficulty arose when he was supposed to interpret the expression, assuming that subject of the sentence was the literal expression (B + M) on the right side of the sign; in this case, the quantity “$B + M$” was clearly greater than or equal to 5, but the inequality sign lost its peculiar direction for Francesco (Video 2).

[1] Videos, in Italian, are included on the DVD.

5.4. Difficulties with the graphical representation

At the end of the teaching sequence, in the study of the last problem situation, students were able to represent on the Cartesian plane intervals that they had previously constructed in algebraic form, for the involved variables.

For instance, in Problem 4, relations $1 \leq M \leq 8$ and $1 \leq B \leq 7$ were represented in the Cartesian plane as stripes running parallel to the axes, whereas the relation $B > M$ was represented as a half-plane under the bisector of the first quadrant. However, students got to this high achievement after overcoming serious difficulties, mainly referred to the transition from the concrete level of single numerical cases to the generalization which goes with the use of algebraic forms (Video 3).

Another relation, $F + Z < 10$, regarding Problem 1, was initially studied starting from the pairs of numbers suggested by the problem; later, students moved to filling the Cartesian plane with a high number of points that were satisfying the mentioned condition (and this implied an enlargement of the number field from natural numbers to decimal numbers and induced a discussion about the numerousness of possible solutions). In this way, the boundary line $F + Z = 10$ emerged spontaneously from the class and this achievement could be extended to the study of the next problem situations (Video 4).

The Cartesian representation has the advantage of enabling students to compare conditions imposed by the problem to their algebraic representation, but this is not the only form of graphical representation spontaneously proposed by students. In Problem 1, for instance, Nicolò produced a graphical representation made of a system of axes in which, on the horizontal axis the different particular cases proposed by the problem were represented and numbered sequentially, whereas, on the vertical axis, the quantity $y = F + Z$ was considered (Figure 3). After drawing the line $y = 10$, he associated a point in the plane to each pair of values F and Z (on the horizontal axis), drawing a distinction between points under the line (acceptable situation) and those lying on the line or above (non-acceptable situations) and marked the latter with a frame.

This strategy was shared with the class and this allowed for the construction of a histogram to represent that situation: this enriched the class discussion and students got to the conclusion that different types of graphical representations can be chosen alternatively.

Figure 3: Spontaneous graphical elaboration by Nicolò with relation to Problem 1.

5.5. Difficulties with the argumentation supporting solution strategies

The proposed problem situations present progressively more complex conditions on the variables involved. While in the first problems the main difficulties emerged when relations had to be formalized, as well as in the shift from an arithmetic solution strategy to the use of algebraic symbolism, at the end of the teaching sequence students experienced difficulties in considering the given constraints jointly (Video 5).

For example, discussion on the particular cases presented in Problem 3 imposes that the number of puffs bought by one of the characters should be put into relation with that character's expenditure and, only later the constraint on the second character's expenditure should be studied and interpreted in terms of the number of puffs bought by the latter. Such solution strategy requires a gradual sequence of logical steps and a solid ability to read the constructed tables of data. Difficulties encountered by students in this phase were actually linked to a scarce familiarity with the use of tables (students were able to construct them but not to use them effectively to deduce new information); moreover, students encountered difficulties in using existing information in consequential form in order to produce a logical answer without drawing on additional calculations (Video 6).

In Problem 4 some students clearly showed they found it hard to consider all the required conditions together to identify the possible solutions.

Andrea M., for instance, left some of the given conditions out and modified some others (B is greater than M became $B = M+1$) with the purpose of providing a solution to the problem, but his logical difficulties clearly emerged when he attempted to argue about his solution strategy. He translated the need to have more butter than margarine by writing sums of numbers where the first addendum was 1 unit higher than the second, without keeping in mind that the total should be between 5 and 10 anyway. When his classmates pointed out that the solution $2 + 1 = 3$ was not acceptable, Andrea erased that line justifying this with a non-mathematical argument: "yes, because the pie is no longer good."

These episodes confirm that asking students to produce arguments for the solutions they propose, besides getting them used to expressing their ideas with words, has the advantage of bringing to the surface failing strategies linked to logical weaknesses, and distinguish them from mistakes due to distraction and superficiality.

5.6. Difficulties with communication

Unlike issues related to argumentation, where difficulties are to be related to the capacity of grasping links and connecting information and deductions, communicative difficulties can emerge clearly when students are not able to express verbally their thinking process in the enactment of a solution strategy.

This is a very common difficulty when students are not used to exchanging ideas with peers and expressing verbally their behavior. It is sometimes difficult to decide whether a communicative difficulty hides a difficulty in reasoning or not: this is the reason why we propose an example from which it is clear how a student constructed a solid argumentation without being able to express it.

In problem 3, the question "If only 3 out of the puffs bought by Pietro are with chocolate, how many puffs at most can Marco buy to spend less money than his friend?" can be answered quickly, using the already completed expenditure tables. Pietro always buys 8 puffs, so, if 3 are with chocolate, the remaining ones will be with cream; after getting the expenditure from the tables it will be necessary to verify that Marco does not exceed it, as he only buys cream puffs. Without using tables, all the calculations will have to be repeated from the beginning.

Luca C., following his reasoning at an abstract level, proposed a refined solution that did not entail the use of tables and limited the necessary calculations; on a basis of 5 common puffs for both Marco and Pietro, Luca C. calculated the maximum number of cream puffs with total cost not exceeding that of the remaining 3 chocolate puffs. Exposing this reasoning, made the student get stuck and he could only communicate his own strategy to the class with the aid of a classmate as well as of the teacher who, interpreting his thoughts, helped him complete the sentences he was only sketching

(Video 7):

Luca C.: If Pietro has bought 3 chocolate puffs, Marco can buy ... to spend less money than he did ... [he gets stuck] then for the rest they are all with cream, aren't they?
Mentor: For Pietro it's so, there are 3 with chocolate and the rest with cream.
Luca C.: To spend less money, Marco must get …4 chocolate puffs…from 5 onwards…
Mentor: Marco doesn't get chocolate puffs!
Luca C.: I meant with cream, I get confused…
Mentor: Are you answering randomly?
Luca C.: No, because I counted….so…he, no?...I mean his friend buys 3 chocolate puffs, then, since he only buys cream ones ... I made a calculation ... I did...up to 5 cream puffs they are equal, after that I counted ... those 3 chocolate puffs ...3 x 0.90 is…
Francesco: 2 euros and 70.
Luca C.: Ok, thanks, and 4 x 0.60 is…
Francesco: 2 euros and 40.
Luca C.: …and 5 cannot be done because…
Mentor: …you deduced that…? He can buy four…
Luca C.: Yes, four more than 5.
Mentor: …because they would cost less …
Luca C.: Than the 3 chocolate ones.
Mentor: What if there were 5 cream puffs?
Luca C.: No, because they become ..30[2]… and therefore you can't…
Mentor: …they would exceed the maximum expense.. so…[to the class] did you follow the reasoning? Try to repeat it properly…
Luca C.: If Pietro buys 3 chocolate puffs…the last remaining 5 are with cream, aren't they? If Marco gets 5 more with cream again …since he cannot buy chocolate puffs ... he can buy at most 4 cream ones more ... because when you add them up...
Mentor: Luca is saying that…let us imagine that both get 5 cream puffs, Marco clearly spends less money, because Pietro also bought the 3 chocolate puffs. Now try to increase Marco's expenditure …it's clear that this is what counts …. Luca is comparing the cost of the 3 chocolate puffs to how many cream puffs you can buy with the same money ... he got to see that …
Nicolò: Basically he can only buy 4 cream puffs more …
Luca C.: Er, yes.

5.7. Relational difficulties

Episodes referred to this teaching sequence and based on a continuous exchange between students as well as on an indulgent and not censorious attitude by both mentor and teacher, were characterized by a gradually increasing level of participation by weaker and less confident students, who contributed to the solution of problems and proposed their own ideas.

The study of communication flows in the classroom (Nasi, 2008) allows for a detection of possible relational difficulties experienced by students, besides the different roles they play in interpersonal processes.

A desirable communication style is one where the teacher plays a mediatory role in the discussion and students are able to exchange their ideas without overwhelming their own interlocutors; the use of the second person to address classmates during a discussion is a symptom of good communication in the classroom. This is strictly linked to our own objective, as teachers, that is, to favor actual exchanges among peers, where the use of the third person points to the choice of the teacher as interlocutor.

An opposite example is provided by Alex's attitude in the following analysis of the discussion flow about Problem 1 (Figure 4). Once he started to talk, Alex clearly shows his preference for a direct exchange with teachers, with interventions that are beyond the context of his classmates' discussion. These deviations from the theme tend to

[2] They actually become 3 euros, to be compared to 2.70 euros of Pietro's chocolate puffs

block the previously enacted process and might highlight the student's scarce maturity (and therefore his will to be the centre of attention) and his little self-confidence, which make him escape from a direct exchange with peers on a risky ground.

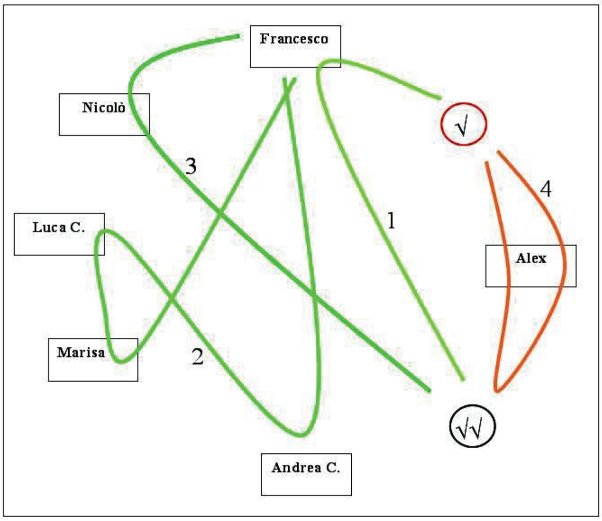

Figure 4. Phases of a discussion:
1 – Start
2 – Debate
3 – Conclusions
4 – Deviation
√: Teacher
√√: Mentor

6. CONCLUDING REMARKS

The teaching sequence about the study of linear optimization problems, proposed to students, aged 12-13, enabled the whole class to apply a heuristic methodology in the search for solutions to problem situations, based on the interpretation of texts in natural language, on the translation into symbolic form and on both graphical and tabular representation of the mathematical relations presented in those problem situations.

Exchanges with peers during whole class discussions carried out frequently, led to the achievement of conclusions shared by everybody and, at the same time, highlighted single students' different behaviors, much more explicitly than the analysis of individual worksheets brought to light.

These behaviors often revealed difficulties at mathematical level, regarding formalization of algebraic expressions and coordination of representation instruments, and at linguistic, logical and expressive levels. In addition, some relational difficulties sometimes influenced classroom-based processes in the learning phase.

In this work, we meant to emphasize difficulties encountered by students in completing the teaching sequence because we believe that a careful reflection on the latter can provide useful tools to design and plan more effective and purposeful teaching interventions. On the one hand, the description of the emerged difficulties highlights the complexity of the teaching intervention we implemented; on the other hand, it provides useful hints about the possible issues to be dealt with by a teacher who wants to carry out a teaching experiment based on a collective construction of shared knowledge. For example, a clear outcome was that it is necessary to give more space to, and intervene more accurately on, interpretative aspects which anticipate representation-related aspects.

The class responded positively to this kind of activity, which did not provide a unique strategy to solve a problem, but rather encouraged students to make proposals about both representation and solution of problem situations.

Students were increasingly motivated by the challenging nature of the proposed situations; therefore, during the whole teaching experiment students kept a high level of interest. The fact that the teacher accepted all the answers provided, including partial and incorrect ones, and proposed them to the class for further discussion, so that they could be validated or refined, also led weak students to participate in the activities with interest.

Individual difficulties were thus gradually overcome in collaboration with the whole class and the identified processes highlighted students' respectful attitudes towards ideas of others.

As concerns teachers, the ongoing comparison of partial results of the activity allowed us to introduce changes to the initial project and to focus on initially neglected aspects. This methodology let us develop issues we proposed here in other parallel classes, achieving similar results in the different cases: this confirms the potential reproducibility of the teaching project.

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AN INTRODUCTION TO PROOF IN ELEMENTARY NUMBER THEORY: TWO TEACHERS AT WORK

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ABSTRACT
In this paper we will present a teaching experiment carried out in two of our classes (grade 10) during the third year of our participation in the PDTR project. The teaching experiment was planned on the basis of teaching sequences and related worksheets designed after experiments implemented in previous years within the project. We will analyze the excerpts of two class discussions proposed during different phases of the teaching experiment. The aim of our analysis is to highlight the different roles teachers assume during different activities in order to stress the importance of teachers' behaviors in helping students grasp the need for a joint work on arithmetic and algebraic aspects and develop a flexibility in both using the algebraic formalization as an instrument to generalize numerical situation and in interpreting algebraic expressions. This paper is also meant to highlight the positive influence of the work of analysis and reflection that characterized our activities within the PDTR project on our teaching profession.

1. INTRODUCTION

The traditional approach to algebra, in which manipulation often obscures linguistic aspects, brings about in students' collective imagery the idea that algebra, with its arcane notations, is a sort of weird "alchemy of letters and numbers" one learns to operate with, only in order to simplify complex expressions. For this reason, many students do not reach a good level of awareness of the meaning underlying algebraic manipulations. For instance, students are hardly aware of the fact that syntactic rules of literal calculus correspond to the application of properties of the operations they already met in the study of mathematics. Students who do not know whether $x+x$ is $2x$ or x^2 clearly show this. Moreover, many students do not think of an algebraic expression as a representation of the formalization of a property (for example that the property of a number of being "multiple of 4" may be translated by means of the expression $x=4k$) or even that a suitable manipulation of a certain algebraic expression may highlight one or more numeric properties (for instance, the equivalence $8k=2\cdot 4k$ highlights the fact that $8k$ is twice a number which is a multiple of 4, whereas $8k=4\cdot 2k$ highlights that $8k$ is four times an even number; or rather, the equivalence between $3k+12$ and $3(k+4)$ allows us to grasp the property of the former to be divisible by 3).

For these reasons, we deem extremely important that mathematics teachers aim to lead their students to view symbols as instruments with an attached meaning and become aware of the fact that algebraic language may be an effective instrument to explore situations, generalize, communicate and prove. The experiences we document here, referring to our third and last year of participation in the PDTR project, lie in a wider research project and concern proof in elementary number theory (for a detailed description of the theoretical framework, see Cusi, 2008).

The activities we propose here may be located within those aimed at making students develop some essential competences indicated by the OCSE/PISA project as typical of the mathematical activity. In particular, we work on valuing competences related to *thought and reasoning*, *argumentation*, *communication* and *representation* (with reference to groupings of both connections and reflections). Our idea of dealing with exploration in elementary number theory as a field of experience for an early introduction to proof, is based on our belief that "the most formal and abstract ways for both representing and reasoning emerge, over time, as a consequence of an active engagement in activities designed to help the development of informal ideas" (PISA 2003-Program for International Student Assessment, 45).

Working in the intersection between arithmetic and algebraic settings helps students develop a sense for algebraic representations, which not only translate hypotheses: they also support subsequent reasoning aimed at the achievement of new properties revealed by the interpretation of obtained results. During this activity, students have the chance to achieve the indicated skills and are led to reflect upon the sense attached to the use of algebraic language. The latter becomes an instrument for translation, investigation and communication results as well as for generalization and proof. In this way, students project themselves onto mathematical activities that turn out to be more complex and less usual than those related to grouping of connections.

Another objective of the proposed activities is to convey the need to justify the validity of one's conjectures and avoid merely trusting one's intuition. Expressions like "you can see it" or "it's obviously so" should be abandoned in favor of the need to create chains of reasoning that confirm one's intuition. This objective is transversal to other disciplines as well: when students understand the importance of learning to communicate their own reasons and related justifications, mathematical language is kind of re-qualified as an effective instrument to develop and define competences that can be used in other contexts.

In the present paper we illustrate a teaching experiment carried out in parallel classes for the first time, on the basis of teaching sequences and related worksheets designed after experiments implemented in previous years within the PDTR project.

2. PRESENTING THE CLASSES

Students who participated in the project attended the second year of a Liceo socio-psico-pedagogico,[1] whose students are not particularly fond of mathematics: in the last decade this school specialized in linguistic, historical and sociological areas, thus offering a great number of courses in related subjects. In this framework, participation in the PDTR project becomes particularly meaningful for both the classes and the school, because the activity implemented in the classes goes beyond its main objective (teacher training) by valuing mathematical language as an indispensable and effective instrument to both develop and define transversal competences, like argumentation in different subject areas. Moreover, taking into account the role played by this type of school (i.e. to form future primary school teachers), it is clear that practicing a new mathematics teaching may contribute to a more effective teaching of this subject in primary school,

[1] A Liceo socio-psico-pedagogico is a higher secondary school with a social and pedagogical specialisation. The involved school is the Institute "Matilde di Canossa" in Reggio Emilia.

thus breaking a vicious circle which tends to consolidate a negative image of mathematics in society.

The involved classes are two tenth-grade classes of the school, with social specialization (28 students) and socio-psycho-pedagogical specialization (20 students). In both classes, the teaching experiment was proposed at the beginning of the school year. The classes, which worked simultaneously, never followed mathematical extra-courses and showed some logical and linguistic gaps. Nevertheless, students were interested in the novelty of the proposed topics and in the working modalities: they were all willing to exchange ideas when they worked in groups and trusted their teacher's choices.

3. METHODOLOGY OF WORK IN THE CLASSROOM

Our work with students was structured through moments of work in small groups, followed by collective discussions on the results of the activities, orchestrated by teachers. As we shall clarify later, this work was carried out in 6 phases, each characterized by some worksheets that were used by the different groups in their work. We chose to split the classes in homogeneous groups (based on competences and motivation) so that student might find their own space in the interventions, without being afraid that the most clever would precede or correct them. During the subsequent collective discussion (for the theoretical framework and a detailed description of the adopted methodology, see Cusi, 2008), teachers reported all the different solutions formulated by various groups on the blackboard: they also included incorrect proposals from both the cognitive and linguistic viewpoints, in order to make no one feel excluded or not considered enough. In the discussion, the objective "correcting the worksheet" was not achieved through a frontal work by the teacher, but rather after a careful collective analysis of the solution strategies produced by students. Teachers worked in order to make the results of the collective discussion be experienced as a conquest by the whole class group.

4. THE PROPOSED TEACHING SEQUENCE AND ITS DIDACTICAL OBJECTIVES

The teaching sequence we proposed in this third year was essentially split into 6 phases: (1) activity of translation; (2) activity of interpretation of expressions with variables; (3) analysis of the truth/falseness of statements and justification of provided answers; (4) exploration of numerical situations, formulation of conjectures and related proofs; (5) analysis of proving strategies; and (6) construction of proofs for given theorems.

For the *first phase* two worksheets were elaborated, concerning translation from natural to algebraic language and vice versa. The objective of this phase was to make students aware of the fact that algebraic formalism is not only a set of operations between numbers and letters: it is mainly a symbolic language, which reveals numerical properties with immediate clarity. The *second phase* concerned the search for conditions on variables that were written in algebraic language and that needed to satisfy particular properties, already formalized in the previous phase. In the *third phase,* students were invited to reflect upon the truth or falseness of some statements with a further request of "justifying their answers." The objective was to lead them to use algebraic formalism as an instrument to explore numerical properties and to argue about their validity. Another objective was to make students reflect upon the validity of numerical examples in refuting statements and upon the fact that the same example is not sufficient to investigate the truth of propositions, since it is impossible to make an infinite number of numerical tests. The introduction of the *fourth phase* was necessary to foster in students the pleasure of conjecturing, making them feel the need for proving. In the *fifth phase,* students were required to analyze four different proofs of one single statement (analogous to that proposed by Healy and Hoyles, 2000), so that they could reflect upon the correct translation of the hypothesis, as well as upon the acquired awareness that numerical cases only just confirm but do not prove anything. Besides analyzing the correctness of proposed proofs, one of teachers' objective was to call students' attention to the difference between the argumentative and the proving activity: arguing is viewed as a useful modality for seeking the solution to the problem, but not an exhaustive one, if it is not organized in a consistent deductive process, regardless of the use of either natural or algebraic language. Objectives of the previous phases could be applied in the last and *sixth phase,* dedicated to the proof of statements provided by teachers.

Particularly interesting were Phases 2 and 4, both for the activities proposed and for the didactical achievements, due to an active participation by students and a relevant degree of relational exchanges between teachers and students.

4.1 Activities of Phase 2

Through the activities proposed during Phase 2, students learned how to grasp the link between values assumed by variables in the algebraic expression and the properties of the expression itself. Students were required to investigate the properties of some particular algebraic expressions (like those reported in Table 1). The three exercises did not explicitly ask students to "justify their answers," the aim being to work with students towards a collective construction of a justification for the answers provided during the subsequent whole class discussion.

1) Determine for which values of k (natural number) the following conditions are satisfied:
$k+3$ is a multiple of 3
$k+3$ is even
2) Determine for which values of b (natural number) the following conditions are satisfied:
$3b$ is a multiple of 3
$3b$ is even
$3b$ is a mutliple of 6
$3b$ is odd
3) Determine for which values of n (natural number) the following conditions are satisfied:
n^2 is even
n^2 is odd
n^2 is a multiple of 4
n^2 is not divisible by 9

*Table 1***.** Activities proposed during Phase 2.

4.2 Activities of phase 4

During Phase 4, students were given 4 problems (see Table 2). Each problem is characterized by the request to investigate particular numerical situations in order to produce a conjecture to be proved later.

1) Consider a natural number; determine the difference between its square and the predecessor's square. Which regularity can you observe? Would you be able to prove your statement?

2) Consider natural number; determine the sum of this number and the two subsequent natural numbers. What can you observe? Would you be able to prove your statement? Add to the natural number the three subsequent numbers. Can you observe the same properties? What can you say if you add the four subsequent numbers to the natural number?

3) What can you say about the difference between the cube of a number and the number itself? Give some numerical examples to support the requested conjecture. Would you be able to prove your statement?

4) Write down a natural number with two digits and the number you get from this by swapping the digits. Calculate the difference between the greatest and the smallest number. Repeat this procedure starting from other numbers with two digits. Which regularities can you spot? Would you be able to prove your statement?

Table 2. Activities proposed during Phase 4.

The correct construction of numerical examples that translate procedures written in the task, confirms that some objectives were achieved in previous phases, whereas promoting exploratory activities fosters in students' awareness of the importance of numerical examples as an instrument of investigation in the formulation of conjectures. Previous collective activities (Phase 2) concerning the construction of justifications underlying one's answers, enabled students to trust their conviction that they need to justify their intuition: this helped teachers to shift the focus of the collective discussion onto the identification of the correct proving strategy, either through the analysis of routes proposed by students or by proposing other routes. Regularities spotted that students could not justify, represented another chance to make them aware that the simple numerical testing was not suitable for an actual proof. Teaching students to organize a correct deductive procedure to validate the intuited regularity required students themselves to translate verbally expressed propositions into algebraic language.

A careful reading of the identified regularity allows for an algebraic translation of the hypothesis and, through suitable transformations and remarks, enables students to reach the expected result: they not only have to construct simple proofs in algebraic language but also discover the pleasure of "giving a universal and irreproachable justification" to their own intuition thanks to an effective and communicatively clear language. The problems also offer a chance to make students acquire metaawareness about algebraic language as a powerful proving instrument that may interpret (in the case of hypotheses' translation) and reveal (after the elaboration of the hypothesis) algebraic properties, becoming a useful and convenient vehicle of reasoning paths that can be hardly carried out through natural language only.

5. ANALYSIS OF SOME SIGNIFICANT MOMENTS: SOME EXCERPTS FROM COLLECTIVE DISCUSSIONS IN PHASES 2 AND 4

We report here two excerpts from discussions: the first one relates to the work carried out by teacher T1 in the class with social specialization and the second one relates to the work carried out by teacher T2 in the class with psycho-pedagogical specialization.

5.2 Example 1

This first example refers to the first problem of the worksheet related to Phase 2: "Determine for which values of *k* (natural number) *k*+3 is a multiple of 3." Initially, various groups exchanged ideas on the answers they had provided to this request and the class reached the shared conclusion that "*k* must be a multiple of 3 in order for *k*+3 to be so as well." The excerpt we present here relates to one specific moment of the work within the second part of the discussion, aimed at enacting algebraic representation as a modality to express generality and to support one's reasons.

1 T_1: Now we have basically decided that this should be …it seems to be the condition on *k* so that the quantity *k*+3 is actually a multiple of 3. So how can we be sure about that? [The class looks at T_1 as if they are saying "we are already sure"]
2 Students: Let's try to put a multiple of 3 instead of *k*.
3 T_1: Ok. What should I do to try?
4 C: Like 9+3.
[T_1 writes 9+3 on the blackboard; whispers of disapproval come from the class; T_1 therefore stops and looks at them]
5 T_1: Shall we work on a particular numeric case then?
6 A: No, we are making an example here; *C* took a multiple of 3 randomly, we might take another one.
7 T_1: Therefore if we want to be sure …
8 A: Ah! Mister, I understood! Let's take a number which is not a multiple of 3 and let's see if that doesn't work. If it doesn't work it means that it's right.
9 T_1: So you are saying that in order to see that multiples of 3 work we show that all others don't work.
10 C: For instance, let's take 5…
11 T_1: Yes, but if all others don't work, who tells you that those, i.e. the multiples of 3, work?
12 A: Let's try.
13 T_1: Can I try with all others?
14 A: No, Mister, we have no time!
15 T_1: I wouldn't have time enough in all my life, for long it might be! Ok? [they agree]. Let's get back, A. suggested we could try with all, but, since all others are an infinite number, we cannot and, however, we were saying that even though with non multiples the quantity *k+3* is not a multiple of 3, nobody guarantees that it is so for values of *k* that are multiples of 3. So, what shall we do?
16 B: Let's try with the multiples of 3.
17 T_1: I try with 6, 9, 12, 15, etc…
18 A: Mister, we must do something!
19 T_1: Yes, all right, but the ways you suggested are not feasible! Sorry, but we understood that we should try with all the multiples of 3 and that the multiples of 3 are infinite, but we have found an algebraic expression that included all those numbers [in the previous discussion students had got to formalize the property of k of being a multiple of 3 through the expression 3x]
20 C: Yes, $k=3x$.
21 T_1: So, instead of trying with 3, 6, 9, …what could I do?
22 A: Ah! $3x+3$ then you take out 3 then…
23 T_1: Correct! Why not using the expression $3x$? Let's use the expression $3x$ instead of *k* then let's apply a minimum of algebraic calculations! Do you mind if I substitute *k* for the generic value it should get to verify the requested condition? [the class nods the teacher to carry on] … Ok, so the expression $k+3$ becomes $3x+3$ [T_1 writes on the blackboard $k+3=3x+3$] and now what shall I do?
24 A: Let's take out 3!
25 T_1: Ok, let's take out a factor 3 [T_1 writes $3(x+1)$] and can I say that $3(x+1)$ represents a multiple of 3?
26 Students: yes.
27 T_1: Why?
28 C: Because it's 3 times…
29 T_1: Times something which value is not important now! Ok?
30 Students: yes.
31 T_1: Have we shown that $k+3$ is a multiple of 3 if *k* is any multiple of 3?

Table 3. An excerpt from a discussion on activities of Phase 2: the use of the representation in general terms to construct proofs.

The teacher at the beginning of the discussion (line 1) tried to raise doubt in students about the correctness of the remarks produced up to that moment. Her objective was to stimulate them to search for a justification to support their remarks. Students, probably due to the self-evidence of the result they got to, did not seem to feel the need to justify their statements, but nevertheless proposed a strategy. Some students actually suggested to consider some numerical examples with the aim to prove that if k is a multiple of 3 then also $k+3$ is (lines 2-4-6 and 17), whereas a student, A, suggested that examples might be used to prove that if k s not a multiple of 3 also $k+3$ is not (lines 8-10). Both strategies entail a totally inductive view of the concept of proof. The teacher underlined then that it was impossible to consider all the infinite numerical examples (line 15) and suggested (line 19) to use the previously constructed generalization ($k=3x$). A immediately grasped this input (lines 20 and 22), remembered the expression $3x$ previously constructed, she mentally substituted $k=3x$ in the examined expression and suggested the correct syntactic transformation to be operated on $3x+3$ in order to highlight the property to be proved (taking out a factor 3).

At this point, T_1 re-voiced what A noticed (lines 23 and 25) and underlined the importance of the syntactic operations carried out (i.e. substitution and subsequent taking out of a factor as application of the distributive law) asking students to make explicit the interpretation of the expression $3(x+1)$ as a multiple of 3. Some points need to be clarified with relation to the teacher's statement "then we apply a minimum of algebraic calculations," in line 23. This statement clearly subtends the basic idea that all might be reduced to pure algebraic calculations. Students thus ran the risk not to grasp how syntactic transformations were consequences of properties of the elementary number theory setting considered (for example, taking out one common factor is an application of the distributive law). Another point that can be picked relates to the sense of transformations: they were neither random, nor consequences of "a minimum of algebraic calculations," they were rather finalized to an objective and thus guided by anticipatory thinking (Boero, 2001). The teacher's statement was obviously unintentional, but it was a starting point to underline the need to pay attention in thinking over words to be used during class discussions.

We chose this excerpt to underline how, since the first activities teachers' aim was to convey the idea that algebraic expressions turn out to be instruments for generalization and, as a consequence, for providing a general justification of the regularities observed in correspondence with particular cases.

5.3 Example 2:

The following excerpt refers to problem 1 set during Phase 4, by teacher T_2 in the class with psycho-pedagogical specialization. We chose this excerpt since it highlights the value of collective comparisons of the different strategies emerged during work in small groups.

1. T_2: I remind you that you must interrupt me if you don't understand or if you want to point out something, to clarify something different to what the various groups will say.

[V talks for his group]

2.V: We wrote that the difference between the square of a number and that of its predecessor is always the predecessor of twice that number, that is, basically, x is our number, so x^2

minus the square of the predecessor $(x-1)^2$…

4. L: We also solved the problem in this way, but we made a mistake because we only put x squared $(x^2 – 1)$ ...and therefore when we read it, it was not the square of its predecessor, but the predecessor of the square

9. T_2: Yes, there was a mistake in the initial translation

10. L: ... and then we interpreted the result differently

12. V: So, then we have solved this operation …

13. T_2: Question: so, didn't you start from numerical examples?

14. V: No

15. T_2: But you started from the formalization of what was written in the text ...

16. V: Yes

17. V: Making calculations you eliminate x^2 and get $2x-1$ so we got to the conclusion that the result is always the predecessor of twice a number. We have seen that basically we have x^2 to start, which is what we have already written, we subtracted the square of its predecessor and if we analyzed the single addends of the binomial [with the term binomial V refers to $(x-1)^2$], we'll see that the addendum x [points to x in $(x-1)^2$] is squared, therefore if we take it out of the parenthesis we have $–x^2$ so we can eliminate it with the initial x^2, rather, for the second addendum [pointing to 1 in $(x-1)^2$] which will always be 1 for every x that we find, and therefore for any natural number we'll always have -1

18. T_2: … inside the parenthesis

19. V: Yes, yes inside the parenthesis, then if we take it out we'll get -1, right? Then, after that it becomes -2x that turns into +2x once you take it out

20. T_2: So you have developed the square of a binomial and simplified whatever you could …

21. V: It would become $2x-1$ and so we got to the conclusion that this subtraction always gives the predecessor of twice the natural number as result.

22. M: We have done the same way

23. T_2: Just one moment! So, the conclusion, the regularity, let's call it so, observed by V's group is that doing what is indicated by the text, one gets twice the starting number -1 as equivalent value, i.e. the predecessor of twice the starting number.

24. T_2: I would call G to propose their way of tackling the problem in order to see whether we get to the same conclusion

25. G: We have done three numerical examples, i.e. we took three pairs of subsequent numbers [she writes the considered pairs in columns on the blackboard], 3 and 4 then 4 and 5 then 5 and 6. [she thinks about it for a while] Eh…hold on! We have done the square of 3 which is 9, the square of 4 which is 16, the subtraction between the two squares is 7 and we found the same regularity of odd numbers with the other examples.

26. T_2: They have done a number of numerical examples in which they observe the result of the subtraction.

[G. carries out the subtraction in the three examples written on the board]

27. G: So, we have noticed that we got odd numbers

28. T_2: This was your first observation

29. G: Then we have seen that the difference equals the sum of the two numbers we started from and then we saw that adding up the two numbers their sum was equal to the difference of the squares of the two initial subsequent numbers, as a matter of fact 3+4=7, 4+5=9 e 5+6=11.

30. T_2: This is a different aspect again. You noticed that the result equals the sum of the two initial numbers. This is interesting. This is a result that you, V. did not highlight when you interpreted the result of the difference as the predecessor of twice the initial number. Now, I'm interested in knowing how you justified this!

31. G: You mean how we carried out our proof? We wrote $a^2-(a-1)^2=b$, with b belonging to odd numbers and then to natural numbers … [she writes $b=a+(a+1)$ on the blackboard]

32. T_2: You have written the same formalization as V's group, but then I don't get the conclusion that equals b ...

33. D: [D is in the same group as G] In the sense that the square of a minus the square of its predecessor that gives b odd as result and then we wrote that their sum is b itself [pointing to the equality] $b=a+(a-1)$) that is $a^2-(a-1)^2=a+(a-1)$.

34. T_2: Did you mean to formalize what you noticed?

35. G: Yes.

36. T_2: I can see something understandable, but I want to understand how aware you are of what you wrote. Starting from our premise [the statement], that "equals" there puzzles me … It

seems to me that you only wanted to formalize, that is to write in algebraic form what you noticed, but writing a^2-$(a$-$1)^2$=a+$(a$-$1)$ is not a proof.

37. D: Yes we only expressed what we got with letters.

38. T_2: They have noticed that this result, besides being twice a minus 1 is also the sum of the initial numbers, so if this is true, there must be a link between what is written at the left-hand member and what is written at the right-hand member; the question is: can we make this link become evident? Try to work on this… Can you factorize, somehow, the difference of squares? … [unease in the class because of the request] … Let's start from another viewpoint, let's compare $2a$-1, i.e. the result of the first expression, to a+$(a$-$1)$, the sum of the two values. This is also a point on which you can make your remarks …

39. A: If we erase the parenthesis we get a+a-1 that is $2a$-1 equals to the first written quantity.

40. T_2: So, if you have obtained this type of result from the development of your initial situation, and they have identified the sum of the numbers, we got to the same result. The difference lies in the fact that in the first manipulation, that of V., this feature was missing, perhaps it was possible to see that it was an odd number and it was interesting to see that it was twice the number minus 1, but certainly the other aspect was not visualized, i.e. that it was also the sum of the two numbers of which the difference of squares is made. You can see the same thing by breaking $2a$-1 in the expression a+$(a$-$1)$, but they are not the same thing, they are rather two equivalent expressions that say different things.

Table 4. First excerpt of a discussion on activities of Phase 4: value of the comparison of different strategies.

This discussion allowed teacher T_2 to highlight the different approaches to the problem of searching for regularities and subsequent proof tackled by different groups. An initial highlighted approach was that by the group of V, who "skipped" the phase of exploration with numerical examples and immediately got to the formalization of the expression to be examined, in order to manipulate it and interpret the obtained result. It was noticeable that the words used by V may lead to analogous remarks as those made with reference to the previous discussion. V's use of terms like "solve an operation" (line 12) or "make calculations" (lines 17, 19 and 21) revealed a procedural view associated with a lack of awareness of the fact that syntactic transformations must not be merely fruit of pure calculations: they should be guided by reasoning. The correct interpretation by V of the syntactic transformations carried out is to be nevertheless underlined, together with her ability to describe her group's approach to the examined problem to both class and teacher. The teacher remarked (lines 13 and 15) that the group explored the situation not through the analysis of particular numerical examples, but rather directly through the analysis of the constructed algebraic expression. The risk of this approach was associated with the lack of a goal, which might lead to be rigid in the interpretation of the algebraic expression obtained at the end of the manipulations carried out.

T_2 tried to highlight the importance of a synergy between numerical and algebraic aspects by calling in G's group, who had a sort of complementary approach, engaging in the numerical exploration only. As a matter of fact, the teacher (line 30) tried to highlight how the approach based on the analysis of particular numerical cases enabled G to interpret the result in a different way.

The moment when G illustrated to the class how her group "justified" the noticed property, gave T_2 a chance to point out how the formalization of a conjecture was not a proof for the conjecture itself (lines 34 and 36).

After having the proof that students were aware that setting the equality a^2-$(a$-$1)^2$=a+$(a$-$1)$ was a consequence of the formalization of what they noticed (line 37), T_2 tried to stimulate the class to prove the correctness of that remark (line 38) through the algebraic manipulation of the left-hand member of the equality. The class got stuck after the first stimulus provided by T_2 (she proposed to factorize $a^2-(a-1)^2$ as the difference between two squares). We notice how students, although they studied notable identities, are not able to draw on them to solve cases that would require a particularization to not any number: in fact, they are generic but linked by a relation, since the second is expressed as a function of the first one. At this point T_2 tried to change the point of view and suggested to compare the expression $a+(a-1)$, which should be the goal of the algebraic manipulation to be carried out, with the expression *2a-1,* obtained by V's group. This second approach turned out to be fruitful, because the class grasped it and exploited it to prove the examined identity (line 39). Another point that needs to be highlighted relates to students' difficulties in providing different interpretations of one single algebraic expression *2a-1*: this block was probably due to the dominance of the representation of *2a-1* as an odd number and therefore of *2a* as an even number, instead of being viewed as sum of *a* with itself. T_2 concluded the discussion (line 40) underlining the different interpretations of the result pointed out during the discussion: *2a-1* can be simultaneously interpreted as both an odd number and the predecessor of twice the initial number, but the equivalence between the expressions *2a-1* and *a+(a-1)* also allowed one to highlight that it was the sum of the considered number and its predecessor. In this way, the teacher emphasized the importance of flexibility in interpreting the algebraic expressions progressively obtained.

6. SOME CONCLUDING REFLECTIONS

We devote this paragraph to the exposition of some of our reflections:
(1) the first group of local reflections aims to point out some of our main remarks as well as to propose a comparison between the two transcripts we have analyzed;
(2) the second group of reflections, more global, aims to highlight some of the main results referring to the meaningfulness of the proposed teaching sequence, focusing on effects on students as well as on the importance of the role played by teachers during the various phases of the sequence itself.

6.1 Some brief remarks on the presented episodes:

With reference to the two episodes we presented and analyzed, we propose two remarks: the first one relates to the different approaches the two teachers used with their classes, whereas the second one aims to underline the sense of these activities within our project.

Reading the protocols we proposed, it is possible to detect the different attitudes of the two teachers in the discussions reported in previous paragraphs. T_1, in fact, seems to be more directive, whereas T_2 tries to make students bring to the surface some points for the evolution of the discussion. The different approach to the management of a class discussion is due to the fact that the two transcripts refer to different moments of the teaching sequence, occurred at distant times. While T_1 tackles one of the first activities proposed to the class and, therefore, plays the role of model for her students, T_2 must test what the class learned from the previous activities and hence makes a step backwards, acting as an interlocutor who wants to be convinced and keeping the right distance not to inhibit students.

We believe that the two proposed discussions highlight the importance of an in-depth work in every phase of the teaching sequence. In both protocols, emphasis is on the shared work aimed at making students develop a flexibility in both using the algebraic formalization as an instrument to generalize numerical situation with proving purposes, and in interpreting the different algebraic expressions, equivalent to one another, that are progressively obtained through syntactic transformations carried out on the initially constructed expression as a generalization of the situation under exam. The second excerpt, in particular, underlines the importance of comparing different strategies in order to enable students to grasp the need for a joint work on arithmetic and algebraic aspects.

6.2 Some general remarks emerged from the analysis of the whole teaching sequence

The remarks we propose in this paragraph are finalized to underline the fruitfulness of this sequence for students as well as to highlight some points related to the role played by teachers.

Our analysis of the collective activities in the different phases of the project enabled us to highlight the difficulties we encountered in trying to help students get rid of the idea that numerical examples they analyze might guarantee the truth of one's remarks. Convincing students that numerical examples are not sufficient as a validation instrument, is a fundamental step towards the acquisition of new awareness about the fundamental role played by algebraic language as an instrument to generalize, communicate and prove. Understanding the need for justifying, independently on the context and the used instrument, becomes meta-awareness which makes students aware of the importance of arguing about their reasoning, so that the others might not only understand but also appreciate its validity. This qualitative leap helps students grasp the clarity and communicative immediateness of algebraic language, as used within proofs. In this way, a proof becomes a *transparent argument,* where all the information used and the reasoning rules are clearly expressed and open to criticism (Hanna, 1995).

Another important aspect relates to the influence of the work of analysis and reflection that characterized our activities within the PDTR project on our teaching profession. We actually realized that sometimes some of our students' failures can give a clue about our inappropriate teaching attitudes, as clearly highlighted in the analysis we proposed earlier. In this sense, an essential role was played by both an a posterior discussion in the class group, and the review of the class process. Living again one's lessons as a spectator stimulates teachers to reflect upon both the modalities for managing the different teaching and learning activities and teachers' role as a guide towards a goal.

A problem we particularly felt as teachers is the involvement of the whole class group. For this reason, we decided to start our discussions from the productions of weaker students, carrying out a collective reflection upon the reasons for possible failures. This enabled us to analyze and deal with critical points that might have led to further gaps in the future. With the aim of stimulating interventions to favor an active and productive participation, we did not interrupt students who were justifying their reflections through an incorrect use of specific terms, providing precise comments and corrections in other, more appropriate, moments. A mistake is thus turned into a resource, because it becomes an opportunity to make the whole class group reflect, giving them the chance to get a deeper insight on the discipline and consolidate acquired knowledge. Working with mistakes brought about explicit interventions by stronger students, who disentangled problematic knots and facilitated communication. Different interventions encouraged by teachers, as a basis for a subsequent discussion on the related justifications, promote critical and reflexive skills in students, who become aware protagonists of their own knowledge, not only in terms of minimal competences but in terms of connections and reflection.

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PROFESSIONAL CHANGE IN TEACHING ACTION: CASE STUDIES ON FUNCTIONS AND ALGEBRAIC MODELING

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ABSTRACT
The professional development of mathematics teachers is a hard but necessary goal to achieve, in order to improve the teaching-learning process, as emerged by the PISA assessment results. It needs that teachers act in their daily didactic action according to a theoretical reference frame, with respect to research results in educational science and mathematics education and, not less important, that teachers train themselves to deeply reflect on their own practice as knowledge mediators. In this paper we want to report on our change as school teachers, during our three-year participation in the PDTR European project, presenting and discussing some excerpts from our class activities about linear functions and both equations and inequalities.

INTRODUCTION

The PISA assessment results on mathematics literacy of 15-year-old students at an international level presented evidence that our students have great difficulties to reach the mean level of literacy with respect to the competences fixed by OECD. These circumstances widely impressed the authors, school teachers for several years, who began wondering where the school failed. To be sincere, the surprise, or the worry, due to PISA results was soon replaced by a sort of sense of familiarity when we had the possibility to read the PISA tests. Indeed, in PISA 2003 Italian students lost a considerable part of test scores on the open-constructed response items, where also non-mnemonic, or simply applicative, abilities are involved, but we were used, and perhaps it would be better to say that we were almost resigned, to accept that our students get stuck in front of a problem, even if they were able to handle the mathematical techniques fit to solve the problem (equations, linear systems, and so on). Indeed, too often, in teaching mathematics, teachers limit themselves to identify mathematics with a list of contents, as indicated in national curricula, in a sequential and temporal order, while OECD defines mathematical literacy as "the capacity to identify, to understand, and to engage in mathematics," and, more explicitly, Niss, who individuates eight specific competences, affirms: "Mathematical literacy cannot be reduced to mathematical knowledge and skills. Such knowledge and skills are necessary prerequisites to mathematical literacy but they are not sufficient" (Niss, 2003). It is by now widely recognized that the difficulties experimented by students in doing mathematics are not only due to the intrinsic nature of mathematics; rather, it is necessary to reflect on the way mathematics is taught and to invest on teachers' professional development. Also in Italy things are changing, and official guidelines[1] have been released in order to focus teachers' attention on the need for changing goals and methods in mathematics education, in accordance with research results (see, e.g. Malara, 2003; Mason, 1998, 2002; Jaworski, 1998, 2003; Ponte, 2005; and Shoenfeld, 1998).

By now we believe that teachers cannot decide to modify their teaching practice only on the basis of naïve beliefs, but they need a concrete innovation in their knowledge in educational science in general and in mathematical education in particular. In this paper, we are going to describe our own change due to the participation to the three-year project PDTR (Professional Development of Teacher-Researchers, www.pdtr.eu), which has given us the opportunity to reflect on our own practice as school teachers and to greatly modify it, on the basis of our own cultural enrichment. In the next section we illustrate the theoretical frame which we refer to, and then we will focus our attention on our choices and our changes on the basis of the theoretical frame, by means of some class excerpts, in order to witness our professional change.

THEORETICAL REFERENCE FRAME

The great number of variables which contribute to the success of classroom interaction between teachers and students do not allow for searching for a universally valid didactic strategy, with respect to contexts and situations. On the other hand, one of the aims of the PDTR project is the "research-based transformation of classroom practice," and we are by now convinced that it could be achieved only when teachers are able to set the micro- and macro-decisions necessary in a specific didactic context in a general theoretical reference frame.

For a general theoretical framework concerning mathematics education research, we refer to Tall (2004), as we believe that wide knowledge of the main psychological theories in cognitive growth in mathematics, from Piaget onward, until the recent results obtained by means of brain imaging techniques, constitutes the first objective for a teacher, according to Guidoni (2005), who indicates four ingredients to plan and evaluate a teaching-learning path. The second step is to improve our model of cognitive dynamics, both natural and "forced" by the teaching mediation.

It is worthwhile to remind us that several authors in mathematics education make a distinction between procedural and structural aspects of mathematics, as an intrinsic dichotomy of the discipline. In Skemp (1971) a distinction is made between the instrumental understanding of mathematics, consisting in learning formulas by heart and in the ability to apply them to solve exercises and problems, etc.; and a relational understanding of mathematics, which consists in being able to reason to solve problems, recognize connections, etc., while in Sfard (1991) there is an onto-psychological duality between operational conceptions (related to processes, algorithms, necessary actions to understand mathematics contents), and structural conceptions of a mathematical notion, but Sfard herself affirms "operational and structural conceptions of the same mathematical notion are not mutually exclusive. Although ostensibly incompatible, they are in fact complementary" (1991).

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[1] Retrieved on 2008, February, 17th, from the website www.pubblica.istruzione.it/normativa/2007/allegati/dir_310707.pdf

A similar, more general, duality is described in cognitive/neurophysiology research in order to model cognitive behavior. Bruner (1986) distinguishes between "narrative" thought and "paradigmatic" thought, each of which determines a method to arrange our experience and to organize our knowledge. According to Bruner, these two terms are irreducible one to another, but it is necessary to develop both of them in a complementary way in order to have a complete, correct vision of the reality. In Iannece and Romano (2008) these two dichotomies in mathematics and in cognitive research are re-read by means of recent developments in cognitive sciences and experimental neurobiology (see Houdé, 2000; Changeux, 2002). They distinguish between procedural and structural thought and affirm: "the procedural thought (in mathematics), like the narrative one, is favorite from a neurophysiologic point of view … the structural thought (in mathematics), instead, like the paradigmatic one, substantially is a cultural acquisition and requires an appropriate teaching mediation." On the basis of this assumption, teachers have to modulate their didactic choices with the aim of developing both procedural and structural thought in their students. But this goal can be "chosen" and then achieved, only if teachers already engaged in their own professional development dedicate a lot of time to improving their capacity of observation and reflection. According to Mason (2002), "noticing as an intentional stance towards our profession is enhanced by various practices which serve to support both 'picking up ideas' and 'trying them out for ourselves."

OUR DIDACTIC CHOICES AND SIGNIFICANT CHANGES

In our PDTR team, reflection on our own practice is supported by a strict collaboration among us: we are used to discuss students' protocols in order to analyze both our own and our students' attitudes and reactions, in order to increase sensitivity to noticing what was previously neglected and to develop the needed awareness to design future choices. We learned that it is necessary to observe, by means of recordings, the way we propose any activity to our students, to reflect on the range of possible variations of the activity, and to observe students' behavior: we are by now much more interested to know the cognitive processes which guides students' answers than before, when we simply looked for the correct ones; furthermore, we let them be free to question and we *want* to listen to their answers, even incorrect, as "the knowledge grows up from the answers to the questions" (Postman & Weingartner, 1973). We compare our experiences and materials just as we expect our students to do, and in this way we got convinced that sharing and comparing each other yield and facilitate reflection about contents and about individual ways of thinking (Vygotsky), and improve the capacity of looking at things from different points of view. This approach produced better implementations of the activities in our classes, and, above all, improved our self-awareness as teachers, as we want to show in this paper. We have chosen to present, among the events which we experienced and which we discussed, some scenes drawn from the job on linear equations, led by two of us in the first two years of upper secondary school: "Liceo Scientifico" and "Liceo Psicopedagogico."

The activity was preceded by explorations on natural numbers, aimed at improving students' competences in making conjectures, argumentations and proofs. In fact, students had to communicate their discoveries and to validate the hypotheses they put forward, and the algebraic language turned to be a valid instrument to support their attempts. For a detailed description of the didactic reasons which led us to this choice, we refer to Iannece and Romano (2008). Strictly connected with this kind of activities are the concepts of relation and function, so we decided to deal with linear equations and inequalities by means of functions. In this way we continued asking our students to look for relationships, not only in a numerical context.

In the classroom

The first three scenes are excerpts of the activity on linear equations; the last one refers to the activity on inequalities.

Scene 1

We decided to introduce students to linear equations, starting from the analysis of three problematic situations, intentionally simple and similar in their structure.

1) About… pocket-money!

One of my friends, Roberto, has got a job and no longer needs the weekly pocket-money that by now amounts to 25 Euros. His father doesn't agree with him, so they decide that Roberto's pocket-money will lose 1Euro every week, and meanwhile Roberto will contribute to his little brother's pocket-money as follows: every week he will give him € 2 less than half the amount of the pocket-money that he still receives.

a) Try to represent the relationship between Roberto's pocket-money and the gift to his brother in a symbolic way, as you prefer, also showing its domain and range.
b) When will Roberto give € 10 to his brother?

2) About… investments!

Mr. Smith, one of the owners of a hotel chain, is monitoring his company's gains. He proposes to his older partners to invest in the refreshment sector half of the gain obtained in the previous year. His partners suggest bigger caution in the investment, so they agree to invest half of the gain obtained in the previous year less € 200.000.

a) Try to represent the relationship between the previous gain and the next money investment in a symbolic way, as you prefer, also showing its domain and range.
b) When will the company invest € 100.000?

3) About… jeans!

During a meeting of Benetton retailers, it is shown an analysis of the sales of the last model of jeans. The question is if it is plausible to affirm that each of the retailers has sold 3 jeans less than half of the numbers of jeans he bought.

a) Try to represent the above relationship between the bought jeans and the sold ones in a symbolic way, as you prefer, also showing its domain and range.
b) According to this, how many jeans did a retailer sell if he bought 10 jeans?
c) Some retailers cannot recognize themselves in the above analysis. Can you explain their reasons?

Most of the students quite simply translated the above situations in algebraic language as follows:

(i) in the case of the weekly money-pocket and the related gift: $g = \frac{p}{2} - 2$, where g is the value of the gift and p the value of the pocket-money;

(ii) in the case of gain and related investment: $i = \frac{g}{2} - 2$, where i is the value of the investment, and g the preceding value of the gain;

(iii) in the case of jeans: $s = \frac{b}{2} - 2$, where s indicates the jeans sold, and b the jeans bought.

Then the question was to establish when the gift amounted to 10 Euros, the investment to 100,000 Euros and the jeans sold are 10.

Valentina: In the case of the weekly pocket-money, in order to know when Roberto gives 10 € to his brother, we must add 2 € to 10 € and then multiply by 2. In the case of investments, add 200,000 to 100,000 and then multiply by 2. As for the jeans, we must add 3 and then multiply by 2. It is all the same!

Maria: We have to go "the opposite way" of what is written in the problem.

Francesco: Yes, the opposite of the operations above.

In the following, we directly report considerations and comments by the teacher:

> I notice that the students use the term "opposite" (*contrario* in Italian) to refer to inverse operations. At this moment, I realize that the discussion is turning away from the traditional direction, but in an interesting way. The "functional" approach to equations is greatly influenced by the previous activities on numbers, as the majority of students are looking at the arithmetic operations involved. I let students talk about the problem and in this way they led me to treat the equations as an "inverse problem," which can be naturally solved by inverting the shown operations. The whole class unanimously accepts a method of resolution of what I have called equations, and myself too am going to watch to linear equations in normal form, i.e. $ax + b = c$, from a different point of view. Indeed, in the usual teaching practice I would have justified the resolution techniques for the equations through the so-called equivalence principles, focusing the attention on the equality sign; and this is the way equations are treated in the majority of school text-books; this is a structural way to look at an equation: a static object, where the couple (x, y) is rather hidden and therefore difficult to be seen by the students. I always noticed that with this kind of approach students act as in a ritual scheme, according to Vinner (2000): they learn practical rules, and each mistake is attributed to inattention, and I observed total lack of control about the exactness of the solutions.

At the end of lessons I was a little worried: this feeling, experienced as a conflict between the old and the new didactic practice, may be interpreted as the need, difficult to achieve, to merge the structural and procedural aspects related to equations. So I decided to propose to my students the following problem:

> The swimming pool
>
> Andrea and Barbara go to the swimming pool, but they have chosen two different options: Andrea pays an annual fee of € 140 plus € 2 for every admission; while Barbara pays € 20 for a membership card and € 8 for every admission. They are going to spend the same amount this year. How many admissions can each of them have?

My students described the problem using the following equation: $140 + 2x = 20 + 8x$, where x is the number of admissions. They had some difficulties with choosing the operation to be inverted, because of the presence of x at both members, so I posed the following question:

T.: Does the description of the problem change if, at the first admission, they also buy a swimming cap there?

Zaira: No... because of the invariant property of equalities… that we have also studied in geometry.

The word "invariant" helped me introduce the first equivalence principle, and then the second one, with further questions of the same kind. Well, the possibility to look at the equation in a double way does not constitute a problem for the students, as it is manifested, in our opinion, by the following answer:

Zaira: When the unknown is on both sides of an equation, I use the equivalence principle to obtain an equivalent but simpler equation as, for example, $3x + 7 = 22$; then, the solution is obtained by inverting the arithmetic operations.

Scene 2

The students discussed some more questions on the problem of jeans.

T.: A retailer bought 15 jeans. How many jeans does he sell, according to our previous formula?

Elena: 4 or 5.

Antonio: No, it is not possible to give an answer in this case: 15/2 isn't a natural number, and such a situation is meaningful only for natural numbers.

Elena: But the equation can always be solved...

Antonio: Yes, but if you want it to match the story, you must use the appropriate numbers, *even natural numbers* in this case!

Anna Laura: Further, also if a retailer bought 4 jeans, this model cannot be applied, because neither of negative numbers is significant in this case.

Antonio: So, we can use even natural numbers greater than 4.

T.: Have we the same problems about the appropriate numbers for the investments and for Roberto's pocket-money?

Roberto: No, we have no problems with money, neither from rational numbers, nor from negative numbers that can represent a debt.

In this scene, students wondered which algebraic structure was better to describe the various situations. I noticed that the study of the three problems, in different contexts, naturally led students to formulate hypotheses. I tried for years to explain from the desk the difference between hypothesis and thesis, without appreciable results, while this time my students realized that in every situation there are some choices to be done before, and some consequences to accept afterwards. More and more often, I realize that noticing the methods students use to face problems helps me choose and adopt different teaching strategies, in order to solve the difficulties due to conceptual knots of the discipline itself, and this is a direct consequence of my new awareness about the theoretical frame.

The following scene describes another classroom activity on linear equations, which opened the way to inequalities. Once again we want to focus our attention on the teacher's actions in conducting the activity. This time, the students were more interested in a graphic approach to the problem.

Scene 3

We report the reflection of the teacher:

I ask my students to represent the function $y = 2x + 6$, letting them freely choose any tool to represent it. A small group of students fills a table, while another group decides to use also the Cartesian plane, but most of them assign to x only positive values.

Tommasina: If I use only $x > 0$, the ordinate increases too much! So, let me try to give x negative values!

Maria: It seems that all the points are lined up!

I realized that the use of the Cartesian plane and of a graphic representation opened the door to negative numbers; in the meantime, the two different approaches strengthened the concept of one-to-one correspondence between the ordered couples of real numbers and the points on the plane. The relationship $y = 2x + 6$ was perceived as a constraint for the points, all lying on a straight line.

T.: If I ask you $y = 0$?

Rossella: $y = 2x + 6$, therefore $2x + 6$ must be = 0... That is... the ordinate is zero... I must give a value to x in order to get 0.

Tommasina: …but it is like to solve the equation $2x + 6 = 0$.

Rossella: Yes, $y = 0$ for $x = -3$. It coincides with the intersection with the x axis.

T.: And if I ask you $2x + 6 > 0$?

Rossella: If $x > -3$ the straight line "goes upward" and I get positive numbers, while if we give values smaller than -3 the line "goes downward."

The graphic approach allowed students to consider in a dynamic way the straight line: by imaging to move on the straight line, they have created a link between the sign "+" and the term "onward," and between "–" and "downward." I was surprised that they were able to distinguish between the role of the variables x and y, i.e. to understand that solving an inequality, like $2x+6>0$, requires to determine the values of x corresponding to $y>0$, which used to be a difficult conceptual knot in the past.

Scene 4

Again the study of inequalities is intertwined with that of functions. We again gave students the problem of the swimming pool, opportunely modified in order to induce students to face inequalities with the unknown variable at both members.

In order to go to the swimming pool, Andrea can choose between two different options:
A) Annual fee of € 140 plus € 2 for every admission;
B) € 20 for a member-ship card and € 8 for every admission.
For how many admissions is the second option advantageous?

Rossella: If the times are x, I have two possibilities: 140 + 2x and 20 + 8x.

They decided to represent these two expressions as two lines on a Cartesian plane, but encountered some difficulty because of 140, so I suggested using a non-monometric reference system. After a brief resistance due to the strong idea of the "same unity of measure" on both axes, my proposal was accepted, since it was recognized as useful for operational purposes. They observed that the two straight lines intersected and discussed the meaning of this point in the problem.

Rossella: Let me solve the equation: 140 + 2x = 20 + 8x
Miriam: Why "=", if I want the smallest one?
Rossella: I can establish the convenience if I know when I spend the same amount of money!
Maria: When the two quantities get the same value, this means that I spend the same amount, therefore for 20 times I spend the same and afterwards the expense B goes upward.
Maria: The expense B goes upward compared to the A one, therefore… Andrea spends more.

I want to specify that Rossella obtained the value 20 from the graph and not from an algebraic resolution. Frequently, I was on the point to intervene and to shorten discussions, but the curiosity to listen and to reflect on the conjectures that students put forward prevailed. I was quite surprised that the students focused their attention on the point where the quantities are equal and only afterwards they explored the convenience. For me it was "natural" to directly translate this kind of problems into an inequality, but I chose to follow the students' hints, so I had the possibility of noticing the variety of their reasoning. It is worthwhile to emphasize that a similar scene was experienced in another class. In the same way students naturally searched the solution of the equation, but they explored the convenience by means of a tabular representation of the linear function, while my students' strategy was strongly supported by the graphic representation of the straight lines.

At this point I proposed, not without an evident forcing, to study the sign of the function $y=\dfrac{3x+2}{4x+8}$.

Roberta: I see two equations and therefore two straight lines.
Maria: Then we must find what happens before –2 and compare with what happens between –2 and –2/3, and then with what happens after –2/3.
Tommasina: For example, for x = –1, I see that a half-line is above and the other one is under the axis. So the fraction is negative.

To solve the inequality $\dfrac{3x+2}{4x+8}>0$, the students immediately decided to use the graphic representation of the two linear functions y = 3x + 2 and y = 4x + 8, and they successfully caught the meaning of a fractional inequality, with a graphical version of the signs rule, without explicitly naming it. They realized that the points of intersection of the straight lines with the x axis mark the critical points, but for me it keeps in shadow the graphic representation of a fractional function, with the problem of its point of discontinuity. Indeed, as the discussion was about functions, I would expect a proposal of drawing a graph of the fractional function, and I was surprised with their pragmatic approach. At this point, I decided not to intervene anymore, and the lesson came to an end. The continuous game between functions and inequalities appears interesting, but it is a puzzle for my management task, and it has been an occasion to weigh the difficulties we meet when students are free to follow their cognitive processes.

CONCLUSIONS

The difficulties in school mathematics education revealed by the PISA assessment results in many countries, and in Italy, in particular, naturally got each of us, school teachers, involved in a process of innovation of the usual didactic practices and in the change of didactic goals. But the possibility that changes will really occur and expand to the majority of teachers is strictly connected with the need for a process of updating for teachers themselves with respect to the newer research results in education science in general, and in mathematics education in particular. However, this is a necessary but non sufficient condition: we believe that it has to be accompanied by a deeper reflection on one's own practice as didactic mediator.

In this paper we have described some scenes experienced by the authors in their classrooms, focusing our attention on teachers' reactions and reflections. On the basis of the theoretical reference frame, we decided not only to observe the behavior of students but to monitor our own performances as teachers, i.e. the way we conduct our lessons. In this way, we discovered that in our past experiences, we often induced the expected answers in our students or underestimated some of students' arguments, simply because we were conditioned by our traditional formation or because we are too concerned with syllabus constraints. We have also realized how much time is needed for self-analysis, for the analysis of class processes and, most of all, to elaborate and make innovative choices. We cannot avoid to say that we are often tempted to go back to old didactic methods: this happens when we face the difficulty of combining a well thought-through and motivating mathematics with the need to produce valid, objective test results, or when we worry about being compared to other colleagues or having to face parents' claims, but we are sustained by the awareness that every significant change needs a long time.

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EXPLORING THE PROPERTIES OF ARITHMETICAL OPERATIONS WITH CHILDREN

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ABSTRACT
When teaching becomes research, every didactic situation becomes "rich soil" for reflection on the multiple forms that characterize the dynamic interaction between the subject and the object of education. In this paper we discuss cultural tools that a teacher needs, to become an autonomous reflective practitioner. We illustrate how the theoretical framework influences the meaning itself of reflection in action, and we present three brief episodes of the training process we experienced within the PDTR project.

INTRODUCTION

One of the main expected issues of PDTR project is expressed by the following sentence

> … engaging classroom teachers of mathematics in the process of systematic, research-based transformation of their classroom practice… to… the transformation of mathematics education towards a system which, while respecting the standards and contents of the national curriculum, should be more engaging and responsive to students' intellectual needs, promoting independence and creativity of thought, and realizing fully the intellectual capital and potential of every student and teacher.

But we discovered, comparing ourselves with our colleagues from other countries that the meaning of required transformation can be very different for different people. In the paper we want to propose an interpretation given by the Naples group. At first, we have to clarify that we refer to teachers' image drawn in Malara (2008): teachers cannot be simple knowledge conveyors any longer but they assume the more complex character of decision makers. In particular, they have to plan teaching trajectories that foster students' conceptual constructions, to create an environment that enables the development of students' argumentation and the sharing of ideas; to choose the communicative strategies to be adopted in classroom interaction (Malara, 2008); to listen to various voices in the classroom, catching intuitions, emotions, difficulties, to analytically observe themselves in the development of the action, to consciously take micro-decisions.

Many authors (see e.g. Schön, Mason, etc.) identify the attitude of reflecting on one's own classroom practice as the main tool for changing from "knowledge conveyors" to "decision makers." But reflecting on a process requires some a priori "lens" or reference theory about the teaching-learning process and its global goals. Moreover, it is influenced by teachers' beliefs on what mathematics is as a discipline, what kind of links mathematics has with reality, with natural cognitive processes, and so on.

The work of the Naples teachers' group is deeply rooted in two previous (lasting more then ten years) national projects[1] on curricular innovation and teachers training in scientific and mathematical areas in compulsory school. Successful strategic choices for the transformation of both mathematics education of students and teachers' classroom practice, have been firstly shaped within that multi-year, multi-classroom action-research, and they are currently refined, by classroom research correlated to progressive adjustment of teachers' training, as ways-to-look-at cultural transmission.

We experienced the process toward Schön-like *reflective practitioner* in the PDTR project that was drawn on the basis of this cultural background. In the next section we will synthesize the main theoretical issues of previous research. In section 3 we will present some excerpts from the professional training process of three different teachers in three different classrooms. Our aim is to illustrate the role of our specific theoretical framework in the enrichment and deepening of teachers' aptitude to reflect on their own practice.

2. THEORETICAL FRAMEWORK AND METHODOLOGICAL CHOICES

A reasonably satisfactory model of cognitive dynamics, adjusted to non-specialist knowledge levels, plays a key role in both teachers' training and the teaching process. The model we refer to, can be summarized as follows: (i) understanding is a complex, long-term (longitudinal along years), and wide-range (transversal across disciplines) process; (ii) understanding is critically mediated by purposeful adults' involvement, to make credible and achievable the resonance between individual cognition, social culture and world's facts; (iii) understanding requires at any level a constructive interference of the various dimensions of "natural thinking" i.e. perception, language, action, representation, planning, interpretation, etc.; (iv) progress in understanding in particular in science and mathematics areas is critically correlated to a progressive metacognitive awareness of the modeling nature of any "scientific" knowledge: in different but correlated senses, our culture, in fact, successfully splits into "formal" and "factual" model-structures with their ability to account for, and to control, the variety of phenomenic correlations.

These assumptions (Guidoni, Iannece & Tortora, 2005) lead the authors to assign to teachers the main role of "resonance mediators" (Iannece & Tortora, 2008). In other words, teachers' first task is to recognize the always present different dimensions of the learning process: the actual potentialities of individually developing cognitive structures, the framing patterns supplied by implicitly as well as explicitly codified cultures, and the constraints of reality. Then, on the basis of this analysis, they have to draw cognitive paths efficiently addressed and controlled in their meaning-driven dynamics. In Guidoni, Iannece and Tortora (2005) some crucial strategies for teachers' training are indicated in these directions; such strategies have been refined within the frame of PDTR project. Now, let us collect some of the key points of our work.

First of all, we planned the work of all teachers involved in the project on the same disciplinary content, the numerical structures, in a longitudinal manner for students aged 5-13, with the prevailing aim to analyze how the approach to numbers by students and by their teachers changes as the age varies, and how the didactic mediation has to

[1] Projects "Capire si può (It is possible to understand)" and "Modeling to understand".

consequently change. The first outcome of this choice was the possibility, for teachers of younger children to become aware of where they want them to go; and for teachers of the older ones to understand the roots of some learning difficulties.

The second choice regards the learning environment that was socio-constructive in the sense of Vygotsky for students as well as for teachers. In particular, the collaboration among us (co-workers and mentors) was supported by the use of an e-learning platform that gave us the possibility to share audio and video files of the class activities; as well as our reflections, doubts and perplexities, and the solutions to problems. The multimedia platform was crucial also because it gave everybody a shared access both to specific disciplinary remarks by instructors, and to methodological comment by mentors. Last but not least, we have stored nowadays an extremely rich historical memory of our route.

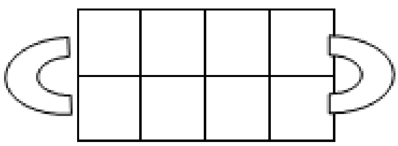
With respect to the learning environment for children, we chose to be inspired by Hawkins (1979) and Postman models (1973). This approach, closer to children needs, allows them to participate actively in the process, employing their natural thought strategies.

Activities based on problem-solving in circle-time, stimulated the curiosity, memory and perception of the children, who can develop the abilities needed to discuss their representations of mathematical structures, so they get used to reasoning and they have to try their best to express them correctly. Moreover, the relaxed, competition-free, game-like atmosphere lets all children to express themselves freely, outside of the traditional class work schemes.

Finally, all our activities are planned taking into account that the contents of the verbal and motor experience help to improve linguistic, logic-mathematical and symbolical skills.

3. THREE EPISODES FROM RESEARCH IN ACTION

The first activity was planned for students, aged 5-6, to explore and improve their natural knowledge about operations on natural numbers. They had already experience of addition in N, so I decided to bridge them towards multiplication. As usual, because of the very young age of children, I presented the classroom activity through a story, "the Gluttonous King"[2] that was told and dramatized. I wanted to explore if the perception and the graphic representation of the "structure" of the story (two sweets at a time…), might bridge students from addition to multiplication. Assuming a Vygotskian perspective, I proposed a semiotic mediator of the bidimensionality of multiplication, i.e. a tray divided into eight parts, where they had to put sweets during their dramatization.



When the children were asked to represent the story with a drawing, most of them performed the sequence of the 4 "trips in the woods," making clearly visible the sweets taken each time in this way:

[2] The Gluttonous King story was written by Marina Spadea and it is reported in the Appendix.

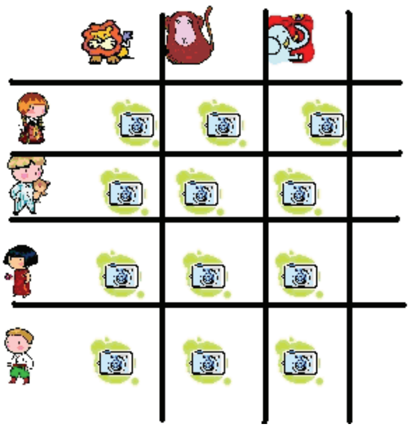
but they did not use Cartesian representation, which I took for granted. In fact, I was sure that "the tray" should work as a semiotic mediator (Vygotsky, 1931) toward the interiorization of the Cartesian representation. By reflecting on students issues, I discovered that there is a big difference, from a cognitive point of view, between working with a proposed representation and proposing one for one's own purposes. Moreover, I realized that perhaps *the failure of mediation* is due to the fact that the tray acted as *a closed structure* related only to that specific situation: the disposition of sweets on the tray was for children just a representation of their story, not a symbolic icon, as I supposed.

To solve the problem, I went back to our theoretical framework. Starting from the observation that *the temporal sequence* of events belongs to children's conscious experience (before/after, day/night), I decided to construct the need for a Cartesian representation, as a tool to tell about a temporal repetition of identical actions. I worked on performing different types of rhythms (the haunted castle from the ArAl[3] plan). Working on rhythms in different ways, that is, playing with another natural cognitive structure, the "narrative thought," children soon and naturally recognized the Cartesian representation as a tool to record patterns invariably repeating in the course of time.

A final observation: I already proposed to my children to work with rhythms several times, but only now I discover that I can promote their algebraic thinking through a graphic representation of the experience. As Rizzolatti and Sinigaglia (2006) say, the origin of the comprehension of mathematical operations has to be searched within the action schemes.

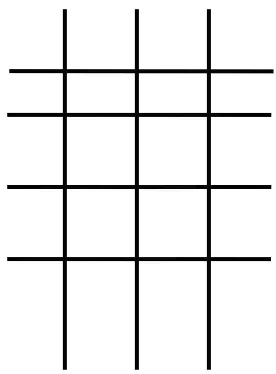
The second activity was proposed in a fourth-grade class. My aim was twofold: from a didactic point of view, to allow students to reflect on properties of the multiplicative structure, and from my own research point of view, I wanted to analyze if and how children utilize the Cartesian representation as a semiotic mediator. According to my habits, I proposed a "real" problematic situation, whose modelization brings in arithmetic structures. In this particular case I invented a story of photos at the zoo. I then suggested, in agreement with the colleagues of the project and with the mentors, to use a scheme that makes evident the two dimensions of the multiplicative structure. The initial scheme includes the characters of the story: 4 children that go to the zoo to snap some photos to 3 animals.

[3] The ArAl Project promotes a linguistic approach to algebra and generates motivation and awareness towards the study of the objects of elementary algebra (relations, functions, equations) with a special emphasis on its relational and structural aspects. (www.aralweb.it).



In the next phase the images disappear and we obtain the following scheme. We used such a scheme not only to tell the initial story but also a lot of new stories invented by the children.

Things



Times

Then I decided to address the role of zero in multiplication. As I usually do, I looked for "real" stories to support children in their construction of a meaning for the operation. So, I proposed a movement activity where children were required to go back and forth on the number line, but when I said: "Go forward 4 steps 0 times,"

Anna replied: I am not able to do this, I can't move... But what is the sense of the words "4 steps zero times"? I would never ask someone to make 4 steps zero times.

Giovanni: 4×0 is not really a multiplication! A multiplication needs repetition of an action; it needs the "times."

Alessia: The word *multiplication* is obtained putting together two words: *multi* that means a lot, and *action*. Then multiplication means to carry on a lot of actions. But, when there is zero there is no action, so we have to choose a new name... Perhaps we can still decide to call it multiplication but with a different meaning...

Simone: 4×0 is as if the 4 were waiting to cross some line but the intersection doesn't happen and then the result is zero.

In a sense, my students tried to convince me about the uselessness of my search for a concrete situation that constitutes a metaphor for this operation. To my great surprise, students revealed a natural aptitude to change their point of view, jumping from reasoning supported by observation to a logic argumentation. Alessia even analyzed the structure of the word “multiplication” looking for the sense of the operation.

When analyzing with mentors my students’ reasoning, I became aware that not all mathematics can be discovered starting from observation of the reality. In fact, there are some rules that can be justified only by the necessity of an internal coherence of mathematics as a discipline. But a still greater surprise was that my difficulties to leave concrete motivations for mathematics rules were not shared by my students: for children the acceptance of a sort of game rules led to generalization of already established meanings.

The sequel of the activity designed and guided in collaboration with my colleagues in the project and with the mentors, brought me and my students to successfully facing a problem that I had never solved before: why is it impossible to divide by zero?

In conclusion, during my participation in the project, I became aware of my need in deepening my knowledge of mathematics, and in improving the cultural tools that allow me to design and to manage a didactic mediation centered on natural cognitive dynamics. Understanding, as different from learning, and motivation, as different from acceptance, is strictly correlated, for students as well as for teachers: based on feelings and feedback of competence in dealing with increasingly complex situations (Guidoni, Iannece & Tortora, 2005).

The third activity was proposed to six-year-old children and aimed at exploring properties of the additive structure of natural numbers. As usual, I proposed a problematic situation which can be firstly faced through the body, along the number line: “the cats’ and shrimps’ game.” Benedetta gave the orders, Mattia moved as a cat (forward) or a shrimp (backwards) on the line, Diego represented actions on the blackboard.

The first problem the children encountered was the choice of a starting point.

Mattia: Where do I start from?
Stefano resolutely says, indicating the wall: We have to start from zero, otherwise we go wrong!
Benedetta: Make four steps as a shrimp and then three steps as a cat.
Mattia: I’m sorry, but I can’t do it…

At this point the children looked at me for an answer, but I worried about being compelled to introduce negative numbers. So I suggest: “Mattia you can start in the middle of the room, so Benedetta’s order can be executed. You know, beyond the zero there are some particular numbers, called “negative numbers” you’ll meet them later on, now it is too early.” The children agree with the proposal, but they remain a bit confused.

Talking to another teacher who took part in PDTR, I realized that delaying the exploration of negative numbers and solving the problem of the starting point, I lost at least two good didactic occasions. In fact, when she proposed the same activity to her students of the same age, they autonomously found the solution of the starting point problem that I had considered too complex. It is not good to keep children away from the knives: it is wiser to teach them from the beginning to hold the knives by the handle! It is useless and dangerous that children ignore some thorny problems: they would have to know them, instead, and to learn to master them as soon as possible.

During the game, the students encountered by chance some particular situations in which the sequences of steps showed important mathematical properties.

Monica: Marco, move five steps as a cat and then two steps as a shrimp.
Marco: I've reached the number three.

Benedetta: It is as before, when Simona was a cat for seven steps and a shrimp for four steps… she stopped on number three!

[Diego who was reporting on the blackboard all the actions as operations concludes]: It's the same to write 5−2 or 7−4, because I always arrive onto the number three.

Benedetta: Make five cat steps and three shrimp steps, stop for a moment and then add other two cat steps.

Francesca: He arrives where he was at the beginning. It is as though…look…he starts from five and he goes back to five.

Stefano: It's true, but he has made some steps and we can't forget it.

Virginia: He returns onto the number five, but he has walked! He returns on his starting point because he has made the same number of steps in two opposite directions.

Teacher: Now, look at the operations. Do you understand what Virginia said?

A lot of children had some trouble when I suggested to them to focus their attention on the operations.

Listening to the recordings and through discussion with my PDTR colleagues, I realized that it would have been better to reason upon actions (and not upon mathematical symbols). Going away from the experience of the steps, I caused disorientation in most children, who began to loose their attention.

My choice caused confusion, not being resonant, or rather not blending with the children's cognitive processes and knowledge, and not being in tune with their expectations. I think that learning is not produced if students and teachers are not in tune reciprocally, or rather if there is not a continuous and mutual adjustment of the cultural mediation to the cognitive structures of the child and of the child to the new knowledge. Only teachers who reflect on their own educational practices can realize a resonant mediation. "Not to change, modify, experiment is to be stuck in the rut of the habit, ending up where those habits lead. If you do not change your direction, you may end up where you are headed" (Mason, 2002, 7).

Reflecting on my choice, I realized that moving too soon children's attention from the "actions" to the "operations," I *pushed* them too quickly toward the use of symbols.

I agree with Piaget when he affirms that symbols loose their utility if they are presented too early. It is necessary that children freely play with quantities until they spontaneously need symbols.

The syntactic analysis necessarily follows the semantic one. A conscious construction of the mathematical meanings is necessary: the symbol has to be a tool of which children naturally enter in possession to express their own thought. If this does not happen, students refuse it in the greatest part of the cases. It is like to ask children to learn by heart the words of a sonnet prepared in alphabetical order preventing them from tasting the meaning of the work.

During the discussion, my students analyzed another problematic case and they tried to give a particular representation of it.

However, a few children kept on their discussion about operations.

Diego compares the operations 5−3 = 2 and 2+3 = 5 and says: Here (pointing to 5−3 = 2) the number five is the first one, while here (pointing to 2+3 = 5) it is the last one; moreover, the number two here (5−3 = 2) is the last one, here (2+3 = 5) it is the first one. So I can write this way:

$$5 \longrightarrow 2 \longrightarrow 5$$

Before there is the number five, *then* the number two and *at last* once again the number five! Number two is *in the middle*.

Diego's representation was so interesting that it induced the teacher to linger over the meaning of the arrows:

Teacher: Well Diego, what do these arrows mean?

Francesca immediately answers: One arrow adds, the other subtracts.

I did not notice that the children used the same arrow to indicate different acts, so I intervene saying:

Teacher: Very well, and what happens if you subtract and then add the number three?

Francesca: That I start from number five and come once again to number five!

I have wasted a good opportunity by not asking the children: "Can we use the same symbol to say different things?" This way I would have encouraged the natural need for symbols, instead of shooting ahead by inducing the children to use the operators "+" and "−." It is important that teachers do not pass their representation like the only possible one: students have to understand that it is not the only right way to represent an idea and that it is necessary to distinguish a concept from its representation.

Duval asserts that there is no *noesis* (that is, the conceptual learning of a mathematical object) without *semiosis*: only the use of many different representations registers permits to reach the noesis (Duval, 1998).

The modeling of a concept is a good way to help students remember and visualize it. The symbols represent mathematical concepts and I learned to be more sensible to children's ways of symbolizing. If the symbolic expression does not rise like a tool to communicate one's thought, students just learn how to accept meaningless answers to mathematics problems. The numbers children wrote are representations of their thoughts, not substitutes for it: just when this happens, I feel that my work is not useless.

CONCLUSIONS

Our participation in the PDTR project allowed us to assume a different way of looking at cultural transmission and at the role of the teacher. In particular, the critical awareness and the responsible assumption of our role as "resonance mediators" between students' cognitive strategies and mathematics played a crucial role for our disciplinary and professional training. We became aware of the importance of assuming a research aptitude about different dimensions of the learning process, and of having access to research results in mathematical education. We think that the *discipline of noticing* (Mason, 2002) we acquired thanks to the cultural support of our mentors and colleagues from other countries, is the beginning of the path toward our autonomy as teacher-researchers.

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APPENDIX

King Glottonous (Greedy) knew that a girl with golden hands lived in a small house in the wood in his kingdom. She made delicious strawberry and chocolate cakes. When he went to see her, he ate a lot of her cakes and was so happy that, when he left, he gave her a bag of gold coins.

Some time after, the king was sad in his castle, but nobody could make him happy. “Come on, kids, how can we make the king happy?”

The queen wanted to cheer up the king so she sent the servant to get some of those delicious cakes.

After many difficulties, the servant arrived at the baker’s house but he couldn’t buy many cakes because the oven was too small; it was only big enough for two cakes. The servant bought them and went back to the castle as soon as possible.

“What a delicious smell coming from those cakes!”

As only the king could eat the cakes, the queen became unhappy.

“What can we do? We have to go back to the baker!”

The servant went back to the baker to buy the cakes for the queen. He got through the same difficulties and he bought two cakes: a strawberry cake and a chocolate one. The queen ate up the two cakes heartily but she didn’t offer any of them to anybody. Unfortunately, like the king, the little prince and the little princess started crying for cakes, so the servant had to go to the baker again.

“How many times must the servant go to the baker’s house to make everybody happy?”

“In the end, how many cakes did the servant buy to make everybody happy?

“What a small oven!”

II

As for the baker, she started earning a lot of gold coins so she could buy a bigger oven in which she could cook four cakes at a time.

“Now, how many times has the servant to go to the baker to satisfy everybody?”

DEVELOPING THE ABILITY TO SUBSTANTIATE AND ARGUE AS A PREPARATION FOR PROVING MATHEMATICAL PROPOSITIONS

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ABSTRACT
This article is an account of a teacher – the author – describing the course of teaching-research concerning the development of the abilities of justification, reasoning and understanding of proof with the usage of general arguments or refuting the sentence through giving a counter-example. Teaching-research was conducted in the normal course of mathematics teaching, in a class of students, aged 16-17, who were not interested in mathematics very much. Through the choice of appropriate tasks in the introductory diagnostic test and certain didactic procedures together with the selection of appropriate tasks in the realization of teaching in full accordance with the curriculum the teacher achieved her intended goals with a significant group of students, a fact which was visible in the final test. A teacher who plans to realize such teaching goals can use the described experience and tasks.

STUDENTS' CHARACTERISTIC

There were 33 students, aged 16-17, in the first year of the three-year upper secondary school ("Lyceum"), in which my experiment began. When analyzing their written solutions of three PISA questions ("Apple trees," "Antarctica," and "The Carpenter"), and talking with them I realized their difficulties in describing and justifying facts that seemed obvious to them. They often insisted that what they noticed does not require any justification.

OBJECTIVES

The objective of the teaching experiment was to develop competences strongly linked to educational purposes and tasks of the school included in the Polish national curriculum (2007): (1) getting used to typical elements of mathematical reasoning, in particular, using concepts like assumption, conclusion, proof (also indirect), example and counter-example; (2) developing the ability and need of critical assessment of a reasoning carried out; (3) developing the custom of independent acquiring, analyzing, and sorting information, making conjectures and finding methods of verifying them; and (4) ensuring education, which promotes independent, critical, and creative thinking.

The curriculum I chose ("M+") assumed and obligated me to implement educational goals such as: (a) developing the custom of logical and correct thinking as well as applying the principles of logic; and (b) developing the ability of making conjectures and proving them and distinguishing between a conjecture and a fact that has been proved.

I also considered what Niss (2002) called: (i) *thinking mathematically*: distinguishing kinds of mathematical sentences, implicative (if ... then) and quantified expressions, assumptions, definitions, theorems, hypotheses, and particular cases included; and (ii) *reasoning mathematically*: understanding and assessing a sequence of arguments proposed by others; knowing what is (not) a mathematical proof; proposing formal and informal mathematical arguments and transforming them to correct proofs.

INITIAL DIAGNOSIS

Its purpose was to find out information on: (1) attempts and ways of formulating by students justifications (proofs) in a situation when they are required to assess truthfulness of a mathematical sentence or notice themselves some property and try to formulate it in the form of a corollary or theorem; and (2) the attitude of the students towards a proof carried out and its correctness.

The diagnostic tool:

Test I

I. Estimate if the following sentences are true and justify your choice.

1. If the sum and the product of two numbers are positive then the numbers are positive.
2. If the sum and the product of two numbers are positive then the sum of their squares is positive.
3. If the sum and the product of two numbers are positive then their sum is a rational number.
4. The square has exactly 2 symmetry lines.
5. The square has exactly 4 symmetry lines.
6. The square has a point of rotational symmetry.
7. The number of symmetry lines of a convex quadrilateral can be 0.
8. The number of symmetry lines of a convex quadrilateral can be 2.
9. The number of symmetry lines of a convex quadrilateral can be 4.

II. Consider the following examples.

$$\begin{array}{r} 95 \\ -\ 59 \\ \hline 36 \end{array} \qquad \begin{array}{r} 32 \\ -\ 23 \\ \hline 9 \end{array} \qquad \begin{array}{r} 64 \\ -\ 46 \\ \hline 18 \end{array}$$

1. Propose similar examples.
2. What common property those examples have?
3. Could you substantiate properties that you noticed?

III. Acquaint yourself with the following theorem and its proof.

Theorem: An arbitrary number a equals a smaller to it number b.

Proof:

Because number a is greater then b, there is a positive number c such that $a = b + c$.
We multiply each side of the equation by $a – b$.
Then: $a(a – b) = (b+c)(a – b)$
$a^2 – ab = ab+ac – b^2 – bc$
or equivalent
$a^2 – ab – ac= ab – b^2 – bc$.
Drawing common factors out of the parentheses we get
$a(a – b – c) = b(a – b – c)$
We divide both sides by $a – b – c$
getting $a = b$,
which was to be proved.

1. Estimate if the proof is correct. Substantiate your answer.
2. Is the theorem true? Why?

Results of the diagnosis and comments.

Question number	Number / percentage of students who correctly determined the logical values of sentences		Percentage of students who substantiated answers by commentary and numerical examples or picture		Percentage of students who substantiated answers with numerical examples or picture		Percentage of students who substantiated answers by commentary	
1	14	52%	8	30%	8	30%	0	0%
2	24	89%	19	70%	16	59%	3	11%
3	6	22%	3	11%	1	4%	2	7%
4	25	93%	23	85%	16	59%	7	26%
5	25	93%	24	89%	20	74%	4	15%
6	26	96%	23	85%	19	70%	4	15%
7	16	59%	8	30%	1	4%	7	26%
8	12	44%	7	26%	1	4%	6	22%
9	11	41%	6	22%	1	4%	5	18%

Table 1. Scores in the first question of Test I

If the question addressed an object well-known to students (the square) they were able to estimate correctly the sentences and substantiate the answer (questions 4, 5, and 6). If a generalization of a sentence was required (examples 7, 8, and 9), a quarter of the students could do it. What was it that they did not understand? It was the meaning of "in each convex quadrilateral" or how to substantiate it (a half of those that understood correctly could also propose a substantiation (column 3, questions 7, 8, and 9).

1st question	2nd question					3rd question		
Percentage of those who proposed adequate examples	Percentage of those who formulated the property as a conjunctive sentence	Percentage of those who formulated the property by the assumption only	Percentage of those who formulated the property putting the conclusion only	Percentage of those who wrongly formulated the property (results positive so these are differences)	Lack of solution	Percentage of those who substantiated with examples	Percenta ge of those who repeated the property	Lack of solution
67%	11%	15%	48%	19%	7%	19%	19%	62%

Table 2. Scores in the second question of Test I

The students were able to give similar examples but encountered difficulties with formulating their findings in the form of implication (question 2) and any general substantiation (in any form; e.g. "If a and b are digits different than zero then $10a+b-10b-a=9(a-b)$") was outside their capacities.

1 task								2 task							
Number/ percentage of those who did not attempt any answer		Number/ percentage of those who said the proof was correct		Number/ percentage of those who put the correctness of the proof to doubt		Number/ percentage of those who found an incorrectness in the proof		Number/ percentage of those who did not attempt any answer		Number/ percentage of those who said the proof was correct		Number/ percentage of those who said the theorem was false, without any justification		Number/ percentage of those who said the theorem was false, and substantiated it	
13	48%	13	48%	1	4%	0	0%	11	41%	4	15%	7	26%	5	18%

Table 3. Scores in the third question of Test I

Nobody found an error in the proof and one student only expressed their doubt about its correctness. Almost 50% of the students did not answer the first question. Substantiating is very hard for them. Formally carried out, the wrong "proof" with comments convinced the students of truthfulness of the sentence, which should have been found as obviously false. Less than 1/5 of the students said that the theorem is not true, with justification.

The diagnosis allowed partial answers to my questions. The students encountered difficulties in correct formulating of a property they had noticed. In most cases substantiation was conceived as giving an example. They did not see the theorem as an implication. Their attitude towards a proof and its correctness did not help, though, as the example proved to be too difficult for them. An encouraging phenomenon occurred: some students explained why they were unable to answer ("I do not know what a convex figure is like").

In such a situation I decided to carry out, in a short time, a second initial diagnosis in order to get a more complete answer to the questions. Though, before that I discussed the results with students. During the discussion the structure of a theorem (assumption – conclusion) as well as ways of transforming the predicative sentence to a conditional one (if – then) were recalled.

SECOND INITIAL DIAGNOSIS

I intended to check how well students were able: (i) to utter and put down in writing some properties given in the conditional format; (ii) to notice and formulate common property based on the analysis of given (or self-invented) examples; and (iii) to substantiate a given simple numerical property.

Test II

I. Put down the given theorems in the form "If ... then ..." Then point out in each of them the assumption and the conclusion.
   1. Number *n* whose last digit is 5 is divisible by 5.
   2. A number divisible by 9 is divisible by 3.
   3. In the equilateral triangle all angles have 60^{o}.
   4. Each rectangle is a parallelogram.

II.
   1. From the set of positive integers select a number, multiply it by its inverse, and from the answer subtract the number opposite to 1/3. Add 2/3 to the last result. Write down the result.
   2. Now replace the positive integer consecutively by a negative integer, a fraction, an irrational number, and carry out the calculations above. What did you notice?
   3. Write down this situation using a formula, without forgetting the assumption. Formulate the adequate theorem.

III Prove that the sum of two even numbers is even.

Outcomes of Test II and conclusions of the teacher-researcher

This time the number of diagnosed students was 31. Quantitative results are in table 4.

Number/percentage of those who formulated the first theorem in the form of implication		Number/percentage of those who formulated the second theorem in the form of implication		Number/percentage of those who formulated the third theorem in the form of implication		Number/percentage of those who formulated the fourth theorem in the form of implication	
28	90%	25	81%	23	74%	17	55%

Table 4. Scores in the first question of Test II

About 50% of the examined managed to transform the theorems faultlessly. Putting down the last theorem in the conditional format was most difficult. The students attempted to add “by force” some properties of features of a parallelogram or rectangle, coming up with a new theorem. It happened that they wrote an inverse theorem with respect to the given one. Only 1/3 of the examined pointed the assumption and conclusion in the given theorems.

1st task						**2nd task**						**3rd task**					
Number / percentage of those who answered correctly		Number / percentage of those who committed a mistake		Number / percentage of those who did not attempt to answer		Number / percentage of those who answered correctly		Number / percentage of those who committed a mistake		Number / percentage of those who did not attempt to answer		Number / percentage of those who correctly wrote a theorem or formula		Number / percentage of those who wrote a theorem or formula adequately to the results of the task		Number / percentage of those who did not attempt to answer	
7	23%	23	74%	1	3%	3	0%	21	68%	7	23%	2	6%	8	26%	21	68%

Table 5. Scores in the second question of Test II

Above 2/3 of the examined faultily executed the numeric operations, which made it more difficult to notice a common property in all the examples. Two persons only (6%) made correct transformations on numbers, were able to write down the property with a formula, and formulated the theorem. Additionally ¼ of the students noticed properties and wrote a formula according to their calculations. Almost 2/3 of the examined did not attempt to write down a generalized property, which resulted from errors in calculations or too much attachment to operations with concrete numbers.

Number /percentage of those who produced a correct formal proof		Number /percentage of those who gave examples and additionally substantiate verbally		Number /percentage of those who gave examples but could not comment, substantiate		Number /percentage of those who gave examples and saw the need of a generalization but could not do it		Number /percentage of those who did not attempt to answer	
0	0%	8	26%	20	65%	1	3%	2	6%

Table 6. Score in the third question of Test II

Almost all students based their substantiation on numeric examples. ¼ of the students substantiated correctly. No one, though, was able to write down a formal proof of the theorem (they did not see the need of using the language of algebra).

Discussing the results of Test I with the students and a short recollection of the knowledge on theorems before Test II resulted in students coping with formulating and writing down a property in the conditional format rather well. So, in the future I will have to consistently continue those exertions: insert simple tasks during the class, for example pointing to the assumption and conclusion in occurring theorems, counter pose a theorem with its inverse, indicate their differences, and practice making inverse theorems.

Students encountered difficulties with correct execution of numeric calculation, which contributed to erroneous conclusions of lack of them. The second initial diagnosis showed that student has serious difficulties with generalizing and formal proving properties “For all numbers...”

CORRECTIVE STRATEGIES

In the planning the corrective actions I reached for professional literature. In the domain of my interest, i. e. substantiating and arguing, the following publications proved helpful: Krygowska (1977, 1989), Konior (1989), Lietzman (1958), Nowecki (1978), Siwek (1974), and Vinner (1994). Analysis of the results of the initial diagnosis, as well as a study of the literature brought me to the following decision. Corrective actions connected with substantiation, arguing and proving should start with an improvement of students' understanding of the notions of theorem, axiom, definition, proof, "for all," "for an arbitrary," and their role in mathematics.

Having considered the above conclusions, and observations made in both initial diagnoses I conceived certain corrective actions. They were adjusted to the obligatory curriculum and capacities of students and implemented continuously in the teaching process. I was reassured by Siwek (1974) who consistently investigated the understanding of theorems in the conditional format and of definitions, in 7 experimental and 7 control groups of secondary school students (359 students) who took a logic course. She said:

> Even very modest and intuitive elements of the logical theory remain defunct and are mostly forgotten if during the course of mathematics they are not come back to, if mathematical expressions (definitions, theorems, and proofs) are not analyzed in the logical aspect. On the contrary, if the teacher creates situations when the student gets conscious of the logical structure of mathematical texts, and consciously applies his knowledge of logic for mathematical topics – this knowledge grows firm and more concrete.

Understanding of the expressions "for each," "for any number," "not for each," "one can point such...," "there exist" is connected with understanding theorems and definition, generalization and specification of a theorem. It is also important that students know the meaning of "true" (on a theorem) as derivable within a theory, "false" as such whose negation is derivable, and "hypothesis" as a sentence, of which it is not known if it is true or false (Krygowska, 1977; Nowecki, 1978). I reassured myself that I have to talk with my students about it.

Corrective efforts applied during class with my research group: (1) frequent classroom discussions aiming at clearing up the notions belonging to methodology of mathematics (i.e. definition, theorem, axiom, and proof); (2) formulating and proving, based upon definitions, of facts concerning the newly acquainted notions of increasing and decreasing sequences; (3) analyze with students a selected excerpt of the textbook in order to disclose "hidden" definition and theorems; (4) exercises of the following kind:

I Write down the given sentences in the form of implication**.**

1. The square of an even number is divisible by 4.
2. The sum of two rational numbers *a* and *b* is a rational number.
3. A natural number is divisible by 10 when its last digit is 0.
4. The product of an even number and an odd number is an even number.

II Write down theorems inverse to the sentences you just wrote.

III Which of the sentences below would you prove?

1. If we multiply (or divide) both sides of an equation by the same number different than zero or by the same formula, which does not change the domain of the equation and whose value is not zero, then we get an equation equivalent to the initial one.
2. Degree is an angle whose degree measure is 1.
3. For each angle α such that $ctg\alpha$ exists

$$\frac{\cos\alpha}{\sin\alpha} = ctg\alpha$$

4. Zero of a function is called an argument for which the value of the function is zero.
5. If $a > b$ and $c > 0$ then $ab > bc$.
6. The radian measure of angle is the number equal to the ratio of the length of the arc, on which the angle is standing, by the length of the circle's radius.
7. In a right angled triangle the sum of the areas of squares built on the catheter equals the area of the square built on the hypotenuse.
8. For each angle α

$$\sin^2\alpha + \cos^2\alpha = 1.$$

9. Sine of angle α is called the number being the ratio of the ordinate of point P and its distance from the origin of the coordinate system.

$$\sin\alpha = \frac{y}{\sqrt{x^2 + y^2}}$$

IV. Check if the inequality below is true for any real numbers.

$$(a + b)^2 > 2ab$$

(5) forming the notions "for each," "for any number" by encouraging students to formulate symbolic expressions and proving simple theorems about integers, e.g. "Prove that the sum of two consecutive odd natural numbers is divisible by 4." "Prove that if the sum of two natural numbers is an even number then their difference is also an even number;" (6) analyzing selected proofs in the textbook; (7) students' analysis of various erroneous solutions of a selected problem (examples from students' papers); (8) proving with students basic trigonometry relationships and other trigonometry identities; (9) adding one or two problems to assessment tests, which, besides finding the answer, would require its substantiation; and (9) using figures that suggest something, but when you reason you arrive at something else.

FINAL DIAGNOSIS

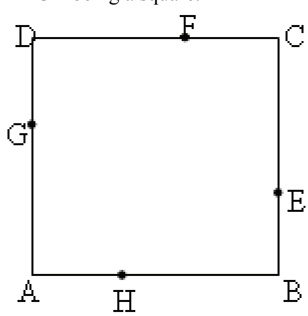
After all the corrective efforts, I made a final diagnosis. The diagnostic test presented here sums up our work on the posed research problem in this student group. It aimed at the verification of expected improvement in the understanding of the notions of theorem and definition, and of the way of substantiation. Besides I added to the test a problem that would check if for students a figure provides satisfactory substantiation (problem 2).

Test I

Problem 1**.** Prove that the sum of two consecutive odd natural numbers is divisible by 4.

Problem 2**.**

1. Complete the sentence: *The square is a quadrilateral ...*
2. In square *ABCD* segments *AH, BE, CF, DG* of equal lengths were laid out as shown on the figure. Give reason for the quadrilateral *EFGH* being a square.



Problem 3. Is the following sentence true?

If $a^2 = b^2$ then $a = b$.

Problem 4. Among the following sentences indicate: which is a theorem and which is a definition. Which one would you prove?

1. If the line $y = ax + b$ passes through two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ and $x_1 \neq x_2$ then $a = \frac{y_2 - y_2}{x_2 - x_1}$.
2. Each function whose arguments and values are points of the plane is called a geometric transformation.
3. The rest of division of polynomial *W(x)* by the binomial *x – a* equals *W(a)*.
4. The radial measure of a central angle in a circle is the number equal to the ratio of the length of the arc on which the angle stands to the length of the radius of the circle.
5. For each angle α the following equality holds $sin^2\alpha + cos^2\alpha = 1$.
6. If point *P'* is assigned to point *P* through a geometric transformation then point *P*' is called image of point *P* in this transformation. Figure *F'* resulting from transformation of figure *F* is called image of figure *F*.
7. Number *a* is the root of polynomial *W(x)* if and only if the polynomial is divisible by *x- a*.
8. If the abscissa of point *P* belonging to the other side of angle α positioned in the coordinate system is not equal zero then tangent of angle α is called the ratio of the ordinate of point *P* to the abscissa of this point.
9. The angle whose radial measure equals 1 is called radian.
10. The segment joining midpoints of two sides of a triangle is parallel to the third side and it is twice shorter.

RESULTS OF THE DIAGNOSIS AND COMMENTS

Number/ percentage of those who gave examples and a correct verbal substantiation		Number/ percentage of those who gave examples but found editing a commentary difficult		Number/ percentage of those who gave examples, commented on them and tried to generalize		Number/ percentage of those who did not attempt to solve the task	
3	16%	10	52%	6	32%	0	0%

Table 7. Scores in the first problem

Students still had difficulties with formal expression of the proof, but their substantiation was not reduced to giving numeric examples (as it was diagnosed at the beginning). There was also a commentary describing the situation. Almost 1/3 of the examined students tried to construct a proof using the algebra language.

Percentage of those who gave an incomplete definition (all sides of equal length)	Percentage of those who gave a correct definition	Percentage of those who pointed at the figure as a reason	Percentage of those who provided a formal reason
53%	47%	89%	11%

Table 8. Results of solving the second problem

As one can see, for students the figure was a sufficient argument to consider the polygon *EFGH* a square. They did not consider the case where a figure suggests a different answer. If a wrong definition of the square appeared the students restricted their comments to the sides, without discussing equality of the angels. Lack of knowing the definition contributed in “insufficient arguments.”

Results of solving the third problem show that over 2/3 of the examined students said the theorem was not true, but they encountered difficulties in substantiating this decision and giving an adequate example. A comforting fact is that 70% of them knew that in order to refute truthfulness of a theorem it is enough to give an appropriate counter-example. Others tried to formally prove the theorem, so they transformed the equality $a^2 = b^2$ by taking square roots on both sides. When they received $a = b$, the conclusion was a positive answer on truthfulness of the theorem.

Distinguishing definitions and theorems			
Percentage of those who correctly sorted all sentences	Percentage of those who committed no more than 2 mistakes	Percentage of those who committed 3 or 4 mistakes	Percentage of those who committed 5 mistakes
5%	16%	42%	37%

Table 9. Results of solving the fourth problem

The average fraction of correct answers of the problems was 60%. An analysis showed that students encountered difficulties in distinguishing between definitions and theorems, but it seemed to have been caused by not knowing the given theorem or definition or by the structure of the task, which may have suggested a wrong answer. An encouraging fact is that most students knew that a definition did not require proof. This meant progress in comparison with the initial state when definitions were not distinguished from theorems and proofs.

As only 19 students participated in the test (out of 33) and for some organizational reasons the moment for the test was not fully suitable, the results could not provide a full answer. So a second final diagnosis was necessary.

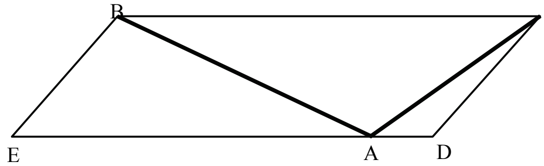
<u>Test II</u>

Problems 1 and 3 were linked with the second initial diagnosis. Problem 2 aimed at finding out about students' attitude to figure; at the same time it made an introduction to the topic of substantiating (initial diagnosis). Krygowska (1977) suggests that teaching substantiating a figure could be used in such a way that wrongly suggests what can later be disproved by reasoning.

Problem 1. For the given theorems make inverse theorems.

A. Number n whose last digit is 5 is divisible by 5.
B. A number divisible by 9 is divisible by 3.
C. In an equilateral triangle the measure of each angle is 60°.
D. Each rectangle is a parallelogram.

Problem 2. Point A belongs to side ED. Is segment AB longer than segment AC?



Problem 3. Prove that the sum of two arbitrary even numbers is an even number.

RESULTS OF THE LAST TEST AND COMMENTS

Number of participating students	Number /percentage of those who formulated the inverse for the first theorem		Number /percentage of those who formulated the inverse for the second theorem		Number /percentage of those who formulated the inverse for the third theorem		Number /percentage of those who formulate the inverse for the fourth theorem	
29	12	41%	21	72%	20	69%	20	69%

Table 10. Scores in the first problem of the final diagnosis II

Almost 2/3 of the examined students showed that they knew the principle of constructing the inverse theorem. Comparing with the initial diagnosis, there was an improvement of the ability to indicate assumption and conclusion, which is indispensable for a correct construction of the inverse theorem. Some students, unable to complete the task, only wrote the given theorem in the form of implication. It was interesting that a few of the students added a property in the fourth sentence to obtain a true inverse theorem; for instance, "Each parallelogram that has all right angles is a rectangle."

Scores in the second problem show that only one student correctly substantiated their answer and considered the possibility of a change in the position of point *A.* Two students did not answer this problem. And 26 (90%) of examined students substantiated their answer using only the *given* picture.

Number of participating students	Number/ percentage of those who carried out a correct formal proof		Number/ percentage of those who gave examples and additionally correctly substantiated verbally		Number/ percentage of those who gave examples but could not comment or substantiate		Number/ percentage of those who gave examples, comments and attempted to generalize		Number/ percentage of those who did not attempt to solve the task	
29	7	24%	6	20%	6	22%	10	34%	0	0%

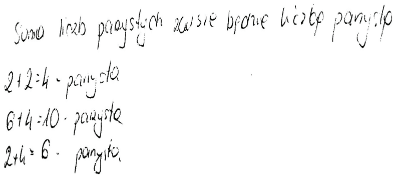
Table 11. Scores in solving the third problem of the final diagnosis II

CONCLUSION

Comparing to the initial diagnosis, there was a great improvement in the need of writing a proof in a formal way. 24% of the students were able to write a proof using the algebra language and 44% correctly proved the theorem. In addition, over 1/3 of the students did not content themselves with giving numeric examples but tried to use a symbolic language for building an argument. Most of them knew that giving examples is not a proof and an arbitrary even number does not mean a selected concrete number but is a representation of each number of the set of even numbers.

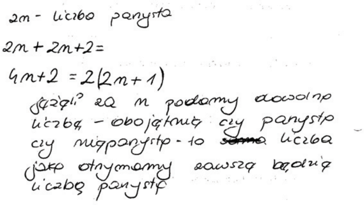
Comparing papers of the initial and final diagnosis of the same student one can see that in the period of 8 months of work progress was achieved first of all in the way of formulating and substantiating facts. Problems were not left without answers; attempts were always present, not always correct, though.

To illustrate the changes that occurred in the way of thinking of some students, I present below excerpts of papers from the initial diagnosis II and final diagnosis of the same student. It is her answer to the task "Prove that the sum of two arbitrary even numbers is an even number."


Suma liczb parzystych zawsze będzie liczbą parzystą
2+2=4 - parzysta
6+4=10 - parzysta
2+4=6 - parzysta

Sum of two even numbers is an even number
2+2=4 – even
6+4=10 – even
2+4=6 – even

Figure 1. Excerpt of a paper – initial diagnosis II


2n - liczba parzysta
2n+2n+2=
4n+2 = 2(2n+1)
Jeżeli za n podamy dowolną liczbę - obojętnie czy parzystą czy nieparzystą - to ~~sama~~ liczba jaką otrzymamy zawsze będzie liczbą parzystą

2*n* – even number
2*n*+2*n*+2=
4*n*+2=2(2*n*+1)
If for *n* we put any number, even or odd; then the number we'll receive will always be even.

Figure 2. Excerpt of a paper – final diagnosis

The student is proving that the sum of two consecutive (so not arbitrary) even numbers is an even number. But she uses a general name for "arbitrary even number."

Carrying out the whole experiment was a valuable event for the whole class and, above all, an "effective lesson" for the beginning teacher-researcher. We gained a lot together with the students. My practitioner teacher's knowledge got stronger and the process of teaching in class became better. The work which I put into the whole cycle of research (the choice and edition of the tasks, the analysis of results, the planning of the "corrective actions") as a beginning teacher-researcher was indeed rewarded by the results of the students. Many students changed their attitude to mathematics. The understanding of what is justification and reasoning in mathematics was improved, together with what is theorem, converse theorem, proof and definition; the same happened with the understanding of notions "for each number" and "for any number."

The presented "corrective actions" helped the students recognize when they should use a general argument and how they should write it down. They also understood when it was enough to give an example (the role of counter-example). The students' way of formulating justifications changed visibly. Furthermore, thanks to this experiment we gained something which cannot be measured and examined by the means of test and which was influenced in a positive way by a bond between the students and the teacher – the atmosphere of work during the lesson changed. Students started to have a different attitude to mathematics. They preferably expressed their thoughts and, in the case of difficulties with the task solving, they wrote down things which they did not understand. Carrying out the tests for which the students were not graded allowed a more informal expressing of opinions and conclusions. The students' questions and utterances became a valuable source of information.

Having used my own experience, which I gained during the presented experiment, in the domain of research and having seen the effects of the work I put in, I will continue research concerning the problems of justification and reasoning in other classes using the presented approach, despite the fact that these processes require an effort from the teacher. The choice of the appropriate tasks which help the achievement of the planned goals and which are interesting for students is not easy. This is how I wish to share my experience with other teachers who try to achieve the same goals.

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THE ABILITY TO CONSTRUCT COUNTER-EXAMPLES BY STUDENTS AGED 15-17

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ABSTRACT
The paper presents a piece of work of a teacher-researcher who consciously and purposefully attunes her teaching so as to develop in her students the ability to construct counter-examples. All research was carried out as part of the standard teaching process, in full accordance with the curriculum and teaching plan. The teaching-research investigation addressed the question of constructing mathematical counter-examples to false theorems by students in her classes in a secondary school, of finding learning improvement strategies and assessing their effect.

1. INTRODUCTION

Verbalizing conjectures independently, verifying them within mathematics, and proving theorems belong to the most important abilities to be attained by learners of mathematics. Krygowska (1977) wrote:

> No matter whether a conjecture will prove true or not, whether it presents a mathematical fact of a greater or smaller importance – its formulation, justification, and attempts to prove – with the teacher's approval – develop student's intellectual courage, which is one of the most important goals of teaching in general and teaching mathematics in particular.

From my school experience I know that students in secondary school encounter great difficulties when solving problems that require discovering a theorem, formulating a conjecture, and verifying it within mathematics. Developing elements of mathematical ability mentioned above is neglected in school. Textbooks hardly provide problems aiming at the development of those abilities, while teachers think that instead of guiding the process of discovery it is better to transmit ready knowledge, and instead of proving they only mention that the proof is in the textbook. In their opinion, time saved in this way could be better used for solving more problems.

Unfortunately, this approach results in secondary school graduates – even those taking an extended mathematics course – who cannot distinguish theorems from definitions, cannot understand the role of the proof of a theorem. Among others, this kind of difficulties among university mathematics students was described by Hawro (2006).

The objective of my research was to identify my students' difficulties in discovering, formulating, and proving theorems, then to undertake an attempt to develop this ability and deepen students' knowledge in this domain. I was aware that the development of those abilities was not a simple task, so adequate didactic actions should be applied by the teacher, consciously and consequently.

The initial diagnosis was carried out in 2006/7 school year among 22 students aged 16-17 during their first year of secondary school, who took an extended mathematics course. The students studied over 20 lessons on logic. They learned principles concerning determining the logical value of composed sentences and of negating sentences. They were trained to indicate assumptions and conclusions in theorems of various verbal structures, to form inverse theorems; sometimes they (together with the teacher) tried to decide if the analyzed sentences were true (proving them was not always possible).

To conclude the topic of general properties of a function I presented a list of 10 simple problems on properties of a function. The task of the students was to write down in the form of implication three true and three false theorems, using some of the given conditions (properties):

1. f jest even
2. f jest odd
3. f possesses two zeros only
4. D_f = <0;10> (D_f is the domain of f)
5. f is increasing
6. f is decreasing
7. $x = 0$ is zero of f
8. point $A(0,0)$ belongs to the graph of f
9. graph of f is symmetrical with respect point (0,0)
10. $f(x) = x^2$

Outcomes showed that the biggest difficulty for my students was to assess the logical value of the sentence they built. Over a half (45) of the theorems they proposed as true were, in fact, false. So, I decided to reduce my research problem and focus on the ability to construct counter-examples for false theorems.

I assumed a restricted objective in the form of two questions: (1) How do students understand counter-examples; are they able to build them? and (2) What exertions could I apply in order to develop their ability to build counter-examples?

In this paper I present a piece of the investigation concerning the restricted objective.

2. FROM THE LITERATURE

As a very important didactic strategy for a functional grasp of mathematical definitions and theorems in students' thinking, Krygowska (1977, 109-110) points to *contrasting*. In the former case it means that to make students aware of the operations that have to be conducted according to a definition in order to decide if a given object *is* among those determined by the definition, one should also show how to decide if it *is not*. In the latter case (theorems) it means to have students know that a theorem *is not* true when all assumptions hold but the conclusion does not. Students often forget to check some essential assumptions; conclusions they come up with are either false or right, but their reasoning in this case is not correct.

A special form of contrasting is to provide examples and counter-examples. Looking for a counter-example requires creativity and discipline of thought. If the examined example affirms the rule an impression is made that there is a rule. One may get a hint how to prove it or what example should be examined to refute the rule (Polya, 1962).

Klymchuk and Grunwald (2006) say that one of such strategies might be

> Giving the students wrong statements and asking them to pay more attention to the concepts, conditions of the theorems, properties of the functions, and to reasoning and justification. Practice in constructing their own examples and counter-examples can help students enhance their creativity and advance their mathematical thinking. Counter-examples deal with disproving, justification, argumentation, reasoning and critical thinking, which are the essence of mathematical thinking. These skills will benefit students not only in their university study but also in other areas of life.

3. REPORTING THE RESEARCH

Organization of research

Test papers were based on the subject matter of classes in the regular teaching course and errors selected from students' products in the initial diagnosis. The tests were administered when students mastered the new topic rather well and felt comfortable using it.

Between tests *corrective strategies* were applied, such as: (1) recalling the results of the initial diagnosis, exemplary assessment of the logical value of one theorem and building various counter-examples; (2) recalling the rule saying when a sentence in the conditional format is true, and when it is not; (3) grouping students with various levels of mathematical skill in order to have those who can find a counter-example explain it to those who cannot; and (4) including elements of "work on a theorem" in current topics; where possible, theorems discovered and formulated or counter-examples were found false.

First stage of research

During a lesson, having discussed results of the initial diagnosis, I wrote on the board one of the theorems formulated by a student and qualified as true. It was "If $D_f = <0, 10>$ then the function is increasing." I asked for its logical value. Students quickly said the theorem was false and sketched several function graphs for which it did not hold. During the lesson we recalled the rule when a theorem is true or false and what should be done to decide it. A theorem is true when each object satisfying the assumption also satisfies the conclusion. If one points even one object (a function in our case) that does satisfy the assumption but does not satisfy the conclusion, or gives a so-called counter-example, then the theorem is false.

Test 1

I supposed that finding a counter-example would be easier if students knew that the sentence they were given was false. They would concentrate on looking for an example that would not satisfy the conclusion. I prepared tests with five false theorems that students created themselves in the previous test, qualifying them as true. I asked them to construct counter-examples for at least three such sentences. A function not satisfying the theorem could be presented in any format, for instance, by sketching a graph or writing a formula.

Task sheet 1.
Examples of false theorems:

1.	If $D_f = <0, 10>$ then point $A(0,0)$ belongs to the graph of f.
2.	If f is even then f is not odd.
3.	If zero of f is $x = 0$ then f is odd.
4.	If the graph of function f is symmetrical with respect to point (0, 0) then zero of function f is $x = 0$.
5.	If point $A = (0, 0)$ belongs to the graph of f then f is even or f is odd.

Quantitative analysis of test 1

Number of correct counter-examples	Number of students
5	4
4	3
3	2
2	3
1	4
0	5
Total	21

*Table 1.*Scores in Test 1

During the third stage of my research there were 19 students present. 13 (2/3) gave at least three correct counter-examples.

Unfortunately, 5 of 19 students did not participate in the former test (they were absent then). I compared the outcomes of 14 students that participated in both tests on finding counter-examples. It proved that 9 students improved their results. There were three persons among those nine that did not provide any counter-example correctly the first time, while now found respectively one, three, and five of them. There were students, though, who scored worse in the second stage. A plausible reason is an insufficient command of the subject matter involved in the recent test.

Qualitative analysis

I will present six identified categories of difficulties encountered by my students while constructing counter-examples.

(1) Substantiating of the kind "it can but does not have to."

The task required finding a counter-example. But part of the examined students substantiated falsity of the theorem with verbal description. Sometimes this description made me believe that students knew which assumption an example of function should fulfill and what part of the conclusion it should not, but examples were not given.
Descriptions of this kind were classified as "it can but does not have to." For example, falsity of the first theorem was substantiated as follows:

Student 10: Because point (0, 0) does not have to belong to the graph if the domain = <0, 10>.
Student 20: No, because 0 can but does not have, for example (0, -3), (0, 2).

In other statements conditions for the value set of the function occurred. For example: "We do not know what is the value set;" even if we assumed that 0 belongs to the value set the theorem would remain false as it is; not necessarily must be $f(0) = 0$. It was better expressed by Student 6, "Because the value set does not have to contain 0." Of course, if we consider a function that fulfills the assumption but 0 does not belong to its value set this would be a counter-example.

The above quotations demonstrate students' difficulties with understanding the role of a counter-example, despite their understanding of conditions that it should satisfy. In paper 15 I noticed another difficulty. Despite the information "examples of false theorems" a student attempted to determine their logical value. He decided that the second and fifth are true, the third and fourth false because "it does not have to be so." He decided that the first sentence was false, which he later crossed out leaving only the argument: "It can but does not have to because the graph does not have to begin at point (0, 0)." Crossing out the answer suggested that he was not sure if the theorem was really false. Durand-Guerrier (2003) explains: "How can we understand that some good pupils declare, concerning a conditional statement, that ‘they cannot decide if it is false or true,’ while teachers think that it’s obviously false?”

2. Errors connected with the understanding of what a counter-example is.

Research by Nowecki (1974) clearly shows that in the first two years of upper secondary school (lyceum) most difficulties that students encounter are in understanding that a theorem occurs when it is in the predicative format. The conditional and conditional-functional formats cause much less difficulties. As my research question was about how students understand counter-examples and whether they can produce them, all theorems in tests were in the conditional format. The assumption and conclusion being visibly exposed, it helped to observe which students had difficulties with the logical aspect of creating a counter-example.

Among answers to problem 3 (“If a zero of function *f* is *x*=0 then *f* is odd”) I found the following: “if *x*=0 is in a point different than (0, 0) such a function is odd;” but in this case the assumption was not satisfied. Student 27 wrote: “For an even function *x*=0 may also be a zero,” which did not contradict the given theorem. Student 14 twice sketched, as a counter-example, graphs of functions that satisfy the conclusion but not the assumption. Below is his “counter-example” for theorem 3.

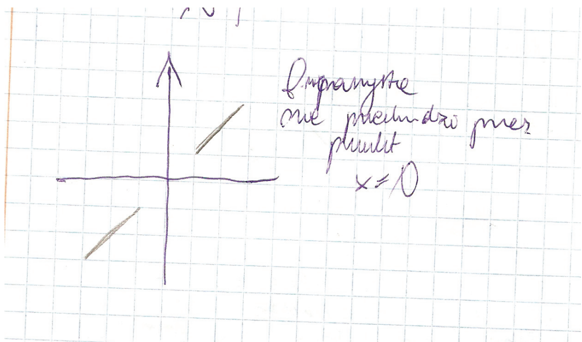

Figure 1. [Even – does not pass through point x=0].

As a counter-example here a function could be used, whose zero is in *x*=0, but which is not odd. The student’s graph presents an odd function (it is symmetrical with respect to the origin), for which *x*=0 is not a zero.

3. Deficiencies of previous knowledge.

Four students said that a function cannot be both even and odd. They argued, for example, that the graph cannot be symmetrical with respect to both OY axis and point (0, 0). This example showed that many facts discussed in class were ignored or forgotten if they were not reinforced from time to time. A function was considered whose graph consisted of two points only: (-3, 0) and (3, 0). Students were surprised noticing that it is both even and odd. We looked for other examples fulfilling the conditions. Students found, for example, function $y = 0$, and functions, whose graphs are composed of line segments or points belonging to axis OX, disposed symmetrically with respect to the origin.

Unfortunately, deficiencies turned up not only in the area of functions but also among more elementary topics. Errors of two students consisted in confusing closed and open intervals. They said that if the function domain was (0, 10> (instead of <0, 10>) so point (0, 0) would belong to the graph. Another student’s answer showed his lack of some knowledge concerning central-symmetric figures. It goes: “The graph is symmetrical with respect to (0, 0) so this point does not belong to it.”

4. Lack of critical thinking.

One of the students, as his counter-example for theorem 4, produced a graph, which did not satisfy the conclusion, was symmetrical with respect to (0, 0) (so satisfied one condition in the conclusion), but unfortunately it was not a graph of any function. (The same error was made by another student addressing the theorem 3.) This student is able to construct counter-examples, but he lacks a critical view of the result of his work.

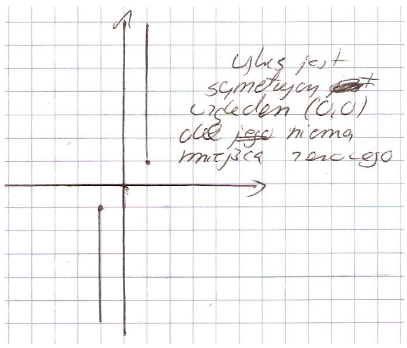

Figure 2. [The graph is symmetrical with respect to (0, 0) but has no zero].

5. Difficulties with using mathematical logic.

Theorem 5 (“If point (0, 0) belongs to the graph of f then f is even or f is odd”) was more difficult because of the alternative in the conclusion. Some substantiations focused exclusively on that alternative and were detached of the assumptions: “An even function does not have to be not-odd,” “A function can be even and odd at the same time.” Also, difficulties appeared with negating the alternative and the quantifier “there exists.” In mathematics, “there exists a zero” (in singular) does not mean a unique zero; negation of this sentence is not existence of many zeros. Student 12 understood the conclusion of theorem 4 (“If the graph is symmetrical with respect point (0, 0) then zero of this function is x=0”) as saying that 0 is the unique zero:

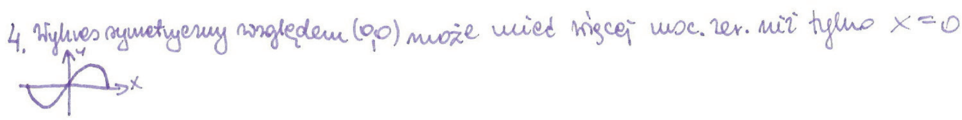

Figure 3. [A graph symmetrical with respect to (0, 0) may have more zeros than only x=0.]

If the term “only” was used in the conclusion the counter-example would have been constructed correctly.

6. Difficulties with using the language of mathematics.

While reading students’ expressions one cannot miss many linguistic errors: “A(0, 0) may not belong to ZW_f” [The set of values of f], “this point can, but does not have to lie in that domain,” “A function with graph (x)=0....” Those few examples manifest students’ misunderstanding or incorrect use of notions such as: ‘argument, value of a function, point, graph, and function formula.' It is also an impulse for me to come back to those notions within future topics.

Summing up

Almost a half of my students did understand a theorem in the conditional form and were able to construct counter-examples for given false theorems. But the other half of them did not attain a sufficient command of it. Besides, the study disclosed deficiencies in students' knowledge, difficulties in applying mathematical logic, and using the language of mathematics.

Corrective strategies

Students were grouped in teams in such a way that in each team there were persons who coped very well with finding counter-examples; their task was to explain to colleagues how to build counter-examples to false theorems. Each received a sheet with examples of false theorems (Work sheet 1) and sheets with their own solutions. Students who failed to give counter-examples or their counter-examples were erroneous, were asked to provide counter-examples to the given sentences. This was an opportunity to see their errors once more and discuss them in a more accessible language.

I also made a plan of corrective strategies, which I applied in the course, within each subject matter being taken.

(1) I gave students a mixture of correct and incorrect statements (Klymchuk & Gruenwald, 2006).

(2) I asked students to spot an error on their worksheets (Klymchuk & Gruenwald, 2006).

(3) I gave students extra marks for providing excellent counter-example during the lesson (Klymchuk & Gruenwald, 2006).

(4) I asked students to identify the assumption and conclusion in theorems they learned, to negate those conditions, and look for examples that fulfill the assumption and do not fulfill the conclusion (Krygowska, 1977).

The second test

In order to learn if the work in teams brought positive effects I prepared a new set of six false theorems. Because we had just finished linear functions, the theorems addressed that subject matter. Students were asked to find counter-examples for at least three theorems. I did not want to force them to do it for all theorems as the test lasted usually about 15 minutes, which was not a sufficient time for all students. Working hastily they would construct counter-examples without carefully analyzing the content.

Task sheet 2.
Below are six false theorems. Find counter-examples.

1. If a function possesses more than one zero then it is not linear.
2. If a function is partially linear then it is not monotone in its domain.
3. If b in the formula $y = ax + b$ is a positive number then this function is increasing.
4. If $x = 1$ is a zero of a linear function then its graph passes through 1^{st}, 3^{rd}, and 4^{th} quadrant of the coordinate system.
5. If a function possesses one zero then it is linear. If a linear function is increasing then its value is always greater than the respective argument.
6. If a linear function is increasing then its value is always grater than the respective argument.

Quantitative analysis

Number of correct counter-examples	Number of students
6	1
5	5
4	2
3	5
2	4
1	1
0	1

Table 2. Results of the second stage of study

In this stage only 19 students were present. Of them 13 (about 2/3) gave at least three correct counter-examples. Unfortunately, five of those 19 students did not participate in the previous test (they were absent). So I compared results of 14 students that worked with both series of false theorems. Nine of them improved the result. Among the nine there were three persons who failed to construct even one counter-example in the first series, while now 1, 3, and 5 counter-examples respectively. There are persons, though, whose result worsened. A plausible reason is they insufficiently acquired the necessary knowledge.

Qualitative analysis

I will compare results of the second test with the first one. To begin with, I will analyze a few papers with improved results in order to identify errors, which were eliminated thanks to working in teams. Then I will describe several typical errors I found in the papers of students, who had not made any progress.

One of the persons, who in the first test gave only one counter-example is student 14. In the first test he attempted to solve four problems but he only managed to sketch one function graph as a counter-example for theorem 4. Addressing the first theorem he wrote "If D_f = <0, 10> then the function may but does not have to pass through point A(0, 0)." He failed to give any concrete example. He twice presented functions fulfilling the conclusion and not fulfilling the assumption as counter-examples (details – see above Figure 1). In the second test the same student gave four correct counter-examples.

In the case of the first theorem, it was necessary to give an example of a function having more than one zero and linear. The student drew the following figure:

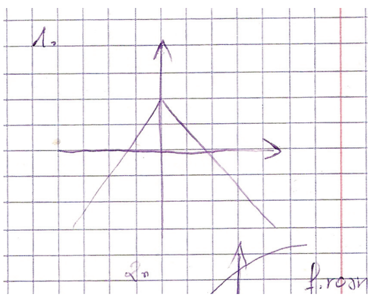

Figure 4.

This function fulfills the assumptions but is not linear. Error consisting in handling a partially linear function as linear appeared in numerous papers. It signals the fact that the definition of linear function, whose formation starts in elementary school, is not correctly applied by students in problems.

Student 14 substantiated the third theorem as follows: “Whether a function is increasing or decreasing is determined by the direction coefficient a.” Of course, it is true; but it was required to indicate a concrete example of a function with positive coefficient b, which is not increasing.

Summing up results of student 14, I think that work in teams considerably contributed to his correct understanding of counter-example. He did not repeat the error of negating the assumption.

I observed a considerable progress of student 27, who failed to construct any counter-example in the first test. Instead, he wrote a verbal comment to each theorem, which revealed his ignorance on constructing counter-examples. Theorem “If a zero of function f is x=0 then f is odd” he refuted with the following commentary: “Not true because for an even function $x = 0$ may also be a zero.” Besides, he wrote earlier that “a function can be simultaneously even and odd,” so his reasoning did not aim at negating the conclusion.

Difficulties with using mathematical logic occurred also in the last example because there was an alternative in the conclusion: “f is even or f is odd” that he negated: “This function can be even and odd at the same time.” In general, his expressions were imprecise or quite unclear. He often used the expression “this function,” “its domain” when it was evident what function is addressed or whose is the domain.

In test 2 this student correctly built five examples of functions being counter-examples for consecutive theorems. In the case of the second theorem he drew in one coordinate system graphs of two linear functions (Figure 5).

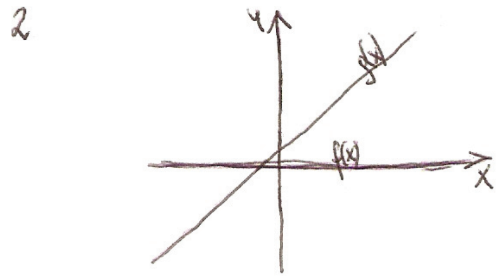

Figure 5.

He also found counter-examples for four theorems and drew adequate function graphs. For example, for the fifth theorem “If a function possesses one zero, then it is linear” he drew the following graph (Figure 6).

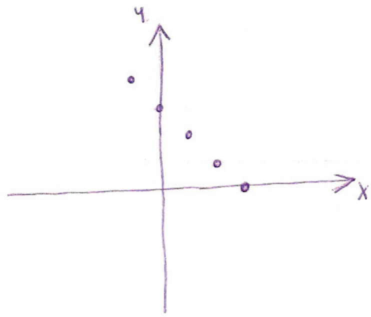

Figure 6.

I will present one more solution of his for the third problem: “If *b* in the function formula $y = ax + b$ is a positive number then the function is increasing” (Figure 7).

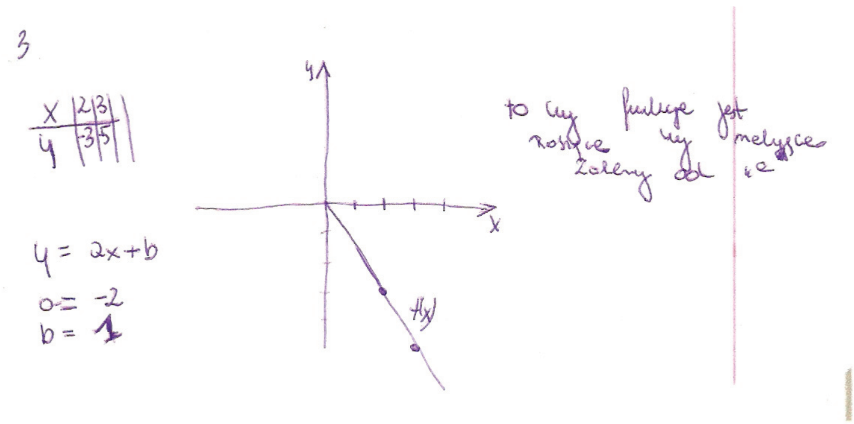

Figure 7. [Whether a function increases or decreases depends on “a”].

Like student 14, he remembered that monotony of a linear function depends on the *a* coefficient, but in contrast to the former he gave an example of concrete coefficients of the linear function used as a counter-example. In order to draw its graph he calculated coordinates of two points and linked them by hand with a line. The sketch is so imprecise that the line passes through point (0, 0), which contradicts $b > 0$.

Of course, in mathematics we use drawings that are more or less precise. Yet we try to get students accustomed to precise marking characteristic elements that in the case of linear functions are zero and intersection point with *OY*-axis. When in a problem where the assumption $b>0$ is essential a student sketches a graph suggesting $b=0$, it makes one think that he is not conscious of the influence that a change of *b* exerts on the change of the position of the graph. For me, it is one more reminder that knowledge concerning the linear function should systematically be reviewed while solving problems belonging to a current subject-matter.

As it was said earlier, 9 persons improved their results. The remaining 5 persons, who solved both tests, worsened their performance or stayed at a low level.

Many errors concerned the definition of linear function (they consisted in missing the condition of R being the domain) and the fact that a partially linear function was not linear. Among counter examples for the first theorem occurred functions like the following.

1. y=|x|-2
2. (see Figure 8)

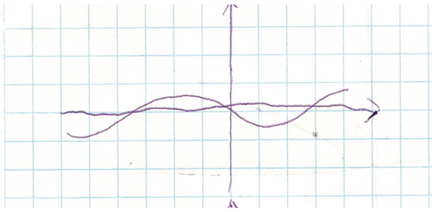

Figure 8.

Of course, they are not linear so they are not counter-examples.

Similarly, counter-examples for the second theorem, which were supposed to be partially linear, were sketched by some students as consisting of isolated points or partly curved (see Figures 9 and 10).

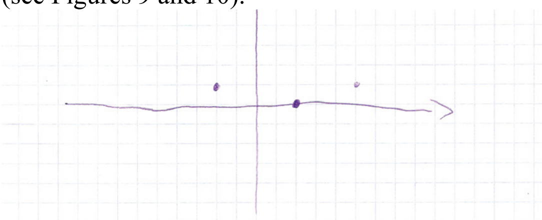

Figure 9.

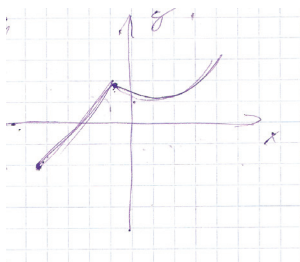

Figure 10.

When comparing results of tests 1 and 2 with respect to the categories of errors identified in the first stage of study, we notice a progress mainly in the first two categories, i.e. "it can but does not have to" and understanding the role of a counter-example. In the second test those errors were much rarer but were not totally eliminated. More errors appeared, though, caused by poor acquirement of the subject matter.

4. SUMMARY

For refuting a theorem (sentence) it is enough to construct one counter-example. Refuting false theorems is – in my opinion – easier than proving theorems (true sentences) that enclose all the objects they speak about. I tried to teach it to my students in the regular school practice, upon various mathematical subject matters. Excerpts of this work were presented above. Didactic exertions I applied in this process consisted in the following.

1. Students *themselves composed true and false sentences in the conditional format*.
2. They were shown what counter-examples that refute such theorems consisted in: *fulfilling assumption and not fulfilling conclusion.*

3. Students were given *false sentences and asked to find counter-examples*. Those false sentences were selected from students' own production.
4. Short tests assessing their ability to find counter-examples were given within each kind of subject matter in the course, at a moment when they had dominated the notions and language of the topic rather well. I was able to see that students tended to specify true theorems and look for "examples that confirm the rule;" but their awareness that is does not prove the theorem "for all" grew. This activity contributed to a better understanding of concepts because they were finding various objects fulfilling the defining condition.
5. Students were given sentences in the conditional form and asked to assess their logical value. Those sentences either were simple corollaries of theorems they knew, or required constructing counter-examples to substantiate their falsity.

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DIFFICULTIES WITH THE CHANGE OF BAD HABITS AND BELIEFS: THE DIDACTICAL PHENOMENON OF IMPRINTING

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ABSTRACT
The commonly used model for calculation with percentages is additive. This model is difficult to apply, e.g. when knowing the price with VAT, we want to know the original value of commodity without VAT. In general, the additive model expressing change of value using percentages is difficult, when the reverse operation is needed. The multiplicative model using decimal numbers is better. Even though students were shown the advantages of multiplicative model, they persist in using additive model and fail. This didactical phenomenon is called imprinting. It is analogous to the phenomenon of proceptual divide of Gray and Tall. A remedial teaching was applied using graphical representation of percentages. It was only partially successful.

INTRODUCTION - GENERAL REMARKS

I am a mathematics teacher in secondary school ("Liceum") in Siedlce. In this paper, I will present a teaching experience with a group of students aged 16-17. I often have the impression that my work as a teacher is effective: students learn according to expectations. Then, after some time and in a concrete situation, very much to my regret, it turns out that their skills disappear and unexpected fundamental mistakes occur.

It began with my concern over percentages. The usual model commonly used for calculation with percentages to represent growth, enlargement or decrease, is additive. We add tax, VAT or calculate a discount by addition or subtraction of percentages. On every 4-operation calculator, there is a special key helping to use percentages in the additive way. It works with one operation problems. This additive model "the operation of adding a percent to a value" is difficult to apply, when inverse operations are required: e.g. when knowing the price with VAT of say 22%, we want to know the initial value of a commodity. The multiplicative model for percentages is much more versatile. In place of adding, say 22%, we simply multiply by a decimal number 1.22. Then the inverse operation is "divided by 1.22." The students, who were taught percentages only the additive way, disregarding the inverse operation, may have difficulties, even though they were shown later the multiplicative model and its advantages. They usually stick to the first model learned and fail. Because of my participation in the PDTR, I got especially sensitive about such problems, where the skills learned at early stages of education turn into deeply rooted habits, detrimental for further mathematical development. This phenomenon of students not being able to accommodate new better fitting model was called at our seminar *imprinting*.

We discussed several such didactical problems at our PDTR seminar in Siedlce, but for this particular one with percentages, I wanted to find some way of remedy by exposing students to a specially designed series of problems and using pictures along with formal symbolic calculation. Then students divided into two groups, those few, who reacted well to the treatment, and used pictures and the easy to apply multiplicative model, and those who did not, and failed, trying to use the familiar old model, not compatible to the situation and leading to difficult calculations. This didactical phenomenon was analogous to the *proceptual divide* described by Gray and Tall (1994).

The common beliefs are that: (1) calculation with percentages is important at every educational stage; (2) it is essential that students should understand percentages well, since they occur in everyday life; (3) the most important thing is that students should understand well the notion of "percent," interpret it correctly and be able to apply it in concrete situations; (4) we should remember that the "per cent" is always "a per cent of something;" (5) we should avoid teaching rules without proper relational or structural understanding (Skemp, 1976; Krygowska, 1977).

My conclusion is that we should add one more point: (6) percentages are good for pictorial representation; the corresponding calculations should be done without percentages, using decimal numbers.

CHARACTERISTICS OF STUDENTS' GROUP

The research group consisted of first grade students, aged 16-17 of a Polish high school, where I am a mathematics teacher. There were 30 students. They chose mathematics and information technology as their specialization. Most of them had good achievement in mathematics by the end of the previous educational stage in middle school, and they got quite high marks in mathematics and science part of the post middle school test. At the beginning of a new school year in September 2006, there was a test which showed that the level of their math competency was rather low but they were active and willing to do extra work. They did not avoid asking questions, on the contrary – they often asked "is it possible to do it differently?" "Is this way all right?"

I noticed that the group had problems with percentage calculations involving more than one step computation. At first, I relied on their experience from secondary school which, as I found, was not satisfactory. I explained how to use the multiplicative model for percentages. Then, I designed a series of activities in order to see, if they *accommodated* that new model.

The participation in the PDTR Project required that I assess the effects of my work not only by implementing popular standards, but also because I care about conceptual development of my students and look for *possible improvement of my teaching*. To work as a teacher-researcher means to me, first of all, being sensitive for developing key competencies and at the same time controlling the structure of the mathematics which is taught. More often than before, I ask myself the question, if what I teach is good for further conceptual development in mathematics or blocks this development by the mere structure of the subject matter. So, noticing fixation of my students on additive model for percentages, I worried what to do with this didactical problem.

First observations:

The first lessons provided observations, which were essential for my future work with the group. I noticed that: (i) my students knew from the previous stage of school (in middle school) some ways of solving several types of problems involving percentages and they were strongly accustomed to them; (ii) they do not feel at ease using new ways of calculations; (iii) at a lesson in school, they use new methods but while doing the homework they very often return to the old ones; (iv) some students, actually those with average marks, feel in general more confident when using earlier known ways; and (v) some students (a rare event) are fast to notice advantages of the new ways of solution, and they apply them willingly.

At the curricular topic of "real numbers," according to the program of study, I spent a part of the lesson on problems with percentages. We solved some typical and some unusual tasks. The majority of students had no difficulty with solving typical tasks. The usual standard test confirmed that students can do typical tasks. At this moment I wanted to probe that ability a little.

Research – stage 1

Students of the first grade of secondary school got the following task to solve (the problem was suggested by the PL2 PDTR team from Rzeszów):

> Read, answer the questions and write your comments to the following text:
> The management of the SIGMA PLUS factory announced the reduction in the salary of all workers by the 10%. Dissatisfied workers struck. During the strike their salaries were fixed at the announced level. After three months the management announced the pay rise by the 10% and the strike was ended.
> 1. Should workers be satisfied with such pay rise? Justify the answer.
> 2 . Mr. Kowalski who works in this factory had monthly salary of 1,200 zloty.
>   a) calculate how much he earned after the reduction.
>   b) calculate how much he earned again after the pay rise was announced to end the strike.
>   c) how much per cent should the pay rise be, so that Kowalski gets as much as before.
> 3. Mrs. Nowak is a member of the board of trustees of the factory. It was settled that the reduction by 10% also includes salaries of people from the board of directors (board of trustees + director).
>   a) after the reduction Mrs. Nowak earned 9,900 zloty. How much did she earn before the reduction?
> b) how much per cent was the pay rise of the members of the board of trustees, if finally Mrs. Nowak earns 11,385 zloty?

My remarks after checking solutions: (1) answering these questions was rather easy for the students; (2) in the group of 30 students almost everybody solved the problems correctly; (3) can I be sure that students are OK with percentages? (4) should I close that topic and take another one?

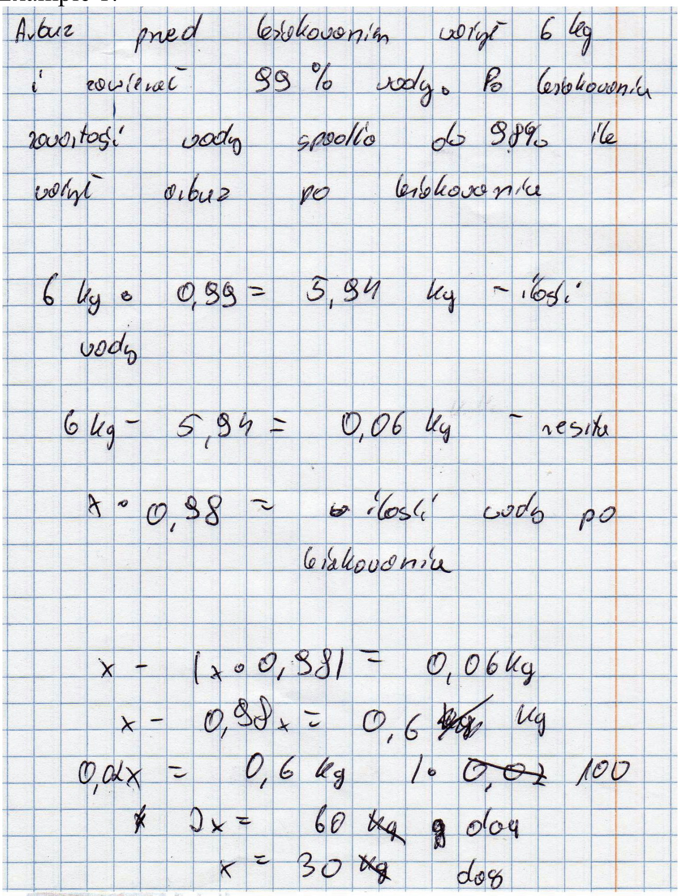
Research – stage 2

I asked students to solve a problem called "Ripening of a Watermelon." The story was as follows:

> A watermelon weighed 6 kg and contained 99% of water. After some time of ripening, the content of water fell to 98%. How much did watermelon weigh after ripening?

I didn't make comments on the task. Students were supposed to suggest the solution. I only observed them work. The majority worked independently, they could only speak with a desk partner. Result: Only 1 answer was correct, 29 were wrong. Here are some papers written by the students. First, the correct one (with explanation):

Example 1:



Arbuz przed dojrzewaniem ważył 6 kg i zawierał 99 % wody. Po dojrzewaniu zawartość wody spadła do 98%. Ile ważył arbuz po dojrzewaniu

6 kg · 0,99 = 5,94 kg – ilość wody

6 kg – 5,94 = 0,06 kg – reszta

x · 0,98 = ilość wody po dojrzewaniu

x – (x · 0,98) = 0,06 kg

x – 0,98x = 0,6 ~~kg~~ kg

0,02x = 0,6 kg | · ~~0,02~~ 100

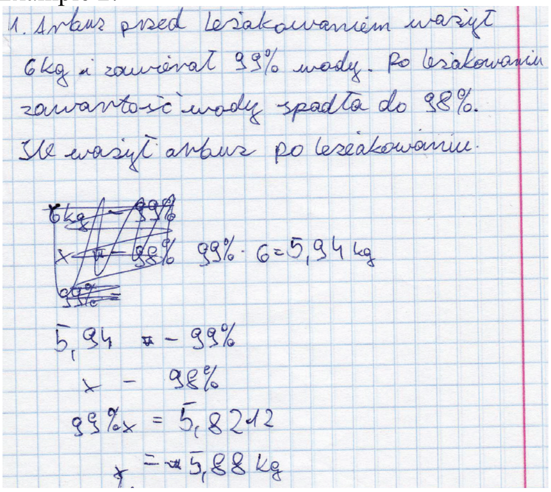
2x = 60 ~~kg~~ dag

x = 30 ~~kg~~ dag

Transcript: Before ripening, it weighed 6kg and it contained 99% of water. After ripening the content of water fell to 98%. How much the watermelon weighed after ripening?
6 kg times 0.99 = 5.94 amount of water
6 kg – 5.94 = 0.06 kg the rest
(it was) x times 0.98 = the amount of water after ripening
x – (x times 0.98) = 0.06 kg
x – 0.98 x = 0.(0)6 kg [one zero is missing]
0.02x = 0.(0)6 kg
2x = 60 dag [should be 6 kg]
x = 30 dag [should be 3 kg]

This student made small miscalculations, but the structure of the solution was correct. The other students made the following structural mistakes: They tried to apply proportions. They fell into difficulties and were not able to get it right. They incorrectly interpreted the content of 98% of water in watermelon after ripening.

Example 2:


1. Arbuz przed leżakowaniem ważył 6kg i zawierał 99% wody. Po leżakowaniu zawartość wody spadła do 98%. Ile ważył arbuz po leżakowaniu.

99% · 6 = 5,94 kg

5,94 – 99%

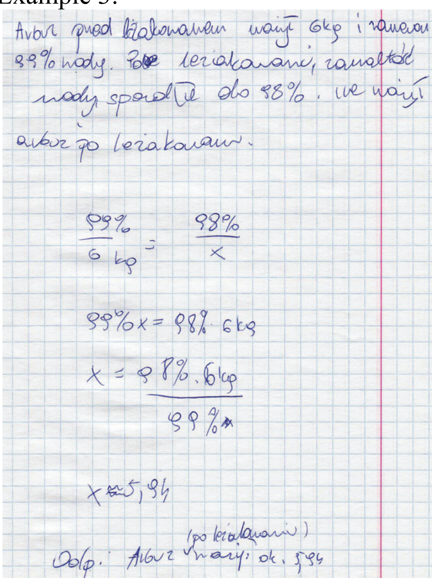
x – 98%

99% x = 5,8212

x = 5,88 kg

Transcript: [after the text of the problem, the calculation is as follows]
99% times 6 = 5.94 kg
5.94 is 99% ["= –" was used metonymically meaning "is"]
x is 98% ["–" was used metonymically in place of "is"]
99% x = (5.94 times 98%) =5,82.12 = 5.88 kg
5.88 + 0.06 = 5.94 kg
[note that this particular script reminds the old "*regula tri*"]

Example 3:


Arbuz przed leżakowaniem ważył 6kg i zawierał 99% wody. Po leżakowaniu, zawartość wody spadła do 98%. Ile ważył arbuz po leżakowaniu.

$\frac{99\%}{6\ kg} = \frac{98\%}{x}$

99% x = 98% · 6kg

$x = \frac{98\% \cdot 6kg}{99\%}$

x ≈ 5,94

Odp. Arbuz (po leżakowaniu) waży ok. 5,94

Transcript: [after the text of the problem, the calculation is as follows]
99% / 6 kg = 98% / x
99% x = 98 % 6 kg
x= (98% 6 kg) / 99%
x ~ 5.94
Answer: The watermelon weighed around 5.94 after ripening.

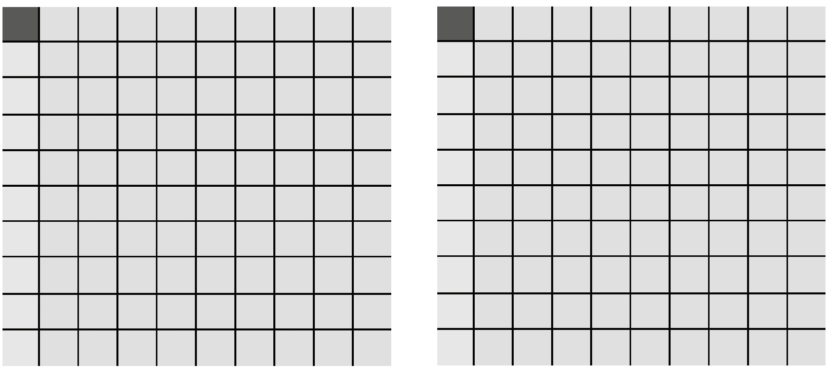
The next common mistake was an incorrect interpretation of the content of 98% of water in watermelon after ripening, and it was rather the calculation of "the content of water in water." I do not include here these papers. It turned out that students who earlier correctly solved the typical, even very difficult tasks concerning percentage calculations, this time they failed. I asked myself the following questions: (1) do only my students fail at this task? (2) perhaps the task was very difficult? (3) maybe it was inaccurately stated? (4) perhaps it contained a trap which was hard to notice for students. If so, why?

At our weekly seminar of the PL2 team, we discussed what to do next with my students. There was a suggestion (quoting Kilpatrick, 2007; Sierpińska, 2007) to give students a certain "graphic" clue. Just to introduce into the play *another representation along with the algebraic one*.

Research – stage 3

After a week, I returned to the same task without commenting on students' solutions. I asked students to solve the same task but in the following form:

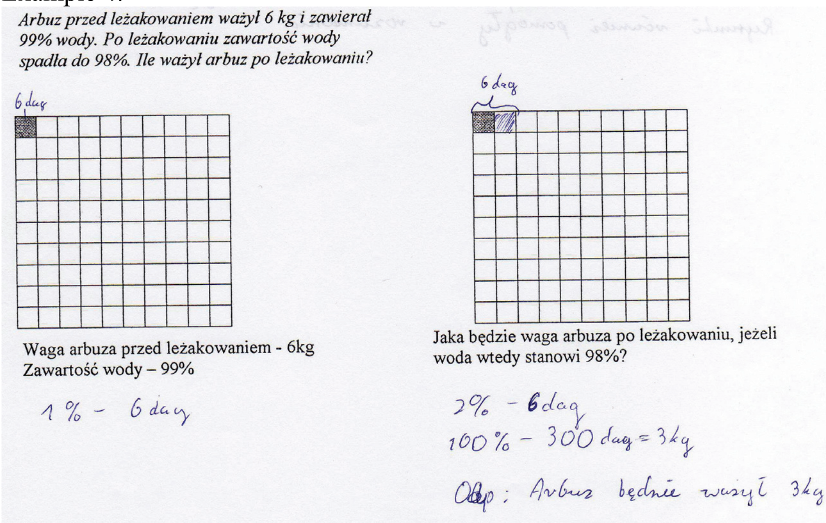
> Watermelon before ripening weighed 6 kg and contained 99% of water. After ripening the content of water fell to 98%. How much did watermelon weigh after ripening?
> The weight of watermelon before ripening – 6 kg.
> Content of water – 99%
> What will be the weight of watermelon after ripening, if water then is 98%?



I gave no hint and I did not explain why I gave two drawings. After a moment, I collected their papers with solutions. This time students gave 8 correct solutions. All of them tried to use drawings for solving the problem. I observed their work all the time. I noticed that the "small black square" was treated as a "dry substance" in the watermelon.

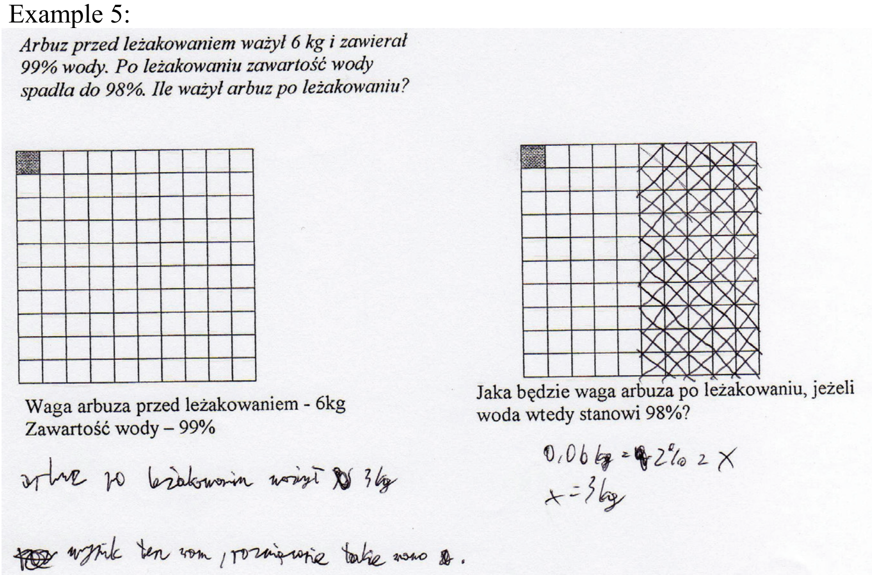
Students' comments: "water will evaporate, not the dry substance of watermelon; after ripening there will be less water; after ripening a dry substance of watermelon won't change;" [It should probably be "during ripening"]. Here are some examples of correct solutions.

Example 4:



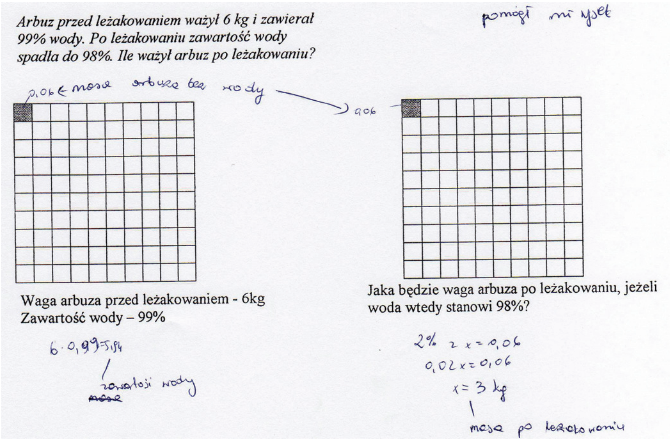
Solution correct. No translation is needed here

Example 5:



Solution correct. No translation is needed here, unnecessary area was crossed out.

Example 6:
“Pomógł mi rysunek” i.e. “The drawing helped me.”



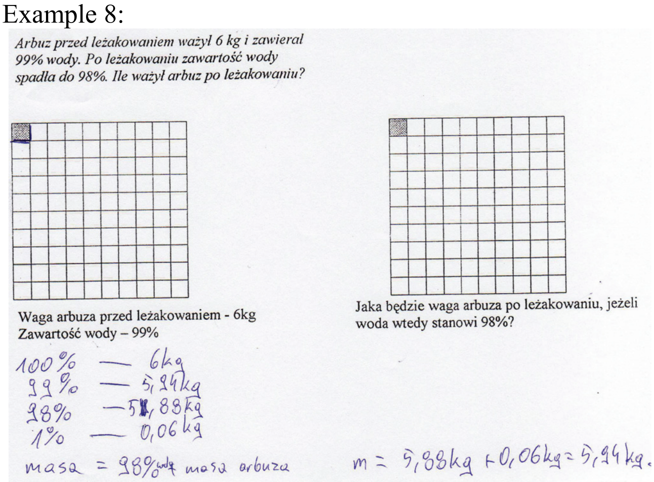
In the top right corner it says: “the drawing helped me” but at the drawing on the right hand, it is only indicated that the small black square means 0.06 of dry substance. The 98% of water is not marked; it was probably seen only in the students’ “minds eye,” because the calculation is correct.

In both Examples 5 and 6, students noticed that after ripening the mass of watermelon is 2% of mass of the whole watermelon. I asked students to justify their answer, and if they got different solution from the previous one, *why* they had changed their mind. They said that *the drawing helped* them.

Example 7:

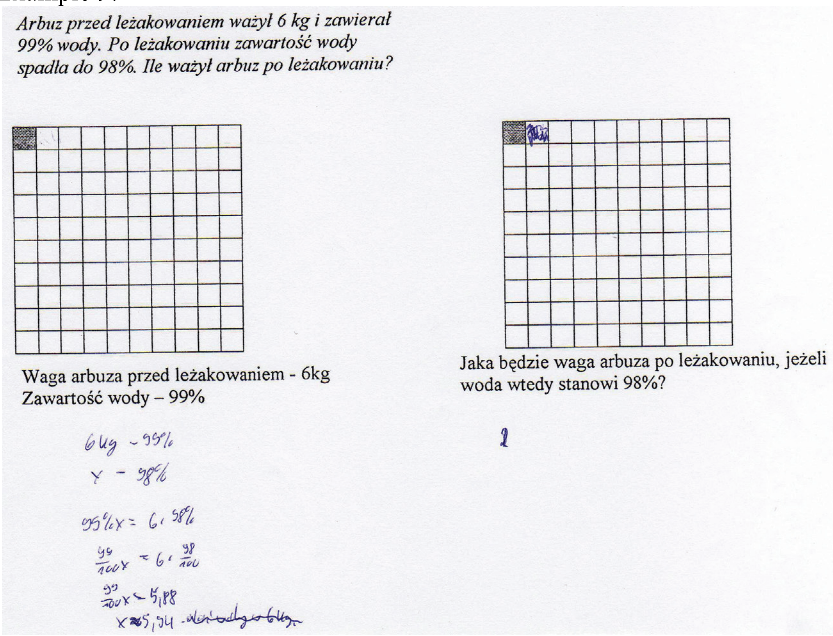
Again solution was correct and it was confirmed that the drawing helped, but it was not visualized at right hand grid what was that 98% of water. (The facsimile is skipped). However there were still incorrect answers:

Example 8:



No translation needed here. Visualization is wrong, the answer is wrong.

Example 9:



The visualization here seems correct, but the calculation is wrong, and the answer is wrong.

There were also some misinterpretations of the drawing. This means that we have to be very careful when giving hints. Students may take it the wrong way. One such example is below:

Example 10:

Nothing was marked on the grid, so the facsimile is skipped.

Transcript: The weight of the watermelon remained unchanged. (Here evidently the drawing at the right grid was understood that "nothing changed")

Still, I did not analyze these answers with students. Before analyzing the solutions, I wanted to ask students to solve this task for the third time without letting them know my opinion about the correctness of former solutions. I edited the task differently in order to simplify it.

At the same time, I wanted my students to describe how they understand the previous text describing the task before they start solving the same task for the third time. The aim was to check whether the difficulty in the solution was caused by the way that the story was told, the narration itself, or by the difficulty with choice of a mathematical model which would fit to the story.

Research – stage 4

Students received next task (I give only one drawing and the additional information. The story was modified a little):

Watermelon before ripening weighed 6 kg and contained 99% of water. While ripening water evaporated and in that process its weight decreased. After some time, the content of water in watermelon fell to 98%. How much did watermelon weigh at that time? How much would be the weight of the watermelon, if the content of water is then 98%?

You may sketch an auxiliary drawing. The weight of the watermelon at the beginning was 6 kg. How much it weighs now, if the water is now 98%? The weight of watermelon at the beginning was 6 kg, and content of water 99%.

This time 50% of solutions were correct. The remaining papers were still incorrect because of the attempt to solve a problem with use of "proportion" in a way which was not compatible with the story. Students were supposed to explain the conduct of the task.

Observations:

Students who solved a problem correctly were actually able to explain the conduct. However, in other cases descriptions of the task were incorrect. Students adapted interpretation of the task to their way of modeling and solution.

The main reason of such a large number of incorrect answers was that students did not understand the story. Graphical interpretation together with the plot of the story caused better understanding of the problem, and consequently correct, and a considerably shorter way of solution, a short-cut. This time I discussed with the students their previous answers. We analyzed correct answers and the most frequent mistakes.

Research – stage 5

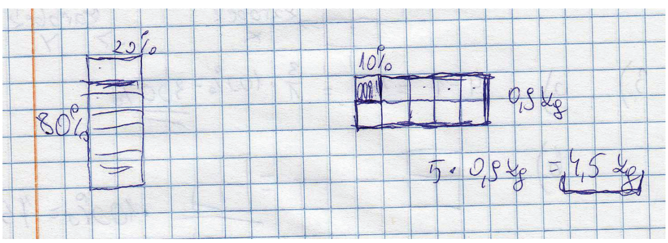
After a few weeks students received a new set of tasks involving percentages. I gave an additional hint: For each task there should be a drawing which helps you find the answer and avoid inconvenient calculations. One of tasks was the following:

A certain amount of mushrooms which contained the 80% of water was collected. As a result of drying, 1 kg of mushrooms contained only a 10% of water. How much mushrooms was collected at the beginning?

This task was similar to the task about a ripening watermelon. I got 60% of correct answers to this problem. Students used drawings, which illustrated the stories involved and the questions asked which were similar to the watermelon one. So this method, using two representations, one pictorial and another one algebraic, turned out to be quite effective.

Examples 11, 12 13:

I only give here the facsimile of one of them. The others are different as drawing but the effect is similarly seen right from the drawing, so that translation is not needed:

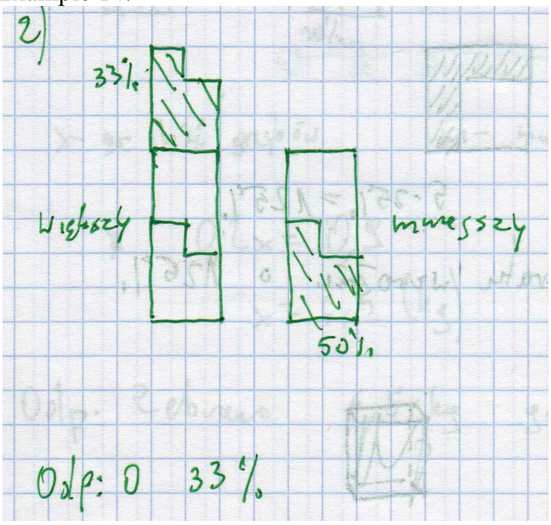


Another task:

Mr. Bigger earns 50% more than Mr. Smaller. How much percent less does Mr. Smaller earn than Mr. Bigger?

The task was easily solved by the majority of students with the use of a "drawing method." I give one facsimile, the others have different drawings, so that it was clear that they were independent, but the numerical effect was the same and was correct:

Example 14:

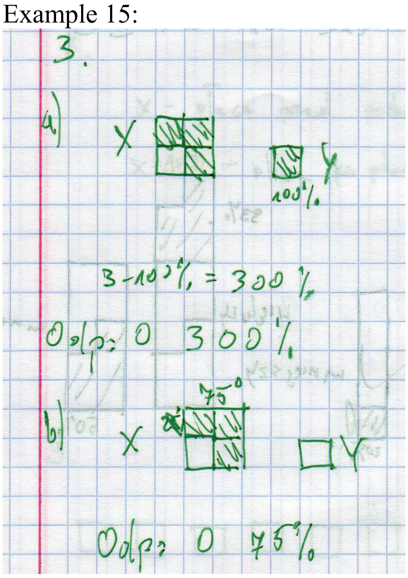


Similarly, without any trouble the next task was solved:

X earns 4 times more than Y.
a. how much per cent more does X earn than Y?
b. how much per cent less does Y earn than X?

An example of a solution:

Example 15:



Some of the students solved the problem incorrectly in spite of clear analysis of previous mistakes with the "Ripening Watermelon." They stubbornly practiced the method of the proportion or the additive model for percentages and ended up with errors. However, there were only a few.

CONCLUDING REMARKS

(1) I am satisfied for the time being with the fact that students are able: (a) to interpret mathematical problems graphically; (b) to read answers out from the drawing; and (c) to shorten ways of solutions. (2) However, I must think what I have to do to convince students that *drawing and sketching is good when it goes along with algebraic model which is proper and fitting*, otherwise it may not be effective. (3) I have to encourage students to make drawings of the "word problems" which explain or go along the plot of the story. This way it is *syntonic* in the sense of Papert. It is often said that a good picture is more than a thousand words. (4) It was often noticed at our seminar that using *parallel representations*, i.e. synchronic use of *two syntonic* representations at the same time, is a powerful method in problem solving. I think that my research confirms it. (5) Mathematics as a language is visual. In mathematics, we extract meaning out of arithmetic or algebraic formulas visually, not acoustically, by "reading aloud in our mind." Using pictures while thinking in the mathematical style is natural (Sierpińska, 2007).

It was evident in my trials that the first model learned for percentages was very deeply rooted habit of handling percentages. It was additive treatment without much care for reverse operation. It was very difficult to correct this habit, by showing in various situations the advantages of the multiplicative model.

The pictorial representation was better for remedial treatment. But the didactical phenomenon of staying with the first model learned was so evident that we called it at the PDTR seminar in Siedlce the imprinting. This phenomenon was mentioned in (Krygowska, 1979). She warned that we need to be sensitive about such situations where at one stage we teach some skills which at later stage are detrimental for further mathematical development.

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WHY OBSERVATION OF TEACHER-TRAINEES IS USEFUL FOR TEACHER-RESEARCHERS

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ABSTRACT
Several classroom episodes are reported here which helped draw our attention to some features of teachers' behavior, which are difficult to notice when we observe ourselves. Each of these features is given its name. Developing a language to communicate about these episodes and selected features made us more sensitive to noticing some problems which belong to didactical phenomenology of mathematical structures as well as to pedagogical human communication problems. Differences between school mathematics register of the Polish language and academic mathematics language register are a frequent source of the classroom communication problems. Some conclusions are formulated.

It is not surprising for active teachers that their students have problems with mathematics. They often do not know the reasons and consequently, they do not know how to act. But first of all, to solve students' troubles effectively they have to *notice* them and *be conscious of them*. The precise analysis and reflection concerning behavior of both students and teachers is essential.

It is obvious that students can be observed, tested, graded, prized, and reproached. But what about ourselves? If we want to take our professional improvement in our own hands, then the first step to do is self-observation. *Self-observation* is "directing the attention to one's own behavior and internal experiences" (Grab). Like any observation, it should take place in everyday situations in the classroom. To make it a part of the research program is especially difficult. One cannot prearrange situations and plan such research moments which depend on prearranged circumstances. Each and every moment must be real. So, how teachers who teach can make it, i.e. consciously direct attention to their own behavior?

Every classroom unit at school should be carefully designed with a prepared script. The script should be a part of a short weekly plan, and this should fit into a master plan of school program of study. It seems at first essential to plan the place and time for a particular issue that arouses our anxiety. However, in the actual dynamic of classroom events situation may change almost at random, even though superficially everything goes according to the plan. Students are all at work. Teachers watchfully scan their students for expected questions, they respond and make notes. But when students are engaged and start to be inquisitive and situation goes out of hand, it is difficult to respond and make notes at the same time. In such situation, the presence of other observers is helpful, especially when they are experienced at such an activity. They can extract from what happens in the classroom some essential features for analysis. Also some devices for recording the course of the lesson might be helpful. These things, however, make record of everything, important or not. And later analysis is time consuming, and essential moments are often blurred or missing. If such research aids are not available, then teachers must revoke the episodes in question all by themselves at a later time and make some notes.

It is much easier to notice some problems in our own behavior as teachers, while watching somebody else that replaces us in our role as teachers. Upon seeing mistakes of somebody else, we should ask ourselves how we would react in the same or analogous situation. It is also easier to observe our students, when somebody else is in the role of the teacher. Such situation arises when student-teachers from university come to a practice in school, a teacher-trainees under direction of the teacher.

Teachers have then an opportunity to watch both a trainee and their own students. A lot of interesting events are easier to notice that way. We are often in such situation. Our attention may be directed towards (1) questions prepared for students; (2) word problems and the way they are used during the lesson; (3) two language registers and students' reactions; (4) some not obvious gaps in understanding the subject matter, both of our students and teacher-trainees; (5) the discrete interventions of students.

1. Questions for students should be clearly and explicitly formulated. It is a frequent event that student-trainees ask questions in a rather academic language that students are not able to answer. Then, they pose new supporting questions and do not notice that they open a different problem, and students do not know, which question they should answer first. Another kind of situation is when teacher-trainees, without waiting for the reaction, start answering themselves, without noticing students' reaction to it. Being excited or perhaps even irritated, they often take some things as obvious and also do not see obvious misunderstandings or even errors of their own.

Classroom Episode 1: *Lack of sensitivity to differences of the school mathematics register of Polish language and a commonly used current Polish*.

At the start of a lesson about similar polygons, a student-trainee defined similarity of right angle triangles. Next, he expected students to use analogy and express the conclusion concerning similarity of the rectangles. However, students were "resistant" and silent. Then, after a brief lapse of not waiting for a response from students, teacher-trainee asked them to make notes about similarity of rectangles in their notebooks. He dictated: "Two rectangles are similar if the ratio of the longer sides is equal of the ratio of the shorter sides." He was surprised when a student asked: "And what about squares? And as he was confused, he could not answer satisfactorily.

2. THE WORD PROBLEMS

Tasks which are supposed to be solved with students during the lesson should specially be selected – aimed at the achievement of goals set by the program of study and related with earlier subject matter content. Activities are supposed to be diverse and reasonable and each of them should be educationally useful. Tasks must be relevant to all students' abilities but ways of solving them should not be all "imposed" by teachers.

Classroom Episode 2: *Disregarding of a good remark from the class.*

A lesson was aimed at "solving word problems with the use of first order equations with one unknown." A word problem for the whole class was:

> There were 120 cars and motorbikes at a parking lot.
> How many cars were there
> if all vehicles have 366 wheels (without spares)?

A teacher-trainee wanted to direct students' attention in solving this task to the fact that the obtained result does not always fit the question posed. He tried to show it. The students denoted by the letter x the number of cars. He asked for the answer to the question “how many cars and how many motorbikes were there?”

Students insisted that he said the question concerns only cars, whereas he told them to consider both cars and motorbikes: “there are x cars and y motorbikes.” One of the students then shyly suggested formulating the task a little bit differently: “There are 120 cars and motorbikes in the parking, *all together 120 vehicles*. How many cars are there if *all vehicles* have 366 wheels (without spare wheels)?”

The teacher-trainee did not react. Perhaps he did not understand what the student actually meant. And he proposed a next task. A remark was missed that an alternative method of looking at the problem exists and could clarify the situation.

Classroom Episode 3: *A Hint that complicates the task.*

At a lesson in middle school with a curricular topic “solving word problems using sets of equations of the first order with two unknowns” a teacher-trainee prepared six word problems on separate sheets. Beside the text with word problem, there was a free space in a frame for possible calculations justifying the answer. However, he did not say what it was for.

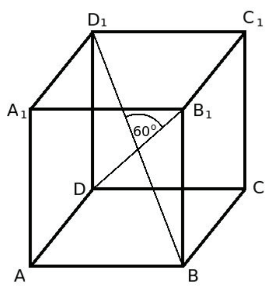
The problems were of the closed type and students could mark the answer by putting a cross in the appropriate space. So students, without the instruction clearly formulated, guessed without explanation why they chose one rather than another answer.

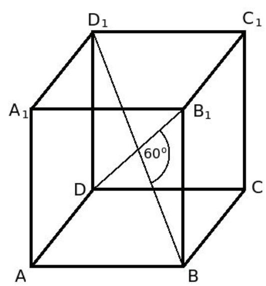
Classroom Episode 4: *Hints that complicate tasks*.

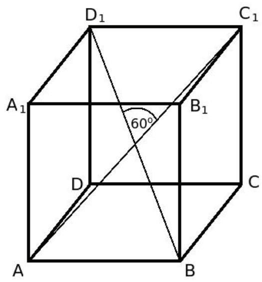
A teacher-trainee found a task which seemed interesting to him and he wanted to solve it with students:

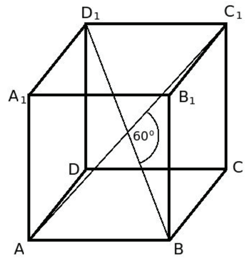
> In the quadrangular regular prism the length of the edge at the base is 10 cm but diagonals of the prism cross themselves at the angle 60 degrees. Calculate the volume of this prism.
> Hint: consider two cases.

The teacher-trainee did not solve this problem beforehand. It seemingly was too easy for him. Therefore, he gave that additional hint. Students solved the problem. But one of the students had doubts – he noticed four different cases.





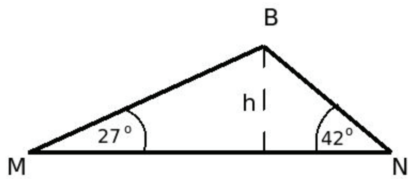


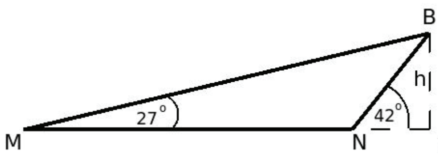


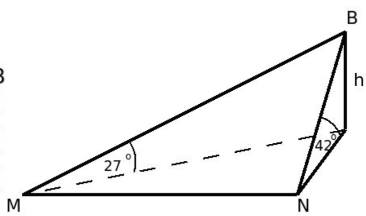
A similar situation with another problem called “The balloon.”

The balloon is seen at the angle 27 degrees from point M and from point N – at the angle 42 degrees. At what height is the balloon, if the distance between points M and N is equal 400 meters? Consider the second case.

It was not clear which the “second” case was. The two cases were significantly different. In the first situation we had one solution, in the second infinitely many.







It was not clear which of the three cases the “second” one was. We can admit that the problems were difficult. The hints were rather confusing.

3. TWO LANGUAGE REGISTERS: CAN STUDENTS UNDERSTAND TEACHERS?

There is lack of sensitivity to the differences of school mathematics register of the Polish language and the common current Polish. There are other often overlooked language difficulties. Difficult language may hamper communication and is one of frequent problems both in teacher-student and student-student relation. Difficulties are sometimes connected with some dysfunctions of the auditory processes related to “listening” and “hearing.”

Classroom Episode 1: *Out of step – being too slow or too fast.*

During a lesson under the title: “what are algebraic expressions for?” a student was supposed to solve a problem: “Write down the perimeter of the polygon formed with n-regular hexagons tessellating tight.”

A teacher wanted to stimulate a generalization of the pattern. He suggested that students should calculate the perimeter of the figure considering first simple cases of one, two, or three hexagons. Next, he asked about the perimeter of a figure composed of 10 hexagons. But students, instead of giving the perimeter for the figure mentioned above, gave the formula (incorrect) for the figure consisting of n-hexagons. It shows lack of careful listening and providing the answers to the question asked.

5. CONCEPTUAL PROBLEMS

Classroom Episode 2: *What is an equation really? Is it only a riddle? While solving an equation some problems appeared of conceptual character.* A student solving the equation

$x + 10 = 2 (x – 4)$

wrote on the board the solution $18 = x$.

To other students’ question: “How did he solve it?” The teacher-trainee explains:

$x + 10 = 2x – 8$,

and after “flipping sides” and changing signs, and unknown to the left side, and known to the right side:

$x – 2x = – 10 – 8$,

and further

$– x = – 18$ and “doing it one more time” we “fix it”

$18 = x$.

It was application of rule of thumb only *without* a remark that in general we “do the same thing to both sides of the equation” so as to leave the solution invariant, or so as to the solution stays the same under the change. This phrase “do the same thing to both sides of the equation” requires a further explanation of course but it was not given there at that moment.

Classroom Episode 3: *Lack of sensitivity to the difference between the equation and the equality*

A student, while solving the equation $3 (x – 2) + 6x = 6 (1.5x – 1)$ transformed it to $9x = 9x$ and left it at that. He was evidently stuck. The teacher-trainee came to the board and as the explanation that “this is identity equation,” he wrote

$9x = 9x$ /: 9, that is, $x = x$.

Another student reacted with the question: “Is it right, because after *shortening* in a different way, I received 0 = 0?”

The teacher-trainee without seeing the problem, explained: “Both are right, because both $x = x$ and $0 = 0$ are [equal]. And the dialogue was left at that for the moment, without mutual understanding on the both sides of the dialogue.

Classroom Episode 4

A solution of an equation may be considered “a set of numbers,” “an individual number,” “a point on the number line,” etc. strictly speaking all these concepts are different but at the same time related by “a family resemblance” (Wittgenstein).

During a lesson devoted to solving inequalities of first order with one unknown a teacher-trainee tried to introduce all the new concepts connected with this problem. Nevertheless, after solving the first inequality $3x –2(x + 1) < 0$

he asked students to write the answer as the full sentence: “The solution of this inequality is the set of numbers smaller than 2.” The problem emerged: what is “the solution”? The whole “set of numbers less than 2,” or each individual such number, which is less than 2. No conclusion was reached.

Classroom Episode 5: *What is inequality why not “inequation”? Are there idioms in the language about algebra or algebra as a language*?

Another teacher-trainee in secondary school prepared a lesson on “simple trigonometric inequalities.” He chose a sequence of examples of the type: $sin\ x<1,\ tg\ x > 1$ and asked to solve them and write the answers. He emphasized that the solution is an interval on the number line, or a set of intervals or a set. Students looked bored.

CONCLUSION

Apart from special episodes taken from the observation of teacher-trainees that direct attention at our own improvement, we may conclude that in general: (1) A deep knowledge of the subject matter and its didactical phenomenology as opposed to pure mathematical content is essential. At the same time, it is important to consider what students already know from the earlier stages of education. It is also essential to know what they will need to know at later stages. An individual learning history of students is more like a *relay race* than a *lonely run*. (2) Tasks for students must be carefully selected by the teacher and earlier solved, possibly in several ways. (3) Students must have an opportunity to give their opinion on a given topic and the freedom of choice of the way of solving a problem. In order that this really happens a special time allotment is necessary during the lesson. Haste is not welcome. (4) Communication is very important in both directions teacher-student and the student-teacher but also student-to-student and teacher-to-teacher. In cases of unclear or wrong answers, it is necessary to stop and give time for looking for reasons, explanations from both sides, and mutual understanding. (5) Young teachers, even though they know that cooperating in a team gives a better chance for discovery learning by students, they do not usually use this form of the work during the lesson. The frequent reason for that is “I am not able then to assess students.”

We should finally emphasize that almost all lessons prepared by teacher-trainees were lessons in the old, traditional style. There was almost no element of experiment in them even though during the classes at university, teacher-trainees were prepared for such attempts. They always revoked the style in which they were taught while they themselves were in school age. With greater cooperation between schools and the university that emerged as a result of the PDTR Project, there was a change in the role of practice in math teacher education at university. As teacher trainees, university students are now obliged to conduct teaching experiments during their practice in school.

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DEVELOPING A NEW ASSESSMENT CULTURE

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ABSTRACT
This article aims to help reflect on the most relevant elements for the development of a new assessment culture in mathematics. In particular, it focuses on what students think about assessment and how they respond to innovative assessment practices. Those practices are sustained in one assessment contract that focuses on the formative and regulatory goals of assessment and six different assessment tools and methods. It uses a qualitative and interpretative research method and is an investigation into my professional practice. The data collected includes my research journal, interviews with students and questionnaires. The analysis focuses on the case study of Sara, one seventh-grade student of one of my classes. The results show that the student actively engaged in learning and enjoyed the assessment methods. The assessment culture instituted in the classroom reduced her feelings of anxiety towards assessment practices. The scaffolding and feedback given to students throughout the school year helped them learn mathematics. Students stressed the underlying formative and regulatory function of assessment and showed support for the continuing implementation of this assessment process. Students acknowledged the efforts to establish an environment of trust and confidence made throughout the assessment process and viewed the feedback they got as a means to help them learn and improve their skills. Finally, the study suggests that experiences of similar assessment processes, in an atmosphere of dialogue and mutual support between teachers, students and parents can contribute to an improvement in students' conceptions towards assessment and mathematics.

INTRODUCTION

School mathematics has been moving its emphasis from basic skills (definitions and procedures for calculating) to critical thinking, reasoning, problem solving and investigations. Most of the curricular documents argue that *knowing* mathematics is especially *doing* mathematics (APM, 1988; ME-DEB, 2001; NCTM, 2000; NRC, 1989). To develop their mathematical power, students need to have opportunities to communicate and reason mathematically in different situations and contexts (NCTM, 2000). These guidelines for mathematics teaching involve change in the way teachers conduct the curriculum management and assessment.

In this context, a pedagogical proposal focused on students learning emerges as relevant, in which the assessment is an integral part of the teaching-learning process. This means to focus on its formative and regulatory function (Santos, 2003), looking to the assessment as a process which should support the learning of mathematics. Also, it means that this must be based on evidence from multiple sources, not only in the data from a single assessment tool and method, from which teachers and students can get useful information on the development of the teaching-learning process (NCTM, 1999, 2000). It also means addressing their key players, teachers and students, to face the assessment process as a natural and meaningful way in which students play an active reflective role (Santos, 2002). This article aims to help reflect on the most relevant elements that can help develop a new assessment culture in mathematics.

ASSESSMENT IN MATHEMATICS

The main function of the assessment is to help promote students' learning in mathematics, involving interpretation, discussion, information and decision in the teaching-learning processes. The formative assessment has therefore particular importance. In general, it is a valuable tool in decision-making in the teaching process. In particular, the information on students' assessment can help them build self-confidence, define goals and give clues for their progress. Throughout this process, the role of the teacher is crucial. The dialogue between teachers and students and the feedback that students receive allow them to understand the successful steps and the difficulties founded (NCTM, 2000; Nunes, 2004).

Assessment should be diversified and happen in formal and informal situations, with the active participation of its players, contributing to the progress and success of the students' learning (Santos, 2002). In this process there are several stages of planning and use of results. The first one includes the definition of the criteria that are the basis of the data collection, processing and communication and a careful selection of the assessment tools and methods. The second one, which refers to the use of results, includes how they are transmitted, how interpretations are made and the regulation of teaching and assessment practices is undertaken on the basis of these interpretations.

What is assessed are students' mathematics learning and skills, defined in the national curriculum (ME-DEB, 2001). In particular, the assessment process as an integral part of the mathematics teaching process can contribute significantly to the students' learning. In this sense, it should be a means and not an end. Furthermore, assessment should be made with the aim of reporting on the development of their learning (NCTM, 2000).

In order to guarantee a high quality of learning, assessment and teaching should be integrated allowing the regulation of the teaching-learning process. Therefore, the development of assessment should try to agree on three broad principles: (i) compatibility between assessment tools and methods and the different components of the curriculum – goals, purposes, content, mathematical processes and learning experiences; (ii) the diversity of methods and instruments, to collect converging data from many sources; and (iii) the adequacy of the assessment methods and practices in relation to the kind of information desired, for the intended purpose and the level of the student development and maturity. Only then, information from the teachers' assessment may help to regulate the teaching process and thereby promote a solid and qualitative learning for all students (NCTM, 2000; Santos, 2003).

Assessing in mathematics includes collecting different evidence on the evolution of students' learning: mathematical knowledge, their ability to use it and their predisposition for mathematics. Furthermore, the process is only complete with the establishment of inferences, from these evidences, for different purposes, in particular the promotion of students' learning. To guide the assessment practice it is essential to follow three basic principles: consistency, diversity and transparency (Abrantes, 2002; Leal, 1992, NCTM, 1999; Santos, 2003). The principle of consistency refers to the assessment processes, the learning and required skills. The principle of diversity relates to the variety of learning environments and assessment tools and methods, so that information about all students' learning and development of skills is real and consistent.

Finally, the principle of transparency is related to the clarification and explanation of the assessment criteria used for assessing students' learning.

METHODOLOGY

This article is based on a study of a qualitative, essentially interpretive, nature (Bogdan & Biklen, 1994). It is also research on the author's own professional practice (hereafter referred to as "the teacher"), in which the reflective component had a crucial role in all of the working phases (Nunes, 2004). According to Ponte (2002), "research on the teacher's own practice aims to solve professional problems, and increase knowledge on these issues with references not to the academic community, but to the professional community" (12).

The participants in this study were 27 students of a seventh-grade class with 11 girls and 16 boys, aged 11-14 and the majority of students was 12 years old. The work done during the school year was characterized by: (i) a good relationship between the teacher and the class; (ii) receptivity of the students to new tasks and use of a variety of assessment tools and methods; (iii) existence of group work habits in class; and (iv) no serious cases of indiscipline.

The data collected for this study were obtained from students' input, the teacher's journal, interviews, questionnaires and analysis of different documents. The analysis of data was performed in two stages. The first one, during data collection, involved a previous analysis to organize and interpret the collected evidence. The second one, deeper, was done after the fieldwork, sought to answer to the question under study. This paper presents the case study of a student – Sara.

PEDAGOGICAL PROPOSAL

Setting a rating system, based on well-defined assessment criteria, and putting it into practice is an important task for the teacher. The criteria and processes must be known and explained to students and parents, ensuring transparency and clarity of assessment. However, the understanding of what is intended, by these, can be achieved only after a certain period of experimentation, requiring from the teacher a strong consistency of behavior (Nunes, 2004).

On the first day of classes of the first trimester the teacher presented her proposal for a method of working as well as the assessment tools and methods to be used until the end of the school year. This proposal was based on the three basic principles mentioned above (consistency, diversity and transparency). Table 1 presents in an organized way the assessment tools and methods offered to the students.

	Written form	Written and oral form	Oral form
Individual	1.Portfolio 2. Two phases test 3. Synthesis		4. Self-oral evaluation
Group		5. Research reports 6. Project work	

Table 1. Assessment tools and methods

There were three reasons for choosing these six assessment tools and methods: (i) the consistency and goals of the entire pedagogical proposal; (ii) the fact that in Portugal, research on some of these instruments is low or nonexistent; and (iii) they have many possibilities in the framework of current educational curriculum guidelines for the teaching of mathematics. To ensure the transparency of the assessment process, the teacher informed students on the criteria approved by the subject group of mathematics which were established since the beginning of the school year.

The methodology of work followed in class sought to foster an environment for a relaxed and constructive learning, in which students could feel welcome to ask questions and issues, to be aware of the development of their learning and feel motivated to learn and "discover" mathematics. Thus, the units were planned in order to contain: research tasks for introducing new concepts or to apply for studied concepts; problems solving and exercises from the textbook; moments of reflection, regulation and reformulation of strategies aiming to improve and enrich the teaching learning process using the feedback from the teacher and students self-assessment; search for information and mathematical curiosities in the media, to improve the reports of research tasks; ludic games and small informal conversations on students issues or concerns.

SARA

Sara is 12 years old and is the youngest of four brothers. Her father is a woodcutter and her mother is a homemaker, and they live in a village five kilometers away from school. Her long curly hair hides a serene and very timid expression. However, as a student she is very decisive and shows maturity in relation to her age. She goes to school since the first year of basic education. She has a group of good friends with whom she likes to participate in activities outside the school.

Involvement in the teaching-learning process

Sara likes math, but the way she is involved in the teaching-learning processes is very much influenced by the work methodology of the teacher. She speaks with some bitterness about lessons and assessment at the first trimester: "I did not understand the subject very well and I had a bad result in the test and now, I think I understand better, I pay attention. ... The other teacher did not explain the subject so well" (E1). She indicates what a good teacher is:

> He explains the matter well, the students who have difficulty ask for help and I also think that those teachers who do not know how to explain very well, the subject, should thus give new ideas, the portfolio, as you did. And you did the investigations. (E1)

In class, Sara gets involved in the proposed tasks with great commitment and satisfaction. When the teacher states that they will make a new task, she reacts in this way because she feels she will learn new things. She likes to work in groups because it allows her to exchange views on the work she is doing and "feels that she gains more" (E2). However, she works fine individually and has a great ability to concentrate and become "almost invisible" in the eyes of the family.

She considers that it is good to have lessons on the computer "for those who still do not know very well how to use a computer and I think it is different and they all like it" (E2). Moreover, she confesses that mathematics has never been well presented and she finds it much easier to learn, because she feels more autonomy and freedom to explore.

Overlooking her involvement in the discipline, Sara admits that she likes to work on this educational proposal and says she studies mathematics in a different way: "in previous years I did not study so much because I was not interested [and this year] my study became more regular" (E2).

Vision of assessment

To illustrate her vision of assessment, let us see one of the moments she held her self-assessment at the end of the second trimester:

> T: Sara, say what's on your mind!
>
> S: Teacher, I deserve a four,[1] because I did everything you ordered, I am strongly committed to the discipline, I always got very satisfying grades, my portfolio is good and I have responded to the suggestions of the teacher, the summaries of the subject are good, the statistics work group was very good, I had a good grade in the test, I participated in class, I behave well, arrive on time and didn't miss a class, I always work at home and improved since the first period. I think I didn't forget anything.
>
> T: Well, congratulations on your self-assessment. I agree with you 100%. Your level is four! But just for curiosity, why do you think you had level two in the first period?
>
> S: Because I had two negative tests! (Teacher's Journal 13)

The teacher found Sara's arguments brilliant. The student has awareness of her work. The conviction with which she presents her arguments shows a high maturity and awareness of her learning, in addition, it shows an involved and rich concept evolved and enriched on mathematics and assessment.

For the teacher, this was the best response of all the students to the pedagogical proposal and assessment. Initially, she thought of giving her level 3, since the student had level 2 in the first trimester. After the arguments of the student, I felt that giving her level 3 would be unfair. After all, taking into account all elements of assessment collected and applying the criteria defined, the student deserved, beyond doubt, level 4.

This rating system makes her more at ease. More important than that, Sara points out that at any time, she would assess herself in relation to the development of her learning much better, because she believes that this rating system keeps her informed about her progress in the mathematics learning already made. The assessment tools and methods used and the teacher's comments on her work contributed to this. In this aspect, she says: "I think they were all important, but the most important were the portfolio and the investigative tasks" (E2).

The portfolio because "we learn better things ... we improved the grades and I think it was good" (E2). The reports of research tasks because there were several ways to answer the same question and "we were there trying to figure things out and trying like that ... is a different way of learning" (E2).

However, Sara also stresses other two methods and tools for assessment, the summaries of the subject "helped because sometimes I did not understand something about the subject and when I made the summaries I understood" (E2) and the self-assessment, which was a way to become aware of their difficulties and their developments in learning.

About the comments that the teacher made on her daily work, Sara comments that they helped her understand her strengths and weaknesses, promoted the improvement and development of her learning because "when the teacher says something I try to improve ... I try to respond to your comments (E2).

Vision of Mathematics

Sara considers it important to study mathematics because "it is a subject very different from others ... and I think in order to get any degree mathematics is always needed" (E1). She considers herself to be an average mathematics student and thinks

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[1] In a 1 to 5 scale.

there are good students in mathematics because “they are interested, like and strive” (E1). Moreover, there are students with difficulties in this course because:

> These students are not interested in the subject, if they begin to take an interest as I did because in the first trimester I was also uninterested because I could not understand what the other teacher explained, and so on. In the second and third trimesters I had the investigative tasks, the portfolio and the teacher explained better so I was more interested and got a better grade (E1)

Normally, she studies mathematics doing exercises and rewriting “some things of research tasks and tests” to do by herself. According to her responses to the questionnaire mathematics is mainly understanding and implementing a set of rules to solve exercises, the most important being calculation and reasoning. Moreover, she believes that mathematics is not to be learned by heart and that the students can be creative and original. She is aware that not all problems in mathematics have a single answer and can be solved in a few minutes.

Like her colleagues, she seems to agree with the roles assigned to each of the players in the teaching learning: the role of the teacher is to transmit knowledge and evaluate students; students’ role is to acquire knowledge of mathematics and demonstrate what they learned.

Summary

Sara has a very strong personality that marks the progress of her ideas. Since the beginning of the study, she demonstrates a high degree of maturity and awareness of her learning and its assessment. The way she regards assessment is connected with either the assessment tools and methods that were used to collect information on the students’ learning and performance; or the way that information was used to give the final grade. Moreover, she states that all assessment tools and methods were important, but gives more emphasis to the comments the teacher made to her work as promoters of the development of her mathematics learning and motivation for the subject.

Reluctant to reveal herself, as always by reserved students, Sara never offers to express, in class, her views and/or opinions of the group she works with. However, she follows discussions at the end of the tasks with obvious attention and speaks with relevance whenever she thinks it is necessary. Sara expresses that she liked investigative tasks, the work group and the tasks that allowed her to have more autonomy and freedom to explore. Moreover, she valued them as ways of learning mathematics and gave them a significant role in the assessment.

Her taste for mathematics comes from the first year of school. The experience she had in this school year was negative due to her relationship with mathematics in the first trimester, and positive (in the second and third trimesters) because it enhanced her taste for this subject and enriched her vision of mathematics.

CONCLUDING REMARKS

1. Involvement of students in the teaching learning and assessment processes. Given the results of the class in the first trimester, the arrival of the teacher was awaited with anticipation and anxiety by the students and their parents.

The way the teacher presented herself and her work proposal to develop in the 2nd and 3rd periods were a surprise to all but it marked decisively how students approached and became involved in the assessment process. Initially, students demonstrated reserve and anxiety about how the whole process would be achieved in practice.

This is not surprising, since both the majority of the work environment or the ways and tools for assessment had never been tested by them. I witnessed, however, that in the course of work, students were involved in a positive way and with commitment in the whole process of teaching learning.

The case of Sara was surprising, since her high involvement in learning was strongly influenced by the teacher's work methodology. In the end, the student said that this school year she studied mathematics in a different and more regular way.

Thus, the interviews show that the pedagogical proposal and assessment procedures pleased both students and parents. One of the most appreciated aspects was the concern to diversify the assessment tools and forms, seeking to cover different areas, the oral and written form, the individual work and the work in groups. Its use has contributed to a positive change in the environment surrounding the assessment and the final ratings. Moreover, the assessment culture practice helped students' anxiety disappear on the initial pedagogical proposal.

2. Formative and regulatory function of assessment.

The testimony of the student on the process of teaching learning and assessment shows how she realized the function of regulatory and formative assessment, when these two functions received emphasis on regular work in the classroom.

There is clearly a point across the six assessment tools and methods used in this study – the importance and significance of the teacher's comments on the work that the student was developing. In her view, the comments helped her not only to understand the strengths and weaknesses of her work, but also to achieve new learning with understanding and correction of errors and improving the work already done. Furthermore, from Sara's viewpoint, with this rating system she remained better informed on the progress of learning and helped improving her work.

Thus, the comments and feedback given to students throughout the year played a key role in the development of their learning and in the regulation of the teaching learning process. The importance that students attached to the regulatory and formative assessment function draws particular attention if we take into account that at the end of the study, all expressed interest in continuing with this assessment system.

3. Reaction to the different assessment tools and methods.

Students reacted differently to the different assessment tools and methods. It can be clearly stated that both the portfolio and the research reports were the assessment instruments with more weight and meaning for the majority. In contrast, what took less weight and meaning for the majority of students was the subject synthesis.

There are several reasons that can justify these results: the particular characteristics of each of the assessment tools and methods, and how they were managed, in the gathering of information on the evolution of students' learning, and the personality of each student, their beliefs and their previous experiences, and an historically installed assessment culture that either like it or not, serves as a reference framework for the school community.

4. Influence of the assessment tools and methods in students conceptions.

First, the assessment culture that resulted from the practice of a diversified assessment system, consistent and transparent appears to have contributed to the change in how the students approached and are related to the assessment practices.

Initially, their reactions to the teacher's assessment proposal showed a strong influence of their previous experiences, marked by a focus on little more than the information from a single assessment instrument – the written test. The teacher's position in relation to the assessment purposes and functions and the way she explained the assessment goals and criteria marked decisively the students' receptivity and their involvement in the learning process.

In the end, Sara said that all assessment tools and methods were important, but she gave more emphasis to the feedback that the teacher gave to her work, as a promoter of reflection on the evolution of her mathematics learning and motivation.

Thus, the results seem to show some progress in the way the students see the assessment. All are unanimous in considering that this rating system keeps them better informed on the progress of their learning, so they feel more at ease and less anxious.

Moreover, they feel that all their efforts and productions were considered for the determination of their final ranking. Finally, the students seem aware that there was an effort to make the assessment happen in an atmosphere of trust and transparency, in which the comments and suggestions made to their work were seen as natural with the aim to help them evolve in their learning.

Secondly, these students vision of mathematics is strongly connected to the characteristics of the tasks they performed in class and the mathematical experiences they lived. One way or another you can see in Sara's testimony that she sees mathematics as a science strongly connected to calculation, as a set of rules and technical routines. But Sara has a broader view when she considers that reasoning is the most important in mathematics. For this student, mathematics is important for their future lives, according to its social role as selecting and/or conditioning element of the future vocational options and profession.

To Sara the role of the teacher is to transmit knowledge and to assess the students. In turn, they must be carefully studying in class and to learn mathematics. But at the end of the study there is one thing that all value, the role that each of these players (teacher, student) must play in the communication and reflection on the evolution of students learning and in the regulation of the teaching learning process.

One can therefore conclude that there is a strong relationship between the assessment culture applied and the development of students' conceptions on assessment as a result of their experiences and learning results. The same is true, although not so clear, with the students' conceptions on mathematics and their learning.

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GEOMETRY LEARNING: THE ROLE OF TASKS, WORKING MODELS, AND DYNAMIC GEOMETRY SOFTWARE

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ABSTRACT

We present several learning experiences that illustrate how three aspects of the geometric competence, constructing and analyzing properties of figures, identifying patterns and investigating and geometric problem solving were developed by students that participated in the implementation of an innovative geometry teaching unit in grade 8 dealing with properties of two dimensional figures, Pythagoras Theorem, loci, translations and similarity of triangles. This development was clearly supported by the dynamic geometry environment but unfolded in different ways, depending on the way students reacted to the different types of tasks.

INTRODUCTION

In the last decades geometry and its teaching have regained importance within the mathematics curriculum. However, Portuguese teachers of grades 7-9 (teaching students aged 12-15), according to the study *Matemática 2001* (1998) consider that geometry topics should be simplified or, in some cases, excluded from the curriculum. We wonder if the introduction of technologies in the teaching of geometry may help to modify such pessimistic vision concerning this topic.

In fact, recent curriculum proposals for geometry teaching give special emphasis to the use of technology. For instance, in *Principles and Standards 2000* (NCTM, 2000) technology, particularly computers, appear as one of the principles for mathematics teaching. Dynamic geometry software is highlighted because it allows for building the basic elements of Euclidean geometry (points, lines, segments and circles) and the relations among them. The constructions made with this kind of software are rigorous and the users can transform them by dragging their basic components. In addition, this software allows to measure lengths, angles, perimeters, areas and also to make calculations with these measurements.

In Portugal, dynamic geometry software caught teachers' attention and led to a multiplication of workshops. At ProfMat2001, the Portuguese National Meeting of Mathematics Teachers, organized by the Association of Teachers of Mathematics (APM), the Geometry Working Group carried out a survey of 228 teachers. About 75% of the teachers participated in at least one workshop to learn how to use dynamic geometry software. However, only 34% of those who responded used it in their geometry classes (Veloso & Candeias, 2003). This shows that introducing this kind of software in the classroom was a difficult process.

This paper concerns an investigation carried out with a group of eighth-grade students when they learnt geometry topics with using dynamic geometry software, the Geometer's Sketchpad (GSP). We intended to study how students developed their geometric competence when they used this kind of software to solve problems and carried out exploration and investigation tasks. We considered the following aspects of geometric competence: constructing and analyzing properties of figures, identifying patterns and investigating and geometric problem solving. Table 1 shows how these aspects were considered, based on the orientations of the Portuguese Curriculum (ME-DEB, 2001). Also following the curriculum orientations, the working mode gave especial attention to students having an opportunity to solve tasks on their own, work collaboratively in pairs, and present and discuss their results and strategies in all class discussions.

So, the main purpose of this study was to investigate how students developed their geometric competence when using GSP to solve geometric problems and undertake exploration and investigation tasks. More specifically, we aimed to answer the following question: How does dynamic geometry software associated to these kinds of tasks and to an innovative working mode, contribute to the development of students' geometric competence?

Constructing and analyzing properties of figures	The skill to make geometric constructions, namely polygons and locus, allowing the recognition and analysis of their properties.
Identifying patterns and investigating	The tendency to find invariants, to explore geometric patterns and to investigate properties and geometric relations.
Geometric problem solving	The skill to solve geometric problems through constructions, justifying the used process.

Table 1. Aspects of geometric competence.

A TEACHING UNIT

Tasks

The tasks used in the study were adapted or inspired by Bennett (1995, 1996), De Villiers (1999), Bully and Baldaque (2003), Key Curriculum Press (1995, 1997) and Lopes et al. (1996). Tasks 1 to 8 referred to the curriculum topic "From space to the plan," tasks 9 to 17 dealt with "Decomposition of figures and Pythagoras Theorem," tasks 18 to 21 were related to "Locus," tasks 22 to 24 concerned "Translations" and, finally, tasks 25 and 26 referred to "Similarity of triangles."

The tasks can be classified in two different ways. The first classification concerns their nature: explorations (tasks 1 to 6, 16, 18, 22 and 23, 25), investigations (7 to 10, 17, 19, 20 and 24) and problem solving (11 to 15, 21 and 26). We distinguish between exploration and investigation tasks, because:

> Many times we do not distinguish between investigation and exploration tasks, calling all of them "investigations." That happens probably because it is complicated to know the difficulty degree that an open task will have for a certain group of students. However, once we attribute importance to the degree of difficulty of the tasks, it is preferable to have a designation to open tasks that are easier and for those that are more difficult. (Ponte, 2003, 28)

In this teaching unit exploration tasks were most prominent because one factor that we considered in this study was the use of computer software that students did not know well. There were also a considerable number of investigation tasks "to give students the responsibility of discovering and justifying their discoveries" (Ponte, 2003, 32). Problem solving activities were also present in this teaching unit, as well as in the textbook tasks that students had to do outside the classroom.

The second way of classifying the tasks is to consider the aspects of the geometric competence that each one contributes to develop. Constructing and analyzing properties of figures was present in 18 tasks since the geometric topics taught emphasized the construction of figures. Identifying patterns and investigating, included finding invariants, exploring geometric patterns, and investigating properties and geometric relations, was presented in 11 tasks. Finally, geometric problem solving, allowing students to develop skills to solve problems through constructions, was presented in 8 tasks.

Working mode

The first author was the teacher of this class. There were 18 students in class and they worked in pairs, chosen by them, which corresponded to nine groups. The lessons were 90 minutes each and lasted approximately for four months. The classroom was recently equipped with 14 computers and a multimedia projector by the Portuguese Ministry of Education.

At the end of each task there was a small discussion about students' results, problem solutions and processes. Sometimes there were also discussions on the difficulties related to the use of the software. This phase of the investigation process was very important, because it is when "a reflection about the work done is carried out. Finishing an investigation task without reflecting on it is somehow not to have concluded it" (Segurado, 2002, 58).

Each pair of students had only one worksheet where they had to write down the solutions for each question and, when asked, the processes that they used. Sketches made by students with GSP were saved in the computer used by each pair. Problems from the textbook related to the topics under study were suggested to students as homework or to be solved in "Supported study."[1] The purpose of solving these problems was to clarify some doubts that students felt during the lessons.

INVESTIGATION METHODOLOGY

Design

Due to the nature of the problem and questions formulated, the study followed the interpretative paradigm (Bogdan & Biklen, 1994). The main instrument of data collection was one of the investigators, who was also the classroom teacher. The research focused then on his teaching practice. In this kind of research teachers think and reflect on their own practice, inquiring about their students' learning (Alarcão, 2001). Sierpinska and Kilpatrick (1998) argue that teachers who investigate their own practice are in a privileged position, because they can reflect on what their planning and teaching. This reflection can be done using existent theory or, sometimes, creating theory from practice. On the other hand, this research was based on the exchange of ideas between the two authors.

1 "Supported study" is a component of students' curriculum devoted to the study of subjects in which they have more difficulties such as Portuguese, English and mathematics.

Data collection

In a qualitative investigation it is important to gather information from several sources because as that helps to answer the proposed problem. The use of different instruments allowed different approaches to the problem and helped to give a more complete answer (Bogdan & Biklen, 1994). In this study we used as instruments to gather data students' written records (tasks, investigation reports, problem solutions and homework), teachers' journal (reflection notes and dialogues occurred between students and between students and teachers), two questionnaires (one at the beginning and other at the end of the study) and interviews carried out by teachers with each pair of students at the end of the research project.

Participants

The students who took part in the study belong to an eighth-grade class of a school near Lisbon. The school had about 700 students in grades 5 to 9. The teaching unit was carried out with a class of 18 students, nine girls and nine boys. Fifteen of them were 13 years old, two were 15 years old, and one was 12 years old. The first author taught most of those students in the previous year and they were for the first time in grade 8. In general, these students had success in all subjects, having more difficulties in the Portuguese language.

In this paper we focus our attention on a single pair of students, André and José, who had a remarkable performance in this teaching experiment. André was 13 years old and was very reserved. He was born in Guinea, an African country, and came to Portugal when he was 8 years old. He lived with relatives, since his parents stayed in his homeland. His expressive look, last generation mobile phone and cap were his trade image. Inside the classroom, he behaved well but had some learning difficulties in mathematics. He was a quiet boy who did not intervene much in class. He was slow in doing the exercises and just copied what was on the board or in his classmate's notebook. During group work, he hid himself behind the work of his colleagues, because he felt some difficulties in speaking Portuguese. In spite of everything, he was successful in school.

André had a great friendship and respect regarding José. They started working together in grade 7, when André had failing grades in eight subjects that he needed to improve. José started helping him in a "supported study" and continued doing it in several other subjects. This partnership allowed André to improve considerably his school results and consequently to pass to grade 8 with no failing grades.

José was a brilliant student in all subjects, except physical education, in spite of practicing several sports. In the remaining subjects he usually had all the answers correct and was very concerned when that did not happen. In his interventions in class, always at a high level, he used a brilliant reasoning and a quite advanced vocabulary for his age. He did not turn down a challenge. José had great expectations about the kind of work that we asked them to do: take part in a study, in which students would use software to learn geometry for a considerable period of time.

DEVELOPING GEOMETRIC COMPETENCE

Constructing and analyzing properties of figures

In task 19 students studied the properties of the midpoint and the perpendicular bisector. The third question was the following:

Construct a segment and its perpendicular bisector. Draw quadrilaterals with opposite vertices on the perpendicular bisector and on the endpoints of the initial segment as the other two vertices. Identify the quadrilaterals that you find and justify your answer.

The students drew a horizontal segment, so the perpendicular bisector was vertical, and made it more difficult to identify the name of the quadrilaterals. The first one was a rhombus, but they did not make any attempt to justify it. Then, they observed and justified correctly both the square and the kite. They also observed the parallelogram. When dragging the points constructed in the perpendicular bisector they found a figure that they did not know. The teacher told them that they could name it "boomerang" because of its shape. However, they did not mention that it had two pairs of consecutive equal sides.

But the exploration of this task did not end here. As José and André finished earlier than their colleagues, the teacher asked them to help their classmates. Minutes later, while José was helping a group, one of his colleagues asked him how he could be sure that the figure on the screen (obtained when the points constructed on the perpendicular bisector were at the same distance as the endpoints of the initial segment) was a square. José accepted the challenge and a few moments later called the teacher to show him a sketch:

José: I constructed the square.
Teacher: How?
José: I used the Rotate menu to do a rotation with 90 degrees of this point [he pointed to one of the endpoints of the segment AB that had been drawn by a colleague].

José considered the midpoint of the initial segment as the rotation center. Afterwards, he drew the segments between that point and the endpoints of the initial segment. With an analogous process José obtained the other two sides. To confirm that the figure was really a square he measured the angles. He was excited when the 90 degrees for each measurement appeared on the screen: "Here it is! I managed to build it!"

Identifying patterns and investigating

We analyzed the way students evolve in finding invariants and exploring geometric patterns. In the second lesson, André and José felt some difficulty to understand what a conjecture about vertical opposite angles was. The difficulty emerged in question 1b) of task 2:

Drag point B. The angles $\angle$BAC and $\angle$EAD are vertically opposed angles. The angles $\angle$EAB and $\angle$DAC are also vertically opposed angles. Write a conjecture about the measurement of these angles.

After reading the problem, José got up and addressed the teacher:

José: What is a conjecture?
Teacher: You must write what you observe regarding these angles. Have they any relation?
José: Like an affirmation about the opposite angles?
Teacher: Yes!
José: So, vertically opposite angles are always equal.
Teacher: That's it!

This first difficulty with the term conjecture was overcome, and the students wrote the following: "The measures of vertically opposite angles are always equal, regardless the position of the points."

The students went through other learning experiences that helped them to develop their skill to explore geometric patterns. That happened in task 9:

Construct a quadrilateral and the midpoints of its sides. Then, construct the quadrilateral with vertices in the four midpoints. What quadrilateral did you obtain? Search for other quadrilaterals: square, rhombus, parallelogram, etc. Write your conjectures and try to justify them.

Based on their previous experiences, André and José started full of energy working on this task. They copied the quadrilaterals they already built in task 8 and started their investigation. They followed the following order: rhombus, square, rectangle, kite, parallelogram, isosceles trapezoid, rectangular trapezoid and scalene trapezoid. After, they extended their investigation to triangles: equilateral, rectangle, isosceles and scalene. At the end of this task they wrote the following conclusions:

We conclude that in spite of the several relations among them, there is a rule. When we connected the consecutive midpoints of any quadrilateral that does not have a formation rule, they form parallelograms. Because rhombus and kits belong to the same family regarding the diagonals, they form always rectangles. Because rectangles and isosceles trapezoids have opposite parallel sides and consecutive equal angles, they form rhombus. The square forms another square due to its unique properties. That happens in some regard with triangles, because when we connect their midpoints, they form triangles with the same properties as the original. In another class are the remaining trapezoids that form parallelograms. The parallelogram forms another parallelogram, since it does not have equal consecutive angles like the rectangle and the isosceles trapezoid.

The classification that they presented was different from the one the teacher expected. Nevertheless, this is a possible way to classify some quadrilaterals. They could have written more conclusions, but this was one of their first investigations. They already showed some progresses writing the conjectures that they arrived at.

At the end of the investigation it was possible to observe once again the students developing their capacity of searching invariants. In task 26 we asked them to write conjectures relating perimeters and areas of similar triangles. Regarding the first conjecture, José and André did not have difficulty relating the perimeters, because they suspected that it would be necessary to divide them as they did to find the ratio of similarity. So, they wrote that the ratio of the perimeters of similar triangles was equal to their ratio of similarity. Regarding the areas they carried out some attempts to find a relation, but they were having difficulties. So, they called the teacher:

José: We discover that the ratio of the perimeters is equal to the ratio of the sides. The problem is to find a relation between areas!
Teacher: Do you think that there is a relation with the ratio of similarity?
José: I think so! There should be some relation, as it happens with perimeters.
Teacher: Compare the ratio of similarity and the ratio of the areas.
José: We tried to find a relation, but we could not.
Teacher: Drag a vertex in the first triangle and compare the two values.

José followed the teacher's suggestion and, together with André, tried to find a relation. Some time later, he got up and talked with the teacher:

José: We found it! The relation is the square of the ratio of similarity! I dragged the vertex and with the calculator did the square of the ratio of similarity. And it was equal!
Teacher: Well done!

Their effort had been rewarded and the GSP tools were of important help. The possibility to measure and to calculate in a dynamic way and the dragging of the triangles allowed the students to realize the relation and to formulate the right conjecture. We can see improvement in finding and writing conjectures.

Geometric problem solving

In this third aspect of the geometric competence, we try to analyze how students improved their geometrical problem solving ability with constructions, and how they justified the processes that they used.

In problem five of task 21 the students also did some interesting work. This problem was composed of two questions:

Draw rectangle *ABCD,* in which *A* and *C* are opposite vertices, $\overline{AB}$ =10 cm and $\overline{BC}$ = 6 cm.

a) What is the locus of the points that are less than 3 cm from vertex *B*?

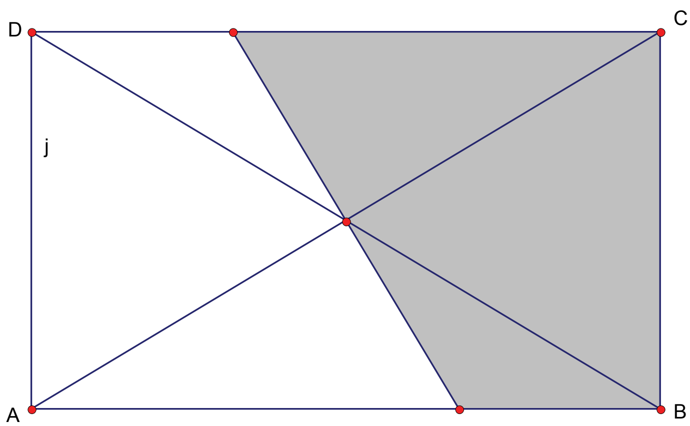
b) What is the locus of the points that are closer to point *C* than to point *A*?

When they finished and because they were not sure about their solution, José called the teacher to confirm it. While he was explaining the solution process he reread question b) and verified that in the sketch the points *A* and *C* were not opposite vertices. The solution that they had found was correct if *A* and *C* were adjacent vertices. André changed the vertices letters and both students continued looking at the screen, while another pair of students called the teacher. Some minutes later they called the teacher again. José had already drawn both diagonals and the intersection point, or rectangle midpoint as they called it. Then, they constructed a circle that passed by that point and with center at *C*. They thought that the solution were all the points inside the circle.

The teacher suggested them to place a "free" point on the figure, and measure the two distances between that point and the points *C* and *A*. That helped students to see that there were more points outside the circle that were also solutions to the problem. Then, they drew another circle with center in point A passing by the rectangle midpoint. Once again, there were points closer to *C* than *A* not included inside that circle. Later, the teacher asked them what they had found:

Teacher: Have you solved it?
José: Yes! We drew a perpendicular to the diagonal at the midpoint. It is this segment. The points that are near to C are the ones in this zone. [See figure 1]
Teacher: And, what is the name of that figure?
José: It is a rectangular trapezoid! [He answered quickly.]

5b) This locus is a rectangular trapezoid with a right angle in point *C*, and every point inside of it is closer to *C* than *A*.



Resolution process - We constructed the rectangle diagonal and its midpoint. Then we constructed the perpendicular bisector to diagonal *AC*.

Figure 1. Answer given by André and José to question 5 b) of task 21.

This episode shows how this problem was solved by these students: (i) they found a solution to a simplification of the initial problem; (ii) the solution did not allow them to answer the initial problem, because this was more complex; (iii) the experimental phase in which the students built both circles did not allow to answer the problem, but permitted to improve their knowledge about the points that were missing in the first try; (iv) they got involved in a new phase of construction, in which they drew the perpendicular bisector, since the construction of the circles did not result; and (v) they identified the figure that permitted to answer correctly to the problem.

CONCLUSIONS

We presented several learning experiences that illustrate how the students involved in this study developed the three aspects of the geometric competence, constructing and analyzing properties of figures, identifying patterns and investigating, and geometric problem solving.

The teaching unit was based on three decisive elements: the dynamic geometry software, the tasks proposed, and the way students worked in the classroom. The use of GSP was an important support to students' constructions and discoveries, and its functionalities allowed them to develop the recognition of properties and the analysis of figures. Dragging a geometric construction and verifying what stays invariant and the possibility of trying many cases allowed them to investigate and solve the proposed geometric problems.

All tasks were well accepted by students, in particular the exploration and investigation ones. Their open nature allowed students to get involve actively in their learning and developed the skill to search for invariants. To solve geometric problems, they used constructions. The justifications that students gave to conjectures that they elaborated, their solution processes and the way they improved strategies until they managed to solve a certain problem are aspects that reflect the challenge created by these kinds of tasks.

Working in pairs promoted the learning of both students, although they were so different. The presentations that André and José made of the result that they found to the other classmates and the discussion of the solutions of the tasks permitted to evaluate the students' involvement and their development of geometric competence. Writing in paper or in GSP sketches permitted to evaluate students in a holistic[2] way and to give them a fast feedback about their work, as well as suggestions for possible improvements.

In the final interview students stated that they liked the teaching experiment very much and there had been a change in their perspective about geometry. This topic became more related to challenges and investigations and that happened largely due to the role of the dynamic geometry software.

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EXPLORING FUNCTIONAL RELATIONSHIPS TO FOSTER ALGEBRAIC THINKING IN GRADE 8

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ABSTRACT
This article aims to identify possible contributions of a teaching unit designed to promote eighth-grade students' algebraic thinking. The unit was conceived within a research project and is based on exploratory and investigative tasks about functional relationships. Using a qualitative and interpretative methodology, data collection included participant observation of classes by teachers, registered in their journal, and gathering written records from students. Results indicate that working in the proposed tasks and combining group work and classroom discussions, promoted the development of meaning for the algebraic language. Working within this unit in this way also encouraged students to widen the number of strategies that they used in exploring situations involving relations between variables, to reason in a more general way, and to express their generalizations using a formal language.

INTRODUCTION

In Portugal, algebra is an important topic of the eighth-grade mathematics curriculum, addressing the use of symbols with different purposes and in different situations such as representing functional relationships, solving problems, 1st and 2nd degree numerical equations and literal equations, and generalizing and proving numerical properties (ME, 1991). The students of this grade had former experiences with an algebraic language, particularly in the study of 1st degree equations (in grade 7). However, the use of algebraic language and the construction of the concept of variable usually create significant difficulties for many students. This paper is drawn from larger research that studies how solving exploratory and investigative tasks, involving functional relationships, may contribute to the development of their algebraic thinking (Matos, 2007). Our main goal is to identify the main contributions of a teaching unit based on this curriculum strategy in students' learning.

ALGEBRAIC THINKING

Kaput (1999) argues that algebraic thinking appears when – through the processes of conjecturing and arguing – one establishes generalizations about data and mathematical relationships expressed in an increasingly formal language. This generalization process may occur from arithmetic, geometric, and mathematical modeling situations. The author identifies five phases of algebraic thinking, intimately related: (i) generalizing and formalizing patterns and constraints; (ii) manipulating formalisms; (iii) studying abstract structures; (iv) studying functions, relations, and joint variation; and (v) using multiple languages in mathematical modeling and control of phenomena. In this way, Kaput stresses the need for a wider way of looking at algebra teaching and learning. This idea is also underlined by the NCTM (2000) that indicates that middle school students must learn algebra, both as a set of concepts and skills related to the representation of quantitative relationships and as a style of thinking that enables the formalization of patterns and generalizations.

Although algebra is not just a language, it is true that some of its power comes from the use of symbols that allows expressing mathematical ideas in a short and rigorous way (Sfard & Linchevski, 1994). Symbols also allow keeping a distance from the semantic elements they represent, becoming powerful tools for problem solving (Rojano, 1996). Many authors point that symbols can be used to represent different mathematical aspects and can be interpreted by students in different ways (Küchemann, 1981; Usiskin, 1988). Namely, symbols can be used in the representation of unknown numbers, in the expression of generalizations, or as variables. Some authors also identify several difficulties students reveal when they deal with an algebraic language. Booth (1984) indicates three main areas: (i) interpreting letters; (ii) formalizing the methods used; and (iii) understanding notations and conventions. In fact, the multiple uses of the algebraic symbols are a source of potential in algebra but also a source of conflicts and difficulties for the students.

Driscoll (1999) and the NCTM (2000) point out that the exploration of patterns is an activity that contributes towards the development of algebraic thinking and that must be promoted since the first years of school. We also note that the study of functions includes the need to understand the way how two variables are related, based in the identification of regularities. Concerning this issue, Smith (2003) identifies two distinct ways of analyzing a function: (i) understanding the relationship between each value of the variable x and the associated value of y, which may enable to write an algebraic expression that represents it; and (ii) analyzing the way in which the variation of the values of a variable produces variation in the values of the other, that is, analyzing covariation of x and y.

A TEACHING UNIT

The teaching unit involved the study of several topics of the mathematics curriculum: numerical sequences, functions, and 1st degree equations. Aiming to promote the development of algebraic thinking, based on exploring functional relationships, its specific objectives were the development of students' ability to: (i) identify and describe patterns and regularities in situations involving variation and to formulate generalizations; (ii) to represent and analyze functional relationships by means of tables, graphs and algebraic expressions; and (iii) to ascribe meaning to algebraic expressions and to deal in an efficient way with the algebraic language. Students must use letters in different contexts and with different purposes – as instruments for generalizing, as variables in functions and as unknowns in solving problems and equations.

The unit was carried out in 16 classes (90 minutes each). It included many types of learning experiences. The first part of the unit included exploratory and investigative tasks (Ponte, Brocardo, & Oliveira, 2003) in the beginning of the study of each topic, as a mean to foster the construction of new concepts. In the tasks about numerical sequences, students had to explore patterns in several sequences, with or without geometrical representations and with different levels of difficulty. All of these tasks created opportunities for pattern generalizing, which could be expressed in the beginning in natural language but should progressively be expressed in a more formal way, using an algebraic language. In the third task, students worked with numerical sequences graphically represented. In this first part of the unit, letters were mainly used as generalized numbers and as unknowns in simple 1st degree equations.

In the second part of the unit, the study of functions was introduced by two tasks involving relationships between variables. They represented an extension from the discrete case to the continuous case. The first task involved a direct proportion. Both tasks explored different ways of representing functional relationships, moving from one kind of representation to another and addressing the potential of each one. The third task had four distance-time graphs that students should interpret to design a situation adjusted to the information given. Although the letter is used both as a generalized number and as an unknown, in this part of the unit the focus was on its use as a variable and on the notion of joint variation.

The last two tasks continued the study of equations that students begun in grade 7 and revisited in previous topics, solving new kinds of problems and equations with denominators. Literal equations appeared through the investigative activity that students developed as a result of generalizing a relationship between more than two variables. After this initial moment, supported by the context of the situation, students got involved in solving these equations in order to solve for one of the variables. In this phase, letters were mostly used as unknowns and as generalized numbers. All of the tasks allowed students to use different strategies and to draw their own paths of exploration. This approach stimulates their active participation giving them multiple entry points, adequate to their ability levels.

Identifying patterns and regularities, representing, generalizing and particularizing are mathematical reasoning processes that play an important role in exploring functional relationships. Solving investigative tasks may allow students to learn through the mobilization of their own cognitive and affective resources, as they follow their own goals (Ponte, 2006). This approach, on the beginning of each topic, complemented by a small set of instructions, seeks to empower students' intuition, as they get involved in the exploration of the tasks, initially in an informal way. The unit includes also some exercises and problems from their textbook.

These classes included individual work and work in pairs and in small groups. At the end of each task or group of tasks a discussion involving all class took place. In these discussions, the students shared orally their strategies with their colleagues. While they were solving the tasks, they made written records of their work which allowed them to organize their reasoning and supported their performances in the general discussions. In two of the tasks these records where analyzed by the teacher which addressed each group some new questions. These questions stimulated students to go beyond their first explorations in the next class.

METHODOLOGY

This article aims to identify possible contributions of a teaching unit focused on exploratory and investigative tasks about functional relationships to students' learning. We used a qualitative methodology, descriptive and interpretative (Bogdan & Biklen, 1994). The unit was taught to an eighth-grade class from a school of the Lisbon suburban area. The first author of this article was the classroom teacher and she performed simultaneously the role of teacher and researcher. This was an investigation about her own professional practice. There were 27 students (aged 13-16) in class.

Reflecting the increasing immigration rate observed nowadays in Portugal, 10 of these students were from former Portuguese colonies – Angola, Cape Verde, S. Tomé and Príncipe, Guinea and Brazil, or from Eastern European countries, such as Romania. The students in this class had low achievement in almost all subjects and several of them did not succeed at least one year. There existed, however, a good relationship between them and their teachers.

Data for this paper is taken from the teacher's journal, including her daily observations and complemented by data from audio recording of all classes and written documents produced by the students, collected and organized. The data analysis was descriptive and interpretative and was done in an inductive and exploratory way (Bogdan & Biklen, 1994). This analysis considers the main episodes that occurred in the classroom during the unit, regarding the strategies that the students used to explore the tasks and the main difficulties they revealed.

RESULTS AND DISCUSSION

Numerical sequences

Students were puzzled by the first exploratory task. I (the first author) suggested they should work in pairs and make notes from their conclusions. Since this was a new topic for them, they expressed difficulties in interpreting what was sought in some questions. When I noticed this, I suggested they should look attentively at the figures and share ideas with their colleagues. After this initial moment, students drew their attention back to the task and detected the first patterns, which gave them more encouragement to continue. The first strategies that they used were very intuitive. Most of them analyzed the covariation between the variables to determine the number of points on each figure, adding two units to the previous figure. Other students represented all or just some of the figures and counted directly the points they saw. Filipa and Carla chose a different strategy, based on the geometrical decomposition of the figure in two parts (Figure 1). The first one had as many points as the place it had in the sequence and the other part had one point more:

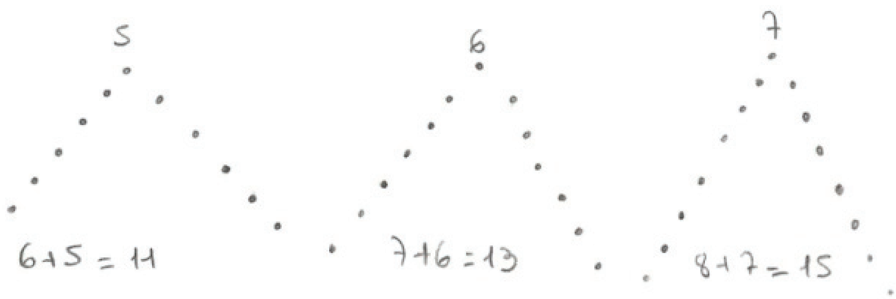


Figure 1. Filipa and Carla – Task 1 – Numerical sequences

When the students had to describe a distant figure and a general rule of formation for the sequence, they used new strategies, considering also the correspondence relationship between variables. This relationship was expressed in two different ways, but was based on the geometrical decomposition of the figure into three smaller parts: two main branches with the same number of points as the order of the figure and the point situated at the vertex. Adding each number of points, they obtained 100+100+1 or 2 x 100 +1, which totaled 201 points. As they did initially, Filipa and

Carla added just two addends, because they divided the figure in just two parts as we already saw. The students explained that the figure had 201 points, calculating: 101+100=201. Supported by the geometrical representation, several pairs could actually imagine the 100th figure, opening the way to generalizing. However, this was also a point where some students showed some difficulties. Jacinto and Carolina tried to use their knowledge about covariation but they thought it was necessary to add 3 points instead of 2. In this way, they could not obtain the correct number of points.

Concerning generalization there were also distinct answers. It was quite apparent that the way students solved the first questions had a strong influence in how they described a general rule. Some, such as Erica and Miguel, made general statements, directly related to the analysis of covariation: "We always add two points to the existing number." Other students based their generalization in the relation of correspondence that they had previously detected. Therefore, students that in the 100th figure had multiplied the number 100 by 2 and added 1 answered as Pedro and Catarina: "It is twice a number plus the higher point." Filipa and Carla, also following their initial reasoning stated: "When we have any number, we add one point more and then, with the result of the sum, we add that number again." In this initial phase of the teaching unit some of the students were not able to generalize and to describe generalizations in an autonomous way.

Students were already familiar with this kind of classroom organization. They knew that during the general discussions, sharing their strategies would be valued, especially when they could add something new to what was being said. In the discussion of the first task, when Ricardo went to the board, he explained:

Ricardo: So, we take this one [the figure's order], we multiply it by two and we add one.
Teacher: But was that the first strategy you used?
Ricardo: No, we saw that we had to add two points.

At this point, the student recognized that his group started by observing the way the number of points varied from one figure to the next. The strategy of using covariation was very immediate for the most of the students. However, this task led students to go beyond this type of reasoning, looking for relations among different variables and describing them in natural language. Although they already had contact with an algebraic language, none of the groups used it in this task as a tool to express the general rule they had found.

During this general discussion emerged the possibility of representing that information in a more succinct way, using an algebraic language. The students suggested the use of the letter x, as they had already done in the previous year to represent unknown values. Taking this idea, I asked the students if they could build a formula to represent the number of points of a general figure. Joaquim wrote immediately on his worksheet the expression 2 x x + 1 and called me, saying, in an enthusiastic way: "Teacher, this is our rule, isn't it?" Before I confirmed this answer, I tried to find out what other students had in mind. I saw that a few other students wrote the same formula. However, most students could not write suitable expressions. Influenced by the reasoning she made before about the 100th figure (to multiply the number of the figure by two and add one), Erica suggested: x x 2= x + 1. After this, Ricardo suggested x x 2+1 completing his colleague's answer.

Filipa and Carla obtained different algebraic expressions, based on the previous decomposition of the figures: they made: x + 1 points to one side and x points to the other. Adding these two expressions, another student suggested that x + x was equal to

$2x$, which lead us again to the formula, so we got the expression $2x + 1$. The students continued sharing their own algebraic expressions. Erica, for instance, suggested again: $x.2 + 1$. Miguel contributed with other possible expressions: $1 + 2x$ e $1 + x.2$. This discussion about equivalent expressions was an important part of the class. At the end, regarding the expression $2x + 1$, Jacinto remembered something he seemed to have learnt before: "Oh, we have to isolate…!" At that moment I saw that the student was referring to the procedures he was taught to solve equations, without noticing that this was only an algebraic expression. In this case, the aim was to generalize and not to determine the value of an unknown. This thinking was corrected by Joana who said: "No, that is only when we have an equal sign …" This dialogue was followed by our first reflection about the difference between an equation and an algebraic expression.

The next task offered students a contact with numerical sequences that were not related to geometric representations. The majority of students again started to find regularities among consecutive terms. The use of tables employed to represent sequences was important because it made the correspondence between the variables favoring generalizations more visible. Students also had an opportunity to work with a graphical representation of sequences and to solve small problems in which they had to find an order of some of the terms. Many students showed they were able to invert their initial operations. This was a first step in formalizing the reasoning, that later became quite important, especially in solving equations.

Linear functions

The study of functions began with the analysis of two contextualized situations that could be modeled by linear functions. Both of them referred to shopping with or without discounts. Students started their work immediately building tables with some concrete cases. This situation was important because it created an opportunity to discuss the values, which could sensibly be used in that context. It was also interesting to verify that the students used the same strategies they had already developed in the study of sequences. One of the main difficulties they revealed was to analyze and describe the variation of the variables. After this initial moment, the students started to analyze covariation, considering equal increments on the independent variable: "They vary with each other. If you increase one liter you increase also 1.1 € in the price you pay" (Jacinto and Florbela). Other students analyzed covariation and also the correspondence between variables: "For each liter more, we add 1.1 € or we multiply the number of liters we buy for 1.1 €.

At this phase, students did not show difficulties in generalizing. When they tried to express it using a more formal language, they started to use the same symbols they used in the study of numerical sequences. In this way, the work of some students in the continuous case revealed the influence of their previous work in the discrete case.

Numerical equations

Students' various experiences in algebra acquired beforehand were one of the aspects more visible in this class, especially concerning equations. Only some students who studied this topic in grade 7 could effectively solve a 1st degree equation. The rest of students, on the contrary, showed they never learnt how to do it, because they had almost no interest in mathematics or because they never studied equations in their previous schools in the foreign countries.

Working with sequences and functions became an opportunity to use an algebraic language as a tool for generalizing and sharing meanings. The study of these topics generated the opportunity to solve simple equations, which was important to create a common understanding among students, allowing them to continue to approach more complex algebraic concepts. In the first general discussion, the study of the sequence with general term $3n + 5$ raised the following dialogue:

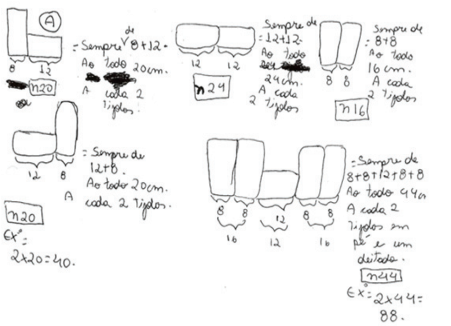
Teacher: So, which was the order in which 300 was placed?
Erica: Teacher, 3 x 100…
Teacher: Ok, but does that give 300?
Erica: No, that is just with $3n$.
Teacher: Oh, but I can't change the rule like that because we would be working with another sequence, different from this one. We just need to know which n makes this expression yield 300.
Sofia: 300 – 5? I don't know. [Students talk with each other.]
Erica: So, we make $3n = 300 – 5$.

Immediately after this episode, some students did not follow the reasoning proposed by Erica, and went on thinking on their own strategies. Pedro claimed with enthusiasm: "3 x 98 + 5 = 299; 3 x 99 + 5 = 302. It will not pass on 300!" This discussion continued with the contributions of Isabel, who finished solving the equation on the board, according to her previous knowledge. The way how the discussion developed allowed the confrontation between Erica's idea, the formal resolution proposed by Isabel and the intuitive process used by Pedro to see if 300 was a term of the sequence and the advantages of all the processes.

Literal equations

The last task was one of the most challenging for me as a teacher in this unit. In the beginning I felt difficulties in dealing with the many paths students could choose. One of the situations involved studying the length of a wall built with yellow bricks. That length varied according to the number of bricks that the students could add to the wall in one of two directions: horizontal or vertical. Before the existence of three variables instead of only two, the majority of students chose to consider blocks of two or more bricks that they could repeat in order to build the wall. This strategy allowed them to know that the total length was a multiple of the length of each block. The first written records from Marisa and Helena showed this reasoning (Figure 2).

The analysis of walls built with two kinds of bricks brought the need, to some groups, to distinguish the length of the wall that was generated by each of them. Firstly, the students represented both the lengths using the same letter, which led them to a situation of ambiguity. That happened to Sofia and Laura (Figure 3). Their initial reasoning had in consideration different possibilities. However, we can see that the 72 cm came from the bricks in the horizontal direction and the 48 cm came from the bricks in the vertical direction. Although they had used x and y as the number of bricks in the horizontal and the number of bricks in the vertical the students used the same letter "C" to describe both of the lengths and formulated the following equation to which they could not find any meaning: x x $12 = x$ x 8. As they were blocked, they asked for my intervention. I suggested them to think again about the meaning of all the expressions they had used. After this they tried to explain those meanings in their worksheet and, considering C as the total length of the wall, they could write the literal equation $C = 12x + 8y$. This was the first generalization the students made with three different variables represented by three different letters. Other groups in the class reached similar conclusions, through the exploration of other possible walls. As the students explored this situation, first in concrete cases and then as a generalization, the meaning of those expressions became clear for them. From my point of view as a teacher, the initial difficulty in dealing with so many paths of exploration gave place to the joy of seeing the different types of reasoning students were able to build and the strategies they could share in the classroom discussion. Analyzing their written records before the general discussion allowed me to be better prepared for that moment.



a) Always of 8+12 (all together, 20 cm) for each 2 bricks: *n*.20
b) Always of 12+12 (all together 24 cm) for each 2 bricks: *n*.24
c) Always of 8+8 (all together 16 cm) for each 2 bricks: *n*.16
c) Always of 12+8 (all together 20 cm) for each 2 bricks. *n*.20
d) Always of 8+8+12+8+8 (all together 44 cm for each 2 bricks in the vertical direction and 1 in the horizontal direction: *n*.44

Figure 2. Marisa and Helena – Task 8

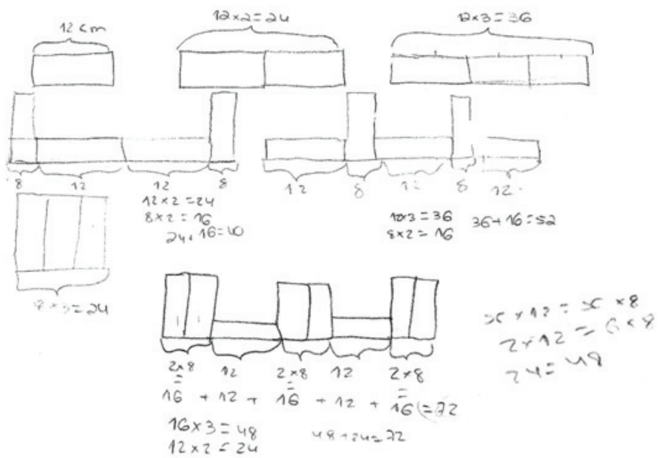


Figure 3. Sofia and Laura – Task 8

CONCLUSION

The work conducted in this teaching unit did not represent a reduction of algebra to just a formal language, although this aspect clearly played an important role. The suggested situations were based on a wider conception of algebra, involving the understanding of patterns and relationships through the exploration of sequences and functions. The initial part of the unit, with students' autonomous work and general discussions, was essential to activate the intuitive resources of students and the knowledge and difficulties they developed in former school years. While sharing their own ideas, the students could clarify their doubts about the meaning of algebraic expressions and become aware of the advantages that the use of this language could offer for expressing generalizations and for solving problems. The first part of the unit was also important because it gave students some contact with reasoning processes they were not familiar with, such as generalizing and expressing generality. Sharing different strategies was an important feature that contributed to the enrichment of general discussions. These moments, especially the discussion on the first task, were very important places to clarify difficulties and negotiate meanings.

All students, even those who most feared mathematics, got immediately involved in solving open tasks. At the beginning, their reasoning was supported by the geometric representation of the sequences but the other tasks allowed them to go further on their reasoning. Some students made some initial mistakes that were discussed and clarified in general discussions. However, most groups solved the tasks in an autonomous way, which gave them confidence and motivation they needed to solve the remaining tasks. This underscores the idea that these tasks yield multiple entry points for students with different ability levels in mathematics. The challenging nature of the tasks involved all students in developing their own strategies.

Working through exploratory and investigative tasks seems to have contributed towards: (i) developing a richer meaning for the algebraic language; (ii) widening the strategies to explore situations involving variables; (iii) using reasoning of an increasingly general nature; and (iv) expressing them using a more formal language. In this way, the work in the teaching unit seems to have contributed in an important way towards the development of students' sense of symbol, yielding opportunities to strengthen their algebraic thinking. The exploratory and investigative tasks included in the unit, the work within the groups and the classroom discussions matched the initial expectations, generating experimentation, autonomous work discussion and, most importantly, assuring that algebra was always a sense-making activity. In this way, students could construct new concepts and enlarge their algebraic knowledge and thinking.

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DESIGNING LEARNING TASKS BASED UPON STUDENTS PREVIOUS KNOWLEDGE

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ABSTRACT

Students, aged 12-16, usually do not transfer their strategies from one problem to another. We tried to analyze how students can construct concepts and procedures by using their own previous strategies or strategies used by others. In this paper we attempt to answer two questions: (1) how can we use students' solving problem strategies in order to design new learning activities? and (2) what kind of activities do we need to introduce in the classroom in order to promote discrimination and argumentation working with connections?[1] Students' group work, coaching students to discover things themselves instead of listening to teacher's explanation, learning students' strategies from their questions and their reasoning are key topics in this study.

INTRODUCTION

Students usually do not transfer their strategies from one problem to another. We tried to analyze how students can construct concepts and procedures by using their own previous strategies or strategies used by others. We designed learning activities where students can show their capacity for facing them. This study presents two different experiences, the first one is related to the construction of students' knowledge, and the second one includes students' discrimination and argumentation when they work on connections.

We began this study with three initial ideas. These ideas have been guiding us along all the study and we used them when we needed to take decisions: (1) students learn better when we propose to them activities that take into account their previous knowledge; (2) if we grasp students' strategy, we can help them advance in the construction of their knowledge; and (3) by working at the same time with different mathematical models (in this case functions) we facilitate students' connection and assimilation of new models in relation to those that they know.

Professional development of the TR[2]

As a teacher, the TR graduated in mathematics in 1977/78 and she has been teaching mathematics in secondary school since 1978/79. She usually collaborates in training pre-service and in-service mathematics teachers. Recently, she has collaborated with the local government and designed tests of basic competences and the new curricula

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[1] In one of the NCTM sense: among different subjects, in this case linear, quadratic or exponential growth.

[2] In the paper we use 'the TR' for "the teacher-researcher Iolanda Guevara."

of middle and secondary school (2006/07). She is also interested in history of mathematics and she is a member of the ABEAM[3] and SCHCT[4] groups.

In her professional development she learned a lot by listening to colleagues and students. Questions like, how to develop theories, how to facilitate students working in groups, how to coach them to discover things themselves instead of listening to teacher's explanation, learning students' strategies from their questions and their reasoning, are key issues for her.

The students

During the last two years the TR worked with two different groups of students. In 2005/06 she taught mathematics to a heterogeneous group of twenty five students, aged 12-13. She did not know the students when the school year began and it was the first time she taught mathematics to young students at this age. The results of some test about basic knowledge were worse than she expected. The majority of students showed, for example, difficulties in understanding the role of whole numbers in time situations.

The following year she did not teach mathematics to this group and she transferred the study to another group of 28 students, aged 15-16. In this case she knew the students; it was her second year with them so she preferred a case study rather than quantitative research. They did not have a habit of doing discrimination and argumentation tasks. In this case the challenge was to present a new concept simultaneously to others which the students managed the year before.

Following her concerns as well as the fact that she taught a different group of students, she designed her second study looking for coincidences with the first one. Correspondingly, these two situations had some common features: the designing method and the class management.

DESIGNING THE STUDY AND ACTIVITIES

In this paper we try to answer two questions:

a) How can we use students' problem solving strategies in order to design new learning activities?

According to constructivist theories people really learn when they can relate a new input in the classroom to something they already know that belongs to their background. Why do some students learn faster than others in the classroom? Why do some students begin reasoning but often they can not complete it? How can we help them to continue their own reasoning? Is it possible to transfer some excellent students' arguments to their colleagues? In order to answer some of these questions we decided to do this study.

b) What kind of activities does the TR need to introduce in the classroom in order to promote discrimination and argumentation working with connections?

We usually forget connections when we program our lectures. For example, we begin with linear functions, later we study quadratic functions and at the end we face exponential ones. The TR wished that her students would recognize each kind of functions in different situations and contexts, so why not introduce some of these functions together and relate them to others studied before?

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[3] The Barcelona Association for Teaching and Learning of Mathematics
[4] The Catalonian Association of Science and Technologic History

The philosophy

We present now some referential ideas about education that we used when we designed this study. Many of these ideas come from the TR's professional development, some have accompanied her since many years ago and others are more recent. We will try to explain how these principles induced a concrete methodology that was used to design and to carry out the tasks of class.

We keep in mind the following principles: (1) the process of students' learning benefits from the relation with other colleagues and with the teacher (Jorba); (2) students learn better when proposed activities take into account a starting point of their knowledge (Vygotsky); (3) if we bring to students tasks where they can show how they understand the situation and later we analyze them, can we design more appropriate questions so that students can construct their own knowledge? (Mason); (4) working in pairs or in groups enables interaction and construction of knowledge because it forces to communicate mathematical ideas and to negotiate results or processes of resolution; (5) it is necessary to design at the same time several activities in order to facilitate students to produce connections between different patterns, and (6) it is necessary to design activities with context about real problems, (although real context must be understood in a wider sense because sometimes real context is not a real context to students).[5]

Choosing contents and a curricular perspective

In 2005/06, in the middle of the second term the TR worked with a class of first-year secondary school students (ESO)[6] on whole numbers. The TR was a member of the test writer team and she wanted to know how her students would solve questions related to numbers and computations with whole numbers included in the test: calculating local hour in a place on earth and measuring time. Here there is a context situation; according to PISA classification it is a personal context.

Calculating local hour in a place on earth and measuring the length of a specified event are two daily situations that give context in the calculation with hours and minutes and exemplify at the same time the need to use a non-decimal system. Calculating local hour in a place on earth is also a subject that students cover in social sciences, when talking about time zones. Including it in a competence test is not a coincidence: it shows mathematics' instrumental facet and how other curricular areas use and need mathematics. It would be necessary to add that PISA test was a reference for CB[7] test writer team.

There were 25 heterogeneous students in TR ESO first-year class. 10 students out of 25 could not calculate local hour in Terrassa,[8] when in Montreal was 21:00 even though they were given the hourly difference between both cities, 6 hours. Moreover, these 10 students could not calculate what time an event began when they knew the duration of it and when it ended.

These results showed little competence skill of the TR's students in questions about hours, so the TR decided that small research was necessary. The study would consist of analyzing students' mistakes in order to discover their difficulties and find the strategies used by successful solvers. The idea was to design new activities to help students with more difficulties to learn this topic.

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[5] For example, we can think of a context about building and prices of houses; it is not an immediate context for students but it is important for their future.
[6] Educació Secundària Obligatòria (Compulsory Secondary Education)
[7] Competències Bàsiques (Basic Competences)
[8] Catalonian city, 30 km from Barcelona

In 2006/07, the TR only taught students in a compulsory education group (aged 15-16) and in high school (aged 17-18). When she decided to continue the study she needed to continue the key idea, designing learning activities based on students' previous knowledge and their strategies. She also considered working with connections because students were in the last course of compulsory education and they knew how many different concepts are related.

She precisely chose to introduce exponential growth working simultaneously with three different types of growth: linear, quadratic and exponential. Usually, mathematics teachers do not introduce exponential growth in compulsory secondary education (the subject is not in the official curricula) but the TR thought it is a usual growth in real world and it is necessary to introduce it to all people, not only to students continuing studying. Students knew linear and quadratic models, introduced in the third course, (first, linear functions are introduced, then quadratic functions – they are never studied simultaneously). We also considered the work context, in this case the so-called scientific context according to the PISA classification. In this sense the TR chose to work with three problem situations: a stone falling down, bacteria population growth and a constant speed movement.

Methodology

We started analyzing the results of 25 students in ESO first-year class in reference to the above-mentioned questions. First, we analyzed tests of the weaker students in order to find out the difficulties. The type of the test did not allow us to deduce which difficulties students had, because the test asked for the results and not for how they proceeded to obtain them. When we tried to discover the strategies from successful solvers we encountered the same difficulty, it was impossible to know how they obtained the result.

So we decided to carry out some interviews. The students with their tests in their hands explained to the TR how they did it. The TR chose three students among those who did very well on the test and three of those who did poorly. Advised by the colleagues of the PDTR group, she did not carry out individual interviews with students who did well or poorly, instead, she formed three pairs of students. In each pair there was a student with a high score in the test and a student with a low one. The TR asked them to explain how they solved the questions. The conversation with the students was recorded.

The TR transcribed the most relevant parts from the conversations with the students, which enabled us to find out which solver strategies were successful. Bearing these strategies in mind, the TR designed a new task for the class. Students solved tasks in other class sessions.

Two days later the students took the initial test again, this time the sheets had enough space to write explanations and calculations. Thus, we could analyze strategies used by eleven students with difficulties, and we could compare their first results with the second ones. This could also suggest conclusions about strategies used and their relationship with those discovered during the interviews. Two months passed between the first the second test and other topics were studied in the classroom.

In the second year instead of asking for the final results, the TR invited students to write their arguments when they answered the questions. In both cases she chose some of their answers and reasoning to design new learning activities. She later investigated how these strategies were incorporated by some students and tried to derive some conclusions on what kind of activities help students build their own reasoning.

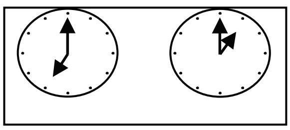
What kind of collected data can we present? In 2005/06 working with students aged 12-13, the TR collected the following data: (i) initial tests of all students of the class. The TR analyzed them and for each question she made a kind of table to classify answers and to decide which answers can be used to help us design new activities; (ii) discussions between the pair members; and (iii) the final test to compare the results with the first one.

In 2006/07 the TR worked with students aged 15-16 and collected: (i) the individual production of students solving a situation problem on their own. She studied all the individual production and chose the most relevant ones. For each selected case she analyzed what was said and what was confusing or not clear enough. This helped her design new questions to guide students to develop or to complete their own reasoning; (ii) a recording of two class sessions when a group, selected at random, talked to the whole class about their conclusions and when other groups completed the explanation. This material helped her follow students reasoning and also showed how all class working together arrive at the final conclusions about the relationships between different types of functions and what kind of criteria they followed to analyze a functional table; (iii) students' answers and reasoning on the final test. This material was very important for the study because it showed how students integrated new ideas and concepts between the initial the final tests.

The activities

Activity 1: When can two people chat?

Marc knew that time difference between both cities is quite big. In order to find an appropriate time to chat with Brigitte he decided to find information in the Internet and found this:



1. What time is in Terrassa when it is 21:00 in Montreal?

2. Every day Marc leaves home at 8:00 to go to high school and he comes back home at 13:30. In the afternoon he goes out at 14:30 and he comes back at 17:30. He goes to sleep at 23:00 and he wakes up at 7:00. Brigitte has the same daily routine but in Montreal. Mark the correct answer of each section with one X**.**

a. Is it possible for Brigitte to chat with Marc from 22.00 to 23.00 Montreal time, before going to sleep?

☐ It is possible, Brigitte and Marc are free.
☐ It is impossible, Marc is asleep.

b. Is it probable that Brigitte chats with Marc from 14:00 to 14:30 Montreal time?

☐ It is little probable, Marc is in the physical education class at this time.
☐ It is very probable, they can chat at this time and they do it whenever they can.

c. Is it possible that Marc chats with Brigitte from 20 to 20:30 Catalonia time?

☐ It is possible, Marc can chat with Brigitte at this time
☐ It is impossible, Brigitte is in the school at this time

Figure 1.

In the 2005/06 school year, the proposed activities came from the following questions which were on the test of basic competences designed by the educational department of the local government. In this paper we will refer to them as Activity 1 and 7 (Fig. 1&2).

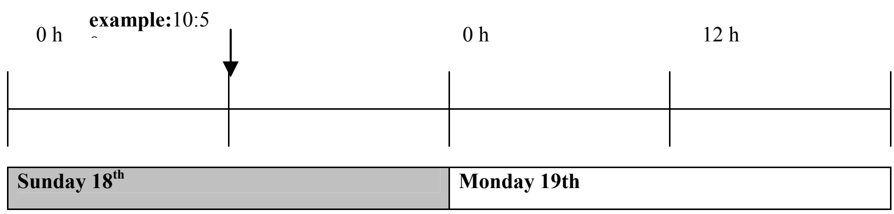
Activity 7: We limit chat time

Some days ago Marc parents were worried because he spent a lot of time on the Internet, sometimes up to four hours. Finally, after a heated conversation with his parents, Marc arrived at the following agreement: “Ok, from next Monday, 19th September, I will use the Internet an hour at the most every day.” In order to check if he did what he promised, Marc’s parents suggested he write in a chart beginning and end of connection time every day. Some days later Marc showed his parents the following chart to demonstrate that he kept his word. Fill in the missing data:

Day	beginning	end	length
Thursday, 15th September	20: 15	21: 45	
Friday, 16th September	19 : 20		3 hours
Saturday, 17th September		12 : 00	2 hours 10 min
Sunday, 18th September	10: 50	14:10	
Monday, 19th September	19:45	20:25	
Tuesday, 20th September	20:15		1 hour
Wednesday, 21 September		21:10	50 minutes

a. Write the following data on into time line, as you can see in the example:

- Beginning of the connection on Sunday (**example**)
- End of the connection on Sunday
- Beginning of the connection on Monday
- End of the connection on Monday



Can you see Marc’s attitude changed after he made his promise to his parents? Explain your answer.

Figure 2.

In 2006-07 school years, proposal activities began with this initial paper:

MOVEMENT AND GROWTH

The description of the following situations:

1. A stone falling down

and the formula $e = 1/2 \cdot g \cdot t^2$

2 A bacteria population growing up

3. A constant speed movement

What is similar in each situation? What is different?

In order to analyze the three examples, you can use

a) A tabular table

b) Can you find some rules to foresee what the following numbers are going to be on the table?

c) Which operations characterize each table? Additions, products…

d) Which movement or which growth is the faster one? And the slower one?

Figure 3.

How did students work in class?

In the first class students worked individually and also worked individually in the final test in both cases, in 2005/06 and 2006/07. Afterwards, students did the news tasks, designed especially for them from the study of their answers, in pairs or groups. The pairs and groups were heterogeneous and we promoted the interaction among them. In 2005/06 students, aged 12-13, worked in pairs. In order to form the pairs each weaker student chose their partner from the better ones. In 2006/07 students, aged 15-16 worked in groups of four. The TR made heterogeneous groups so aware of the fact that each group had one able student at least. She decided who the best students were, according to the results of initial individual tasks.

THE EXPERIENCE AND RESULTS

The first study

In 2005/06 the TR worked with students aged 12-13 and began the process of analyzing the results of the whole class. The TR needed a global idea of the level of the class, on tasks 1 and 7 (Fig. 1&2). Five students had a low score in both activities, two students had low scores in activity 1, five students had low scores in activity 7 and only eleven students had a good competence skill in this situation. The TR chose 4 students for the pairs of interviews. To make all students feel comfortable, two among the chosen students had more difficulties and two did the tasks correctly; to eliminate the discrimination by gender, the pairs were formed by people from the same gender.

The new tasks designed in accordance with the data collected in the interview contained several sections: (i) local time; (ii) simultaneity of hours in two places on earth; (iii) length of an event, the time of beginning and end; and (iv) time representation on a straight line.

In general the TR observed that the students knew where it was necessary to add or subtract but they had problems calculating hours and minutes because they thought in the decimal system and not in a non-decimal one. Some students were a step behind, they did not know when it was necessary to add or subtract.

The TR analyzed a particular case. The student gave all incorrect answers, even though she had a very strong strategy behind, but it was an erroneous strategy.

20.15 B		19.20 B		12.00 E		10.50 B		19.45 B		20.15 B
21:45 E		3 L		2.10 L		14.10 E		20.25 E		1 L
41.60 L		16.20 E		9.09 B		24.6 L		39.70 L		19.15 E

Figure 4. B: beginning hour, E: end, L: event length.

According to this student:

Beginning time + end time = chat length

Beginning time – chat length = end time

The TR observed that these two statements were contradictory because if one of them was right the other one could not be right. She supposed the student did not control relationships between addition and subtraction as complementary operations

end time - chat length = beginning time

The strategy was correct but the student calculated with minutes as if they were figures in decimal basis 10 – 1 = 9

Observation of students' evolution shows that 15 out 25 students improved, 3 remained on the same level and 7 obtained. If we speak only about 11 students who did worse, 6 improved, 1 remained on the same level and 4 worsened. Talking about fairs among 11 students with more difficulties, only one student passed the first test, and three students passed the second test. Taking into account where they were at the beginning, they probably needed more time and some more activities in order to be a little more successful. The students who previously had high scores improved a lot because of the interactions. Twelve students increased their scores significantly.

THE SECOND STUDY

In 2006/07 the TR worked on exponential growth with students aged 15-16. Secondary compulsory mathematics program does not include exponential functions, these functions appear in the first grade of secondary school, but the TR thought an introduction to the exponential growth would be interesting for all students because it is a usual growth one can observe in many contexts in daily life situations. The TR decided to introduce this model of functions in the last grade of Secondary Compulsory Education.

We worked on linear and quadratic functions before and she decided to present this new model as related to the others. The TR and her students spent some days on reviewing what they knew about linear and quadratic functions and in class they repeated the test of linear functions they took in the 2005/06 course. This way, students could more or less review some ideas about functions, tables, formula and graphics. The TR decided to work only with tables and formulas concerning the exponential function but not to work with graphics. Exponential graphs will appear next year in high secondary. Whenever the TR could, she introduced new concepts with a real context, so we looked for three situations in order to introduce exponential growth as a daily life situation in the real world. Figure 3 shows the initial task with three situations where TR invited students to analyze similarities and differences.

Examples of guided questions

What kind of guidelines did the TR introduce in the next worksheet to promote the reasoning according to answers in the initial test?

Some student said in the initial task: "Example 2 is the fastest one because the numbers in the table are the biggest ones." The TR asked students why they believed number 2 was the fastest. She suggested they write answers with argumentation.

Suggestions:

Try to analyze which numbers and operations appear in each table. In each table, write the sequence of operations you must do to get the final result.

Identify the repeated numbers which appear along the table line by line successively.

Figure 5.

Other suggestions meant to develop their analysis of the tables.

t	e
0	....
1	....
2	....
.	.
.	.
.	
.	.

Going to the patterns, vertical and horizontal criteria

Which operation makes connections between two consecutives lines? Additions, products…?

Which operations make connections between left and right column? Additions, products…?

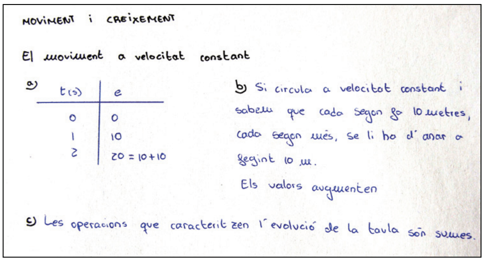
Formula?

Figure 6.

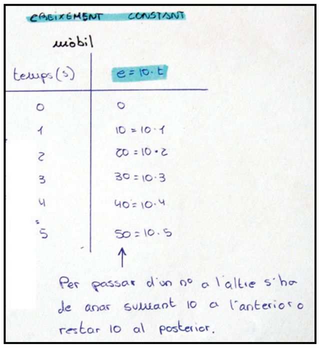
Students worked in groups. The TR talked with the groups and she looked at their worksheets. What were they doing? Did someone need help? What kind of help could she offer them without cutting their own reasoning? She did not like to explain what happened, instead she preferred they discover it themselves.

Following a case

The TR followed a single student, her first production (working alone) in the initial task and her production when working in a group. She was a usual student, sometimes she understood well and sometimes not, but she was patient and she asked the teacher or other colleagues when she did not understand. Her grades in math test varied from 4 to 6. The TR chose her because, although she had some difficulties, she aimed to learn and she tried to follow the teacher's suggestions. We could see her evolution step by step. We have learned a lot from her. Figure 7 shows the initial task and Figure 8 task in group, you can see the differences:



MOVEMENT AND GROWTH
The movement with constant speed
b) If it circulates with constant speed and we know it does 10 m, each second we must add 10 m each time. The values increase.
c) The characteristic values of the table are the additions.



CONSTANT GROWTH
Passing from one to the other one we must add up 10 to the previous or to subtract 10 from the last one

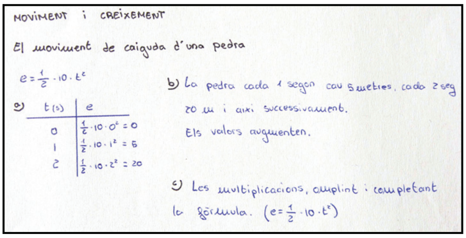
Fig. 7 *Fig.8*

In the group task, her argument was more or less the same, to add or subtract 10, but in the second case she wrote the formula of the function. Now we will consider the next situation, stone falling down; (the three situations are not in order, but we analyze them in the natural order –linear, quadratic, and exponential).[9]

The initial task: here the formula appears in the statement of the problem; it is not something to find. The student applies the formula to find *e* and she explains the operation related to the table is multiplication. After the group task, she is able to explain the relation between the different spaces; she finds the rule of increase.

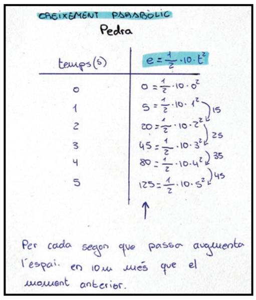
In the next figures we can see the differences between her initial individual task and the group's work:

[9] The situations were not in order for a didactic reason. If we present them in order we push students to relate to the different models mechanically, not in a reasoning sense. We follow a 'natural' order in the analysis for an 'epistemological' reason.



MOVEMENT AND GROWTH
The stone falling down
b) The stone falls down each second 5 m, every 2nd second 20 m and successively the values increase
c) The multiplications, completing and using the formula …..

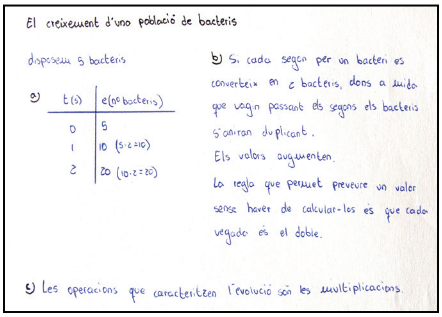
Figure 9.



PARABOLIC GROWTH
A stone
For each second the space increment is 10 m more than the previous one.

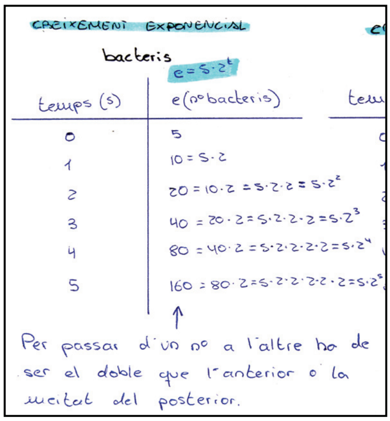
Figure 10.

Finally, we analyze the third example, bacteria population growth. Initial task: the statement of the problem explains the growth; the bacteria population duplicates their number each second. The student characterized the operations by multiplication; she did not speak about the exponential growth. After the group task: she could write the formula, look at the formation of the right numbers and relate to the left one. She also classified the growth as an exponential one. Initial task – Fig 11; group task – Fig. 12.



The bacteria population growing up
we have 5 bacteria
b) If in each second each bacteria becomes in 2 bacteria, time going on bacteria duplicates, the values increase.
The rule which lets us foresee the value without calculus is that each time it doubles
c) Characteristic operations in the evolution are the multiplications.

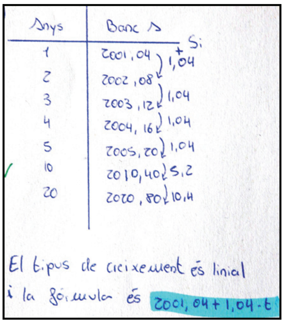
Figure 11



EXPONENTIAL GROWTH
bacteria
To pass from one to the other one it must be the double of the anterior or the middle of the posterior.

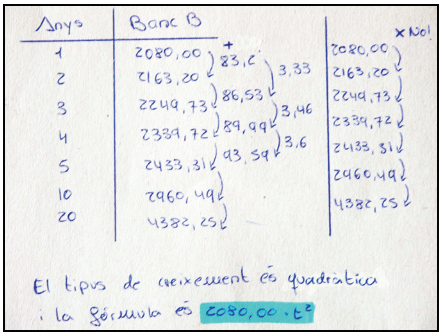
Figure 12

In the final test, the TR presented several situations to students and asked them to analyze and classify the situations and to find special values. Let us consider some answers of the same student, who worked alone, which show that she was able to find results successfully. We need to add that we foresaw some problems because the initial value was not very clear in the statement of the problem. So the formula would be: $e = 2000 + 1{,}04 \cdot t$, but the process to discover it is essentially right in Fig. 14.



The increment is a linear one and the formula is …

Figure 13



The growth is a quadratic one and the formula …….

Figure 14

In the example presented in Figure 14, there was a problem with decimals. The student rejected exponential growth and accepted the quadratic formula as the final solution. Perhaps teachers must insist on the fact that sometime there is no formula or they cannot arrive at it because it is so difficult, otherwise we induce students to write the final formula they can find.

Let us see the same student's production, in another exercise of the test. In both cases the criteria to prove the addition increase and the multiplication related to linear or exponential growth yields to the formula. The problem with the initial value continues.

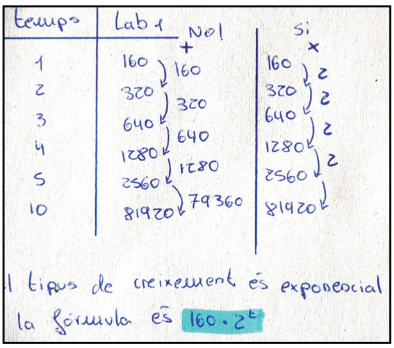
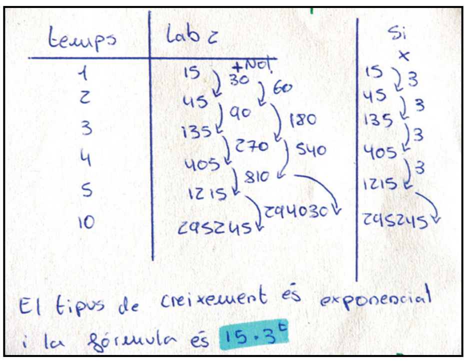


Figure 15.



the growth is exponential and the formula is .....

CONCLUSIONS

We would like to sum up with two kinds of conclusions, first related to our own research question and second, more general, related to the TR experience during the last two years as well as to our own evolution in teaching research.

The research question was how we could use students' problem solving strategies to design new learning activities and what kind of activities we had to introduce in the classroom in order to promote discrimination and argumentation working with connections. Now, we would like to reinforce these three ideas that we introduced as well as some theoretical assertions present in the study.

In reading and analyzing initial students' productions we found a way to introduce strategies when we designed new learning activities. We use some students' statements that they made without any given reason to make them aware of the necessity for reasoning. Interactions among students, when they solved tasks in heterogeneous pairs or groups, help slow students but also improve the reasoning in the more active ones.

The way we preceded introducing exponential growth connected with linear and quadratic growths is very interesting because students understand growth as the key concept and not only as the simple model, for instance, they discover proportionality. The analysis of different types of growth concerns the items of change and relationships, according to the PISA classification.

As far as teacher experience is concerned, the TR drew some conclusions about her role as a teacher and as a teacher-researcher. She is convinced now that it is important, and not a waste of time to make students work and discover relations or arguments themselves. When students ask teacher questions, the most usual reaction until now was to answer them immediately. Now the TR tries to answer their questions with another question in order to enable them to find their own answer. This looks more interesting and sometimes it is not easy to find the adequate question but she tries it. Answering questions with another question is a procedure that the TR has discussed a lot during these two years with some colleagues not involved in the PDTR project but in teacher training.

Another important idea is to know how students evaluate the work of their teacher, this has been important for the TR for many years and she continues the dialogue with students in order to understand more and more about them. It is also necessary to note that students were happy in math class and to be happy does not mean that they do not work, of course!

In terms of the teaching research approach, TR recognizes her evolution from the first investigation in 2005/06 to the second one in 2006/07. In the first one she tried to collect a lot of data, from the entire class, and she did a lot of classifications and charts. After these classifications, she chose four or five proposals to design new activities for the class. In the second investigation in 2006/07, she did not make this extensive classification. She read of course all students' productions but while she read them she chose some of them and analyzed only these ones more carefully. She passed from the quantitative to the qualitative analysis.

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THE USE OF STRATEGY GAMES FAVORS LEARNING OF FUNCTIONAL DEPENDENCIES FOR DE-MOTIVATED STUDENTS

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ABSTRACT
This article refers the way a strategy game, DRAGO, can help students from ESO (Obligatory Secondary Teaching) to develop strategies aimed at solving problems, and, in our case, to develop strategies aimed at solving problems where functional dependencies are involved. More specifically, we want to see how the use of games helps enhance mathematical activity in high capacity students that do not manifest it due to an excess of routine, that provokes in them de-motivation towards discipline. We will show the results of this strategy on two students as an example of what we say. But it is not only for these reasons. Bishop (1998) says: "Game has a close relationship with mathematical reasoning, and we can consider true the statement that says that it is the base of hypothetical reasoning."

BACKGROUND

Many teachers and investigators have dealt with games as an important element for motivating math works at the classroom. We know that all cultures have adopted games as part of their pastime. Thus, Asher (1991) shows how American Indians (the Cayuga) used a wood bowl and 6 disks as a game. The game consisted in throwing the 6 disks in the air at the same time and count how many fell inside the bowl. According to the number of disks that effectively fell inside the bowl, they scored more or less points. This kind of game would be equivalent to dice-throwing or coin-throwing with a high degree of chance and probability.

According to Bishop (1991), there are up to six important mathematical activities that all cultures practice: to count, locate, measure, draw, play and explain. We agree with Bishop (1998) that playing is a universal activity and that mathematics is also a universal area of knowledge.

Another important aspect of the use of strategy games is the specific kind of situations it develops, because it allows for important social interactions between players, for instance, the group or class, if the game is conducted in a classroom.

Strategy games, as Corbalán (1998) points out, can be considered as a particular class of problems, and thus can be treated using the channel defined for problem-resolution by Polya:

PROBLEM	GAME
1.- Understand the problem.	1.- Understand the game rules.
2.- Elaborate a plan.	2.- Elaborate a game strategy.
3.- Execute the plan.	3.- Apply the strategy while playing.
4.- Examine the results.	4.- Review the results of the game.

In this sense, we asked ourselves if students would be motivated to work in a strategy game. Would they develop strategies that could drive them to establish relationships between the different elements involved in the game in order to deduct the best way to play? In other words, would they be able to deduct how to win most of the games?

DEFINING THE EXPERIMENT

We decided to conduct research with two groups of third grade of ESO students (aged 14-15) at our Public Institute (secondary school) “IES de Sant Andreu de Llavaneres,” situated in an upper-medium-class residential town, 30 km north of Barcelona. Though conditions may seem advantageous for students, because of this social profile, and in many cases this was true, the index of school failure is quite high, about 25% for third ESO graders.

We chose for this research two different groups: 3rd –A and 3rd – C out of 6 that comprise the whole of third grade at our institute. Initially, all groups consisted of 16 students and were relatively homogeneous in terms of composition, because one of our school’s strategies is to try to distribute students in equilibrated groups as far as their capacities are concerned. (1) Group A included, among others, two grade repeaters, a student with special teaching needs, another student who reached third grade but did not pass any subjects of second grade, and, finally another student with digestive dysfunctions. When the research work ended, only 11 students out of the original 16 remained in the group. (2) Group C consisted of two grade repeaters as well, one student that needed special attention during the previous two years, and four students that finished the second grade without fulfilling mathematics requirement. By the end of our study, the group was reduced to 13 students.

During the first two years of ESO, mathematics teaching can be considered as “text-book:” explanation by teachers at the blackboard, practical exercises of subject that was explained, and, finally, problems with brief explanations. It is a very typical teaching system. We may say that problem-resolution processes are not dealt with, or, if they are, it is more in a very self-learning way, as many authors point out.

We developed the experiment along five 50-minute sessions. The first session was dedicated to the introduction of goals that students would have to achieve when they finish their work, and also to the introduction of the game, the materials needed and its rules. In this session we also began to work with the game with the simplest possible situations. Students had to work in pairs randomly organized, and they hade to complete activities included in the dossier given to them at the beginning of the session.

At the beginning of the second session, we presented the results of the previous one and discussed them. At the beginning of every session, teachers’ role was extremely important; because they had to make sure that the majority of students participated in the discussion. We ended the second session working with the same pairs formed in the previous one, but with more sophisticated situations. At the beginning of the third session, we again presented the results, tried to involve all the participants, and continued working in pairs in order to deepen the understanding of the game’s mechanisms. At the beginning of the fourth session we presented the results achieved by each of the two selected groups. The fifth session was dedicated to testing practical understanding by means of a competition among all the students.

Data used for an analysis of the experiment results included: (a) a summary of the work done by the students during the five sessions; and b) a video recording of the two sessions where both groups participated.

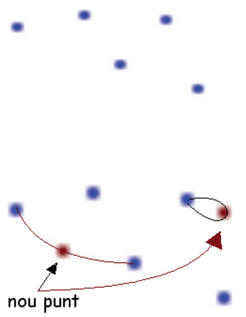
In this article we want to present the results of 2 out of 32 students involved. These students, a boy from 3rd-A and a girl from 3rd-C, were chosen because of their mathematical capabilities, according to their teachers, and also because they presented clear de-motivation trends regarding studying.

The girl, Laura, comes from a well-off family that owns a family business. She is the youngest daughter. During the first six months, she exhibited an increasing de-motivation towards studying, together with incipient absenteeism, especially during first-hours classes, due to staying-up-late sleeping habits. Even with her background, the results were not that bad.

On the other side, Fran is the youngest son of a well-off family. He is an absentee with parental permission; that is, when he does not want to go to school, he is allowed to. Although in first and second grades he succeeded, he is a firm candidate to fail in third grade. He loves to be in the center of other’s attention. During the first six months of school, they both demonstrated an outstanding capacity for mathematics.

The game chosen for this experiment is called DRAGO, and also, PLANÇÓ. It is a game that fulfills all requirements to be considered a strategy game, according to Gómez Chacón’s definition (1992): (1) it can be played by more than one player; (2) it has a set of fixed rules; (3) the rules establish the goals to be achieved by each of the players; (4) the players have to choose their specific paths in order to achieve their goals; (5) the rules clearly establish when one of the players is the winner.

Many other games fulfill these conditions, but we found that this game is very suitable for third-ESO-graders for introducing the first degree functional dependencies.



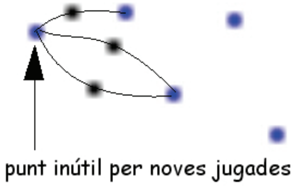
new point

The player that starts has to link two points with a continuous line, or one point with itself. Once the player does this, they have to draw a new point in the arc just drawn.

The rules to be followed are:

1) From one point, no more than three lines can be drawn. (fig 1)
2) Line cannot intersect.(fig 2)
3) The winner is the last player to do a valid movement.

Fig.1



This point is not usable for new movements

Fig. 2

Forbidden movement

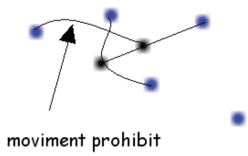


Figure 1.

We will now introduce the game.

GAME DESCRIPTION

DRAGO is a strategy game for two players created by John Conway in 1966. The only things needed to play it are paper and pencil. The game begins by deciding, between both players, a certain number of points to be drawn in a paper. It is advisable that the number of points should be between 10 and 15. If the number is smaller than 10, there is no game, and if it is higher than 15, the game may take too long. Let us suppose we chose number 5. The points are drawn in the paper.

The mathematical interest of the game is based on the two following aspects:

1) The relationship between the initial number of points and the maximum or minimum number of movements that can be done, as also with the final number of points drawn. A detailed mathematical analysis of the game can be found in Mora (1991) who shows this dependency to be a linear one of the type y= ax+b.
2) To find, and be able to explain, which is the best strategy to try to win the highest possible number of games.

RESULTS AND STRATEGIES OF STUDENTS DURING THE SESSIONS

As mentioned above, the first sessions consisted in explanation how to work during the next sessions, and in introduction of the game and its rules. Once this was done, students had to begin to play. They were divided in pairs (or groups of 3) and began playing the first games in order to see if they understood the rules and mechanisms of the game, with the simplest cases; 1 and 2 initial points.

One of the first things to remark was the trend we observed in students aged 14-15 who did not register what they observed, so they just played, as if there was no other goal. Teachers' task at this stage was to make sure that players registered the movements they made. At the beginning of the second session, there was a brainstorming to discuss what was done and to try to deduce some results. Below are the graphics drawn by students at this stage:

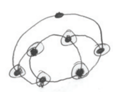
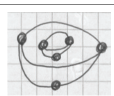
<table>
<tr><td>1 point</td><td>
fig 1</td><td rowspan="3">Obtaining Fig. 1 does not pose a major problem to students. When drawing it on the blackboard, teachers should ask them to consider which elements are especially interesting. Only one student pays attention to the resulting form. As it is that figures 1 and 2 are obtained naturally and immediately, figure 3 requires some more work. To produce this form, one needs to have a winner's strategy.
This is Laura's description of different movements that can be made with 2 points. It is remarkable in two ways:
1) It introduces the idea of unblocked points, basic in a winner strategy.
2) It intuitively establishes a relationship between initial and final number of points. She's already relating these two variables.</td></tr>
<tr><td rowspan="2">2 points</td><td>
fig 2</td></tr>
<tr><td>
fig 3</td></tr>
</table>

Figure 2.

Once this stage was completed, we had to go on with more initial points. At the beginning of session 3, we did some brainstorming to analyze the results of working with 3 or more points. At this point, the task of the teacher was very important in order to organize the ideas that were introduced so far. The way students had to represent results was not the best one to find relationships, so teachers' guidance became basic.

In the following drawing (Figure 3), we can see how they represent their results. Laura drew some of the possible situations that can be obtained in a 3-initial-points game. The work is quite exhaustive, for she established the final number of movements, and the winner. She did the same with 4- and 5-point games.

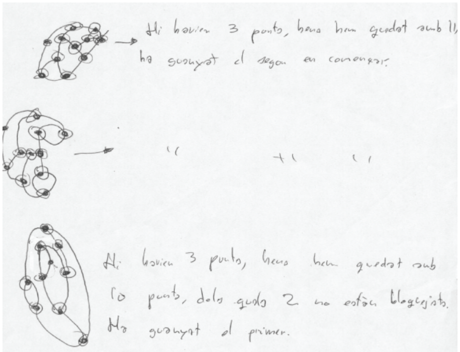


Figure 3.

Harmonization of results became essential for teachers to be capable to orientate the presentation of results in the best possible way.

After harmonization, we obtained the following table (see Figure 4):

With results put in a table, the relationships between different elements (initial points, final points, number of movements, non-blocked points and winner) were more easily established.

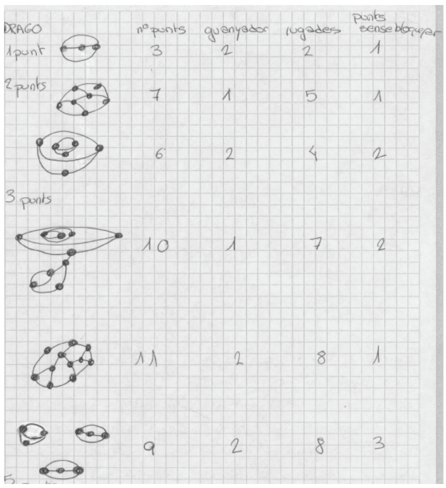
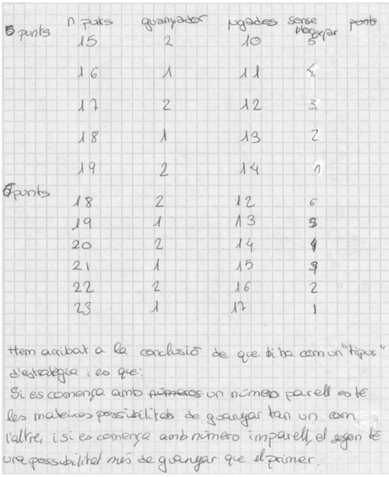


Figure 4.

One of the advantages of having a table was that from now on we could go on completing it without the need to play. We could deduct how games would end just by looking at this table.

That was what Laura did, and she completed a table with all possible movements with 4, 5 and 6 points. And not only this, she was also able to formulate part of a winner strategy. It is to be said, though, that a complete formulation of the winner strategy is quite complicated and none of the students was able to complete it.

In the following drawing (Figure 5) we can see the final part of the table drawn by Laura, based on the previous table shown before. Laura followed this reasoning: With 5 points there are 5 possible endings. To obtain the number of movements we should subtract the initial number of points from the final number of points, and based in this number we can deduct if the winner will be the first to play or the second.



	n punts	guanyador	jugades	sense jugar	punts
5 punts	15	2	10	5	
	16	1	11	4	
	17	2	12	3	
	18	1	13	2	
	19	2	14	1	
6 punts	18	2	12	6	
	19	1	13	5	
	20	2	14	4	
	21	1	15	3	
	22	2	16	2	
	23	1	17	1	

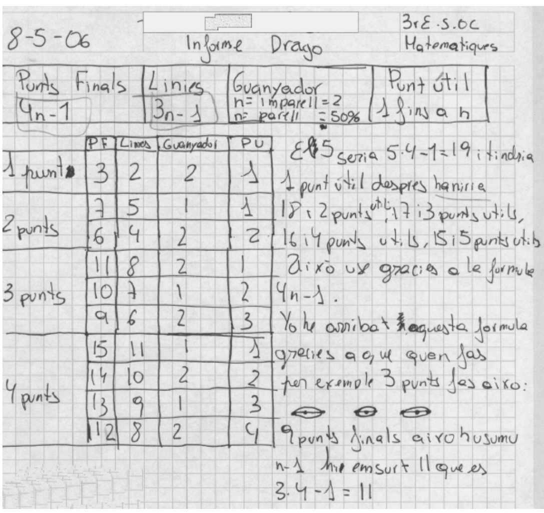
Hem arribat a la conclusió de que hi ha com un "tipus" d'estratègia i es que: Si es comença amb ~~punts~~ un número parell es te les mateixes possibilitats de guanyar tan un com l'altre, i si es comença amb número imparell el segon te una possibilitat més de guanyar que el primer.

Figure 5.

From this, she could establish a small part of a winner strategy:

If the game starts with an even number of points, both players have the same chances to win. If it is not even, the second player has higher probabilities of winning."

But Fran went further. With the same data he was able to obtain an algebraic expression that related initial and final number of points. And knowing this we achieved a complete mathematical analysis. In the following drawing (Figure 6) we can see the formula proposed by Fran, and the reasoning he followed.



8-5-06 Informe Drago 3r E.S.O.C Matemàtiques

Punts Finals	Línies	Guanyador	Punt útil
4n-1	3n-1	n= imparell = 2; n= parell = 50%	1 fins a n

	PF	Línies	Guanyador	PU
1 punt	3	2	2	1
2 punts	7	5	1	1
	6	4	2	2
3 punts	11	8	2	1
	10	7	1	2
	9	6	2	3
4 punts	15	11	1	1
	14	10	2	2
	13	9	1	3
	12	8	2	4

EX5 serie 5·4-1=19 i tindria 1 punt útil després hi hauria 18 i 2 punts útils, 17 i 3 punts útils, 16 i 4 punts útils, 15 i 5 punts útils això ux gràcies a la formula 4n-1. Yo he arribat aquesta formula gràcies a que quen fas per exemple 3 punts fas això: 9 punts finals això hu sumu n-1 i me surt lo que es 3·4-1=11

Figure 6.

Fran found that beginning with n points the game ends with a maximum of $4n$-1 points, and it takes $3n$-1 movements.

The same as Laura, he was capable of guessing and writing down, a small part of the winner strategy:

If starting number of points is uneven, the first player has a certain advantage. If not, both players have the same chances.

Both lack relating this fact to the number of unblocked points remaining at the end of the game. It is possible, though, that they both were conscious, although did not put it in writing that this relationship exists.

In the fourth session we explained to everybody the results that were found. These results included the table with all the data that was written down and the formulas worked out by Fran. In order to establish a first-degree functional dependency, it was needed that the teacher completed the information with a graphical representation. In this way, we used the game to establish the three main elements of a functional dependency: table of values, formula and graphical representation.

The 5th session was devoted to a DRAGO championship between all the students of the class.

CONCLUSIONS

Does the use of strategy games favor the learning of functional dependencies in de-motivated students? I think that strategy games are a very important motivating element that can be applied in a very positive way to help learning functional dependencies. Suitable games could be found to match different subject contents for students in secondary obligatory teaching. In this same sense, we could generalize the use of strategy games to the teaching of mathematics in general.

Due to the parallelism between problem resolution processes and strategy game analysis processes, the latter can be used to establish a set of activities that can be used in classes with high cultural diversity. Games have a universal range, and those games we have qualified as strategy games, are known for not establishing social or cultural preferences.

Concerning students with remarkable capacity but low motivation, according to results, we may say that strategy games allow them to fully develop all their abilities, as opposed to traditional teaching that drives them to school failure due to the abuse of routine.

It is important to also state that it is basic to try not to loose these students with demonstrated capacities, and that this task should be a joint task of all areas involved in teaching.

In our case, in order to emphasize our last assessment, we want to publicize that Fran will have to repeat third grade, because he has not complied with the criteria established by the Education Law in Spain.

It is also very important to guarantee that all students achieve their goals, and that teachers develop the ability to carry out different kind of works in class.

On the one hand, there must be aspects that have to be worked out individually, or in very small groups. On the other hand, it is essential that brainstorming sessions can be held in order to make results available to all of participants, and also to establish order in them. In these sessions, teachers should try to promote participation of everyone involved, and, if necessary, focus on proposed goals and achievements.

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TEACHER-RESEARCHERS AND ENCULTURED NEGOTIATION OF MEANINGS

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ABSTRACT
PISA test scores highlight the low level of competence among Catalan students in mathematics. The reasons for these results varied, but we identify two areas in which work is required: first, the need to return the mathematics curriculum to its cultural roots so as to facilitate the reconstruction of the mathematical knowledge of this diverse student body; and second, the need to enhance the management of the mathematical activity of these students so as to encourage the group negotiation of meanings. Here, steps in enculturation are made in introducing algebra to students aged 12-13; initial results are highly encouraging.

INTRODUCTION

New curriculum proposals in Spain and Catalonia reflect the importance attached to the development of thinking and mathematical reasoning, the ability to analyze realistic situations by employing models at a range of levels, and the capacity to reason and communicate. These target skills can be achieved through enculturation (Bishop, 1988) a process that exposes pupils to rich and challenging tasks and promotes the management of critical and self-producing knowledge. Such an encultured curriculum combines rich, contextualized tasks (Masingila et al., 1996) with the following characteristics: (a) high-level mathematics: challenging, not overly simplistic content; (b) adapted to what students already know; (c) all students capable of participating; (d) best when handling materials that facilitate resolution of problems, which should encourage reflection; (e) generating intelligent questions; (f) gradable according to different rhythms of learning; (g) promoting decision-making; (h) improving personal models; (i) encouraging teamwork; (j) promoting research and discovery; and (k) connecting different types of mathematical knowledge.

THEORETICAL FRAMEWORK

But what might initially be considered a rich proposal, can quickly become impoverished if it is mismanaged. To be successful, a task should be closely linked to an authentic context, and the activities undertaken should involve conceptual discussions, reflection and analysis, and should conclude with explanations, arguments, reasons, and the eventual communication of mathematical ideas. In this context, management of a proposal should be seen as the facilitation of students' interaction with their peers in a process that is mediated by teachers. Moreover, the tasks should be undertaken in a constructive, cooperative atmosphere, in which the proposed activities can be presented and debated, thereby ensuring that the classroom becomes a true community of practice. Here, we argue that the construction of such "shared meanings" (Crook, 1996) should be based on three prime processes: *articulation, conflict* and *co-construction*. The first of these, articulation, involves the explicit public justification of ideas achieved through organized thought and interpreted for the benefit of the joint activity. The second, conflict, emerges from disagreements as participants seek to justify their standpoints and attempts are made to resolve conflicting opinions, which requires processes of re-thinking and cognitive restructuring. Finally, co-construction can be seen as the construction of cognition that is jointly built in the articulation and interweaving of arguments during the exchange of text-targeted collective thinking. In the scenario we describe here, this process of reinvention (Figure 1) is not solely guided by a formal language but thrives upon intra-group dialogue.

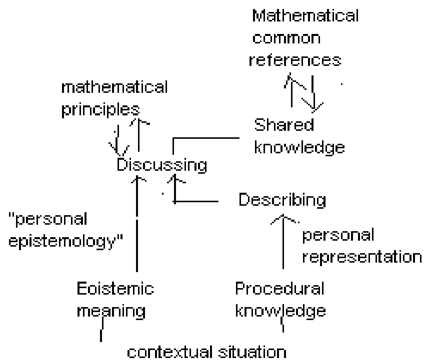


Figure 1. Guided model of teacher-researchers' experience

In this process, a teacher-researcher is more than a simple action-research "tinkerer" (Gravemeijer 1994), but rather someone who seeks to combine and integrate global and local theories (which may have arisen in other domains), to develop a learning pathway and a local theory of instruction for a specific topic. Seen in this light, professional development becomes a process thanks to which reflective perspectives introduce new theoretical frameworks for the interpretation of school practices. Through the acquisition of experience, contextualization not only serves the goal of developing internal representations, but also that of broadening its epistemic terms of meaning. In other words, what is sought is the expansion of the vision of what mathematics is and the purposes it serves. In the study we describe, we hope that our empirical findings also provide food for thought for the wider community of teachers and researchers who perceive the importance of the construction of meanings.

METHODOLOGY

In this study, we seek to present evidence of the value of the construction of meanings when introducing algebra to a group of 25 students aged 12-13, and through the resulting self-reflections of the teacher-researcher. Eight classroom situations (Vilatzara, 2005) were videotaped with focus on the group co-construction of hypotheses and meanings.

Part 1. Starting with arithmetic relations

Our initial aim was for students to recognize three basic types of arithmetic relations and their associated epistemic meanings: (rnr1) referential numerical relationship of proportionality between quantity, unit price, unknown price; (rnr2) relationship part/part/whole, with partial prices and total price; (rnr3) linear equalities type $c = ax + by$ factor quantity and with unknown price. The first question posed was "Here are two pizzas and three salads whose total cost is €19.90. What do you feel you can say about their prices?" This question was designed to promote a certain amount of reflection among students as to how they might be sure about any conclusions reached on the basis of the information given. It was assumed that reasoned arguments were the key element in this task, and so students were prompted with questions such as: "Explain why two pizzas cannot cost €20. Explain why one salad cannot cost more than €6," and so on. Finally students were asked to give five possible prices for a salad and the corresponding price for a pizza.

After considering these questions, we initiated our first debate. The aim was to share the work out among the group, and to compare each of their personal proposals. The debate sought to generate two types of negotiation: one that was purely cognitive and a second that was based on social norms. Our first objective was to ensure that a contextual relationship can be expressed by a widely used arithmetic relation (e.g., an equation with two unknowns), and that, consequently, only one pair of values can satisfy that condition. Our second objective was to negotiate the social meaning that students attached to arithmetic relations, that is, as relations with a single solution that can be verified since it gives a certain numerical result. By contrast, it is not easy to accept a relationship with two variables that might take on an infinite number of possible values.

Part 2. **Generalized situations**

A second set of questions was then used to generate a strategy for resolving common arithmetic problems (similar to that of the Gauss method for solving systems of two equations with two unknowns). The introduction of this de-contextualized generalization is considered important because students cannot be left ignorant of traditional systems of equations and because they need to acquire an awareness of the processes involved in re-contextualization/de-contextualization. A set of questions was, therefore, used to generalize patterns: "The price of three slices of pizza and three drinks is €12. Explain if we can determine the price of a slice of pizza and a drink". In this way a new debate is initiated for sharing personal knowledge, and by presenting arguments and counter arguments students learn the value of dialogic debate. In this process teachers' role is to maintain the principle of equality. It is not their role to establish a relationship of power of one over the others (Skovmose, 1994) in a decontextualized manner.

At this point we introduced a new set of contextualized situations of the type: "Yesterday we bought three pieces of chewing gum and four sweets for 50 cents, today we bought two pieces of chewing gum and five sweets for 45 cents. Can we calculate the price of one sweet? And that of a piece of chewing gum?" The objective underlying the introduction of these situations was primarily to regulate and promote the systematization of processes learned that might apply to other contexts (Talizinia, 1985).

Part 3.

A further *set of decontextualized games* was then used in Chinese form (following Streefland & Van Ameron, 1996) and a third debate was conducted to negotiate the idea of an incompatible system. After this, the traditional system of equations was presented: $2x + 3y = 8$; $2x + 4y = 10$ to promote a new epistemic debate.

Part 4.

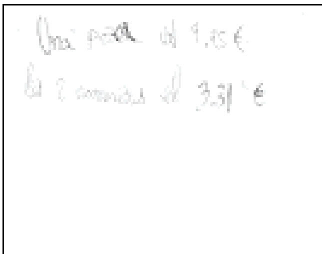
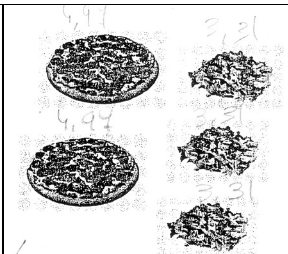
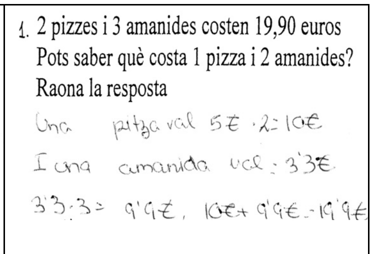
A *decontextualised debate* was then proposed based on a newly introduced question: “Consider the following equations: $4x + 3y = 12.50$; $8x + 4y = 22$. What is the value of x? These tasks were undertaken adopting the following sequence of three steps: a) individual time for reflection, b) sharing and comparing ideas in small groups, c) full-class discussion. During this last stage, the teacher adopted the role of facilitator and moderator of the debate encouraging dialogue, whenever possible, and creating uncertainty (Zaslavsky, 2005) but without becoming directly involved.

Part 5. Collective regulative reconstruction

A discussion was held on one intervention in the contextualized debate led by one of the students.

DATA ANALYSIS

The initial task promoted the construction of personal meaning (perhaps incorrect) regarding the relations observed. Interestingly, initial individual development points to partial arithmetic rules present in the prior knowledge of students. Thus, one student (Figure 2a) divided the full price, €19.90, by two (the number of pizzas) to obtain the price of one pizza and then divided half the total price (€9.95) by three (the number of salads). That is, she interpreted the information separately and appears only to demonstrate the arithmetic relation rnr1.

		1. 2 pizzes i 3 amanides costen 19,90 euros Pots saber què costa 1 pizza i 2 amanides? Raona la resposta
Figure 2(a) Trad. One pizza cost €9.95 and salad €3.31	*(2b)* ...each pizza costs €4.47 and each salad costs €3.31	*(2c)* One pizza costs €5 ... and one salad costs ...

Another student (Figure 2b) recognized the relation rnr1 and interpreted the relation rnr2 as being equal partners. She therefore split the price in half and allocated half the price to the pizzas and the other half to the salads. Then, she divided each half (€9.95) by the number of elements in each group. The third student (Figure 2c) attached a value to one of the objects (here each pizza was given a price of €5), and then gave evidence of having recognized the relation rnr3, and its equivalent $(c\text{-}by)/b = a$, by subtracting the price of the two pizzas from the total and dividing the remainder by three (to give a price of €3.30 for a salad). Then, in the final line of the solution the student offered a check that seems to demonstrate the wisdom of his reasoning.

In the second set of questions, we observed the role of a decontextualized situation, in which students reflected in a different way.

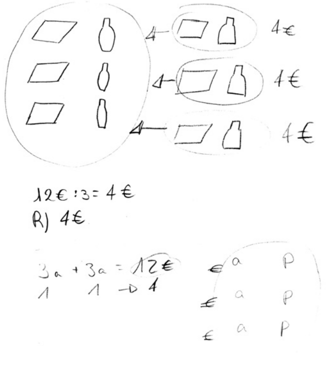


Figure 3(a).

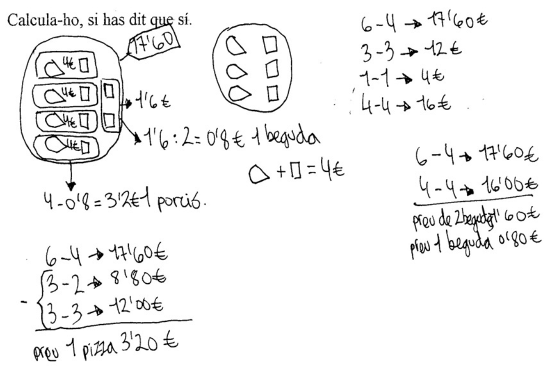


Figure 3(b).

Here one student (Figure 3a) started from the global, and decomposed the question into three sets. But the student did not finish here. He then wrote: *3a + 3a* = €12, indicating a marked improvement on the considerable symbolic representation of the situation. In the case of another student (Figure 3b) the progress made is clearer still. In responding to the final question, she drew a symbol representing a slice of pizza plus (note the use of the + sign at this point) a drink (represented by a symbol of a glass) is equal (note the = sign used without any difficulties) to 4€. During the debate, the students negotiated the meaning of the possible procedural algebraic phrases that can be deducted from a given sentence and the arguments that justified why one phrase does not follow from another. An initial consensus served to consolidate arithmetic elements that had been previously learned (e.g. the proposed prices for six slices of pizza and six beverages, nine slices and nine drinks, etc.).

Guided abstraction and third debate

At this point the teacher needed to clarify that what they were seeking was values of x and y. In the ensuing dialogic debate meanings were negotiated and students were given the opportunity to request help from a partner. The intrinsic motivation of the task is that of a game that promotes self-reliance and perseverance, because everyone wants to solve the problem themselves. Here is an extract from the debate to illustrate the process:

Teacher: They should really have understood. I want you to solve it mentally.

Jordi: But ... What can we do?

Teacher: What is the value of x, and what is the ....? The structure's the same as yesterday's ... I don't want to give you any more clues, please, everyone has to think it out for themselves. And now tell me what goes next and what is x equal to.

Anna [David talking loudly): It's worth 2.

Teacher: [David raised his hand]: Here's someone who thinks he knows the answer.

Several students: Don't tell us!

Some students: How easy!

Teacher: Now we'll get David to explain what he has done to find the value so quickly.

David [after a while): I subtracted the price of two from 4. I considered 2 and 3, therefore I found y ...

Lola: I don't understand anything at all

David: From the 2nd I subtract the 1st because I see one y less, right? When it is I subtract it remains only one y. Therefore, from 10 I subtract and the difference would be 2. With such a number of y, you multiply by the number of y. It would be 4 times 2 is 8. After that, I remove such drinks from 10, and these are the 2 x, and we divide it by 2 and this is a result for x.

As we see, David’s reasoning is still somewhat tied to the context in that he refers to ”drinks,” though in fairness to him he perhaps makes this reference to help his classmate, Lola, who asks for clarification.

The role of a rich situation

Below we examine two different proposals.

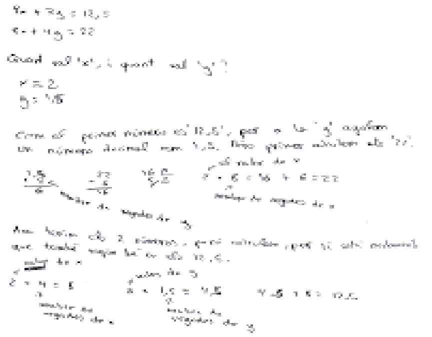
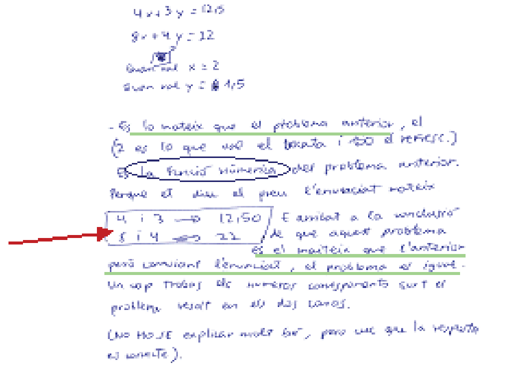
	

Figure 4(a) Laia: Now, we have both numbers. We calculate because perhaps we did a mistake. If the result 12.5 it can be OK”	*Figure 4 (b)* Dani: The problem is the same as the previous one, the 2 is what a sandwich is worth and 1.50 a soda. It is the numerical function of the previous problem. Because the question gives you the price. I conclude that this problem is the same as the one above, although the wording has changed, the problem is the same. Once you find the corresponding numbers you solve the problem in both cases (I haven’t been able to explain it very well, but I think the answer is correct)

Note that Laia’s argument (Figure 4a) shows considerable potential, but is not well described mathematically. In Dani’s case (Figure 4b), he shows little difficulty in comparing the structure of the problem in context using words with a system written in symbolic language, and it is clear they are the same. Likewise, he uses the following representation:

4 and 3 → 12.5
8 and 4 → 22

Below is an extract of dialogue that occurred in class when this task was set for the whole group:

Laia: I made a guess and checked it on the other side. The *x* is any number; in this case it is 2.
Sandra: Why did you put the same numbers in the two operations? *x* and *y* need not necessarily coincide, right? In the two operations ...
Laia: Yes, they have to match, if it was a single operation [meaning equation] there could be many numbers, but when there are two, then are the same numbers in both.

It might appear that Sandra had a conflict of a semiotic nature. But the fact that Sandra did not wish to return to the context suggests she wanted to understand it at an abstract level. She dares to ask Laia because she trusts she will give an explanation. However, Laia seems to be clear that both conditions must be fulfilled simultaneously. There seems to be evidence that the class is making progress towards the establishment of a community of practice.

Final reconstruction of meaning

The teacher presents Jordi's comments to the system of equations (Figure 7, left) and encourages comments from the whole class.

What is written			What the students perceive
4	3	12.5	They recognize that this is the first equation of the system
			They recognize that this is the second equation of the system
8	4	22	They see that the first equation is repeated
4	3	12.5	They realize that the second equation is halved to get an
4	2	11	equivalent in order to reduce the unknown
			They note that there has been a reduction, and one of the
	1	1.5	required values has been obtained
4	3·1.5	12.5	They see that the process of replacement has been made
4	4.5	12.5	The inverse relationship is recognized
4		12.5 – 4.5	
4		8	Finally, the other value is obtained
1		8 : 4 = 2	

Figure 7. Observations from joint discussion

During the debate, we observe the "different needs of the contextualization/ decontextualization processes" to understand what Jordi introduced. Here is an extract from the dialogue:

Jordi: The third is the same as the first and the fourth is half the second.
Teacher: Andrea, is it possible to divide a whole bag by two?
Andrea Yasmine: Yes, and the price too!
Sandra: In the next row, we know the price of a drink.
Jordi : In one bag there are 4 snacks and 3 drinks and in the other 4 snacks and 2 drinks. The difference in price has to be a drink.
Yousef: In the sixth row are 4 snacks and 3 drinks multiplied by their price.
Teacher: What do you think of Yousef's interpretation?
Raquel: It's 1.5 times 3, which is the cost of 3 drinks.
Jordi: In the next row, 4.5 is the price of the drinks, is equal to 12.5 which is the total price, and you have 8 ...
Teacher: This is the 8 some students don't understand ...Let's see where the 8 appears below...
Laia: It comes from subtracting, 12.5 minus 4.5.
Carles: There are 3 drinks, which gave us 4.5 which has been subtracted and has given us 8 and now we find the price of 4 sandwiches, which gives you what divides between 4 and ...

Andrea Yasmine was glad to see that the price can also be divided by the same number as the number of elements in the bag. This can be realized from context, such as buying a packet of pizzas from a supermarket. Sandra concluded that in the next step we could ascertain the price of one of the two unknowns by applying the method of reduction. Yousef was encouraged to interpret one of the lines, something quite unusual given that he normally did not participate at all in the maths class, and was successful in his commentary. Raquel then identified the inverse relationship.

Jordi's interventions show an interest in lowering the level to facilitate the contextual understanding of his peers. He no longer needed it, but they did, and so they continued to develop the community of practice: confidence among its members, shared effort, adjustment of the different paces of learning. No complaints were registered from the more skilled as they had a job to perform in the community of practice. The teacher refrained from establishing the truth in terms of the mathematical points under discussion but preferred to see if the peer dialogue could validate the argument correctly, and in so doing encouraged a rich participation (EMiCS, 2007).

DISCUSSION

Many studies highlight the problems that secondary school students encounter year in, year out, namely the failure to cope with school algebra. The fact that a majority of students in such classrooms present little or no competence in facing situations in which they have to call upon the concepts, skills and values associated with algebra shows the need for alternative methods.

Our staring point was, therefore, the following: allow the students to build their own representations of the situations that we present and observe just how powerful the context is in channeling their efforts correctly. This approach demanded a further premise: if the representations were to be personal, we needed to facilitate peer exchange – initially, so that they might compare representations and, eventually, so that we might encourage a process of consensus towards more homogeneous, shared representations. It was essential that these interactions, which we have termed "participation rich," take place within a proper environment so as to facilitate the creation of a community of practice in the mathematics classroom.

To put this plan into operation we needed proposals for rich activities, the management of which had to reflect our goals. Likewise, the context chosen had to be very close to the students. They needed to be aware of it so that what was learned outside the educational establishment could have its input and role in classroom learning. Students' representations must be considered key to the development of competence. We established earlier that a potential source of failure in the introduction of algebra to secondary school students stems from the imposition of symbolic representations, and at too early a point in time. The path taken here, however, allows students to make their own representations, and we have seen how students developed and started using – sooner than the authors imagined – useful symbols necessary for all.

Having reached this stage, the introduction of x and y is no longer an imposition, and occurs at an appropriate point in time. The possibility that each student has, if necessary, of going back to the context avoids the initial confusion surrounding the formal language and so students can focus their attention on understanding the equations and their possible manipulation in order to solve new situations.

The evolution in the representations depends largely on the negotiation of group meanings. Little benefit would be derived from such an experience if students were not able to compare and share their performance with those of the others in the group. The community of practice is the context in which this negotiation is played out. Mutual confidence, group identity, a shared goal, the sense of achievement are all factors that facilitate dialogue in peer-rich participation. When taking this alternative path, the first steps in teaching algebra should pay particular attention to the handling of errors, ensuring that they become a learning experience, while the validation of mathematical truths should occur, as far as possible, without the direct and explicit intervention of teachers.

Thus, we root students firmly in a context in which initially they are encouraged to develop their community practices and, subsequently, to take their first steps in algebraic thinking, but after this introduction we turn to a purely numerical approach which appears early on following the reaching of a consensus as regards the use of symbols x and y. Students move constantly between the two contexts and rely on either to solve the situations they face. By this point, some students no longer need the specific context provided by the starting point and work in the abstract. But at no time during this initial sequence are students forced to abandon the standard context and to depend solely on the formal.

The introduction to algebra continues with the presentation of new contexts in which the students continue to develop their skills and certain mathematical skills. The rates of mobile telephones (used to introduce the tool and spreadsheet functions), a police chase, castaways lost in the ocean (used to present reference systems and coordinates), the discovery and study of an Iberian village that is more than 2,000 years old are just some of the contexts that are presented to the students so that they we continue our task of rebuilding algebraic thinking.

FINAL REMARKS: LEARNING FROM RESEARCH

Contextualization is not only a need, but it can also form part of the mathematization-demathematization process (Gellert & Jablonka, 2007). In our experiment, we observed how dialogic debate provides opportunities for working collaboratively in such a process. It was possible to achieve formal solutions, included in the usual "foreground of the students" (Skovmose, 1994) expecting to solve equations. Such a path was drawn from the practices of teacher-researchers in constant interaction with the teacher mentor.

This approach, initiated several years ago, meant the enrichment of teacher-researchers' work in the classroom. Observing what actually happens in the classroom, above and beyond the intuitions built up over years of work, is an important element. We are unable to state what is not actually tested, as testing is so complex. Only by reflecting on our own teaching practice can we hope to find clues to the way forward in improving the teaching of mathematics combined with the reflections of faculty experts who can undertake parallel and rigorous analyses. The placing of greater value on the observation of the mathematics classroom leads to another important point: assessment as a process from day one through to the end of the course and not simply as a final certification. The problems faced by the faculty in order to assess where we want to get, beyond a simple final review of the issue that has become irrelevant to the learning of the student body, can be solved with continuous observation. So almost naturally appears the need for observation, which in turn demands a process leading to the identification of important points to be born in mind, including what is observable and what not, in the giving of mathematics class.

A second aspect of importance in this study is the value that it attaches to dialogue. Although in the case of these authors, dialogue is a frequent resource in the mathematics class, we have experienced a change in terms of how it is assessed and in terms of the goals set for it in class. On the one hand, the value we attach to dialogue in mathematics class has increased because it permits the validation of a large part of mathematical knowledge without our direct intervention. On the other hand, the goal of such dialogue is not simply the negotiation of meanings, but also the guarantee of the development of a true community of practice, so that ultimately learning is more than just the result of the work completed by each student individually.

An additional aspect to bear in mind is the development that occurs in a teacher-researcher as a trainer of trainers. Collaborations of this nature with the world of research lead to a reorganization of what we already know, based on theory and in practice, and exponential increases in knowledge enhance a researcher's ability to train other teachers. Teacher training is a task that depends largely on self-confidence and knowledge, and in both respects the study presented here has succeeded many times over.

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