Handbook of Mathematics Teaching Improvement:

Professional Practices that Address PISA

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INTRODUCTION

This volume presents the results of the three-year long (2005-2008), international Professional Development of Teacher-Researchers (PDTR) Project*,* bearing the name of Anna Sophia Krygowska.[1] The project was coordinated by the University of Rzeszów (Poland), and supported by the grant from the Socrates Program of the European Community (no. 226685-CP-1-2005-1-PL-COMENIUS-C21). There were six teams of mathematics teachers, apprentices in the craft of teaching-research, participating in the project: one team from Debrecen (Hungary); one team from Italy including teachers from Modena and Naples; two Polish teams: one from Kraków and Rzeszów; and one from Siedlce; one team from Barcelona (Spain); and one from Lisbon, Portugal. They were supported in their work by teaching-research mentors, comprising experienced teacher-researchers and academic researchers.

> The central aim of the project has been to engage classroom teachers of mathematics in the process of systematic, research-based transformation of their classroom practice while producing evidence-based innovative instruction and contributing to research knowledge of the profession; it is to initiate, using teaching-research as the leading methodological agent, the transformation of mathematics education towards a system, which, while respecting the standards and contents of the national curricula, would be more engaging and responsive to student's intellectual needs, promoting independence and creativity of thought, and realizing fully the intellectual capital and potential of every student and teacher.[2]

An important incentive to create the project was the poor outcome of the PISA international test. "All five countries represented in the project scored below average in the recent PISA 2003 international test of mathematical and problem solving competencies. There is urgency in the need to successfully address mathematics-learning issues of expanded Europe."[3] For mathematics educators it was evident that there is a need for a deep change of the very concept of learning mathematics in the classroom with teacher‘s guidance: transmission of knowledge has to be replaced by facilitating learning. For this teachers need to know and understand the students' ways of reasoning and errors made. The teaching-research methodology[4] seems an ideal solution.

The teachers' work in the first year addressed issues and problems of the mathematical component of the PISA international test. Results of that aspect of the work are presented in Part 2.

In the second phase the PDTR apprentices designed, with the help of their mentors, classroom teaching experiments, collected data, observed their classrooms with a new eye of an investigator, analyzed and discussed the data with their team members. The experiments focused on the observed phenomena and problems in their classrooms of mathematics; some issues for investigation were suggested by the academic researchers, mentors of the teams. Reports of that work are gathered in Part 4.

The book opens with Part 1 devoted to two general issues of innovation of mathematics pedagogy: communication in the classroom and assessment of students' performance. It has been widely recognized that learning in the classroom takes place

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[1] Anna Sophia Krygowska (1904-1988), professor at the Pedagogical University of Krakow (Poland), was an internationally recognized founder of the modern Didactics of Mathematics.

[2] Comenius 2.1 Project Desciption, Section 4.

[3] Ibidem.

[4] See *Handbook of Teaching-Research*.

mainly during and through an interaction among students and between students and the teacher. So the format of this interaction, or communication, is of utmost importance. On the other hand, written tests, applied now much more extensively than half a century ago, often as the unique tool of assessment of students' performance, influence the content and teaching proceedings. Without continuous improvement of those aspects of the education system genuine changes in the teaching/learning of mathematics will not be possible. Directions in which changes could go are proposed and illustrated.

Finally, Part 3 is devoted to what is in a sense a generalization of the direction given by the PISA test: elementary applications of elementary mathematics. If the principal objective of teaching the subject is to make students able to apply their mathematical knowledge and skills to everyday problems, they must acquire the ability of modeling the real problem in the abstract mathematical world, and demodeling the found solution of the abstract problem, or interpreting it back in the reality of its origin. This activity is elaborated and exemplified here.

I am due to express an admiration to all teachers participating in the three-year activity, in particular to the authors of this volume, for their involvement, often demanding a sacrifice of weekends with family or late night work. It is in them and their followers that there is hope for essential improvement of the teaching of mathematics.

Stefan Turnau

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PART 1

COMMUNICATION AND ASSESSMENT

COMMUNICATION, CONNECTION AND REFLECTION HAVE TO BE LEARNED BY STUDENTS AND THEIR MATHEMATICS TEACHERS

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ABSTRACT
This contribution deals with the consequences of PISA results. The PISA results mention lack of students' communication skills. Authentic tasks are considered as an important learning environment to solve problems by using communication skills. The PISA category connection *stresses the necessity of manipulations with representations of mathematical concepts to construct, to understand and to use/communicate different representations of mathematical concepts. Besides the PISA category* reflection *indicates the necessity to learn how to look for and find information about different ways of solving a problem and to communicate that to others. This contribution reports Dutch teachers' training research results with the focus on teachers' support to stimulate students (aged 12-15 as well as at upper secondary level) learning processes considering the combination of communication and reflection activities.*

KEY WORDS:
Communication, connection, reflection, teacher intervention, abstraction

THEORETICAL FRAMEWORK

Problem definition and research questions

The problem is how to learn the connection between communication and reflection skills to solve problems and how to understand mathematical concepts.

Research findings about communication and reflection skills accentuate metacognitive instruction in solving mathematical authentic assignments. A characteristic feature when solving metacognitive assignments is that the approach is not immediately obvious (Darling-Hammond, 1992). In authentic assignments, students are confronted with excess information so that they have to make choices in order to get started (Cobb, 1994). The solution cannot be found through application of an algorithm or a standard solution (Prawat, 1998). It is difficult to establish links with problems already solved. Students find authentic assignments difficult (Kramarski, Mevarech & Libermann, 2001; Verschaffel, Greer & de Corte, 2000). According to Verschaffel et al., weaker students in particular have difficulty with abstraction into sub-problems, as they are unable to separate the information between relevant and irrelevant. They therefore soon give up, also due to algorithms which do not offer a solution (Anderson, 1990). These problems go beyond the skills of solving mathematical problems. They concern the skills of solving problems together with others, for example Cardelle-Elawar (1995).

Based on these findings, teaching methodology research aims at developing instruction methods to support teachers in training (TTs) in the activation of students' metacognitive learning processes (Lester, Garofalo & Kroll, 1989; Mayer, 1987; Schoenfeld, 1987). Basic elements in the development of instruction methods for students are metacognitive questions put to small groups of students, such as: (1) conceptual questions: What is it about? What is the question? What is the meaning of a mathematical concept? (2) relational questions: Does the question resemble…? Does the question differ from a problem already solved? Why? (3) strategic questions: What is the solution strategy? Why this strategy? How does this strategy work? and (4) reflective questions: What have I done? Was it purposeful?

These metacognitive questions which students ask each other and answer jointly can be traced back to Polya's theories (1957). According to Polya, teachers gain insight into the way in which problems can be solved but also into how students can be supported, by analyzing various solution methods, communicating them to others, and reflecting on their effect. Metacognitive instruction is strongly related to cognitive units, connection and compression (Barnard & Tall, 1997). According to Pinto and Tall (2002), students' cognitive constructions occur through reflective abstraction, in which a predicate with one or more variables is conceived as a mental process. Recent research findings indicate teachers' role to stimulate reflective abstraction (Simon et al., 2004). The research study of Ainley and Lowe (1999) suggests different levels of understanding related to teacher interventions:

Cognitive construction

The ability to conceive and to manipulate cognitive units is a vital facility for mathematical thinking. Two complementary factors are important in building a powerful thinking structure: (i) the ability to *make connections* between cognitive units so that relevant information can be pulled in and out of the focus at will; and (ii) the ability to *compress* information to fit into cognitive units by communication. Compressibility of mathematical ideas relies on the nature of the connections from the focus of attention to other parts of the cognitive structure (Barnard & Tall, 2001).

Reflective abstraction

Piaget (1972) emphasized the construction of meaning through different forms of abstraction. One of them, *reflective abstraction,* is a process focusing on mental actions and mental concepts in which the mental operations themselves become new objects of thought (Pinto & Tall, 2002). Later on Piaget (2001) described reflective abstraction as a process by which higher level mental structures could be developed from lower level structures, consisting of two phases: (1) a projection phase in which the actions at one level become the objects of reflection at the next level; and (2) a reflection phase in which a reorganization takes place. Simon et al. (2004) elaborate on Piaget's reflective abstraction: (i) *activity* refers to a mental activity; (ii) *activity sequence* refers to a set of activism in an attempt to meet a goal; (iii) *learners' goals* are not necessarily conscious; and (iv) *effects* are structured by assimilatory conceptions that learners bring to the situation. Bereiter (1985) emphasizes that cognitive advance cannot be directly brought about; rather, teachers promote specific experiences for the development of the intended cognitive structure, a step-by-step outline of how to foster students' reinvention of a particular process of the learner. Underlying is the idea that learners impose mathematical relationships on the situation based on their available conceptions. Bereiter's remarks indicate teachers' role to stimulate reflective abstraction.

Teacher interventions

Ainley and Lowe (1999) defined four degrees of understanding: (1) no apparent understanding, students cannot make a start, no understanding could be identified; (2) procedural (instrumental) understanding, students know how to carry out a mathematical procedure but lack the deeper understanding to recognize when an algorithm had been misapplied or incorrectly remembered; (3) conceptual (relational) understanding, students recognize the constraints of answers, are able to comment constructively on their work; (4) proceptual understanding, students appreciate that the symbol ambiguously represents both the concept and the procedure. Teachers' help-interventions shall be categorized within these levels of understanding. No understanding (1) corresponds to teachers' intervention: *which* concept to use (to find a unit)? The procedural (instrumental) understanding (2) corresponds to teachers' intervention: *how* to use this concept (to find a connection, an algorithm)? The conceptual (relational) understanding (3) corresponds to teachers' intervention: *why* to use this concept (to find related units)? The proceptual understanding (4) corresponds to teachers' intervention: *which choice* of concepts is most effective to use (to find units and connections)?

The hypothesis is that (i) metacognitive instruction has positive effects on students' learning results, rather than merely on the mathematical assignments; and (ii) these teachers' interventions at the actual level of understanding aimed at the next level of understanding are effective (Mevarech & Kramarski, 1997).

RESEARCH METHOD

On the one hand, the research method to support students solving authentic assignments by asking metacognitive questions can be typified as action research, the teacher-as-researcher. This type of research assumes practical learning to be the basis for theory formation (Wang, Haertel & Walberg, 1993). The criticism of this approach confirmed the differences between teaching methodology research and educational research (Kerdeman & Philips, 1993; Kliebard, 1993). In 2003, Wang countered the criticism with convincing, evidence-based practical examples. On the other hand, a case-study in which a TT taught a group of 30 students aged 16-18 in a year before their school leaving exam. Almost at the end of that year the item taught was the concept *integral* with – at the end of the textbook chapter – a specific problem concerned with an arch of a graph. The TT questioned her students in her capacity as assistant teacher and was focused on the highest level of understanding of the concept integral. She taped her dialogues.

Material

There are two types of material: (1) an authentic task with the focus on students' communication, and (2) a dialogue with the focus on the connection between communication and reflection. The first type of material comprised the theory of metacognitive instruction. The material used by the TTs in their own teaching practice comprised an authentic assignment of "Plusses and minuses" for students (aged 15 years). The instruction was: "Write down 15 plusses and minuses in a row, but make sure that there are no more than two of the same symbols after each other. ++--++ is allowed, but +---++ is not allowed for example. How many different rows can you make?" The assignment was not part of the normal teaching material but was in keeping with the subject taught. The second type of material consisted of the following task: the TT presented her students with a graph and the question involved. The question concerned the arch of the function $y = x^2$ at the closed interval [0,1]. Students were expected to use the concept integral to calculate the arch of the graph between 0 and 1. They did not do that before, and it was a new phenomenon for them. They had to take a lot of difficult steps, and some mathematical creativity was required.

Participants

Students are used to work independently in groups (3 or 4 students each).
Class A comprised 25 fourth year pre-university science and math stream students.
Class B comprised 17 fourth year pre-university science and math stream students.
Class C comprised 28 fourth year pre-higher-education students, in the Culture and Society profile, who had difficulty with mathematics.

A TT questioned five students in the pre-latest school year. Arbitrarily, two of them were chosen to be focused on and examined.

DATA COLLECTION AND RESEARCH INSTRUMENTS

The research instruments comprised logbook reports by A, B and C prior to, during and after the research activities.

The data collection of A comprised a logbook report on*:*

(i) explanation on the board, writing out the possibilities for n=1 (2 possibilities), n=2 (4 possibilities), n=3 (6 possibilities), n=4 (10 possibilities), n=5 (16 possibilities);
(ii) groups of students puzzling out to find rows with 15 plusses and minuses in a row;
(iii) a number of students discovering regularity in the row of figures 2, 4, 6, 10, 16, namely that the first two figures give the third figure and that the third and fourth figures give the fifth figure. It then took very little effort for them to find the number of possibilities for a row comprising 15 plusses and minuses. Two students arrived at the answer 1974; they verified this solution and their method and therefore solved the problem. There was no further puzzling;
(iv) analysis of the problem on the board as soon as the students solved the problem.

Rows of figures were used to show that the formula discovered was indeed the correct one. The students were familiar with this subject, as it was the subject of the last chapter. Two different formulas were presented, namely one for the rows in which the final two symbols are the same, O_n, and one for the rows in which the final two symbols are different P_n:

$$P_n = P_{n-1} + O_{n-1}$$
$$O_n = P_{n-1}$$

The total formula was easy to deduct from these formulas, namely that the row of figures is given by the formula $U_n = U_{n-1} + U_{n-2}$.

The data collection of B comprised a logbook report on*:*

(i) the explanation on the board on the basis of a photocopy of appendix 1. The problem was soon understood. Students set to work in groups of four. They were promised a hint after 10 minutes, if necessary. They were also tempted with the promise of a Snickers bar if they found the right answer. After five minutes of puzzling, all the groups began to call out solutions. When asked, they admitted that all the solutions were more or less a

guess and that a formula had been worked out which was associated with the problem, such as for example 2^{15}. None of the groups actually thought the problem through. The solutions called out were more or less guesswork. The class became quiet after about 10 minutes. A number of possibilities were written on the board (by A): a row with a length of 1 (n=1): 2 possibilities, and then n=2 (4 possibilities) and n=3 (6 possibilities). Two groups understood the approach and discovered the number of possibilities for n=4 (10 possibilities). The regularity in the rows 2, 4, 6 and 10 was then recognized and a group produced the right answer after two minutes, i.e. 1974. Everyone stopped puzzling and analyzing;
(ii) the proof of correctness of the result by means of the recursive rows approach. The students were not really interested. They were then given a photocopy with the presented solution and also a solution according to combinatorics.

The data collection of C comprised a logbook report on:
(i) the group size, sticking to the book because they were due for a test the following week. C adapted the task slightly. The total task remained the same but a stencil was provided with some sub-questions: What is the subject of the task? Explain the task in your own words; What other tasks resemble this one? How did you solve the tasks in the previous question? etc. In the end, half of them completed the tasks.
(ii) the overly difficult task, despite the sub-questions. They found it too difficult to formulate the question. It was very difficult for them to recognize the difference between a task from the book (in general that means that there was too large cognitive distance between student knowledge and the level of the question). ZPD approach suggests diminishing that distance in the process of questioning and this task, and the students were very focused on finding a ready-made solution (which they could not because the task was a bit too difficult for them => disappointment). They tried to apply the theories they had recently learned to this task but that too proved to be very difficult.
(iii) the students who completed the task were those more willing and sometimes also somewhat better at mathematics than the rest. In groups with consulted each other.
(iv) the sub-tasks referred to the approach taken by the students, they were not accustomed to this in their textbook and therefore found it "strange."

DATA PROCESSING AND ANALYSIS

Data processing and analysis of A:

A had probably already given too much information for the fourth year pre-university science and math students by writing out the possibilities of n=1 to n=5 on the board. They soon found a general formula without having to work out the problem for too long. Moreover A's approach led to a small number of students quickly finding the answer, after which the remaining students soon adopted that solution. It might have been more useful to have students search for the answer in small groups rather than individually. The students now started working alone, in pairs and occasionally in groups of four, but once they had found the solution, it was called out loud in the class. Only a small number of students actually had the opportunity to understand the solution. Shared knowledge became general knowledge, only for a small number of students.

Data processing and analysis of B:

These students showed little willingness to make an effort and think analytically about a task which was not part of the curriculum. Communication was spontaneous in the groups. The approach taken by one group had too much influence on the other groups. It might possibly have been more useful if systematic problem solving was part of the educational program, including instructions on how students can tackle a problem. The expectation now was that students would find it challenging and fun to work at solving such problems. That was an illusion in the case of this group. The students were accustomed to undertaking tasks as part of the curriculum. By far the largest group considered this adequate and had little need for anything extra.

Data processing and analysis of C:

C expected that this task would be too difficult for the students in this group (and that they would not be interested in mathematics and solving puzzles to the same extent as the 4th year pre-university groups). C adapted the task slightly. As expected, many of the students were not interested in the task. C had indicated that she needed the results for a research project at the university, and the students therefore filled it in for her. In the end, half had actually completed the required tasks. The students found it "strange" to answer sub-questions and especially to have to think about what they were doing, because their standard textbook did not do so. They looked for quick and easy answers, and once that proved difficult they soon gave up. They also started calculating the first sub-questions (part c) instead of the question which really needed the solution.... (part f). That was not yet achievable: to first describe the approach and only then start calculating. A number of students worked on the task in a group. That gave very interesting results, they prompted each other and consulted on a possible solution, nice to see, nice to hear, and useful for the students themselves as well... Such groups had questions to ask and actually wanted to understand the task. The students who did not work with others but rather completed the task individually had fewer questions and were done sooner.

Experimental task see attachment (4)

RESULTS

The situation of A:

Too much pre-chewed information. Once the original task was structured for the students beforehand and any uncertainties removed, it was no longer authentic. The solution was soon passed around. The students did not start out in groups, so there was no optimum communication between the students themselves.

The situation of B:

Too much information was 'given away' beforehand. The original task was no longer authentic, the general solution too obvious. The division into groups had a negative effect on the students' learning processes. Communication was spontaneous within the groups. Students were not voluntarily willing to look deeply into the context.

The situation of C:

This task was too difficult for this group. C hoped to make it simpler through the use of sub-tasks but this might have made it even more difficult. The approach of first solving sub-questions and only then writing down the solution was tricky. The students were not yet too systematic in their approach and the text book (in use *Modern Mathematics*) did not offer any support for that either. As a teacher, you obviously try to stimulate your students to work that way, but the students generally did not appreciate the approach, and instead began to calculate and write down solutions right away. Once they started working in a particular direction, the students could not be diverted, unless they (despite being split into groups) cooperated with others (or happened to find it interesting). They were also not accustomed to reflection, or verification of their solutions (Why should I? I already have an answer, don't I? It was just difficult; etc., see also appendix part g).

Comparison and analysis

The course of TT's questions to K is as follows, the number indicates a level: 2 - no connection – 4 – 3 – 4 – 3 – connection – 3 – 2 – 3 – no connection – 4 – 3 – 3 – 4 – connection – reflection at a sequence of activities – reflection at effects and activities – stop. Because of TT's help at the current proceptual level, K ends with joyful feelings. She attained the highest proceptual level of abstraction through TT's help. K commands a powerful cognitive structure with strong connection and compression.

Conversely the course of TT's questions at L is as follows: 2 – 1 – 4 – 3 – 3 – no connection – 2 – 1 – no connection – 1 – 2 – 1 – 2 – 2 – no connection – 3 – no connection – 2 – 1 – no connection – 1 – no connection – 2 – connection – 2 – 2 – 2 – no connection – 1 – connection – 1 – no connection – 1 – no connection – no connection – stop. L ends without joyful feelings. In this case teachers' help at the procedural level is sometimes not appropriate ('too high a level'). L attains a low level with weak connections without compression. The only question at level 3 is answered by no connection. L commands a weak cognitive structure without connection.

CONCLUSIONS

On the one hand, students have problems with (boring) authentic tasks; the lower the level, the greater the difficulty they have. The goal of the task needs to be meaningful for them. On the other hand, if group work is an accepted working method for students for this type of questions, the effect of mutual communication is more positive among lower level students than among higher level students. Communication is spontaneous and is effective as long as it is structured. The TT learnt to focus on metacognitive instruction as an adequate principle to activate effective communication between mathematics students. They mentioned a focus on general instruction and avoidance of personal help. The TT wanted to be students' amicable coach, to stimulate students' mutual communication.

DISCUSSION

As indicated by critics, this type of research is methodically weak, crucial conclusions are insufficiently precise, it produces contrary results, is reported in incomprehensible jargon, does not lead to improved teaching results and must all take place much more thoroughly (Slavin, 2000). While good practices can certainly be identified and qualitatively described, the generalization questions (will it work for my subject / with these students / in our context / with a different teacher?), the reproducibility question (will it happen again tomorrow?) and the explanation question (what is the underlying causal relationship between "treatment" and results?) are seldom adequately answered (Van Keulen, 2006). Educational research in the form of action research is difficult but is certainly recommended for professionalization of teachers. The conclusions of the case study are challengeable because of the minimum of data.

Two arbitrarily cases have been analyzed. Therefore, the conclusions can not be generalized. Besides, a TT fulfilled the role of the teacher. She was strongly influenced by Tall's theory of cognitive structures. She intended to attain mathematical concept development by questions at the highest level of abstraction. Her approach resulted in positive effects on high level students and negative results on low level students. She assumed the existence of some basic proceptual views. It is recommended that this approach to teachers' intervention be repeated by questions in classroom practice at the actual individual level of understanding in collaboration with other students at lower or higher levels (Dekker & Elshout-Mohr, 2004). The results of this case study support the intention of such a type of research activities to design teachers' intervention aimed at mathematical concept development. It also shows that despite the good results of Dutch students in the PISA contests there is room for further development, by supporting teachers with focused classroom research – focused on essential elements of mathematics like *communication* and *reflection.*

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Appendix 1: Plusses and minuses

"Put fifteen plusses and minuses in a row, but make sure that there are no more than two signs of the same kind next to each other. For instance ++--++ or +-++-- is allowed, but ++---++ is not. How many different sequences can one make?"
Divide into groups of four to five students and ask yourself the questions. Suggestion:
First analyze the problem as if it where a row of one, two, three or four plusses and minuses.

APPENDIX 2: TWO DIFFERENT SOLUTIONS TO THE PROBLEM

A mathematical solution of this task is that the next item of the sequence can be found by adding the previous two items. The first item is equal to two, the second is equal to four. In mathematical language this leads to: $U_n = U_{n-1} + U_{n-2}$ with $U_0 = 2$ and $U_1 = 4$. Using this, one easily deduces that the solution to our problem should be 1974. In order to come to this solution, it is useful to split the sequence in two separate sequences: one sequence with the last two signs similar and one sequence with the last two signs unequal. By taking this extra step and simplifying the required abstract mathematization, students easier understand the way in which the general formula is derived. There is also another way to come to the solution. A row of plusses and minuses can be seen as a row with two types of elements. A single element (+ or -) and a double element (++ or --). Because plusses and minuses have to alter, we only have to look at the number of possibilities we can order single and double elements. Whether an element is + or -, depends on its position. The number of single and double element is not fixed. Therefore, the following sequences are possible: 15 elements containing 15 single-elements and 0 double-elements, 14 elements containing 13 single-elements and 1 double-element, 13 elements containing 11 single-elements and 2 double-elements, … 8 elements containing 1 single-element and 7 double elements.

The sum of all the possible sequences is equal to

$\binom{15}{0}+\binom{14}{1}+\binom{13}{2}+\binom{12}{3}+\binom{11}{4}+\binom{10}{5}+\binom{9}{6}+\binom{8}{7}=987$ The total number of different sequences is equal to: *2 * 987 = 1974.*

Attachment 3: Answers to the added subdivisions

a: What is the subject of the problem?

Answers:

Putting plusses and minuses in a row, trees, probabilities; calculating probabilities; diagrams

b: Describe the problem in your own words.

Answers:

How many sequences are there? One has to make sequences and see where one needs to put the plusses and minuses; I have to put plusses and minuses and a row, without putting three of the same sign next to each other; How many different sequences can we make if we put fifteen plusses and minuses in a row and no more than two signs of the same kind next to each other; We have 15 places to put a plus or minus sign.

c: Do you know other problems that look like this problem? How did you solve these?

Answers:

Draw a grate and count; calculate probabilities with the help of trees or using the function ‘NCR’ on the calculator; a problem with license numbers, but that was different because these were in some order; A problem with red and white balls in a vase; 15 ncr 2

d: How do these problems differ from the problem with plusses and minuses?

Answers:

It is difficult to see; I don’t see any differences; In this problem, no more then two signs are allowed to be next to each other; We deal with plusses and minuses now; The way I

solved these problems does not work for this problem; I have to do the subdivisions first; I can't solve this problem immediately.
e: How are you going to answer this question and why?
Answers:
I will use a tree, but I think that it will be a lot of work; I thought I had to use the function 'NCR' on the calculator, but it doesn't work. A grate also doesn't work. Do we have to write down all the possibilities?; Write everything out, because I can't use 'NCR'. I do not like that!; I'm going to use a grate and count the number of possibilities; I will use a tree. It will cost a lot of time, but the answer is right; I'm going to use 'NCR'
f: Calculate the solution of the problem.
Answers:
15 ncr 2 = 105 (I have seen some unfinished trees, but I haven't seen any other solution)
g: Look at your answers from part E and G. Do you think you are on the right track? Do you still miss some things and do you think you will have to do it some other way?"
Answers:
I don't think I did it the right way, but I don't know what else to do; I think you have to write down all the probabilities; I think I'm on the right track, but it is too difficult for me to solve; You can't calculate this with a formula, because there isn't one. And a tree is too large to calculate by hand; Yes, it is very logical; It is easy to understand, but it does cost a lot of time to calculate everything; I think that there are better ways, but I don't know them; It is possible but hard.

Attachment 4: Experimental task

Climmy presented her students with a graph and the question involved. The question concerned the arch of the function y = x^2 at the closed interval [0,1]. Students were expected to use the concept integral to calculate the arch of the graph between 0 and 1. They did not do that before, it was a new phenomenon for them. They had to take a lot of difficult steps, some mathematical creativity was required.

Climmy's intended solving approach was as follows:
Start with a small piece of the graph. Taking *dx* very small, this piece will be approached by a straight line. So the small piece of the graph is described as a function of *dx* and *dy*, and by the Pythagorean Theorem $dL = \sqrt{dx^2 + dy^2}$.
By using the derivative function *dy* is described as $dy = f'(x)dx$. By substituting this in the equation of *dL* and taking similar factors out of the square root:

$$dL = \sqrt{dx^2 + f'(x)^2 dx^2} = \sqrt{1 + f'(x)^2}\, dx .$$

Result from the total arch at [a,b]: $L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$ ·

In this case the total arch at [0,1]: $L = \int_0^1 \sqrt{1 + 4x^2}\, dx$ ·

Students did not calculate further, it was enough to formulate the integral. In the dialogues with Karin and Lotte separately, Climmy first asked them to formulate the integral for $y = x^2$. Afterwards, Climmy asked both girls to give a common formula.

Below are two of Climmy's literal dialogues including some of her impressions during these processes.

The dialogue between Climmy (C) and Karin (K)

First, I ask Karen's estimation and she immediately asks, "will it be more than $\sqrt{2}$?" Subsequently I say to her, "we shall calculate this precisely." Promptly she guesses it will be an integral. But she doesn't know the first step. She doesn't hit upon any idea.

C: What is an integral?

K: You are able to calculate an area with that.

C: Only areas?

K: Ehm. Oh no, also volumes!

C: Again, what is an integral?

K: [silence] No idea.

C: Could you estimate an area without using an integral?

K: Oh yes! Surely that's possible with small pieces!

I sketch a number of small rectangles in the graph of $y = x^2$. Clearly, Karin shows her recognition. I ask her about some connection with an integral. She thinks for a while.

K: So an integral is the addition of the area, of all the pieces.

C: Or volume?

K: Oh yes!

C: So how we could use this remark?

K: By taking small pieces of that line?

C: Carry it out.

Karin carefully sketches a piece of a graph and she completes it with a triangle! What a step! I ask her to name that horizontal small piece. Without any doubt she names this horizontal small piece *dx*, and the vertical small piece *dy*. She spontaneously notices that the small piece of the graph is approximately straight. She suggests that it is possible to apply the Pythagorean Theorem. Nevertheless, the next step is too large to be taken alone. I draw her attention again to the small triangle with the notations *dx* and *dy* and ask her whether she recognizes anything. She almost immediately says:

K: The slope!

C: But what is the slope?

K: *dy* divided by *dx*.

C: How you can use that?

K: [silence] Ehm… No, I don't see it.

C: What did you do?

K: We wanted to construct an integral of all small pieces of the line. All pieces have a length of the square root of dx^2 plus dy^2. But, should you really place *dx* after that? ….. Oh no, of course not. Eehm… [silence] Oh! The *dy* is the slope times *dx*! So *dy* can be taken out of the square root!

C: Fantastic! But again what is the slope?

K: That is, ehm….. yes! $2x$!

When the expressions are substituted in the square root, the common factor can be placed after the square root. That does not happen without help. But when *dx* is separated from the square root, she remarks, "I am ready." I ask her to give a common rule for the calculation of the arch of a graph. She returns a long way back into the problem solving approach. She doesn't identify the term $4x^2$ equals $f'(x)^2$ here.

K: That term *dy* doesn't change, and that term *dx* of course also not. Only the slope depends on the graph. So the term $2x$ will be $f'(x)$.

Subsequently she follows again all the steps of the problem solving approach, always $f'(x)$ instead of $2x$. That's why she also solves the problem finally. When she has discovered the common formula, she recovers and says:

K: Ah! That is nice!

The dialogue between Climmy (C) and Lotte (L)

I also ask Lotte's ideas to begin the problem solving process. Lotte immediately starts with an integral. But she intends to calculate an area.

L: Yes, I really don't know anything about this, but perhaps… To formulate an integral or something like that, to calculate the underlying area and than the area of this too [she indicates the whole "square" between x-axis, y-axis and the point (1,1)] and yes, than you have.. eh...that doesn't solve any problem.

C: No.

L: Also I remember a paragraph, and than you have to turn around that graph … Perhaps.. no, this doesn't solve my problem too.

C: What is an integral?

L: For example the area between the graph and the x-axis can be calculated.

C: An integral is always an area?

L: Yes, I think so.

C: And volume?

L: Oh, yes.

We consider the area underlying the graph again. I ask her to estimate this area. Directly Lotte proposes to construct a triangle. I ask her another method to estimate the area more precisely.

L: Certainly with all small triangles?

Lotte starts to sketch. Directly she remarks that the triangles are not successful. Now she constructs rectangles, with a small triangle upon each small rectangle. Nevertheless, it is not successful to describe the area of a small rectangle. Finally, I explain how to use the integral to calculate areas and volumes. I demonstrate how to add small areas and small volumes. I ask her to take a step and to calculate a length.

L: The highest point here has x-value '1' and y-value '1'. And then ehm…. yes, if you do the same with each small piece of line, then you take all y-values together, you add them up, then do you get the length?

C: Let us try something. For example what is the y-value of this point? [I mark an arbitrary point of the graph]

L: Standard x comma y?

C: Ok [I write that near the point….] and now?

L: Well, I take another point, and than that x, all the time that x has another value, ehm…

C: Shall we name them x_1 and x_2?

L: Oh yes, and than these are y_1 and y_2. Well, and if you do that many times, with many more points, you add up all the y-values.

C: And what do you calculate than?

L: Eh….yes, not the right one. [laughs] I don't know!

C: Do you choose the right way?

But Lotte doesn't see any solution. Once more, I indicate that we used the small areas to calculate a total area, and small pieces of volume for constructing the total volume.	
C:	How can you apply that?
L:	Yes, search for small pieces of length. But in that case I have to know that length!
She doesn't recognize a connection. I sketch a small triangle near the two points of the graph. She realizes that there is a relation between $y_2 - y_1$ and the vertical small piece of line.	
L:	You have to add something to get the length. Ehm… but yet this is still a triangle? So this can be calculated [long silence]. With Pythagoras?
C:	Very well.
Lotte maintains the notation $x_2 - x_1$ and can't take the step to *dx*, also not after many references to the calculation of the area. Ultimately I sketch the triangle again, but now with *dx* and *dy*. She understands the convenience of this notation. I ask her any recognition.	
L:	First, I think of Pythagoras, because of the triangle.
C:	Try again to construct something in a total different way.
L:	Ehm… because of *dx* and *dy* immediately I think about integrals.
C:	Oh yes. Much more complete, other ideas?
L:	No. [long silence]
C:	The number of the slope?
L:	Ehm, no, I don't see anything.
C:	How do you calculate the slope?
L:	Yes, something like *dx (dy)* divided by *dy (dx)*.

She doesn't make any progress. Finally I ask her to substitute *dy* into *dx* and the slope. She doesn't succeed. When I prompt, she feels silly. Subsequently, we write the length of the small piece of the graph as a function of *dx* and the slope using the Pythagorean Theorem. Finally, I ask her about the meaning of the slope.

L:	The velocity of that piece.
C:	And how do you calculate that?
L:	[laughs] *dy* divided by *dx*!
C:	Ehm, yes. But for example how do you calculate the slope in the point $x = 1/2$ in this graph?
L:	Ehm.. [silence] no.

She doesn't succeed. My advice is to apply the derivative. She seems to understand, but I feel that she doesn't. Clearly, it is hard. She knows directly from being right here that the derivative function is $y = 2x$. Currently, we have *dy* equals $2x$ times *dx*. She succeeds in describing the small piece of line as the square root of dy^2 plus dx^2 and she changes *dy* in $2x$ times *dx*. She realizes the notation of the total length of the graph at the interval [0,1] as follows: $L = \int_0^1 \sqrt{dx^2 + 4x^2 dx^2}$. She doesn't know how to make further progress. For a moment she seems to ask herself whether she has finished, but she notes from my reactions that she has not. What to do?

C: Do you recognize a standard integral being used to solve the task?
L: Ehm, no…do you have to write it backwards?

The answer "to write backwards" means a lot of different actions. She doesn't succeed to extract *dx* from the square root as preferred. First, she declares as follows: $\sqrt{dx^2 + 4x^2dx^2} = dx + 2xdx$. Subsequently she wants to write the square root as a power. She quickly notices her mistake. But what to do now? I suggest she should rewrite the first part under the square root. After a lot of inaccurate efforts she concludes:

L: Collective factor?

She correctly takes dx^2, puts the common term dx^2 after the expression outside the brackets. But the possibility to write a term out of the square root is hard and she does not succeed without help. Finally when *dx* is out of the square root, we have finished. Lotte is very relieved. She doesn't have a real overview of the process. She is confused, so I determine not to ask her to rewrite the process for the general formula. I tell her that. She nods to me, but looks at me as if I did magic with a rabbit in a big hat.

THE DEVELOPMENT OF MATHEMATICAL LITERACY IN OCSE-PISA VIEW THROUGH COLLECTIVE DISCUSSION: AN EXAMPLE

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ABSTRACT

In a sixth grade of middle school, we realized an innovative methodological and didactic path concerning generalization's activities, investigation of regularities in arithmetical and figural sequences. The aim of this work was to develop necessary mathematical literacy to help students take a conscious and active responsibility in society. Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments, and to use and engage with mathematics in ways that fulfill the needs of that individual's life, as a constructive, concerned and reflective citizen. Teachers' didactic methodology is very important to for promotion of such mathematical literacy. Teaching strategies must make students aware of their own thinking processes, learning ways and methods and not only enforce mathematical rules and properties. In our case class discussions about the solution of a problem, after an individual or small group investigated it, were effective in this sense.

KEY WORDS:
Original mathematical approach, reflection and insight, participation, sharing, awareness, collective construction of knowledge

In the assessment line of OCSE-PISA, in a first-year class of lower secondary school (grade 6) a teaching and methodological sequence was structured, focusing on activities of generalization and search for regularities within initially numerical and later figural sequences, in order to develop those mathematical competencies that are viewed as fundamental in enabling students to play an active and conscious role in society.

The development of these mathematical competencies, viewed as the capacity of making evaluations on the basis of reflection, requires a metacognitive methodological and teaching approach. It is necessary to identify methodologies and strategies that lead students not to the mere application of properties and rules, but rather to the conscious, critical and shared reflection upon the enacted logical mathematical reasoning, upon strategies as well as learning methods. In this line, collective discussions within the class group proved to be effective: they were preceded by the autonomous, either individual or in-cooperation learning, analysis of the proposed problem situations.

The collective discussion turned out to be a particularly relevant teaching and learning methodology for the construction of socially shared knowledge, due to its focus on exchanges with peers. During the discussion, each student contributes with individual information and reflections, which come to be part of the shared knowledge, through an enchainment and adjustment of the different contributions, and which are elaborated through collective reasoning, where thinking together does not correspond to a single person's thinking.

In other words, the discussion is not to be viewed as a moment when new contents are learned, but rather as an opportunity for students to recover already acquired knowledge, reflect upon it and use it again, in a new modality. The group discussion is thus a learning situation, which recalls guided discovery, in many aspects: in both cases, students are stimulated, by means of suitable problem solving situations, to recover old knowledge and establish links between notions, discover new and different connections, and draw personal inferences. In addition, in collective discussions, they are solicited to compare their point of view to different ones, to find common elements, to find out new criteria for classification, to draw inferences between ideas and contributions proposed by others. The choice of methodologies and didactic strategies in the implementation of the project was guided by our awareness of the effects they might have on both achievement and affective and social development of the class as a group. The collective discussion is an important phase of socialization for any class group: it stimulates students to participate, to search for justifications or refutations and to perform a productive cognitive exchange with classmates. Often in schools, mathematics and geometry are reduced to the simple exposition or to the solution of exercises. Rather, inducing students to express themselves both orally and in writing provides them with an opportunity for linguistic enrichment, with an effective spin-off on other subject matters. The type of scientific language used by students shows the acquired level of competency. Paying attention to the language used by students is an important strategy for teachers, who become able to follow, check and understand the process of knowledge structuring. Moreover, language also sheds light on students' mental representations and might show misconceptions, preconceptions and alternative methods to represent reality, which may interfere with learning. Collective discussion promotes autonomy and capacity of critically reviewing one's actions, together with capacity of collaborating. This is why it is so important to create an atmosphere in which students feel free to express their own ideas without fearing negative judgments by classmates, because of the possible mistakes.

If the objective is to favor the construction of independent intelligence and personality, we need to know students better, learn to value the diversities they bring to the scene. Each student is original and distinguished from all others by the cognitive and learning style through which he/she absorbs messages from reality, selects and links them to one another in order to construct knowledge and mental competency. When a cognitive style is valued, students trust more their capacities and use them at their best. They must have the right to make mistakes, because a mistake solicits new mental operations and new attempts to search for correctness. Their mistakes help them perform a continuous restructuring of knowledge, disassemble and assemble again their mental arrangements, the net of connections, in a continuous and dynamic process which makes intelligence grow and multiply its potential.

In collective discussions teachers play a particularly significant role, due to the variety of competencies they must be able to enact: they have to value students' interventions and send them back to the class, they must not show explicit judgments, try to keep consistency between verbal and non-verbal language, they must be able to leave room for the group's dialogues, while continuing to guide the lesson, they must be able to provide students with the chance to reflect upon ideas, opinions, mistakes and successful results obtained together.

Collective discussions related to our teaching sequence were audio recorded, with the families' agreement. The subsequent transcribing phase allowed me to appreciate interventions and linguistic mastery of individuals as well as to get to a self-evaluation of the interventions I made in the classroom. Some a priori sketches were elaborated in order to favor a conscious and consistent management of collective discussions: they included objective, presumed duration of the discussion, stimulating questions, formulation of the main question, identification of fundamental ideas for the understanding and development of the topic, and description of the attitude the teacher was supposed to have. The elaboration on the operative worksheets proposed to students represented an important "script" for managing collective discussions and allowed me to be consistent with the planned activities.

The problem situation in question, centered on the key mathematical idea *change and relations*, whose fundamental objective is to foster thinking in functional terms, is proposed by the PISA test as the "Apple trees" task. The task is characterized by the opportunity to tackle the exploration either with iconic or with numerical methods, or rather with the combination of both, to get to the actual algebraic generalization of the relation existing between the quantity of apple trees and conifers and the ranking number of the considered configuration. In particular, the worksheet was enriched from the iconic and graphical point of view, in order to make it more attractive; moreover, we simplified the problem situation by introducing tables aimed at guiding students to an autonomous exploration through an explicit justification of the identified relations.

The collective discussion starts with the teacher's request to reproduce the drawings from the blackboard.

T: Let's see together how our farmer laid out apple trees and conifers. I present them on the blackboard, I don't make comments, and you are kindly requested to reproduce them in your workbook. While I was drawing on the blackboard what did you check to exactly reproduce the layout of apple trees and conifers?

Marika: How many conifers there are on each side.

Riccardo B: I checked how much the number of apple trees increases from one drawing to another and how many conifers there are on each side.

Andrea: How many apple trees there are altogether.

Gianmaria: I checked the rows.

Khalid: In the first drawing there are 9 conifers.

T : In the first drawing there are 9 conifers. How did you determine the correct number of conifers Khalid?

Khalid: I did 3... 3… I got wrong.

T: Try to explain that.

Khalid: They are 8. I considered 3 at the beginning, on the first side, then I added 2, then 2 on the other side, and then 1.

T: 8. Good Khalid. How many apple trees are there Gessica?

Gessica: One!

The apparently simple request "what did you check to exactly reproduce the layout of apple trees and conifers?" opens the way to a shared metacognitive reflection; students participate and intervene in the discussion serenely, since everybody can contribute and provide meaningful answers, regardless of the specific mathematical competencies and not being afraid of making mistakes. Teachers must pay particular attention to relations and communicative patterns in the class group, giving each student the opportunity to correct a possible mistake without necessarily stressing it: this is what happens in the case of Khalid, who immediately corrects his own mistake, after a positive, immediate and consistent stimulus. In fact, the teacher recalls the student's words to guide him towards a constructive reflection. "In the first drawing there are 9 conifers. How did you determine the correct number of conifers, Khalid?"

Teachers must be able to grasp students' proposals and send them back to the class without making judgments but rather letting the group validate the shared reflections through a clear and correct formulation.

T: So, let's explore the other representations as well.

……

Marika: I noticed that the number of rows is the same as the number of apple trees in the rows.

T: What do you think about Marika's remark?

Andrea: It's correct for n = 2 there are 2 apple trees in each row.

Gianmaria: So, in order to calculate all the apple trees in a fence you must do the number of rows times the number of trees in each row.

Khalid: I didn't get it at all.

T: Yes, actually these remarks are all overlapping.

Gianmaria: I meant that in this case, Khalid, if you want to calculate the number of apples you must do the number of apple trees times the number of rows in a fence and therefore 2 times 2.

Marika: Practically you must do the number of row times itself because the number of apple trees in each row is the same as that of rows.

In this way the exploration of the problem situation progresses only when the whole class group can actually share and master the reflections and remarks being made. Nobody feels judged, since it is the teacher herself who asks for explanations, thus reversing the classic role teacher-student, in which the former is generally giving explanations, whereas the latter accepts them with respect.

T: So, let's see if I get it. What do you mean by "the number of rows" and "the number of apple trees in each row"?

Andrea: Those are rows…. (pointing to the drawing at the blackboard).. I mean the number of lines. The lines that are there. The number of apple trees is how many apple trees there are in each row.

T: Ah! I got it now. Thanks, Marika. So how can we write this 4 which indicates the number of apple trees?

Class: 2 times 2!

Adem: For n = 3 then there are 24 conifers.

T: Are you really sure? How did you get that?

Adem: I'm sure. I did 8 times 3. Because the first number there, eight,… after that, in the second place there was sixteen and you added eight, and the third you added 8 again, 24. I counted them.

T: Well, I did not understand why eight times three.

Andrea: Every time you go on with the ranking number, you add eight. Eight is also the starting number and the ranking number is the eight I add.

Giulia F.: I make eight, which is the starting number of conifers, times the number of rows.

Riccardo B.: Eight which is the starting number, times the number of rows, that is the number of how many times eight is repeated.

Marika: So I must do eight, which is the starting number, times the ranking number.

T: Andrea earlier said that eight is the number you always add. How do we call this number we always add?

Giulia P.: Step.

Jessica: Conifers have a step of eight.

Gessica: For n=4 therefore there are 32 conifers. Eight times four.

The teacher must be particularly careful to use a correct and suitable specific language, without being "dragged" by the discussion's evolution. Paraphrasing students, the teacher gradually introduces specific and correct terms, such as "step," for instance, which are appropriated by the class group without feeling them as a senseless imposition.

In this way, students themselves drive observation; the teacher's task is to keep in sight the objective and direct reflections through effective and measured stimulating questions.

T: What about apples?

Khalid: Sixteen. In number three there were 9… you must add 5 apple trees to get to sixteen.

Giulia P.: Can I say something? That 2 times 2, 3 times 3, 4 times 4 can be substituted for 2 to the power of two, three to the power of two, four to the power of two.

Gianmaria: Can I make another one? There are two different ways to count the apple trees: the first one is to multiply the ranking number by itself, the second one is to multiply the number of rows times the apple trees in each row. That is the area of that square over there. After that, to count how many conifers there are, we have two ways again. I mean to multiply the basic number which is 8 in this case, times the ranking number or the row number. That is the perimeter of the drawing.

T: Perfect. So, if I wanted to know for $n = 10$ how can I determine the number of conifers?

Jessica: Eight times ten.

T: What about the number of apples?

Jessica: Ten times ten.

T: Good. Suppose that I'm thinking of a number x.

Andrea: 8 times x to calculate the number of conifers.

Riccardo A.: x to the power of two to calculate apples.

Riccardo B.: I multiply x by itself.

Occasionally, the teacher must renew students' interest and participation by challenging them with little cognitive challenges "are you sure? are you tired?" that students can easily accept, and stimulate the external attributive style, by proposing a symbolic award, such as ten minutes of rest or relaxation at the end of the activity. In view of a desired goal, everyone's efforts and enthusiastic participation get twice as strong.

T: We got there. Can we carry on or are you tired?

[All students want to carry on.]

T: Let's write down this question now: Suppose that the farmer wants to enlarge his orchard with many rows of trees. As the farmer enlarges his orchard, what increases more: the number of apple trees or the number of conifers? Justify your answer.

Mirco: Well ... I wanted to say that conifers increase faster. It's better to carry on with conifers, because conifers start from 8 then become 16, 24, 32 et cetera. Apples start from 1 then become 4 and then 9. The numbers are lower.

Andrea: I don't agree with Mirco. Apples increase faster because it's true, as Mirco says that at the beginning apples go slowly, but then they go faster than conifers because you must do n times n whereas there it's just n times 8.

Riccardo B.: I agree with Andrea. The conifers' growth is faster than that of apples, at the beginning, but then at some point, for $n = 8$, apples go faster, because you increase faster by doing n times n rather than n times 8.

T: How can we verify who's right?

Mirco: Yes, they are right, I stopped too early. Anyway, we can see it if we write down both sequences.

….

T: Let's see. How do conifers grow?

Giulia P.: In the first one the step is always +8

Riccardo B.: In the second one (referred to apples) it always increases by an odd number: + 3, + 5, + 7, + 11,… So there is a point where the step of the second one becomes longer than the step of the first one.

T: Let's see whether I got it. For conifers the step is always the same, while for apples it doesn't seem to be the same.

Adem: It's always longer than the previous, but the first three steps must be shorter than the first three of the other one..

T: Ah! I see. I start little and then …

CL: You get longer!

Giulia P.: Yes, but we can carry on up to infinity … let's put dots over there (at the blackboard)

T: Good. Let's make another effort and then we can relax a bit. When does the apples' step become longer than the conifers' step?

Riccardo B.: The step which from n = 4 leads to n = 5. For conifers it's eight, always. For apples it becomes 11! Longer.

T: And when does the number of apples become the same as the number of conifers?

Gianmaria: $n = 8$. I must see when the two strategies can become the same and 8 times n can be equal to n times n only if $n = 8$.

T: Good, guys! I thank you for your participation and for being so serious in managing the discussion.

At the end of the teaching sequence we verified how, due to the collective discussion, not only students developed those mathematical competencies that concern students' skills in analyzing, reasoning, communicating ideas effectively: they also acquired transversal abilities and competencies, including flexibility, adaptability, capacity of working in groups, evaluating and comparing possible alternatives, being given a chance to experience different teaching sequences, autonomously, to get to a goal, thus making conscious choices. The collective discussion allowed students to develop and empower their participation, search for justifications or refutations and a productive cognitive exchange with classmates. The creation of a good group atmosphere favored shared learning, in which all subjects involved felt protagonists independently on their knowledge or acquired mathematical competencies. Moreover, students had the opportunity to develop linguistic competencies, with an effective spin-off in other contexts, since they were requested to write down and carefully read their own texts.

Transcription of the recorded collective discussions allowed the teacher to evaluate her own work, appreciating everyone's contribution and becoming aware of relations within the class and, at the same time, reflecting upon her own role in the group. In particular, a reflection upon the used language was carried out, with focus on expressive clarity, effectiveness of interventions, paraphrases and cross-references.

Finally, the audio recording allowed me to emphasize students' cognitive and argumentative competencies, and therefore to formulate a more pertinent judgment on the actual capacities of students.

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COMMUNICATION AND GROUP DYNAMICS

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ABSTRACT
In this article I present my investigation on how communication affects the group's dynamic, development of its work and, therefore, the students' learning. I analyzed an extract of a ninth-grade school lesson on systems of equations. From the analysis of that episode we realize different communicational life of the two groups observed and how that difference influenced their work.

KEY WORDS:
Communication, discussion, questioning

INTRODUCTION

Communication is present in every minute of our work and is the leitmotif of the discourse that gives meaning to mathematics. Several times I have wondered if communication that goes on in my classes is appropriated and if it is based on mathematical evidence that leads to the development of my students' learning. The diversity of situations that may take place in a classroom over the established communication between students is huge, unpredictable and challenging. It is impossible for teachers to foresee the arguments' sequence, and the students' interventions, so it is important to adapt the existing framework in order to improve the situation and extend it conveniently. For students communication may mean a unique moment of sharing knowledge, learning and intellectual growth especially when this dynamic is a product of the interactions in small groups. So, students have several opportunities to explore ideas, help each other, to discuss strategies and argue mathematical meanings. But what happens when the group's dynamic only develops around one member and if all the work depends on the person? Will it be a productive work when that member is in the role of a teacher, even though the person does not have the necessary experience in managing communication? In this article I present my investigation on how communication affects the group's dynamic, development of its work and, therefore, the students' learning. Based on this issue, I analyzed an extract of a ninth-grade school lesson on systems of equations. From the analysis of that episode we realize different communicational life of the two groups observed and how that difference influenced their work.

COMMUNICATION IN THE CLASSROOM

The Principles and Standards for School Mathematics (NCTM) notify that "in the classroom where students are encouraged to think and reason about mathematics, communication is an essential characteristic: at the same time they express orally and in writing their reasoning" (NCTM, 2000, 318). Therefore, teachers must implement in their classes a sense of community, so that students feel free to honestly and openly express their ideas without fear of being ridiculed. So, teachers should promote a climate of learning for all individual students to present and explain the strategy they used to solve a problem. Teachers should also use significant mathematical tasks and should let the students' interventions to be evaluated through discussion in the class, thus enabling students to develop mathematical skills at different levels.

Working in small groups (as we can verify in this article) is valued by Ponte and Serrazina (1999) because it "enables students to expose their ideas, listen to their colleagues, questioning, discuss strategies and solutions, argue and criticize other arguments" (21). This type of work provides an opportunity for efficient mathematical communication, as well as teachers have an important role in helping to further the mathematicians objectives to all members of the group. According to the *Principles and Standards for School Mathematics*, teachers must resist when students' try to make teachers think for them "and should" answer in a way that allows them to concentrate on thinking and reasoning, rather then acquisition of the right answer" (NCTM, 2000, 323).

According to Ponte and Serrazina (1999), "students learn in consequence of the activity that they develop and reflection on it" (3). Communication plays an important part in reflection and learning development when arguments are shared and mathematical concepts and processes discussed. The communication process generates mathematical meanings through their trading and use in a social interaction. According to the same authors, the three basic types of communication are exposition, questioning, and discussion.

In the first type, communication consists in interlocutors' exposition of an idea. In the second one, the question presupposes that interlocutor put successive questions with an objective for others. In the third, the discussion involves the communication modes just mentioned, when it allows the interaction between different interlocutors, sharing ideas and questioning each other and, therefore, it is considered the most important mode of communication.

Ponte and Serrazina (1999) subdivided communication through questioning into three types of questions posed by teachers: (i) focus, (ii) confirmation, and (iii) inquiry. They considered focus questions "false" questions, with the aim of providing guidance to students so that they can complete the task. Teachers use the confirmation questions to make sure that students have got certain knowledge. The inquiry questions provide teachers with information that they have not and the authors consider them as the only real questions.

THE EPISODES

The class

The analyzed episode was observed in a ninth-grade class of a school in Lisbon's suburbs. The class consisted of 25 students aged 13-15 and characterized by their main aim to obtain good results. The majority of students were dedicated and responsible, although discreet and reserved. In general, these students preferred to work individually or, sometimes, in pairs. Some of them expressed their dislike every time the teacher proposed working in groups, and tried to accomplish the tasks alone. In class they eagerly performed the tasks, but when they were confronted with a discussible situation they preferred not to participate in discussion orally but rather submit a written answer.

The lesson

The two episodes reviewed were conducted in a class on systems of equations. The aim of this class was to mobilize students' knowledge of linear functions and graphs in real contexts and the capacity of development communication in the classroom. The task consisted in the exploration of a closed situation in a real context, with which the students could have some degree of familiarity. This presupposed that the activity could be rewarding for all students in the use of both mathematical ideas and the potential for presentation, discussion and sharing of their productions. The class was organized into three distinct stages: the first was the presentation of the task by the teacher, the second was working in groups, and the last was general discussion. The adoption of this methodology of work aimed to raise the involvement of students' activity in the classroom through the presentation and discussion of ideas and to establish the teacher in the role of mediator and moderator. The two episodes refer to question 1 of the task that was already mentioned before.

1. While surfing in the Internet, Pedro found a very appealing site entitled "Always fit: the most popular gym." However, the website was not clear about the prices in the gym, providing only the following information:

Package use of the gym

Time to use weekly (h)	1	2	3
Price of the package (€)	15	24	33

Help Pedro evaluate the offer of this gym.

1.1. Represent graphically the situation shown in the table.

1.2. Characterize the graph that you drew and write the corresponding analytical expression.

1.3. For the graph that you drew state:

1.3.1. the slope and its meaning in the context of this situation.

1.3.2. the passing through the origin and its meaning in the context of this situation.

Question 1 (incomplete)

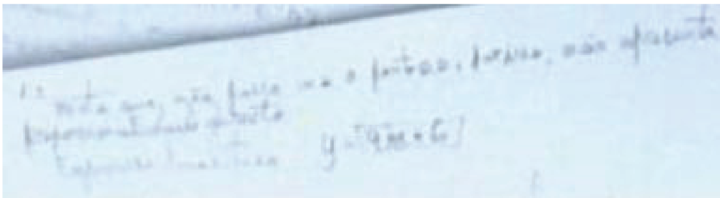
The proposed task was designed considering the mathematical model for the situation under study as a function of the type $y = kx + b$. The first episode refers to the group that we will call A, consisting of the following students: Bruno, Cristina, Ruben, and Susana. The second episode refers to group B, which was formed by Cátia, Luis, Renato, Sara, and Tatiana.

Analysis of the episodes, both in Group A and in B, is focused on question 1.2. The teacher wanted to find out if the students: (1) understood that the equation of the line containing points given in the table is of the type $y = kx + b$; (2) managed to write – from the table and the graph drawn – the analytical expression of the achieved "line;" (3) identified the slope of the line and determined it (i) as the difference between prices of consecutive packages, (ii) across the graph, by the difference between the coordinates of the two points, or (iii) by analogy with the graph of direct proportionality.

Communication between students

The two groups observed had a dynamic of their own, including the fact that in each group there was a student in each group (Bruno and Tatiana), who was recognized as a leader by the other colleagues. Although this situation was acknowledged, each leader's attitude was different. Bruno took initiative to guide development of the work, imposed his viewpoint, and liked to be identified by his colleagues as the one who had the largest “mathematical learning” influence.

Susana: Bruno, you understand this, help us… Ah, sequences, right? So this is a sequence, 9, 9, 9,… right?
Bruno: Direct proportionality is a sequence. This is a sequence 9 in 9. Someone has a ruler?
Bruno: That’s right, girl, you are doing very well.
Susana: And the analytical expression?
Bruno: The analytical expression is easy.



…

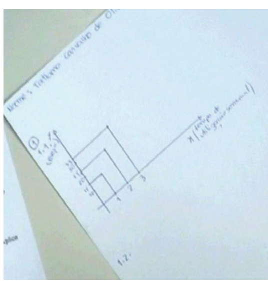
Bruno: This is done wrong.

Tatiana presented many capabilities for mathematics, revealed good communication skills, and explained the work done. But she did not like to put emphasis on herself. Nevertheless, the colleagues accepted her as the natural group leader.

Tatiana: It is 6. It is the entry…… [Laughs]…
Luis: I agree with what she [Tatiana] said. It is the entry in the gym.

Relatively speaking, in the work done in the observed groups there was a participant in each one, who did not contribute to the development of the group’s work. In group A the other participants contributed to a good performance in the work produced because, as can be seen in the classroom’s transcript, there was a continuous interaction between Susana, Cristina and Bruno. In this group Bruno tried to impose his own ideas to resolve the tasks, whether Susana or Cristina did not move forward in the activities’ development, when they did not understand infomation that they considered necessary.

Cristina: The graph is very large. Why didn’t you put it down and to the side [negative number]?
Bruno: It isn’t. There are only positive numbers. There aren’t any negative numbers.
Cristina: So, now we have to mark these points, right? Show this part down. This is not like this, the three isn’t here. The three is suppose to be there.
Bruno: I have outlined it here. 6 is the number for zero. Zero, six.
Cristina: The zero passes in 6, the 1 passes in 15, the 2 passes in the 24 and the 3 passes in 33. I have already understood that. It’s not that difficult.



In group B, the interaction was more frequent between Tatiana and Sara while the remaining participants expressed their opinions sporadically. These were the two students who contributed most to the development of the activity although they had different rhythms of work. While Tatiana did the activities quickly, to present a good reasoning in numerical and algebraic calculations, Sara worked slower, due to her perfectionism in the presentation of solutions.

Tatiana: This is wrong, [to Luis] but it isn't necessary to rub it out.
Sara: So we write the independent variable here [pointing to time in hours] and dependent variable [pointing to price in Euro] here.

Tatiana: Put only x and y.
…
Sara: We say that here is no direct proportionality and explain why.
Tatiana: Write only: There is no direct proportionality in the chart.
Sara: Why?
Tatiana: Because no. People will see that these are mathematical, so they will know…

The role of the teacher

The teacher used questioning to identify the difficulties experienced by students and relied essentially on the pattern of confirmation and focused on them directly to resolve the issues. In both groups the teacher did not answer questions but pointed to guiding issues that once answered provided the students with a new runway that enabled them to build mathematically significant knowledge. The questions that the teacher asked were based on her knowledge in mathematics, teaching, and curriculum. She tried to lead students to actively build their own knowledge.

The intervention of the teacher began in this episode when Bruno asked for her support, because after several interactions with colleagues he could not get rid of his doubts about the graph's characterization.

Teacher: Characterize the graph. What is the chart designed?
Bruno: It is a graph.
Teacher: Yes, a graph. But what kind of graph?
Bruno: Of direct proportionality.
Teacher: Really?
Bruno: There is no direct proportionality, we have already seen it. It doesn't pass in zero.
Teacher: So this graph is what?
Bruno: A graph that represents an analytical expression.
Teacher: A graph that represents the analytical expression, says Bruno. Yes but what kind of graph is it? What type is it?
Cristina: It is a Cartesian graph.

The interaction between the teacher and students in this fragment was characterized by a series of confirmation questions so that the group made the graph characterization corresponding to the situation under review and exactly described the properties of the line found.

Bruno: Positive.
Teacher: It's a Cartesian graph. But what is that of positive?
Students: [silence]

There appeared uttered by a student an expression "positive Cartesian graph." By placing the question "But what is that of positive?" the teacher intended Bruno to clarify it and explain the meaning of that term because it was not mentioned before. The students' silence showed little confidence and conviction regarding the answer.

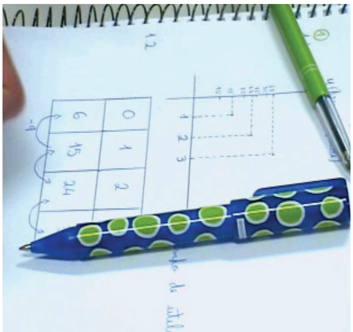
Teacher: You have done this [and, with a sign, she made a line]. What is this [pointing to the line]?
Cristina: It's a line.
Teacher: It is a line, yes. It is a graph that represents a line. It is a graphic formed by a line that… Passes there? [Pointing to the graph's origin].
Cristina: No. It doesn't pass the axes' origin.

…
Teacher: What is the variable that is there?
Cristina: Oh, it's *h*. [Replacing in the expression $6+9n$, variable *n* by *h*]. So, it's a graph that is not passing by zero, which expression is $y = kx + b$,
Bruno: 9 is the *k*. $y = 9x + 6$
Cristina: So it's $y = 9x + 6$, where the slope is *k*, that is 9, and 6 is the y-coordinate in the origin. I agree.
Bruno: So it is a line of equation $y = 9x + 6$ doesn't pass in the axis zero, zero.
Teacher: It doesn't pass in the axis zero, zero, or point (0,0)?

Although students managed to write the analytical expression of the graph, it was only with the use of sign language used by the teacher that they managed to say that it was a straight line. That situation showed that the students did not encounter difficulties in mathematics activity, but encountered difficulties in the interpretation and understanding of the question. It should be noted that the hierarchy of the issues must be treated gradually, from the lower to the greatest degree of difficulty. In the last issue of focus, the teacher paid attention to the language used by the student: "It doesn't pass in the axis zero, zero, or point (0,0)?"

In the second episode examined, the teacher's intervention was requested when the group B (like the other group) had doubts about the characterization of the graph. As can be seen in the following transcript, students interacted, discussed reaching the value of the slope among themselves and could easily write correctly the analytic expression.

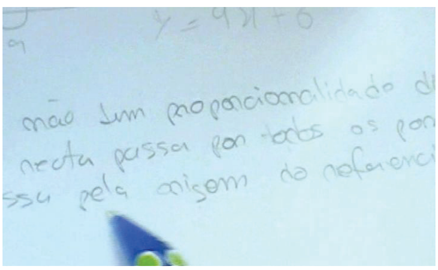
Cátia: Look, here is 9. The difference between them is 9, isn't it? Here is 6… [Looking and pointing to the chart that Sara built].



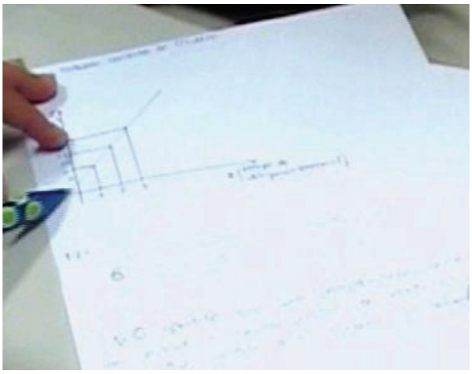
Sara: The analytical expression. You see… it's what Cátia said: $y = 9x + 6$.
Tatiana: Yes, I saw it, only that… all right. Then k is the 9. No, yes…
Luis: k is the 9.
Sara: Exactly, k is the 9 and b will be the 6 because it is under zero, isn't it?
Tatiana: Eh, eh. Yes. And now, characterize the graph, how?
Sara: The analytic expression, it's done. Characterize the graph… [Laugh]…

This group also revealed some difficulty in describing the properties of the graph obtained:

Sara: What do you think of this characterization of the graph? … "The graph isn't a direct proportionality because the line passes through all points, but not passes by the origin."



Teacher: All points?
Tatiana: Of these three [points to the table].
Sara: Teacher, how are these points called? These here [points to the table].



Teacher: All points that are where?
Tatiana: In the graphic. In the table…
Teacher: Oh, ok…
Tatiana: They are in the table and we put them on the graph.
Teacher: Is there direct link between the graph and table?
Sara: So the graph characterization … can we also examine the table? Can we associate one thing with the other?

Following the same way of questioning the teacher in this dialogue raised several questions of focus and confirmation in order to navigate the group to the correct answer. However, in the interactions observed two different ways of representing the situation under study emerged: a graph and table. The students from the data in the table managed to correctly do the graphic representation of the situation, but showed some doubts about the answer concerning the characterization of the graph. In analogy with group A, this seemed to have shown some difficulties in interpreting the issue and not because of the mathematical concepts involved.

CONCLUSION

Students had some difficulty in the interpretation and understanding of the issue concerning the interpretation of the graph of the task, much more than in pure mathematical part of the activity. This raises the question of hierarchy of issues in terms of difficulty, i.e., would it not be preferable to start with the simplest task and, gradually make them more open and complex? In the classroom, the teacher led students through a process of shared communication to build and consolidate their mathematical thinking. She promoted a more detailed analysis of the issues and the formulation of explanations, different kinds of argument, and the reflection on the knowledge of students and on the ideas of others. The issues raised by the teacher were focus, confirmation and inquiry, used in the type of guidance that she wanted to give students. During the episodes she provided guidance to students so that they could reach certain knowledge; wanted to make sure that students fully understood the knowledge and expresses their doubts.

Although the communication in the groups was dynamic, it was not shared by all its members. Two students, Ruben in group A and Renato in group B did not participate in the discussion. In subsequent conversation the teacher of the class indicated that Ruben is a repetitive student, introverted, and that he still did not feel well integrated in the classroom. Also Renato never participated in any activity; he was at risk and worked only when the teacher, during class, sat beside him and motivated him to work. Once the teacher finished individual support, he stopped working again.

In developing the work done by both groups a communication involvement showed up, with the exception for these two students named before. However, in group

A all the suggestions/guidelines of Bruno, the leader of the group, were accepted thus making communication poor; there was no sharing of reasoning, knowledge and strategies. The communication in this group was basically unidirectional and Bruno, which is understandable, did not care about developing the learning of his colleagues but only wanted to transmit the knowledge he considered correct. In group B, it appeared that despite Tatiana being regarded as the leader, the students were involved in sharing strategies, questions and checking knowledge. Tatiana sought to promote dialogue in a confrontation of ideas and went on building knowledge of the group from the contributions of all. While no one can expect a student to have attained the required competency level of communication to promote the learning of colleagues, let us note that there were differences in dynamics in the groups resulting in a large part from different communication formats between their members.

This diversity of environments, communication, and learning takes place systematically in our classrooms. The way of talking and sensitivity of teachers is fundamental to the process of communication between students and is rewarded by its multiplicity. Teacher's mediation is essential to develop work of a group in order to homogenize the discrepancies that inevitably occur within them. The combination of factors, such as knowledge that teachers have of their students, the activities they propose, the questions they pose, and the debate they promote about strategies, mathematical reasoning and meanings that favor building an effective and permanent educational success.

Analysis of these situations can enrich our teaching. The position of an "observer" allows a reflection on the different types of mathematical reasoning and approach. Apparently, communication in the classroom develops intuitively and spontaneously. But in reality there must be a coherent effort of teachers to enhance the communication and engage all students taking into account their specific features and characteristics.

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REFLECTING TO IMPROVE THE COMMUNICATION IN THE MATHEMATICS CLASSROOM

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ABSTRACT
In this paper I intend to illustrate some moments of reflection, when trying to promote the communication between students of a ninth-grade class who habitually resist participation in whole-class discussions. The activity in this lesson elapsed from a semi-reality task on functions, whose characteristics aimed to promote the debate in the class. The students went through the task without great difficulties, but in my opinion the objectives of discussion on their activity were not fulfilled. So I questioned myself about my approach concerning the dynamics of the classroom.

KEY WORDS:
Communication, whole-class discussion, reflection about the practice

For twenty years as a mathematics teacher I have encountered many classes, each one with their own characteristics. The class that I will talk about in this paper stands out because of being – according to the teachers' council – a class, whose main objective is to get good marks. Most of these students are very studious and responsible, though discreet and reserved. At the end of the second term, 18 out of 25 students of this ninth-grade class had positive marks in all school subjects. On the whole, they preferred to work individually or in pairs. For instance, some students, when it was suggested that they work in group, showed their displeasure and tried to carry out the task individually. Usually, these students got involved in the purposeful tasks, but when they were confronted with situations of a whole-class discussion they preferred not to join in and to present their answers in writing rather than orally.

Wondering about this attitude, I suggested that the students do a task in a small group, which at the end should be discussed in the whole class. As I started to work with them on systems of linear equations, I chose a situation connected with functions, taught in the previous year, in which these students had been very successful (Picture 1). Since it was a situation of semi-reality (Skovsmose, 2000), I thought it might be a good starting point to promote a whole-class discussion, because it demanded interpretation and critical spirit. It also seemed that it could be challenging enough for me, which indeed happened.

The situation intended to stimulate the students to analyze the possibilities of using a gym in connection with the price to pay, what I considered could cause the discussion between the students. Taking into account that the main goal of this lesson was to promote the development of communication in the classroom, I chose to present here only the first part of the task, which corresponds with the analyzed episodes.

The task (1st part)

While surfing the net, Pedro found a very interesting site: “Always fit: the most popular gym.” However, the website was not explicit about the prices, giving only the following information:

	Types of gym’s use[1]		
Uses per week (h)	1	2	3
Price to pay (€)	15	24	33

1) Draw a graphic representation about the situation described in the table.
2) Characterize the graphic and write the correspondent analytical expression.
3) Concerning the graphic drawn, point out:

The slope and its meaning in the context of the problem

Thinking initially about the exploration of this task with the group, I conjectured that students would use their knowledge from the previous year to reach the analytical expression of the function (defined in R_0^+) and to identify the slope of the straight line. In the previous year, in similar situations, the students drew a graphic representation of a linear function through a translation from the graphic representation of direct proportionality with the same slope. The question that I asked myself was to what extent they understood the meaning of the slope in this context. So, during the class that preceded the proposal of this task, I suggested generalizing linear patterns ($kn+b$, with n natural), so that it would be obvious that the coefficient corresponds with the difference between consecutive terms. My expectation was now that the students would develop different strategies and reasoning: referring to the previous class or to the way they had solved these situations in the previous year. The expected diversity of students' products could be a very positive contribution to promote a whole-class discussion.

Work in small groups

Most of the groups did not get involved in the interpretation of the situation, as they immediately started to answer question 1: they marked the three points mentioned in the table in a Cartesian referential system and drew a straight semi-line contained in the first quadrant, with its origin in the point (0,6). There was, however, a group that interpreted this situation as if it was a relation of direct proportionality, which caused them difficulties. Therefore, they asked me to help them.

Andreia: Teacher, this is not supposed to join? [Pointing out to the points marked].
Teacher: What is it supposed to?
Andreia: Join. It is not right.
Teacher: What is wrong?
Andreia: If I wanted to draw a direct proportionality straight line… I can't. It does not go through the zero.
Teacher: And do you think it is a direct proportionality?
Andreia. No, it is not. There isn't a constant. But I can't join it.
Teacher: Is it not possible to define a straight line with these three points?
Andreia: Yes. But it doesn't go through the zero.
Teacher: But, who tells you that it has to go through the zero? Is this what we want?
Andreia: OK, teacher, we have already understood…

[1] This situation can be interpreted in several ways, depending on whether or not the possibility of using the gym is considered in fractions of an hour, and the mode of payment during these periods. Note that the task mentions that “the website is not explicit concerning the prices.”

I got a bit surprised by the doubts of these students. Why did they have doubts about the straight line they drew if the points they got from the table were correctly marked in the referential system? Why were the students taken to think that the graphic representation of the situation should be of direct proportionality, when they had already noted that there was no “constant”? During this academic year I still did not work with them on functions. Therefore, my perception was that though in the previous academic year we worked with representations of different linear functions, those that constituted a more outstanding practice for these students were those of direct proportionality.

To my surprise, in question 2 the students generally did not encounter difficulties in writing an analytical expression but they did not understand the meaning of “characterize the graphic.” An example illustrating this situation is the dialogue between students in a group who, after some time of interaction and discussion, asked for my intervention.

Susana: And the analytical expression?
Bruno: The analytical expression is easy.
Cristina: And characterizing the graphic?
Bruno: This I can’t understand so easily.
Cristina: It isn’t direct proportionality because the straight line doesn’t go through zero. But I found the analytical expression.
Susana: Which is it?
Cristina: $6 + 9n$.
Susana: And to characterize the graphic?
Bruno: … I don’t know…. Teacher?
…
Teacher: You have done this [with a sign]. What is it? [pointing out to the line]
Cristina: It’s a straight line.
Teacher: Yes, it’s a straight line. It’s a graphic that represents one straight line … And passes through here [pointing out to the origin)
Cristina: No. It doesn’t go through the origin.
Teacher: And, what is it the analytical expression?
Cristina: So, the analytical expression is this one [pointing out to $6+9n$]
Teacher: Ah! You have already written the analytical expression. Right. What is the meaning of this n?
Cristina: It’s this. [pointing out to the Ox axis]
Teacher: And, what is it? Which is the variable?
Cristina: Ah! It’s h. [replacing on the expression $6+9n$ the variable n by h]. So the graphic doesn’t go through zero, which expression is $y = kx+b$.
Bruno: 9 is the k. y = $9x + 6$
Cristina: Therefore it is $y = 9x +6$, where k is the slope, which is 9, and the 6 is the y-intercept.

Once again the concept of direct proportionality was present when students tried to interpret the situation, stressing that it was not a direct proportionality’s situation. They characterized the function by its analytic expression, since they were not able to identify something in the graphic that in their perspective could characterize it. They only explained that the graphic was “a straight line” when I used a gesture (a straight line). Since students did not understand the questions’ aim, I tried to guide them to the right answer, through questioning.

In a similar way, another group found the slope value and the y-intercept of the line correctly, but they associated the graphic with the fact that it is or not a direct proportionality situation.

Sara: The “analytical expression,” is that what Cátia said: “$y = 9x+6$”?
Tatiana: Yes, I saw that, but….ok. So, k is 9. No,…yes…
Luís: k is 9.
Sara: Exactly, k is 9 and b is going to be 6 because it is below zero, isn’t it?
Tatiana: Eh, eh. Now, “characterize the graphic,” what is it supposed to do?

Sara: The analytical expression is done. “Characterize the graphic”…[laughing]
Tatiana: Supposedly it is the direct proportionality, isn’t it?
Sara: Yes, (to know) if there exists or not a direct proportionality.
Tatiana. So, there isn’t.
Sara: We’ll say that there isn’t direct proportionality and we’ll explain why.
Tatiana: We only write this: there isn’t a direct proportionality in the graphic.
Sara: Because…
Tatiana: “Because,” no…Because 1 to divide ….
Sara: Because the line, the straight line which intersects three points doesn’t intersect the point (0,0), does it?
Cátia: It was what I was going to say…
…
Sara: Teacher, what is your opinion about our graphic characterizing: “The graphic doesn’t represent a direct proportionality because the straight line intersect all points but not the point (0,0).”

In this group, the students seem to identify two reasons for non-direct proportionality. In the first one, they used the table data, and in the second one they did a graphic representation. In these two episodes it is very clear that the students associate the task with the topic of direct proportionality and graphic representations of functions, and that their difficulties are connected with problem’s interpretation, as we can see on the second question in the task.

For me, another goal in this task was that students could identify the slope of the straight line that they drew, as well as they understood its meaning, as asked in question 3. Apparently, the students also solved this question without difficulties. The slope’s meaning was not so different between the groups. The majority of the groups stressed that the slope “was 9 and it is the difference of the prices between consecutive packages” (Tatiana). However, it is interesting to note that in Bruno’s group emerged a different answer:

Cristina: Bruno, the slope is 9, right?
Bruno: Right
Cristina: So, in context of that situation…
Bruno: It is the difference between the points. In Oy axis the difference between the numbers is 3.
Cristina: 9.
Bruno: Yes, 9.
Cristina: So, the slope is 9 and it means the difference, the variation, how can I explain this? [For] 1h [you] have [to pay] 15 Euro, 2h have [to pay] 24 and 3h have [to pay] 33 Euro. So, it is the increase of price by hour.
Bruno: Yes, it has to do with the points’ variation in Oy axis. Between the points the difference in y is 9.

In the written strategies of this group (as well as other groups) it was evident that the slope’s computation was done by the difference of prices between consecutive packages from data in tables. This comes, probably, from influence of the work carried out previously with the numerical sequences. However, it was interesting to note that one student connected the straight line slope with the difference between the ordinates of the points that he represented, which was correct in this situation due to the fact that the difference between the abscissa of the represented points is always a unit.

Whole-class discussion

According to Ponte and Serrazina (2000), the work in a whole-group “is appropriate … to discuss the finished tasks” (127), while the work in small groups enables the discussion of different points of view. Nevertheless, both kinds of works allow the communication between students. Although lack of students’ willingness to communicate in the whole group was a very persistent aspect in the course of the time, as I saw that they were very devoted to the work on the task – I developed the expectation that this time it would be different. But it was not… The students went to the blackboard to write the correct answer and they explained what they had done only when I questioned them. When I asked if there was anyone who would like to add a different strategy, the majority persisted in silence. Only with some insistence on my behalf did they describe what they had done. But as a matter of effect, as all of them answered the questions more or less in the same way, there was not much to discuss.

This lesson could have been different if I realized that there is a strong conditioning for the students' activity: the point that I wanted students to reach so that I could begin solving equations system in the next lesson. As the departure point for the task was not completely clear (it said that "the website is not clear as for the prices practiced in the gym"), it would have made it possible to promote a true discussion with the students on the nature of that situation. If students, before solving the problem, thought of its interpretation, being critical to this situation, and if different solving strategies appeared, the discussion moment could have been much more fruitful.

Sharing and confronting ideas with others would have been a very important aspect in increasing classroom communication. In that circumstance, my intervention could be crucial: to change the lesson's path and to manage the discourse involving the class in mathematics communication (NCTM, 1994). Thus, when students wanted to do a certain characterization ("Is it or not a direct proportionality graphic?"), they would have found unexpected answers and at the same time it would have been an opportunity for them to go to unfamiliar paths.

CONCLUDING REFLECTION

The reported episodes constitute part of a moment of my professional life, in which I had the opportunity of thinking about my practice and questioning it. By definition, classroom is a learning place for students, but it can be also for teachers. So that learning takes places, it is essential for teachers to reflect on their practice. I usually identify students' characteristics and lack of time to focus on all points of the syllabus as constraints to my practice. The exploration of this task with these students made me reflect on other aspects, namely, the type of questions to pose to the students, how to sequence them and their subordination to the mathematical topics that I intend to approach.

The purpose of this lesson was to promote students' mathematical communication. I still want to go towards this aim. In this sense, as a teacher, I need to create mathematical tasks that potentially promote classroom communication, but most importantly, I need to be more attentive and ready to change the discourse in the classroom at the turning point of the discussion (or at any other point). I need to create a new environment of learning, asking questions like "what happens if …" (Skovsmose, 2000, 18). Being able to change my teaching style and strategies daily and being flexible to take decisions in the course of a classroom, constitutes challenges that I propose for myself, as a teacher. It means to develop new perspectives on students' difficulties, looking at these not necessarily like intrinsic characteristics but somehow as a reflex of theirs and my own activity.

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QUESTIONS THAT TEACHERS PUT…

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ABSTRACT
This article analyzes classroom communication, particularly the types of questions that teachers pose to students during a group discussion. The analyzed episode appears in the context of the study of sequences in grade 8, when the teacher faces the students' difficulty to find the general term of a numeric sequence. The data were collected by video recording of the class. The questions that the teacher posed along the discussion played a decisive part in the construction and sharing of knowledge among students.

KEY WORDS:
Communication, communication styles, questioning

INTRODUCTION

Communication is a fundamental process underlying the activities that take place in the mathematics classroom involving teachers and students. It has a major influence on the nature of teaching and learning. In spite of my reading of several theoretical papers about the way teachers can promote the discussion among students, for me, this is a complicated moment, during which I feel some insecurity. Therefore, I found important for my professional development to analyze the way a colleague, confronted with a difficulty shared by all students, promoted the construction and sharing of knowledge during a group discussion.

During the study of sequences the teacher proposed to students to carry out in group a task with three questions. In the first 90 minutes class the students solved a worksheet and discussed with their colleagues the first two questions; in the second class they discussed question 3. In the first two questions the sequence was presented as a figurative pattern. In the third question, in contrast, the students were presented a numeric pattern (see Figure 1), with a non-usual aspect, as the first term is a negative number.

The episode that I describe happened during the discussion of the task by the group after the students identified, without difficulty, the next term of the sequence presented. During the discussion on generating expression the students presented some incorrect expressions and they were not able to move forward. As they were thinking about this problem, the teacher made a parallel with question 1, in which they had identified the general term, $n\times3+1$ that originated the formulation of new suggestions, such as: $n\times3+1$, $n\times5+1$ and $n\times3-1$.

3. Here are the first five terms of a sequence that follows a certain law:

-2 ; 1 ; 4 ; 7 ; 10 ; …

a) What is the next term of this sequence?
b) Write a generating expression for this sequence.
c) Determine the 100^{th} term of this sequence.
d) Is the number 994 a term of this sequence? If not, justify the reason; if yes, indicate its order.

Figure 1. Task proposed to the students.

This episode begins with a student introducing the expression $n \times 3 - 1$ which, although incorrect, includes the addition of a negative constant – a feature that later played a fundamental role. In a second moment, the episode shows how starting from this generating expression students discovered the general term of the sequence. And, finally, the episode ends with the explanation of the general term, in the context of the questions put by some students. Data was collected by video recording. Later, the discourse was transcribed, this episode selected and the communication analyzed.

ASPECTS OF THE CLASSROOM COMMUNICATION

Classroom communication depends on the way teachers regulate and promote it. According to Ponte et al. (2007), learning constitutes a process of interaction and reflection. Teachers need to propose tasks "that promote a variety of strategies of problem solving for the students and to take them to share their ideas, aiming at negotiating mathematical concepts and constructing new knowledge" (43).

Concerning the classroom discourse, the NCTM (1994) refers that an aspect of teachers' role is to raise students' reasoning. Teachers may request students to write explanations for their solutions and justifications for their ideas. Another aspect of teachers' role concerns the mediation of the classroom discourse, posing explanatory or provocative questions, giving information and direction to students in their thinking process. The last aspect is related to the control and organization of students' participation. Teachers have to make decisions about each student's participation so that all contribute to work of the group and enforce the rule that each one has to justify their ideas; that is, "although teachers may appear sometimes more inactive and silent, their role is fundamental when creating a positive discourse in the classroom" (NCTM, 1994, 57).

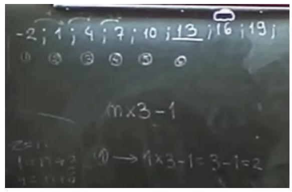
According to Ponte and Serrazina (2000), there are three different ways of communication: exposition, questioning and discussion. Usually, teachers use exposition when they intend to introduce information, to explain a procedure or to systematize a certain idea.

In questioning, we may consider three types of questions: focalization, inquiry and confirmation. Focalization questions aim at helping students follow a determined thinking path. Teachers seek to help students when they are lost, focusing their attention on a specific aspect and providing orientation for the next step. Confirmation questions are used by teachers to become certain of students' knowledge and generally they lead to just one answer. Despite the fact that this type of questions constitutes a good way to internalize mathematical ideas, often students do not understand them this way and regard them as a test of their knowledge. Inquiry questions aim at helping teachers get information on students' work and, at the same time, at bringing to the surface information that students were not aware of. Because of that they are called "true questions."

At last, discussion is characterized by the interaction between students and between students and the teacher, during which the teacher may assume the role of moderator or guide. In the perspective of Matos and Serrazina (1996) the discussion is a very important part of the mathematical activity, because this is the moment, during which students present their results and the teacher clarifies ideas and addresses possible questions.

QUESTIONS POSED BY THE TEACHER

This episode can be divided in two parts: the first concerns the suggestion of an expression for the general term, and the second contains the explanation of the general term that is given based on the questions of some students. In the first part of the episode the thinking process is presented (attempt-mistake) by two students to find the general term of the sequence:

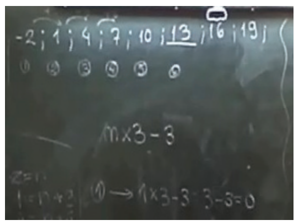


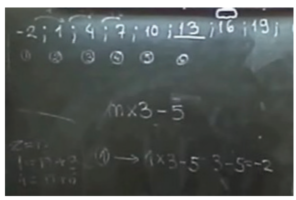
Ana Catarina: Teacher, is *n* x 3-1!
Teacher: You have a suggestion. Let's try. Shall we? It's not a big trouble! Let's go.
Vanessa: The result will be 2 in the first.
Teacher: The first term… "how will it go?"
Joana P: 1x3 -1.
Teacher: 1x3.
[Ana Cristina interrupts the teacher.]
Ana Cristina: It's 2. [Picture 3-1=2]
Teacher: She has already fallen from 4 to 2. Will we be able to reduce it more?
[In the first expression $n \times 3 + 1$ the first term was 4 and in the expression $n \times 3 + 1$ the first term is already 2]

The discovery of the expression for the general term was not easy. Ana Catarina's intervention was fundamental, when referring to an expression that involved a negative constant. In the discussion around the thinking process attempt-mistake to arrive to the general term, the teacher posed different focalization questions ("how will that go? She has already fallen from 4 to 2. Will we be able to reduce it more?") with the objective of drawing students' attention to obtaining the first term of the sequence starting from the general term.

Meanwhile, students suggested other values for the constant to put in the general term of this sequence:

Joana P: -3.
Teacher: Will we put -3 there?
Joana P: Yes.
Teacher: Second experience.
Ana Catarina: It's zero, teacher.
Catarina: -4.
Joana P: -5.
Ana Catarina: Yes -5.
Teacher: How much will it be in *3xn?*
Ana Catarina: -5.
Teacher: Why -5?
Ana Catarina: Because that's the result.
Joana P: It's -2.





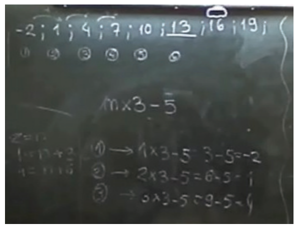
The teacher asked confirmation questions, “Will we put -3 there?” seeking to confirm if students knew where they should put the value -3. Later, the teacher felt the need to make a summary of what happened up to that moment (“Second experience”) so that all of students may follow the next thinking step, during which she continued to pose focalization questions (“How much will it be in *3×n?*”) and inquiry questions (“Why -5?”). In this last case she tried to understand what made Joana substitute -3 for -5.

The second part of the episode is based on a dialogue between the teacher and some students, around the understanding of the general term that appeared in the context of the questions asked by some students.

Patrícia A: I didn’t understand very well -5.
Teacher: Ana Catarina, can you explain why it is -5?
Vanessa: Because the result is -2.
Teacher: Ana Catarina, start to explain.
Vanessa: To arrive to the -2.
Patrícia A: It was for attempts.
Teacher: In this case it was for attempts.
Patrícia A: OK, but what if it was for instance -20, I would never discover the answer.

Facing the question put by Patrícia A., the teacher asked Ana Catarina to explain to her colleague her thinking process, to help her understand. The teacher answered a new question of Patrícia with a focalization question “Imagine the following: imagine that we didn’t have that sequence on the right side, how would we do it?” intending to take the students to elaborate a new thinking process.

Teacher: Imagine the following: imagine that we didn't have that sequence on the right side, how would we do it?
Ana Cristina: n×3-5.
Teacher: The expression will be n×3-5. Let’s see.
Ana Cristina: 6-5 equals 1.
Teacher: And it makes sense. Doesn’t it?
Patrícia A: Why 2?
Teacher: Because it is an illustration number 2. An illustration, not the order. The term order 2. And now the third, does it make sense or not?
Ana Cristina: Yes, teacher!

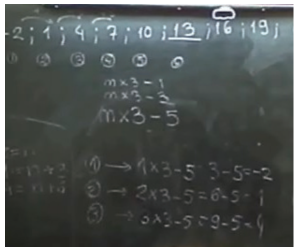


Along the discussion the teacher wanted to assure herself that the students understood what was being discussed and continued to put confirmation questions “Isn’t it? And now the third, does it make sense or not?” However, there were some students that continued asking questions.

Teacher: Yes... Say what you haven’t understood!
Mónica: Is the 1 the illustration?
Teacher: The 1 is the order.
Mónica: 3 is the +3 of the illustration.
Teacher: Yes!
Vanessa: It is what is constant.
Teacher: The 5 was the number about which Catarina (Ana) began, more or less, giving some clues there. She began to give some clues because she… O (Ana) Catarina why have you begun to say n×3 - 1?
Joana P: I said that!
Ana Catarina: Smaller, OK.
[Ana Catarina explains that she tried the expression n×3 -1 for the first term to be smaller.]
Teacher: What was your idea?
Ana Catarina: It came to my head.

One notes that the teacher posed inquiry questions: “Say what you haven’t understood! What was your idea?” in order to understand students’ questions and to make them clarify their thinking process.

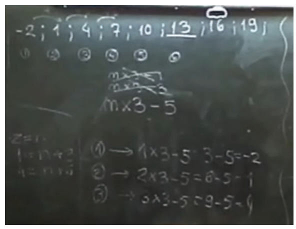
Teacher: It came to your head! But explain it a little bit better. Why has it come to your head? Explain that better.
Student: It starts in -2.
Teacher: Yes! Look there. This here, the first, in what does it begin? In 4 that it is very high. If you wanted it to begin in a lower number, instead of putting +1, what did you think …?. -1. But still it wasn’t enough. Then it was: second experience.
Joana: It was…
Joana P: Ah!!!!
Teacher: What was the second hypothesis? It was n×3.
Joana P: -3.
Teacher: But this, it has also revealed…
Joana P: It was the 0.



In the intervention “It came to your head! But explain it a little bit better. Why has it come to your head? Explain that better” the teacher enquired into the way the student reached the presented conclusion. The dialogue continued with the teacher putting focalization questions that sought to help in understanding of the several attempts that were made until the generating expression of the sequence was discovered.

Finally, the teacher continued making a series of confirmation questions, trying to verify that all the students understood the thinking process:

Teacher: The first result was 0. That wasn’t the answer also! [To Ana Cristina:] Don’t erase anything… Then … conclusion: This [hypothesis] wasn’t good enough, it didn't work, we had to go down. OK? What do you want to understand?
Mónica: I’ve already understood.
Teacher: Have you already understood?
[Mónica nods her head.]
Teacher: Are there any doubts?
Hugo: I haven’t heard a thing of what the teacher said.
Teacher: What haven’t you heard? You were playing with that small piece of paper. What haven’t you understood, Hugo?
Hugo: I haven’t understood what you want me to do.
Teacher: I ask for an expression with the letter *n*, which is to indicate all the terms of my sequence.
Hugo: *n*×3-5.



CONCLUDING REFLECTION

In the discussion with the group, the teacher aimed at sharing with all students the thinking process that led to the discovery of the general term of the sequence. The students posed a lot of questions that mainly sought to understand the thinking of their colleagues. During the discussion of this task, and taking into account the students’ difficulties, the teacher tried to mediate the dialogue among the students and to promote the participation of all of them. The teacher’s questions alternated – confirmation, focalization and inquiry. With focalization questions the teacher wanted to draw the students’ attention to the key aspects of the thinking process. In the course of the discussion, she felt the need to put confirmation questions to realize if students were following and understood the reasoning. The inquiry questions that the teacher put aimed essentially at understanding the questions of some students or at driving them to clarify their thinking.

In my teaching practice I have come across some difficult situations, in which students have difficulties in moving forward with the work. The analysis of this episode allowed me to realize that a possible strategy that I, as a teacher, can use, is formulating questions. It is possible to forward the answers to the students' questions to their colleagues and make new questions in order to make students focus their attention on the main aspects of the reasoning. This strategy, besides helping students to overcome their difficulties, also promotes sharing knowledge and helps us understand the students' questions.

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THE ROLE OF PARENTS IN THE ASSESSMENT CONTRACT[1]

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ABSTRACT
This article presents my experience during a school year with the parents of seventh-grade students. This involved an assessment contract and different activities developed with their parents. The partnership that I established with parents was fundamental and went far beyond my initial expectations and contributed much to the communication and cooperation between me and the students, between me and their parents, and also between students and parents.

KEY WORDS:
Assessment, didactical contract, parents' involvement

School is a complex world that consists of people, all of them with important roles and duties. It is fundamental for the success that teachers, students and parents undertake their roles with confidence. It is a process of joint actions and shared reflections in which the assessment is the integral part. This means emphasizing the formative function of the assessment, oriented at students' improvement. Defining an assessment system with strict criteria and implementing it in a careful and reflective manner, is one of the most important tasks that teachers have to do right at the beginning of school year. This implies not only great care for the criteria but also explanations to students and parents, assuring assessment transparency and clarity. It demands a strong and coherent behavior of teachers, so that learning assessment is compatible with teaching and learning practices.

I think that if parents collaborate in the entire teaching and learning process, then it is easier for students to meet the expectations. In this paper, I present my experience from the school year 2002/03 with the parents of seventh-grade students.

THE ASSESSMENT CONTRACT

Once my work with the students began in the second term, I had a meeting with the teacher that would teach them during the first term. The agenda was the following: to program the topics to be taught for me and her, to fix the assessment forms and instruments that she would use in her working methodology. I decided to attend the assessment meeting at the end of the first term. At this meeting I found out that: (1) students' performance during the first term did not meet the expectations, not only in mathematics, but also in other subjects; (2) there were disciplinary problems in mathematics that were not solved; (3) there were diagnosed difficulties in this group of

1 Part of this work was based on C. C. Nunes (2004). *Assessment as Control Process for Mathematics Teaching and Learning: A Study with Students of the Third Cycle of Basic School* (Master Thesis in Education, University of Lisbon). Lisbon: APM.

students, essentially related to writing and argumentation capacity; (4) the assessment forms and instruments used by my colleague comprised two written tests and five problems as housework. Their marks from zero to five points were added to the arithmetic average from the tests; (5) she students had essentially two kinds of classes – exposition and solving exercises from the adopted manual.

With this information, I decided that on the first day of classes in the second term (beginning in January), I would present to students my assessment proposal for the period until the end of the school year, with a detailed explanation of the methodology. The following table presents the assessment plan and tasks for the students:

	Written	Written/oral	Oral
Individual	*Portfolio* Two phases test Synthesis		Auto-assessment (oral)
Group		Investigations tasks report (open problems) Work project	

As to my proposal, I could see that comparatively to the first term, there would be significant changes in the work and assessment methodology. Students should experience new kinds of teaching and learning, with transversal curricular aspects, like reasoning, communication, formulation and verification of conjectures and presentation of the results. I wanted profound and revolutionary changes in assessment, comparing with what it was like in the first term and in the previous years, not only in mathematics, but also in other subjects.

Since the first day of classes, when I presented my proposal, the students' reactions were immediate, “Are we going to do only two tests until the end of the year? So, if we have a negative grade in one of them, we are going to fail the year!” The same reaction was from their parents, so that in the first meeting in the second term, when they saw the students’ grades from the first term, they showed serious doubts regarding my decisions and consequently regarding students’ final assessment. It became *difficult*!

Without a doubt it provoked a great impact. After all, the tests were used only as the assessment of current practice in all subjects taught in school. Therefore, it became evident that it was necessary to meet with parents and to explain it to them, so as to involve them in the teaching and learning process and in assessment.

TASKS DONE WITH PARENTS

The implementation and success of an assessment proposal and instruments as the one presented here, involves not only the teacher and students, but also, not less important, the parents. Because of the anxiety showed by parents, I decided to meet with them at least twice, once in the third week of the second term, to address the confusion, and for the second time at the end of the third term to talk about the activities done.

The first meeting took place on January 22nd, 2003 and was attended by seventeen parents. I considered parents’ high interest as a sign of the impact made by my assessment proposal presented to students. There were moments that I felt a mixture of anxiety, determination, fear, and blew it hot and cold in turns. But, keeping to my convictions, I began to explain the reason why I summoned that meeting: to clarify the assessment criteria and instruments that were going to be used in mathematics.

I informed the parents about assessment criteria approved by school mathematics teachers and I explained each one of the assessment tasks and how I was going to implement them. I pointed out that in the "students' portfolio" there were documents that explained how each one of the tasks was going to be used. I asked parents to collaborate and to participate actively in this process, suggesting that they should also periodically appreciate students' work. I pointed out their importance in evolution of students' learning style. I mentioned also that this work was going to be included in my master thesis and that the collected data were to be analyzed by me. I informed them that I would make two interviews with each student and make them fill in a questionnaire. The authorization request was delivered to students to be signed by their parents.

I was anxious about their reactions, and I let parents express their doubts, opinions and fears. For example, Francisco's parent said:

> I listened to you with attention and should tell you that I am happy with the enthusiasm and the energy that you have demonstrated in your explanation. I agree with the assessment proposal that you made and I think it is important to change and to do something different. It is sure that the traditional methods (using just tests) do not produce good result. The evidence is the results that our children had and in the general panorama of aversion to mathematics. But, as a father, I have a strong fear: what about the next year? Is this work going to continue? Are we going back to the system that has been being used so far? And what with the work and effort made by students to get used to this new process?

I could understand perfectly the fear of this parent and answered him that I could just assure this style of work for the whole school year. I added, however, that a lot of the work that I proposed to develop could be used in subsequent years, regardless of who would become their teacher. For example, students could continue to build their portfolio and reflect about their development and evolution. I pointed out that parents had a very important role and could help students in this process. Tomás's mother, who was present at the meeting agreed with me and wanted to express herself. Her intervention provoked an intense reaction among parents supporting her:

> I agree with you. And we, the parents, can also suggest that some of these assessment "elements" be implemented and kept in the future. This process should be implemented also in the other subjects.

After this first meeting, I felt that it was possible to work together: teacher-student-parent. The first effect of this partnership was very positive. Right after the meeting with parents, all the students whose parents were present brought a *portfolio* for me to evaluate. I was very satisfied with this surprise. It showed that my message was understood.

I should confess that initially, when I talked with the parents and involved them in the whole process, I had no idea about all the benefits. In fact, parents were pleased by the close communication between them and the teacher, and that they knew the diversity of tasks done and the evolution of students learning. During six months it was possible to see that the working *habits* and their *vision* of mathematics were evolving. We can see that in Francisco's own words:

> I did not study less but I did not make a big effort for tests and for mathematics investigation tasks. ... I have done homework and improved my *portfolio*. ... To fill in a test, I only copied ready examples from my notebook! ... But even when I copied, I studied mathematics and was learning. ... I was accustomed to doing exercises for the test, and we only made tests, practically that was what counted in mathematics, because the rest was always exercises and more exercises; and when it was some new given matter, we just copied to notebook and studied for the tests. But, this year with the portfolio, it was very different. ... Also there were other things: investigative tasks.

My comments helped students to realize what the strengths and weaknesses of their work were and consequently helped them improve their work and learning using my suggestions. As Sara stated, "When the teacher says something I try to improve ... and I try to answer her comments." Without doubt students tried hard to satisfy my expectations and it also was obvious that they were supported by their parents, as it was related by Francisco:

> I have many things to do, go to the computer and write an auto-assessment, answer questions, and a lot to do. ... It is a more demanding system. ... I feel that I am being evaluated daily. ... And my parents are always asking how it is going in my work in mathematics: if there is news. Now, it is a topic of conversation at dinnertime.

The second meeting was held on June 11th, 2003. There were 20 parents. This time, my state of mind was completely different – a much larger participation of parents made me feel good and happy. For me, this mobilization was the effect of the work with students in the last six months and improved communication. The objective of this meeting was to reflect on the work developed and students' learning evolution. I started to do a brief summary of the process evolution: the way students were involved and reacted to the different tasks proposed to them and to the way they were evaluated; the difficulties felt by both parts (teacher and students) and the way how it was overcome; the importance of my *feedback* on learning evolution and the work of the students, that helped them realize what the strengths and weaknesses of their work were, enabling the evolution of their learning; the positive influence of students' working together with their parents, although in most cases this had been done casually. Students were satisfied by the new style of assessment. There was unanimity by parents in their satisfaction. Parents formulated the wish that this style of work should be extended to all subjects. They expressed their regret concerning the following school year, when they learned that I would not be able to assure the continuity of this kind of work, because I was placed in a school in Lisbon.

CONCLUDING REFLECTIONS

The development of this project revealed an excellent opportunity for me to learn. To manage the curriculum and to use assessment as a way of learning was very enriching and gratifying to me, in particular, for my own learning and students' learning evolution. However, it was not easy to select the different tasks and work proposals and to do, in real time, the necessary changes, as a result of the joint reflections and/or individual with students, and to give an answer to students' different questions; to evaluate and to give *feedback* to students on their learning evolution; comment on students' work; to control and to manage the volume of information that resulted from the different assessment events; to reflect and to write about my practices and then make decisions about curriculum and assessment management. I learned a lot about my students and their learning process.

This experience was more than simply six assessment forms and instruments. This study evidenced the importance of a specific culture and a practice of consistent, diversified and transparent assessment, in which there is real intervention of the several factors in the assessment process. Because of this, I consider the partnership that I established with parents as fundamental and that it transgressed my initial expectations. Communication that was established between me and the students, between me and their parents, and also between students and parents was essential.

The results of this experience will be useful to motivate teachers to use one or more assessment manners and instruments to contribute to the diversification of their assessment practices. They also will be useful for students' and their parents' larger involvement in the assessment process, so that a new assessment culture emerges, having the informal communication between teacher and student as the pivotal instrument.

TEACHING AND LEARNING INNOVATIONS IN MATHEMATICS AND EDUCATIONAL PROCESSES: A TEACHER'S REFLECTIONS

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ABSTRACT
This contribution presents the reflection of a teacher about his experiences while participating in the design and implementation of the teaching intervention called algebraic modeling and solution of choice and optimization problems with a group of mathematics teachers in secondary schools of Modena. The chosen problems of the instructional intervention were above all focused on the development of techniques for representation, starting from the use of arithmetic and algebraic symbols, to end up with the graphical representation on the Cartesian plane, going through the organization of numerical data in tabular and set forms. As a result of the intervention, the communication processes in the class showed that satisfactory results were achieved in terms of the relational objectives.

MY EDUCATIONAL BACKGROUND

My experience in teaching mathematics in lower secondary school (students aged 10-14) started recently (in 2005) and up to now I have always been offered temporary teaching jobs and mainly aimed at supporting students with learning problems and at the recovery of students with difficulties working in small groups. When in 2006 I had the opportunity to attend courses in mathematics education offered by the Postgraduate School of Teacher Training (SSIS) at Modena to carry out an educational training within the specialization program and finally to participate in the European "Professional Development of Teacher-Researchers (PDTR)" Project, novelties I met in these contexts contributed to making me question the idea of mathematics teaching I developed in the past.

I must point out that the Italian school system entails a common teaching for mathematics and physics, chemistry and natural sciences in lower secondary school. For this reason teachers come from different undergraduate courses and careers. I graduated in chemistry and initially undertook mathematics teaching without a specific training. However, some didactical experiences at university level in the chemical-agricultural area, before going into mathematics teaching, allowed me to realize how even students with a high level of education in the scientific field often lack basic mathematical notions. These problems are particularly evident in abilities linked to the interpretation and elaboration of data, to the construction of graphs and tables, to the capacity of identifying regularities and making predictions, to the manipulation of numerical results and the conversion of different measurement units and orders of magnitude, or even to the capacity of making estimations and approximations. A dramatically relevant problem is the limited use of logical-argumentative inferences, which involves parts of the social life that go beyond the technical aspects mentioned here.

Being aware that basic mathematical culture is fragile in our society, and realizing that a mathematical-scientific background is considered of little importance in the wider Italian cultural panorama I decided to approach mathematics teaching enthusiastically. As a “non-mathematician,” I have always tried to investigate my students’ difficulties, proposing a strict comparison between possible solution strategies for problems and trying to suggest stimuli that might empower students’ motivation towards a subject, which is often perceived as “mechanistic” and, therefore, “unquestionable” and substantially distant from reality. It was relatively easy to break students’ hostility, through the promotion of the “playful” dimension of mathematics, focusing on the “amusement” implicit in the solution of problems – viewed as riddles – and on the actual solution of exercises through the correct application of the “rules of the game.”

A critical moment for me, due to my fragile background in mathematics, was when some students spontaneously asked me: “What is this for?” A naïve and “utilitarian” answer, which highlights future implications of a mathematical content, is also an inadequate answer; students with preconceptions, or hostile towards the subject, will not be more motivated by the awareness that they will have to keep on studying mathematics for the next few years, nor will they be attracted by professional openings, in which mathematical competencies turn out to be inevitable. At most, they will submit to study the subject uncritically. A pertinent answer to the previous question requires a relevant effort and commitment by teachers: they must reconstruct a context rich of meaning in which any curricular topic might be included. Contextualization might be of a historical-epistemological type, where the emergence of interest in a given aspect of the subject with respect to the evolution of human thought may be traced; more often, a reflection on intra-disciplinary and inter-disciplinary connections will be needed: they will allow teachers to locate a curricular content within a net-like system of relations. The latter will permit the promotion of the development of students’ logical-mathematical abilities, through a comparison to situations experienced as intellectual challenges by students themselves, and therefore as possible answers to real learning needs.

Teachers are thus in charge of hard and complex work and many of the current teachers are not inclined to do that. It is enough to enter an Italian school and exchange ideas with colleagues or rather, and this is my case, to have a privileged observational point of view on their teaching activity as a special needs teacher working in the class jointly with the mathematics teacher, to verify easily how mathematics is often imposed through an old traditional scheme, through a sequence of algorithmic procedures, aimed at getting a numeric result and completely out of context. A clear example of this is the request to solve problems by listing numeric data and ordered calculations to get to the desired answer. The problem’s text, either arithmetical or geometrical, is viewed as a container from which numbers can be extracted to perform operations, and the scarce fantasy showed by many textbooks is certainly not helpful in suggesting alternative routes. A global view of the whole route is missing and, even though teachers might have this view, it is not made explicit enough to the class or discussed with students.

In my opinion, what is missing is the discussion on what is actually being done, possibly to consider alternatives to the mechanical solutions of problems or simply to clarify why one should prefer a blind procedure rather than a thoughtful one. Working on a non-understandable “unsaid” is a habit which becomes dramatically problematic in the approach to algebraic calculus, usually undertaken in an advanced phase of mathematics teaching (students aged 13-14). The traditional approach proposes late introduction of negative numbers and the introduction to literal calculus through the study of monomials and polynomials and related operations.

It is easy to realize how, even for stronger students, this route is deprived of any reflection on the sense of these operations, on the huge implications of algebraic calculus, on the development of mathematical thought, and on the possibilities of generalization and application, which pertain to it. As for weaker students, who meet enormous difficulties with this type of mechanistic approach to algebra, the solution adopted by many teachers is to impoverish the sense of the activity, only expecting of them memorization of a series of rules and definitions.

SSIS COURSES AND PDTR

SSIS courses in mathematics education and then PDTR certainly were to me the first chances to get away from widespread stereotypes related to mathematics teaching and interiorized, if not during my limited professional experience, then during a long scholastic career as a student. In these courses I was solicited to critically think over objectives and modalities of mathematics teaching in compulsory school and this constituted a basis for the construction of my professionalism in this field. However, this is a still fragile basis for me, which needs suitable time to be consolidated, time for an individual reflection and, above all, working opportunities in the mid and long term, which might enable me to construct a continuous teaching plan with classes of students. Methodological inputs suggested by this training and educational process systematically included in a theoretical framework which highlighted the need to adopt teaching strategies that favor a shared and social construction of learning in the classroom. Those strategies will have to be implemented in real situations and deepened within a life-long continuing education, which only can guarantee results in the long term.

One of the first aspects highlighted by PDTR is the need for an international observatory on mathematics teaching practice. Recent surveys carried out by the PISA program worldwide point to the need of a common ground of mathematical and scientific competencies on which our students will have to exchange experiences in the future, to get prepared to a type of society characterized by global communication and by cultural and professional exchanges that go beyond national frontiers. The first results from the PISA test in recent years show how students from countries with a rich cultural tradition, like Italy, hardly prove to have a sufficient level of performance in mathematics and science. These alarming data might be partly explained by the distance between traditional educational models and the new standardized and globalized knowledge, and partly by the ghost of an actual involution in the levels of basic mathematical and scientific instruction.

At the national level recent reforms of the school system as well as of the curricula of compulsory school only partially meet the needs emerging from a comparison at international level; in-service teacher training only involves a marginal part of the teaching staff and it is almost exclusively carried out on a volunteer basis. There are still limited opportunities for cultural exchanges. One of the main vehicles to actually realize these kinds of exchanges, both among teachers and among students, is the linguistic one. In this sense the use of English as an active language in schooled population in Italy clearly shows the existence of a cultural gap with other European countries. In this sense, the new teachers' generation currently entering Italian schools is not generally able to have direct exchanges with European colleagues using a common language. The desired political process, which will tend to uniform European Union's educational systems in the next few years, thus providing opportunities for a wide mobility of both teachers and students, will be a further deepened challenge for Italian teachers.

Another fundamental aspect in mathematics teaching practice, correctly emphasized by PDTR, is the importance of giving a laboratory-like structure to teaching activities. The school is a laboratory itself and it is possible and dutiful to favor moments of sharing and elaboration of educational experiences in it. The model characterized by the mere transmission of contents from teacher to students was harshly questioned in the last decades and, in particular, it is going to lose its effectiveness in compulsory school teaching, especially in a society which goes through radical changes in its connective tissue, and in which the very social perception of school is going through a crisis. On the contrary, leaving aside traditional frontal lessons, viewed as an inevitable step by most teachers, does not always leave room to effective teaching and learning activities. The barriers between mental models teachers would transmit and conceptual net that students should construct remain there and are consolidated where the misconception elaborated by students in the adjustment of their own knowledge are not appropriately identified and questioned.

It is exactly by letting the learning subjects express themselves that teachers will easily manage to possibly detect rooted misconceptions through a teaching and learning action. It can not be reduced to a simple request to students of reproducing what teacher anticipated. It will rather be necessary to set up learning situations, in which one single concept may be tackled from different points of view, letting students "enter the situation," express their own doubts, make hypotheses about the solution process and above all, have a constructive exchange with other actors of the teaching and learning scene, not being afraid of an immediate judgment. The discussion in the classroom is thus a central moment of any teaching and learning activity, and it is effectively performed when it is characterized by a real exchange among peers. In order to achieve this, a long process needs to take place within a class, allowing students to work in an atmosphere characterized by respect and collaboration, as opposed to an environment based on antagonism, and leading to a gradual breaking down of the communicative barriers between teachers and students.

It is clear that, in this kind of class-based processes the very role of the teacher undergoes a transformation: the teacher drops the role of judge or censor and gets closer to the role of a team coach. In this view, complex teaching and learning activities will be favored in order to raise students' interest and challenge them: students thus become actors with reference to the proposed themes. Time scheduling of this type of teaching and learning activities will necessarily envisage a phase of individual reflection on the problem situations shown by teachers: this phase will end when each single student will reach a partial elaboration. After this phase, the actual discussion will follow and it will enable students to compare their solution strategies and consequently construct shared conclusions. Sharing conclusions through teachers' mediation will require students to be able to exchange their ideas, test correctness of classmates' suggestions and finally converge to an agreement on fundamental aspects. It is a route full of obstacles, which requires full control of the process by teacher, but most of all a prior work of reflection upon the potential development and possible expansion of the proposed activity, which might arise spontaneously during the class.

THE TEACHER TRAINING PROJECT AND THE ROLE OF MENTORS

The aspects described above were the basis for the design and planning of a teaching intervention carried out, with reference to PDTR, as training activity within a school where I am not a teacher. This project, entitled "Algebraic modeling and representation of problem situations" is adapted from the first part of a wider project outlined by a group of secondary teachers in schools based at Modena (Roberta Fiorini, Sandra Marchi, Romano Nasi and Paola Stefani) and called as a whole, "Algebraic modeling and solution of choice and optimization problems."

The whole sequence, structured through the proposal of a series of problem situations, is split into phases following the scheme:

Phase 1: Problems with 1 and 2 variables, aimed at the translation from natural language to symbolic and algebraic language; tabular representation and introduction to graphical representation.

Phase 2: Problems aimed at a merged use of algebraic, tabular and Cartesian representation forms.

Phase 3: Problems of choice with organization of data and collection of information.

Phase 4: Optimization problems.

My teaching intervention proposed in a seventh-grade class of a lower secondary school (students aged 12-13) was elaborated in collaboration with two colleagues (Patrizia Dodi and Maria Rizzo), who implemented their activity with eighth-grade students (aged 13-14), and was based on the design and administration of worksheets related to 5 problem situations drawn from phases 1 and 2 of the above mentioned teaching sequence. The teachers who designed this route formed the group of mentors who followed the development of the teaching activity over a series of 12 sessions. It was possible to have a constant exchange with them, both in the planning phase and in the phase of the teaching activity in the classroom.

Problems were adapted and enriched with respect to the initial proposal, so that they might meet the educational needs of the project. In particular, we meant to propose problems with an increasing degree of complexity and stimulate students to use gradually higher knowledge and abilities. The chosen problems are focused, above all, on the development of techniques for representation, starting from the use of arithmetic and algebraic symbols, to end up with the graphical representation on the Cartesian plane, going through the organization of numerical data in tabular and set form.

At the end of the first phase of the project familiarity with merged representation forms might enable teachers to provide the instruments needed to tackle real choice problems (choice between complex alternative possibilities) and optimization (management of variable parameters in a complex situation to get the best advantages, higher savings or any other benefit). At the very beginning of the teaching sequence I found necessary to propose an introductory problem, aimed at testing some pre-requisites entitled "Not much more than that." One out of the 5 subsequent problems was rejected during the sequence, because, together with the group of mentors, we noticed that it did not give enough space for a spontaneous graphical representation of the situation and privileged algebraic representations; the last problem was not tackled by students due to time-related issues. In the frame at the end of the paper, I report the texts of the problem I actually tackled in class.

Among the pre-requisites needed to undertake the sequence the following were identified: (1) numerical field: familiarity with the possible extensions of the numerical field from natural numbers to integers and rational numbers, and possibly to real numbers; (2) operations in N and Q, operations with fractions, percentages; (3)

representation of numbers on the oriented line; (4) representation of points in the Cartesian plane; and (5) inequalities and Order relations.

It is important to mention that the class I worked with had already carried out early algebra activities within the ArAl project (Arithmetic-Algebra), based on the introduction of equations and on the literal representation of variables (naming). This preliminary sequence tackled by the class proved to be extremely important for the success of my own teaching intervention.

It is nevertheless necessary to point out that among the pre-requisites that I considered as fundamental to set up the activity within the ordinary teaching and learning practice of the class there are competencies not strictly linked to curricular areas of the subject: metacognitive, linguistic and relational pre-requisites, linked to students' capacity of reflecting upon their own learning processes, expressing verbally their ideas and comparing them constructively with their classmates' ones. It is a very high-level request, which was nevertheless appropriate in that context thanks to the class mathematics teacher's (Romano Nasi) habit to favor an exchange and collective reflection on the themes dealt with.

RESULTS AND PERSPECTIVES

During my work the communication processes in the class showed that satisfactory results were achieved in terms of relational objectives. The mediatory role played by the teacher was almost always necessary to allow for a clear and logical development of discussions, but in some moments students were able to raise constructive exchanges among peers, being completely autonomous. High achievers often played a leading role during the discussion, but there were no juxtaposed groups and many of them showed they were able to change their own beliefs, after accepting others' argumentation. Low achievers were able to gradually overcome their shyness and, in the last lessons, they intervened more actively and spontaneously in the discussion, showing involvement in and attention to the teaching sequence.

As to the contents, I can point out that aspects related to graphical representation were potentially the most motivating ones for students of all levels; in particular, they refer to experiences already made by students and can be widely used to introduce problem situations rather than as consequences of the latter. Spontaneous graphical representations are not only Cartesian ones; therefore, it is possible to open up a comparison between the different types of graphs and on the adequacy of each one to represent different problem situations. On the contrary, formal aspects that characterize the problems chosen in the sequence (use of symbolic language, naming, order relations, inequalities), also with reference to students' age, caused the main difficulties and would require parallel supporting activities involving simple exercises of construction of numerical intervals, representation on the oriented line, identification of variables and constants, translation from verbal to symbolic language.

As to the latter aspect, I would underline that the relevance of linguistic aspects of mathematics was to me one of the main discoveries during the PDTR meetings. Through the training activity I actually saw the relevance of linguistic obstacles, which make the interpretation of texts with a mathematical content problematic well before their translation into the most typical languages of this discipline (numerical, algebraic, tabular and graphical). For many students this process implies an extremely hard move from a narrative context to a logical-relational one. This aspect is often neglected in the ordinary mathematics teaching activity, whereas it would require an in-depth reflection by teachers, especially in our Italian society, in which the number of foreign students poses problems about the use of language in all the subject matters.

OBSERVING, OBSERVING ONESELF, BEING OBSERVED

One last reflection I consider relevant as to the teaching intervention I described relates to a methodological aspect inherent in the observation of the activity I carried out. I worked with a class of which I am not a teacher and therefore I could avail myself of a constant support by the class teacher, who regularly pointed out the weaknesses of my own teaching approach: scarce communicative clarity in some moments, excessively repeated requests to students, slow rhythms in the class-based processes, etc. This precious contribution was further enriched by the opportunity of a mutual observation of each other's intervention carried out by myself and colleague Patrizia Dodi in our respective classes. This opportunity allowed for an ongoing evaluation of the quality of the project proposed in different contexts, besides helping me to compare my own style to that of another teacher involved in the same activities. It is an experience which convinced me about the validity of activities carried out jointly by two teachers, with the aim of improving one's professional performances.

Finally, since the whole teaching sequence was video recorded, the careful analysis of the recordings strongly highlighted the main features of my own modus operandi in the classroom. It is embarrassing and instructive at the same time to see yourself during a class, to find out that in different moments you were not clear enough, that you did not grasp immediately the opportunities offered by students to guide the lesson towards fertile grounds for a discussion. Moreover, video recordings were precious instruments for a posterior study of the discussion process, as well as for grasping different aspects of each single student's participation in the activities.

PROPOSED PROBLEM SITUATIONS

1. Not much more than that

On a shelf we put some flour and some sugar, as long as we get to 10 kg. How much sugar and how much flour can I put?
Would 2 kg of flour and 5 kg of sugar be ok?
Yes, together they are 7 kg, less than 10.
Would 7 kg of flour and 7 of sugar be ok?
No, because the sum is greater than 10.
Would 6 kg of sugar and 4 of flour be ok?
No, because we must not get to 10 kg.
Let's represent these cases in a Cartesian graph.
a) - Would 7 kg of sugar and 2 of flour be ok?
- Would 6 kg of flour and 10 of sugar be ok?
- Would 9 kg of flour be ok?
- What about 9 kg of sugar?

b) – Suggest other quantities of sugar and flour and decide whether they fit with the request not to get to 10 kg.
c) – Now that you inserted many points in the plane, can you say in what position are the points which refer to acceptable quantities of flour and sugar?
And where are the points to throw away?
– Did you identify the two regions which indicate acceptable points and non-acceptable points? How are they separated?

2. The illustrated book

A book for children features 25 fairy-tales and it is richly illustrated; each fairy-tale is illustrated with a number of images ranging from 8 to 15.
a) What are the minimum and maximum values for the illustrations included in the book?
The book illustrator received from the publisher 30 Euros for each illustration.
b) What is the range of values for the illustrator's pay?

The second edition of the fairy-tales book will prospectively have 300 illustrations altogether; the publisher enters into a new contract with the illustrator, agreeing on a variable pay, depending on the complexity of the illustrations, ranging from 20 to 45 Euros.
c) How much might the illustrator get?
Represent the situations using the language of mathematics in the form you see as more appropriate.

3. Big feed with cream puffs
Some cream and chocolate puffs are displayed in a confectioner's window.
The former cost 0.60 Euros, the latter 0.90 Euros.
Pietro buys 8 puffs, we don't know which type though.
Marco, instead, buys 8 to 12 puffs, all of the cheapest type.
Represent the situation using the language of mathematics in the form you see as more appropriate.
In your opinion, which boy might have spent more money?
If only 3 out of the puffs bought by Pietro are filled with chocolate, how many puffs can Marco buy at most, in order to spend less money than his friend?
If Marco buys 10 puffs and Pietro spends more money, how many chocolate puffs can Pietro buy?

4. The cake
Marco's mum wants to bake a cake with a new recipe. She needs butter and margarine in quantities that may vary within certain limits.
She has small 25 grams butter and margarine packets. Butter ones are not more than 8 and margarine ones are not more than 7.
For the cake she needs 5 to 10 small packets altogether, and she'd better use more butter than margarine.
Represent the situation using mathematical language and ask yourself how many butter packets and how many margarine packets that mum can choose for her cake.

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MATHEMATICAL COMPETITION IN SIEDLCE AIMING AT SKILL TO LEARN MATHEMATICS

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ABSTRACT
In order to arouse interest in learning mathematics as a language to communicate, to predict, and to explain, i.e. in PISA style mathematics, each year a competition is organized in Siedlce, a small town, 100 kilometers east of Warsaw. Each year the competition focuses on a chosen topic outside of the obligatory school curriculum. Two groups of students, juniors and seniors listen to an inaugural lecture on that chosen topic and are given some problems to study and presentation by mail and e-mail. The competition has three stages and the study goes deeper at every stage, up to the final sitting session. It seems that this way special ethos for independent learning of mathematics as a language is aroused in the community and vicinity. A large number of participants every year means that mathematical topics that are left outside of obligatory curriculum and studied as a choice by free will might be attractive for young people and give them satisfaction.

KEY WORDS:
Competition, problems solving, PISA style mathematics, mathematics as a language

The basic document indicating main goals of general education in Poland is a legal act voted by the Polish Parliament (Sejm), and it is called Program Basis (Podstawa Programowa). It is stated there that the main aim of mathematical education is developing of mathematical perceptibility, intellectual skills and attitudes, reasoning, and analyzing.

In spite of that strong intent, there were in recent years sharp topic cuts in external mathematical exams, and as a result, sharp programmatic cuts in schools. In middle school (gimnazjum, students aged 13-16), both trigonometric proportions and functions are now absent, in high school (lyceum, students aged 17-19), there were sharp cuts in geometry, there are no vectors, and only a few spatial shapes, no exponential or logarithmic functions, no steps into direction of calculus, not even derivatives. In spite of declarations about developing spatial imagination, there is practically no motivation for it, as it is absent at the exams.

As a consequence school mathematics has become boring rather than luring students into studying of that subject. The main aim of activity at a typical mathematical lesson is preparation for exams which are narrow topic-wise but very strict in traditional grading in a preconceived system of allocation of scoring points.

Three years ago, together with a group of friends, members of the Polish Association of Teachers of Mathematics in Siedlce, I decided to counteract this tendency. Since that time, we organize a mathematical competition each year, called

"Turniej Wiedzy Matematycznej," TWM, (Tournament of Mathematical Knowledge and Skills) which is different from other such initiatives since it aims at *topic broadening of mathematical skills and competency, and encouraging skill learning in topics outside of obligatory curriculum.* The start of PDTR was an additional incentive for this activity.

The TWM Competition is for students aged 14-19 and it is divided into two categories: junior (students aged 14-16) and senior (16-19). Participation is voluntary. TWM is held annually, and it has three stages every year. It directs students' attention to the topics outside of school curriculum, to independent learning and studying, and building independent inquisitive attitudes toward mathematics in broad semantical context, so as to stimulate mathematical development in both categories of students. This way of formulating the goal of the TWM was chosen in order to take advantage of the young age expressed in the concept of Vygotsky's *zone of proximal development* and our belief that if not used at the proper time in the process of development of young students, then the opportunity given by that zone may not return. By analogy, everybody knows that foreign languages should be taught and learned early in life with good *semantic rooting, developing meaning rather than abstract formal structures*. Mathematics in the style promoted by PISA is less formal than in traditional style and it is much more like a *language used to communicate, to explain, and to predict.* Such style is appealing not only to the able and motivated but also to average students.

The idea of TWM was also to probe the feasibility of introducing by another channel, for those who wish, the important topic areas of mathematics that were practically cancelled out from the school mathematical curriculum in Poland. Each year, we focus our attention at *one topic area*. The topics chosen in the past years were the following: (i) complex numbers (school year 2005/6); (ii) vectors (school year 2006/7); (iii) geometry and visualizing in the problem solving PISA style (school year 2007/8).

Each year the TWM is opened by a lecture, explaining the topic area to the participants, followed by distribution of some written materials for further study and problem solving. After collecting the papers from participants, directly or via e-mail, the next round follows. Those who pass this stage are invited to the sitting session at the local university, the papers are assessed with comments, and those who are successful are invited to the final session for a short lecture, diplomas, and modest prizes.

Every year evaluation is oriented toward checking the validity of our claims that we stimulate interest in mathematics and independent study of it. It helped to introduce some improvements in the management of the competition. We clearly noticed the increase in ethos for independent study. There were motivated participants not only from Siedlce but also from distant areas. The contact was not only with participant students but also with their teachers, and in many cases with parents. From the first year on, the number of participants exceeded our expectations. In 2005/6 – the first year of TWM – there were over 800 applications. Because there was no lecture hall of that size, we were forced to accept only 8-10 students from each school. Still, there were too many students, around 400, and we had to hold the initial session twice, since the biggest lecture hall could accommodate not more than 180.

In the consecutive years the numbers were at the same level. The initial topic lecture and other materials were put on the web in the consecutive years, so that we do not have to deny participation to too many students.

We adopted PISA style grading system and it was much to satisfaction of the participants and their teachers. Visual argumentation was encouraged. Mathematics as a language is visual, not only geometry is visual, even algebraic formulas are perceived visually. Visual communication was encouraged and it was accepted. Students used graphs and sketches freely, more frequently than usual. Evaluation of the TWM competition was by questionnaires and interviews. Also the number of participants was to some extent a significant indication. Here are some positive observations from questionnaires and interviews: (1) Does interest in mathematics grow among students? Yes, it seems so; there are frequent enquiries for more details about the *topic of the year*. A frequent demand is that the initial lecture should elaborate deeper on the topic chosen; (2) Does it stimulate teachers? Yes, many students at the initial lectures are accompanied by their teachers, which initially was a problem because of the size of the lecture hall. There are also growing contacts with teachers of the participants; (3) Is the specific mathematical ethos growing? It seems so. Anyway, communication between participants and their teachers increases, also communication and relationships between participants; (4) Is social acceptance in the community positive? Among parents? Local authorities?

The material gathered by means of questionnaires each year is positive, and each year a little different. For instance, communication by the web increases, and broader questions emerge, much more specific than before. There was a strong interest in the topic area for the following year. The questions about it are more specific and deeper. Parents that come to the final session with the students tell interesting stories about growing interests of their offspring. There are interesting differences among participants from different schools.

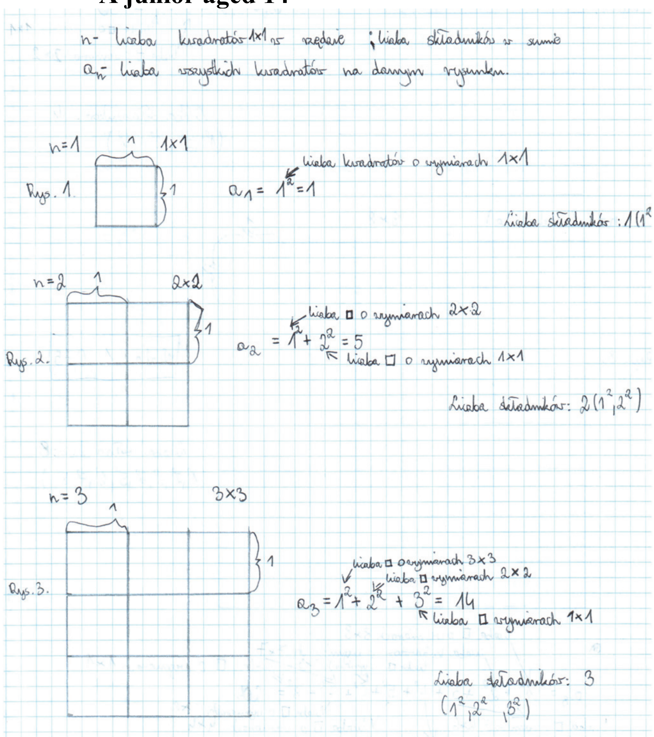
Differences in style between argumentation and visual communication of the students of the two age groups of participants are interesting, especially when they approach the same mathematical problem. Here is one of the problems:

Chessboard

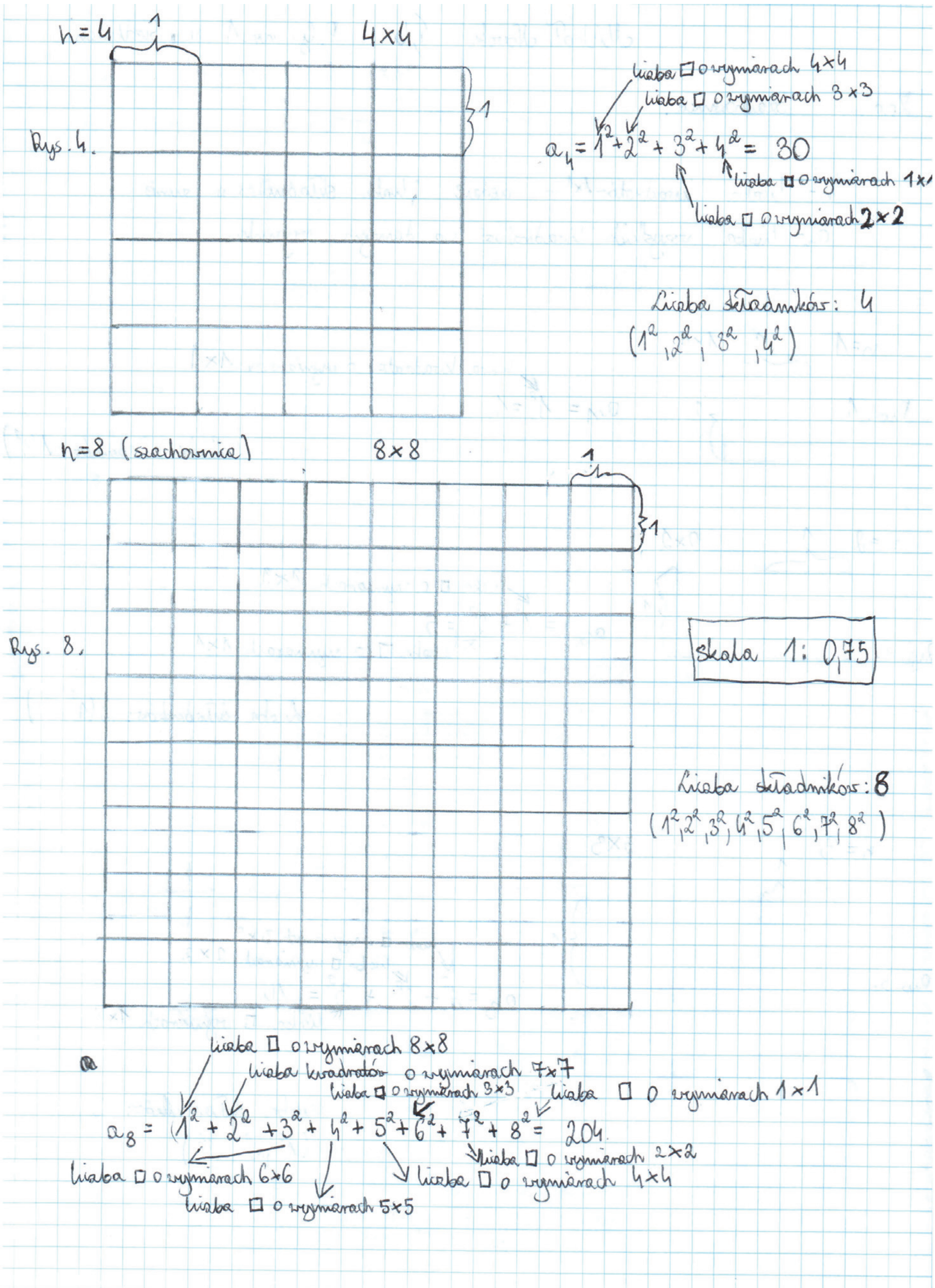
How many squares are there on this picture?
Five!
Can you see it?
How many squares are there on 3×3?
And on 4×4?
How many squares are there on the chessboard?
And how many on $n \times n$ board?

And here are two examples of how it was approached in the two groups, junior and senior:

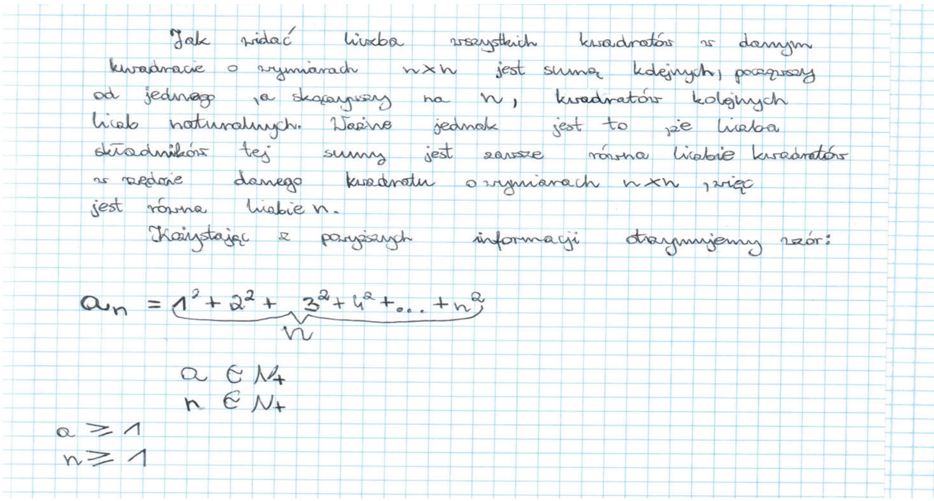
A junior aged 14



He notices the number of squares and gives their sizes. He also notices the number of components of the sum. The smallest squares are unit squares. In case of the chessboard 8 × 8, he remarks that the drawing is on the scale given by the ratio 1:0.75



and ends up giving an algebraic expression for the number of all squares at the chessboard $n \times n$.



Jak widać liczba wszystkich kwadratów w danym kwadracie o wymiarach $n \times n$ jest sumą kolejnych, począwszy od jednego, a skończywszy na n, kwadratów kolejnych liczb naturalnych. Ważne jednak jest to, że liczba składników tej sumy jest zawsze równa liczbie kwadratów w rzędzie danego kwadratu o wymiarach $n \times n$, więc jest równa liczbie n.

Korzystając z powyższych informacji otrzymujemy wzór:

$$a_n = \underbrace{1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2}_{n}$$

$a \in N_+$

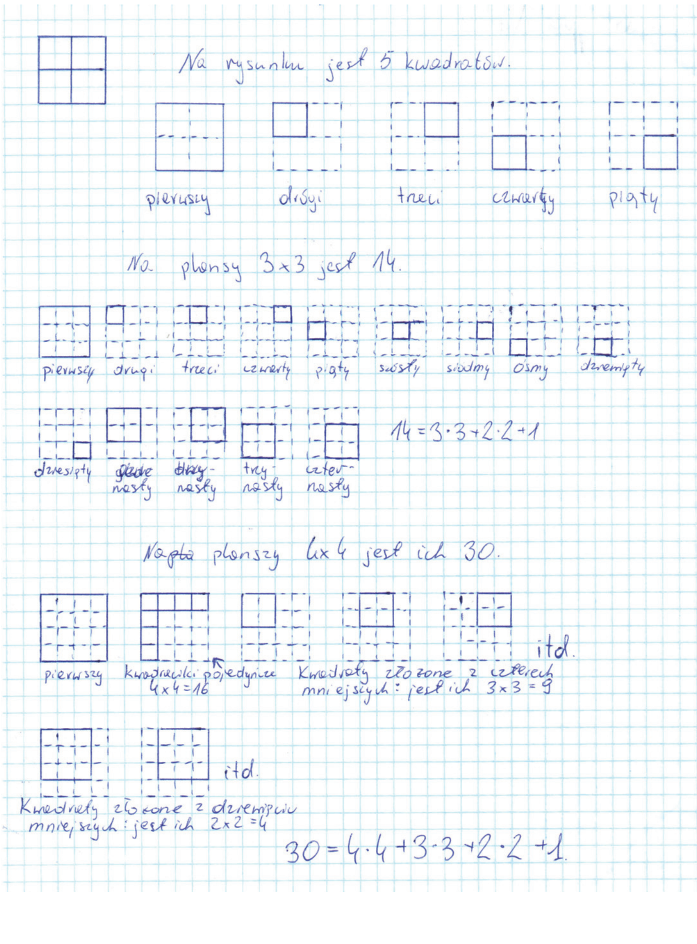
$n \in N_+$

$a \geq 1$

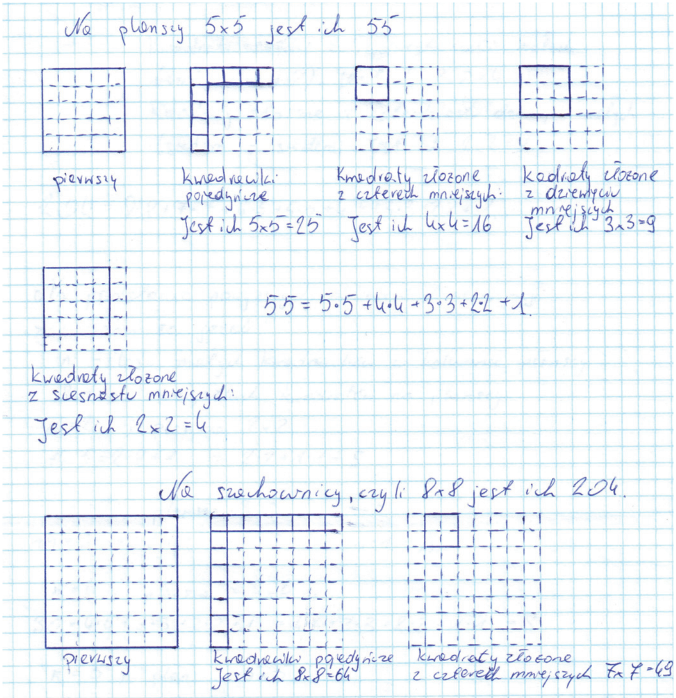
$n \geq 1$

A senior aged 17

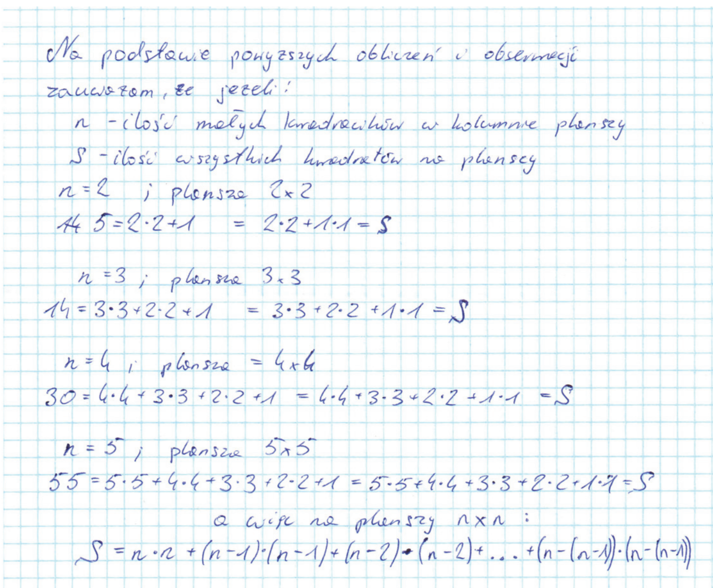
He begins in a similar way, looking visually for pattern, which would give him a hint.



He graphically expresses how he enumerates all the squares, which are seen on the chessboard. He shows some regularity in this activity.



He explains these observations by some arithmetical formulas and finally, he writes the general algebraic formula for any number *n*:



Na podstawie powyższych obliczeń i obserwacji zauważam, że jeżeli:

n – ilość małych kwadracików w kolumnie planszy

S – ilość wszystkich kwadratów na planszy

$n = 2$; plansza 2×2

~~4~~ $5 = 2 \cdot 2 + 1 = 2 \cdot 2 + 1 \cdot 1 = S$

$n = 3$; plansza 3×3

$14 = 3 \cdot 3 + 2 \cdot 2 + 1 = 3 \cdot 3 + 2 \cdot 2 + 1 \cdot 1 = S$

$n = 4$; plansza $= 4 \times 4$

$30 = 4 \cdot 4 + 3 \cdot 3 + 2 \cdot 2 + 1 = 4 \cdot 4 + 3 \cdot 3 + 2 \cdot 2 + 1 \cdot 1 = S$

$n = 5$; plansza 5×5

$55 = 5 \cdot 5 + 4 \cdot 4 + 3 \cdot 3 + 2 \cdot 2 + 1 = 5 \cdot 5 + 4 \cdot 4 + 3 \cdot 3 + 2 \cdot 2 + 1 \cdot 1 = S$

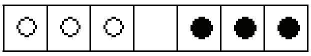
a więc na planszy $n \times n$:

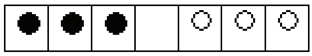
$S = n \cdot n + (n-1) \cdot (n-1) + (n-2) \cdot (n-2) + \ldots + (n-(n-1)) \cdot (n-(n-1))$

In spite of a three-year age difference between them, both students are reasoning in a similar way and arrive at the same conclusion. The senior only explains his formula but he does not try to prove it, as a sentence with the general quantifier: “For every natural *n*, etc.” It seems that he considers *n* in his formula as an *arbitrary constant*, not as a variable under the quantifier. At this stage of mathematical development, it is mathematically correct. It would be interesting to see how these two will develop, and if they show up the next year. The TWM Competition gives us a lot of such comparative data.

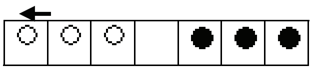
At the first stage of the TWM in 2007/8, one of the problems given to participants for explanation was called “Żaby” [Frogs, sometimes also called in English “Leapfrogs”]. This is a well-known problem for people in Polish Association of Teachers of Mathematics (Zawadowski, 1997; Mostowski, 1999; Mason, 2005). The problem was included in the collection of problems *Ziarenka* [Seed Problems]. All members of Polish ATM know the collection. These problems are sometimes called “Starting Points” to express the idea that they are only starting problems and all possible generalizations and extensions in the mathematical style are welcome. The problem is the following:

> There are seven places in a row and six counters: three white, and three black counters, called “frogs” and a free place in the middle. White and black frog should exchange places, like that:

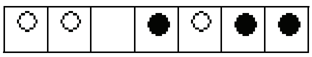




You may shift a frog to a neighboring free place



or leap over a frog to land at an empty place



> Using only these two kinds of moves, you should change the places of white and black frogs. Which is the smallest possible number of moves to this aim? Investigate the problem for different number of counters.

We noticed that both junior and senior students started their investigation by considering concrete cases. Students represented visually the frog movements, and considered cases with two, three or four frog of the same color:

O	O	O		X	X	X	
O	O		O	X	X	X	1 ruch
O	O	X	O		X	X	2 ruch
O	O	X	O	X		X	3 ruch
O	O	X		X	O	X	4 ruch
O		X	O	X	O	X	5 ruch
	O	X	O	X	O	X	6 ruch
X	O		O	X	O	X	7 ruch
X	O	X	O		O	X	8 ruch
X	O	X	O	X	O		9 ruch
X	O	X	O	X		O	10 ruch
X	O	X		X	O	O	11 ruch
X		X	O	X	O	O	12 ruch
X	X		O	X	O	O	13 ruch
X	X	X	O		O	O	14 ruch
X	X	X		O	O	O	15 ruch

Then, they completed a table for these cases showing the smallest number of moves for a given number of frogs of a given color, using ad hoc symbols like 3 × 3 with obvious metonymical shift of meaning.

ILOŚĆ PIONKÓW KAŻDEGO KOLORU	ILOŚĆ RUCHÓW POTRZEBNYCH DO ICH PRZESUNIĘCIA
3 × 3	15
4 × 4	24
5 × 5	35
6 × 6	48
7 × 7	63
8 × 8	80
9 × 9	99

Then, they looked for regularities and tried to find a pattern and fit in a formula.

$4^2 + 2 \cdot 4 = 16 + 8 = 24$

$5^2 + 2 \cdot 5 = 25 + 10 = 35$

$6^2 + 2 \cdot 6 = 36 + 12 = 48$

$7^2 + 2 \cdot 7 = 49 + 14 = 63$

$8^2 + 2 \cdot 8 = 64 + 16 = 80$

$9^2 + 2 \cdot 9 = 81 + 18 = 99$

$10^2 + 2 \cdot 10 = 100 + 20 = 120$

Możemy wyrazić to ogólnym wzorem:

$$\boxed{n^2 + 2n}$$

gdzie n to liczba „żabek" po jednej stronie.

No junior student considered types of frog movements. It was enough for them to discover the formula. The senior students did. Seniors gave not only the number of moves required but also how many of each kind.

w prawo – 1 ruch
w lewo – 2 ruchy
w prawo – 3 ruchy
w lewo – 3 ruchy
w prawo – 3 ruchy
w lewo – 2 ruchy
w prawo – 1 ruch

Pionki:	Ruchy:
2	$1 \cdot 2 + 1 = 3$
4	$1 \cdot 2 + 1 + 2 \cdot 2 + 1 = 3 + 5 = 8$
6	$1 \cdot 2 + 1 + 2 \cdot 2 + 1 + 3 \cdot 2 + 1 = 3 + 5 + 7 = 15$
8	$1 \cdot 2 + 1 + 2 \cdot 2 + 1 + 3 \cdot 2 + 1 + 4 \cdot 2 + 1 = 3 + 5 + 7 + 9 = 24$

Wzór dla n liczb to:

$$n + 2(1 + 2 + 3 + \ldots + n), \quad n \in N$$

n – liczba pionków po jednej stronie.

One of the students noticed separately the number of shifts and of leaps. Another student noticed the symmetry of the situation and used it in a reasonable way.

$n = 3$ (l. pionków białych lub czarnych)

l. przesunięć: 6 ($2n$)

l. przeskoków: 9 (n^2)

l. ruchów dla n pionków białych lub czarnych to l. przesunięć $2n$ dodana do liczby przeskoków n^2

wzór ogólny: $n^2 + 2n = n(n+2)$

The TWM continues to be interesting for students. The number of participants grows, so that we have to repeat the inauguration lecture twice and have to use the web for continuation and *topic refinement*. There are students who are "veteran participants" and some that are new. All of them know that they will not meet the kind of problems from TWM at exams, or at traditional lessons. They know that participation in TWM is time consuming and needs a lot of work. So perhaps our dream that the mathematical ethos of PISA style problem solving will increase in our community slowly comes true.

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THE USE OF VIRTUAL ENVIRONMENTS FOR ALGEBRAIC CO-CONSTRUCTION

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ABSTRACT
The purpose of this paper is to present conversation forums in a virtual environment, as a preliminary study exploring the possibilities that they offer as mediators on the co-construction of learning algebra for students aged 13-14. Co-construction is observed and analyzed.

KEY WORDS:
Algebra, conversation, electronic environments

INTRODUCTION

Traditional instruction begins with the syntactic rules of algebra. Students are expected to master the skills of symbolic manipulation. The mathematical context is taken as the starting-point while the applications of algebra (like problem solving or generalizing relations) come in the second place. It leads to the following four problems: (a) problems related to the content, like the general tendency to consider algebra as part of the curriculum devoted to the development of techniques; (b) problem of lack of motivation among the students (Kaput, 2000, 2); (c) problem of management of the time that usually is granted to students to mature and to express their mathematical ideas; (d) problem of students aged 13-14 who are not used to solving problems using strategies, in which algebraic language and tools are involved.

The purpose of this paper is to encourage teachers to guide students to find out the powers and possibilities of algebra for themselves. In order to achieve it we raise the possibility of using conversation forums to discuss and to look for strategies of solving problems in a collaborative environment. Using forums allows students to become the center of their own learning. Teachers should change their role from control and infallible source of information to the designer and moderator facilitating discourse and scaffolding by providing direct instruction (Anderson, 2001). In this experience knowledge is accepted as constructed through interactions between and among teachers and students. We will see some examples of tasks and tools used as mediators in order to improve the teaching and learning of algebra from the use of natural language to a progressive co-construction of algebraic language. Our purpose is not the analysis of these difficulties, but exploration of possibilities that conversation forums offer as mediators in the co-construction of learning (Mueller, Maher and Powell, 2007). Mathematical tasks determine the behavior of students grappling with them. But tasks are always tasks-in-some-context and the nature of the context also becomes a component of the determining matrix.

FRAMEWORK

Three general aspects are included as framework for this research: algebraic perspective, socio-constructive approach of teaching and learning, and shared meanings using ICT. The co-construction of solutions is to be an interactive process, by which questions and challenges to individual ideas yield responses that produce greater detail and refinement of arguments. The understanding of algebra is a set of affirmations, in which it is possible to produce meaning in terms of numbers and arithmetical operations (Lins & Giménez, 1997, 137). They possibly include equality or inequality, in which six forms of algebraic reasoning are involved: (i) generalizing and formalizing, (ii) algebra as syntactically-guided manipulation, (iii) algebra as the study of structures, (iv) algebra as the study of functions, (v) relations and joint variation, (vi) and algebra as a modeling language (Kaput, 1998).

But we also take into account different ways present in algebraic thinking in early grades: (i) analyzing relationships between quantities, (ii) noticing structure, (iii) studying change, (iv) generalizing, (v) problem solving, (vi) modeling, (vii) justifying, (viii) proving, and (ix) predicting, (Kieran, 2004, 149)

According to Boaler (2003), teachers in traditional classes give a lot of information, while teachers in the reformed classes chose to draw information out of students by presenting problems and asking questions. Our main hypothesis is that the forum of conversation, as a CMO form, can turn out to be a useful instrument to solve or discuss problems jointly. It allows for reflection during the necessary time to have access to different points of view or contributions, to review what is attempted to be communicated before sending it. It combines characteristics of the spoken and written speech that can facilitate collective learning (Balacheff, 1998).

According to Weimar (1998), the nature of the mathematics classroom can be changed through the use of linked technology. This work is a small sample of changes that can take place in the classroom when students can hold conversations about mathematics, receive individualized and personal support, and participate in collaborative problem solving with other partners (Bairral & Gimenez, 2004). The condition of transformation agents assigned to the ICT is worth to be taken into account for conceiving deliberate interventions to change the pedagogical models, the practices in the classroom, and the curricular contents in educative systems in order to lead the students towards a significant and satisfactory learning (Rojano, 2003, 138). Our general project aims to ensure teachers that technology is actually an efficient learning resource, a means to acquire technical attitude and skill required to tackle a problem successfully, not just an optional software module.

METHODOLOGY

An experiment was conducted to recognize, which types of conversation about the developing strategies of resolution of algebraic problems are influenced by such a collaborative environment. The population for the study were three groups of students aged 13-14 from a public secondary school in Girona (Spain). The experiment was developed during two months. The whole class attended the computer classroom once a week during the year 2006/07. This paper analyzes three problems and their conversation forums on the virtual environment *Moodle*, which is new, both for teacher and students. *Moodle* is a course management system, a software package designed to help educators create quality online courses and manage learner outcomes. The design and development of *Moodle* are based upon a particular philosophy of learning, a way of thinking that is referred to in shorthand as “socio-constructive pedagogy.” The problems have been presented in order to solve them after arguing and discussion through the conversation forums designed for it. At the end each student had to present an individual task.

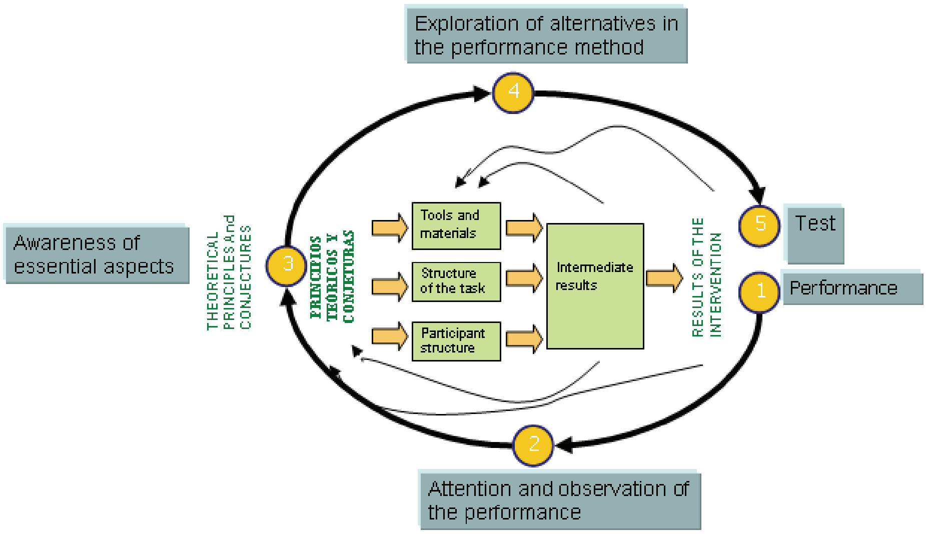


Figure 1. Scheme of empirical experiment as a Design based Research

The general course comprises a combination of learning activities for students, such as Assignment, Chat, Choice, Forum, Glossary, Journal, Lesson, Wiki, and Workshop. The data consists of the dialogues on the forums, the results of a written individual test responded by the students at the end of the experience, a survey written after the experience, and teacher's observations during the process. In this experiment the classroom teacher performed the role of teacher investigating about her own professional practice following a cyclic process. As the diagram shows, it implies an iterative and evolutionary process that requires a series of experimentation, evaluations and readjustments. Let us see the problems used in our analysis.

A ping-pong match
If in a ping-pong championship *n* players play. What is the total number of parties that will gamble?
Hens and pigs
In a farm there are 18 animals, hens and pigs all together. The number of legs is 50. How many hens and how many pigs are there?
Divisions
Which is the smallest number such that when divided by 4, 5, and 6 always returns 3 as the remainder?

All the conversations were observed (Figure 1) and codified in terms of considering types of reasoning, use of concepts, procedures and representations. They are analyzed according to arguments (Mueller, Maher & Powell, 2007).

Re: hens and pigs from Natàlia - Friday, 31 March , 11:52
Merce, I don't understand what you did.

Re: hens and pigs from Mercè - Friday, 31 March, 12:01
I have found the number of pigs by chance. But when you have already found the pigs, you find the hens. It seems difficult but it is not it as much. Try it!!
Re: hens and pigs from Natàlia - Friday, 31 March, 12:05
Anyway, I believe we could put “n” instead of finding the number by chance. Thus, it would be like an equation… is it ok?

Figure 2: A fragment of conversation from the forum belonging to the “Hens and pigs” problem.

DISCUSSION

Some children’s remarks are important to understand their attitude. Children did not perceive seven weeks (once a week) as a long time for doing the task. As they say, it allows you “to take your time.” After this consideration and after the experience the teacher even considers that this time has not been sufficient enough. They also think this way is better than others because they have access to more “opinions.” The coexistence of both types, actual and online interventions, is perceived by students as a better way than only one of them.

Observing the forum conversations we found several important issues. Children read the interventions of their partners much more often than they write in the forum. This can be associated with more reflexive interventions when they do them. Interactions between students of similar and different cognitive levels take place. In the design of these forums teachers have to consider the difficulty of some mathematical expressions. Significant learning is produced by seeing the evolution of writings. The ICT tool is not an obstacle to express arithmetic relations (Fig. 3) even using emoticons.

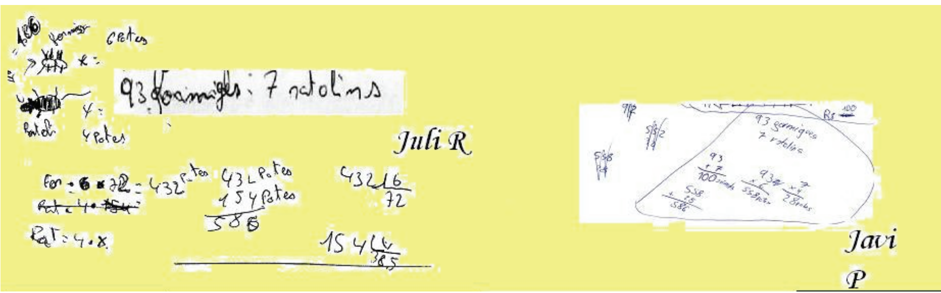
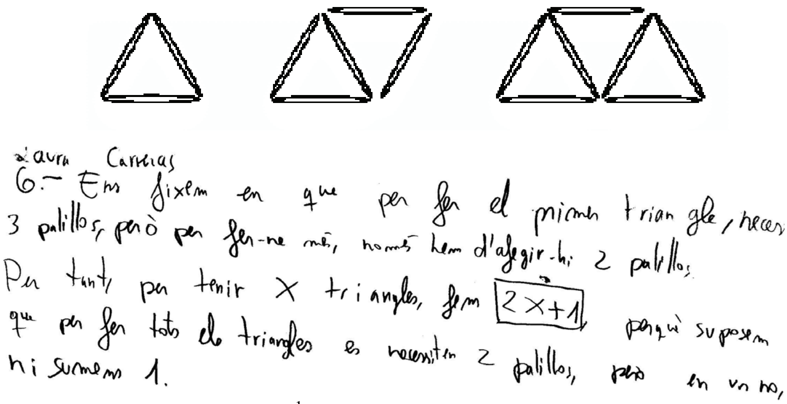


Figure 3. Juli and Javi writings

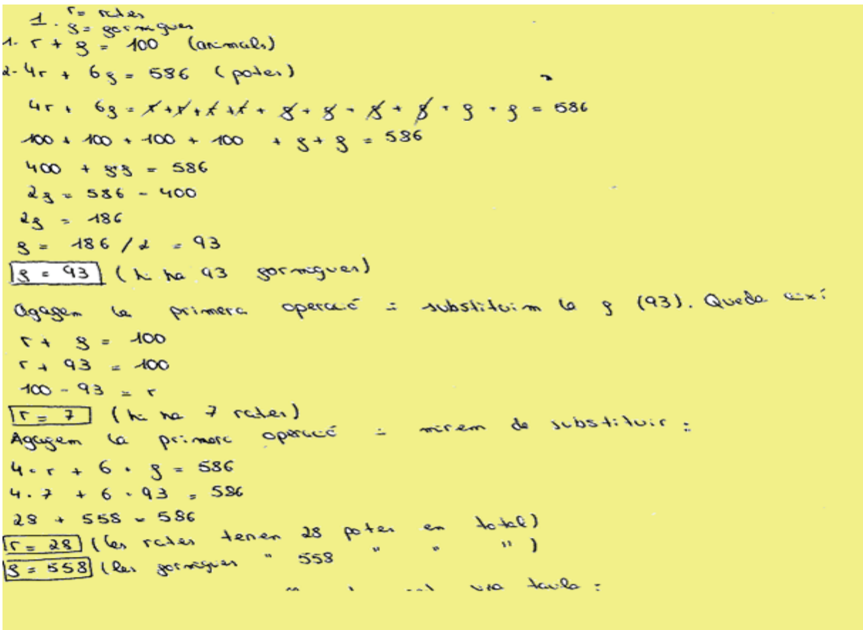
Some explicit generalizations were found, even without a good symbolic manipulation. We can see some examples in Figure 4 and 5.



Laura Carreras
6.- Ens fixem en que per fer el primer triangle, neces
3 palillos, però per fer-ne més, només hem d'afegir-hi 2 palillos.
Per tant, per tenir x triangles, fem $\boxed{2x+1}$ perquè suposem
que per fer tots els triangles es necessiten 2 palillos, però en un no,
hi sumem 1.

Figure 4.[1] Explaining a generalization

Moodle resources seemed not to generate specific difficulties for using particular strategies, because people can attach scanned documents with their comments. Regular writings are used as artifacts to communicate mathematical ideas.



1. r = rates
g = formigues
1. r + g = 100 (animals)
2. 4r + 6g = 586 (potes)

$4r + 6g = \cancel{r}+\cancel{r}+\cancel{r}+\cancel{r} + \cancel{g}+\cancel{g}+\cancel{g}+\cancel{g} + g + g = 586$

$100 + 100 + 100 + 100 + g + g = 586$

$400 + 2g = 586$

$2g = 586 - 400$

$2g = 186$

$g = 186 / 2 = 93$

$\boxed{g = 93}$ (hi ha 93 formigues)

Agafem la primera operació i substituïm la g (93). Queda així:

$r + g = 100$

$r + 93 = 100$

$100 - 93 = r$

$\boxed{r = 7}$ (hi ha 7 rates)

Agafem la primera operació i mirem de substituir:

$4 \cdot r + 6 \cdot g = 586$

$4 \cdot 7 + 6 \cdot 93 = 586$

$28 + 558 = 586$

$\boxed{r = 28}$ (les rates tenen 28 potes en total)

$\boxed{g = 558}$ (les formigues " 558 " " ")

Figure 5: Explaining a particular strategy

Many relations were found and expressed in different languages and representations, even without good final results. Some examples are showed in Figures 6, 7 and 8.

[1] We noticed that 3 small sticks are needed in order to make the first triangle. Afterwards, it is only necessary to add two sticks every time. So, we take 2x+1 sticks to have x triangles, because we suppose 2 sticks for each triangle, and 1 more at the begining.

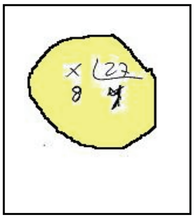


Figure 6.

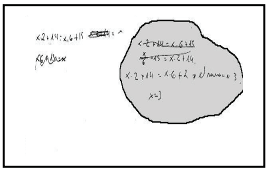


Figure 7.

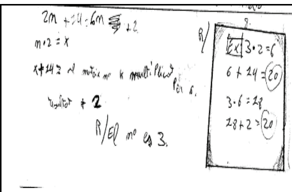
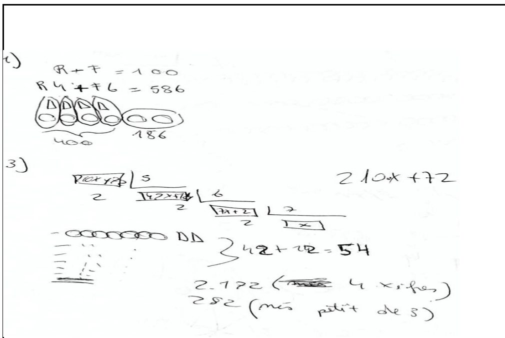


Figure 8.[2]

Students achieve self-confidence in their own strategies, as shown by the examples in Figures 9, 10 and 11. It also means that there is a need for including traditional writings to explicit their personal comments.



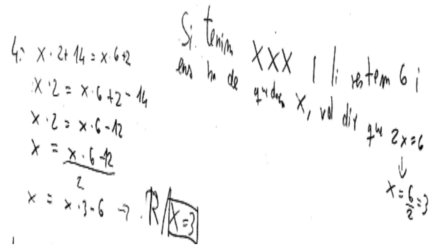


Figure 9 and *10*.

In our preliminary results, we found that many students move from re-iterating ideas of others to expanding on ideas and finally to jointly co-constructing solutions.

[2] Solve the equation *2n+14=6n+2* ; *n.2=x ; x+14=* just like to multiply the number by 6 + 2
R) Eg) *3 . 2 =6 ; 6+ 14 = 20; 3 . 6 = 18; 18 + 2 = 20*
R) The number is 3

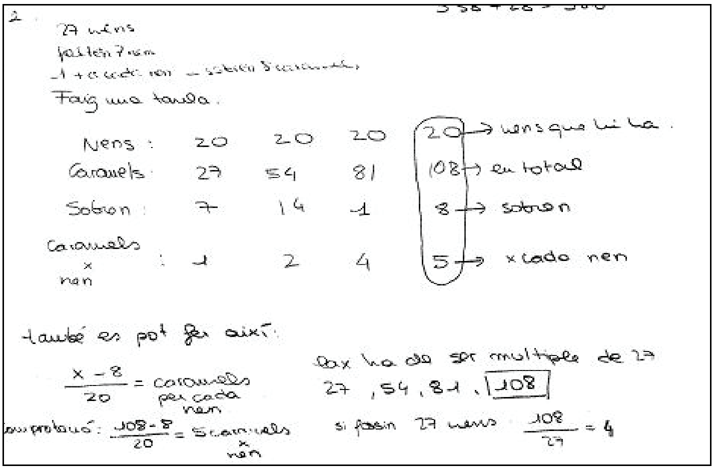


Figure 11.

CONCLUDING REMARKS

A reflection on the process and the results of this study brings to consider that on this level the conversation about mathematics through electronic forums can both affect the motivation of the students and act like mediators in the construction of algebraic knowledge. In addition, the characteristics of their repercussions in the learning and in the role of the teacher deserve to be taken in consideration. Our experiment is one component in a much wider learning setting, which encompasses *Moodle* lessons conveying the theoretical concepts and self-assessment tools,

Teachers need to guide the pedagogical setting towards situations in which relevant aspects are discussed, such as posing questions related to the critical analysis of contexts or the necessity for the generation of new and useful information to promote attention (Ainley & Luntley, 2007). According to the reflection inspired by this study, the roles of the teacher using *Moodle* environment that must be taken in consideration are: (a) design and organization of the learning experience, (b) maintaining proper articulation of activities and conceptual matters by selecting representative algebraic problems to encourage discourse, (c) detection of possibilities, difficulties, etc. (d) guide and scaffolding when necessary, (e) promoting participation of all the students, (f) promoting a progressive way of abstraction and generalization processes, (g) promoting reflection, conjecture and experimentation, (h) legitimize students' directions of inquiry, redirect their attention, encourage certain initiatives and discourage others, (i) assessment and evaluation of the activity.

We found that critical thinking (Bairral & Giménez, 2004) and the increasing use of argumentation through online interactions to develop metacognitive ability are closely aligned with the aims of progressive inquiry. Online asynchronous interaction is not a problem for constituting a community of practice, in which remote experimentation provides a simple way to consolidate knowledge. Furthermore, as Heids (1998) points out, collaborative work with the help of the technological tools can help to increase students' autonomy. This methodology aids students in gaining competence in working both independently and in team, managing time effectively and using computer resources appropriately. The implementation and research experience also act as a technological professional development for the teacher as teacher-researcher but also for the experienced mathematics education researcher. To understand the classroom as a learning community is also to notice observation that improves students' capabilities in mathematical activities. The action-research process acts as a new consideration for artifacts never imagined by the teacher as researcher.

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PART 2

EXPERIMENTS USING PISA SAMPLE QUESTIONS

MATHEMATICS TESTS

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ABSTRACT
The paper presents condensed information on mathematics tests and ways of solving them. This is followed by an analysis of the relation of students' mathematical knowledge and PISA test. Finally, the use of tests as school assessment tool in Hungary is shown.

TESTS

A test is a systematic procedure for comparing two or more persons. There are many different types: (i) intelligence test, (ii) assessment test, (iii) competence test, (iv) neuro-psychological test, (v) behavioral test, (vi) educational test, (vii) mathematics competitions. In mathematical tests there are a *variety of forms*: (1) alternative choice (false or true); (2) multiple choice: (a) one right answer (for example out of five answers), (b) more right answers, (c) searching for the best answer, (d) searching for the wrong answer; (3) connection of answers and facts; (4) choice of response: (a) open constructed response, (b) short response, (c) long response, and (d) essays.

Standardized tests

Standardized test is a test administered and scored in a standard manner. These tests are designed so that the questions, the conditions for scoring procedures are predetermined. There are two types of standardized tests: (1) norm-referenced tests, which are associated with traditional education and knowledge; and (2) criterion-referenced tests, which are associated with competencies.

Advantages and disadvantages of the tests

A well-designed standardized test provides an assessment of students' knowledge, skills or competencies, and it gives useful information. We have to mention the problems of students' and teachers' attitude to tests. Many students occasionally guess. They choose their answer by chance. They mark their answers on the answer sheet without thinking, without knowing anything about the content. We have to pay attention to students' conduct when selecting tests. It is better to choose the multiple-choice form with five answers; here the score does not depend so much on luck. For teachers a test is a model. They teach the subject matter the way it is tested. We have to know that tests are not valid as evaluative tool near the bottom of the score's range.

Characteristics of tests: (a) reliability (b) validity (c) effectiveness.

METHODS OF SOLVING TESTS

Students like to solve tests. Competitors told us that it is a joy, a challenge, a focus on mathematical experiment. The methods and tactics of solving test problems are different: from the typical mathematical problem solving methods through clever guessing and systematic trials, up to guessing, the role of which is growing. The written solutions, where reasoning from the assumptions to the answer is important, are not required. Competitors can work forward and backward, too. They have to make decisions quickly and relatively well. It is important to make computations without mistakes. The knowledge of general mathematical theorems and the exploitation of the symmetry principle are useful. We found another problem in filling the answer sheets: students make mistakes because they have to pay more attention to right marking.

The method of substitution of the choices is a special method to solve multiple-choice test problems. We shall present and compare two different methods for solving a test problem, a traditional method and the above-mentioned substitution method.

Problem

For which k are the roots of the equation $x^2 + 2(k+2)x + 9k = 0$ equal?
(A) 4 (B) 1 or 4 (C) 0 or 4 (D) -1 or 4 (E) 2 or - 4

Solutions

1. Standard method. First, we calculate the value of the discriminant: $D = 4(k+2)^2 - 36k$. The roots are equal if $D = 0$, i.e. $k_1 = 1$, $k_2 = 4$. Answer B is right.

2. Method of substitution

It seems best to try at first the value $k = 4$. Then the equation becomes $(x + 6)^2 = 0$. Possible answers could be: A, B, C. We have to check another value too. We can easily recognize that $k = 0$ is a wrong and drop answer C. For $k = 1$ we get again $(x + 3)^2 = 0$. So we got the proper answer: B.

ABOUT THE PISA ASSESSMENT

PISA presents students with problems set in real-world situations. The goal of the PISA assessment is to obtain results about how students can apply their mathematical knowledge and competencies to solve such a kind of problems successfully. In traditional mathematics education students learn arithmetical techniques, completing an arithmetic computation and solving different types of equations, equation-systems, inequalities; geometric properties and relationships; combinatorics, probability and statistics; some relevant concepts, skills; and methods of proving theorems.

How is mathematics competency measured in PISA?

PISA measures mathematics performance in three dimensions: (1) mathematical content, (2) the processes involved, and (3) the situations in which problems are posed.

Mathematical content

The different problems and questions are related to the following four areas: (1) space and shape, (2) change and relationships, (3) quantity, and (4) uncertainty. Traditionally, this content covers the knowledge of algebra, arithmetic, geometry, functions and probability.

Processes

The processes involved need to connect the observed phenomena with mathematics. Beginning with real world problem students have to organize it according to mathematical concepts. They have to transform it into a mathematical form by making simplifying assumptions. They need to identify the relevant mathematical concepts, then perform mathematical operations, make a mathematical model, and re-translate the result into the original problem, into the real world. In this process various *competencies* are required: (i) thinking and reasoning, (ii) argumentation, (iii) communication, (iv) modeling, (v) problem posing and solving, (vi) representing, and (v) using symbolic, formal and technical language and operations.

PISA distinguishes *three competency clusters*: (1) reproduction cluster, (2) connections cluster, and (3) reflection cluster.

The items of the *reproduction clusters* are relatively familiar and essentially require the reproduction of practiced knowledge, such as knowledge of facts and common problem representations, recognition of equivalencies, recollection of familiar mathematical objects and properties, performance of routine procedures, application of standard algorithms and technical skills, manipulation of expressions containing symbols and formulas in familiar and standard form, and carrying out straight-forward computations. The *connections cluster* builds on solving problems that are not simply routine, but still involve somewhat familiar settings or slightly extend and develop beyond the familiar. Problems used here typically involve greater interpretation demands, and require making links between different representations of the situation, or linking different aspects of the problem situation in order to develop a solution. The problems addressed using the competencies in the *reflection cluster* involve more elements than the others, and additional demands typically arise for students to generalize and to explain their results.

Situations

There are four types of situations: (1) *personal situations*, which directly relate to students' personal daily activities; (2) *educational or occupational situations*, which appear in a student's life at school, or in a work situation; (4) *public situations* relating to the local and broader community. They require students to observe some aspects of their broader surroundings; (5) *scientific situations*, more abstract, involving the understanding of a technological process, a theoretical situation, or an explicit mathematical problem ("intra- mathematical" context).

The PISA assessment items were constructed to cover these different dimensions.

MATHEMATICS TESTS IN HUNGARY

In the years of the PISA tests (2000, 2003) in Hungarian schools tests were not used for assessment in mathematics. In other subjects (foreign languages or biology) the teachers used that form, so tests were not unknown to students, but they were not expert at filling and solving mathematics tests. In this subject only highly able students encountered tests used in some competitions: American Mathematical Competitions (AHSME, AIME, AMC 10, AMC 12 grade 9-12), Zrínyi Competitions (grade 1-8), Kangaroo Competitions (grade 9-12), Gordiusz Competitions (grade 9-12) (from the 1990s). We found correlation between the traditional competencies and competitor's competencies of this kind of mathematics. Since 2005 some parts of the mathematical final examination assumed the test format (grade 12). So our students were not accustomed to solving test problems, they did not know the methods of solving tests. If we look at the history of Hungarian education we find that some educators applied tests in mathematics since World War I (Éltes) and at some schools in physics in the 1930s (in Debrecen: Barra and Tóth). Barra wanted to introduce the test assessment at university entrance examinations too (1948), but at that time it was impossible. As a consequence of the Hungarian state politics his ideas were refused.

Preparing our students for the AMC we solved with them tests of the previous years and drew their attention to these methods. We discussed guessing, too. We advised them to at first read the text of the problem attentively. It is necessary to solve the problem carefully if there are wrong answers between the offered answers. There is a penalty for wrong answers! Sometimes we can find among answers possible mistakes (for example confusing the concepts of perimeter and area).

"Distractor" is a test-makers' term for a wrong answer. The use of this word suggests that the answers are there to trick students. Suppose they are asked to find the diameter of a circle meeting some condition. Now we suppose that the answers include both 5 and 10. The problem asked for the size in terms of diameter, but they figured it out in terms of radius. Students did a lot of good thinking and hard work to prove that the radius is 5. If after all their work they made the small slip of forgetting to convert, they got the answer wrong. Another example: students are asked for the percentage change if a price increases by 20 % and then by 20% again. Suppose both 40% and 44% are among the answers. Students reason that percents add and choose 40%.

In both cases students have been distracted by a "distractor" but the situations were very different. In the first case students were mostly right. In the second case students used a completely wrong approach.

It is important to determine, which aids students could use in solving test problems. At the competitions it was allowed to use ruler and compass. We observed that the competitors made scaled figures, determined the necessary measures, and in this way they could guess the right result.

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STUDENTS' MATHEMATICAL LITERACY IN PISA ASSESSMENT: SAMPLES OF PISA TASKS IN TEACHER-RESEARCHERS' WORK

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ABSTRACT
Teachers, members of the Rzeszów/Kraków team used PISA 2000 and 2003 sample tasks "Apple trees," "Continental area," and "Carpenter" to assess students' mathematical competency in their classrooms, at the same time learning how to conduct research according to the "teaching-research – one task" procedure. Difficulties and errors of students from various schools, various levels and various locations are presented, and also their team analyses and attempts to find corrective methods.

KEY WORDS:
PISA, mathematical competency, mathematical literacy

1. MATHEMATICAL LITERACY IN PISA ASSESSMENT

Program for International Student Assessment Organization for Economic Cooperation and Development PISA OECD conducts assessment of how young people aged 15 are prepared to meet the challenges of life in the contemporary world regardless of the education systems they went through. The assessment is carried out cyclically every 3 years; so far it was conducted in the years 2000 and 2003 in 41 countries and in 2006 in 57 countries.

Elementary literacy that enables the employment of knowledge and reasoning in solving every day problems consists of: (i) reading literacy; (ii) mathematical literacy, and (iii) scientific literacy. Mathematical literacy in general refers to such a level of individual comprehension of mathematics and of the role mathematics plays in the contemporary world (in formulating opinions based on mathematical reasoning) and the command of basic mathematical skills that an individual can efficiently and with satisfaction apply them in a variety of situations created by everyday, social, professional and political life. (*Pisa 2000 Report*, 24; *Pisa Report 2003*, 6)

In selection of tasks that are used to assess mathematical literacy the following criteria are considered: (1) *mathematical content*: (a) space and shape (geometric situations and spatial relations between objects); (b) numbers (numerical values in real situations, calculations, estimates and approximation); (c) change and relationships (changes and relationships between variables represented in a symbolic, algebraic, graphic and tabular manner); and (d) uncertainty (probability and statistics). (2) *mathematical reasoning processes*: (a) *reproduction* (trained skills and very simple objects used in typical problems); (b) *connections* (association of mathematical concepts in less typical problems, a solution of which requires a few steps or a justification of an answer); and (c) *reasoning* (includes a creative approach to a problem, original mathematization, generalization and such an analysis of a situation that enables students to pose a mathematical problem, enables its solution, interpretation, explanation and justification of its solution). (3) *situation context*: (a) *personal* (associated with students' every day life); (b) *educational/occupational* (associated with teaching other subjects/ with professional work of people from students' environment); (c) *public* (associated with communication, banking and environment protection); and (d) *scientific* (pure mathematical situations, physical and technical contexts that require mathematics).

These skills are assessed by means of tests – questions and problems associated with real-world contexts, which are solved by randomly selected students within 120 minutes. Data about learning conditions and methods are collected by means of 30-minute-questionnaires for students participating in the assessment and headmasters of their schools. Only some of the test contents are revealed, the so-called "anchoring problems" which are used for result comparison in the subsequent editions of the research are not revealed.

Scores for each of the assessed skill are presented on a scale calibrated in such a way that mean score for OECD countries is 500 and 2/3 of the scores are placed within the range of 400-600. The procedure of scaling the test scores takes into consideration the number of tasks solved by students, the level of task difficulty and the number of students in the population of a country the students represent. As a result of this procedure a given number is allocated to students of a given country in each assessed skill. As an outcome of the research conducted in 2003 the scale of proficiency in mathematical literacy was divided into 6 Levels and typical students' skills were specified for each of these levels. The highest Level is 6 and the lowest is 1.

General assessment of mathematical literacy in a form of mean score for each country participating in the 2006 research is presented in Table 1. Scores above 504 points are considered higher than average, scores in the range 495-504 points as not significantly different than average, while scores lower than 495 points as lower than average.

Number, country, mean score of proficiency in mathematical literacy					
1. Taiwan	549	20. Germany	504	39. Greece	459
2. Finland	548	21. Sweden	502	40. Israel	442
3. Hong Kong (China)	547	22. Ireland	501	41. Serbia	435
4. South Korea	547	23. France	496	42. Uruguay	427
5. Netherlands	531	24. Great Britain	495	43.Turkey	424
6. Switzerland	530	25. Poland	495	44. Thailand	417
7. Canada	527	26. Slovak Republic	492	45. Romania	415
8. Macao (China)	525	27. Hungary	491	46. Bulgaria	413
9. Lichtenstein	525	28. Luxembourg	490	47. Chile	411
10. Japan	523	29. Norway	490	48. Mexico	406
11. New Zealand	522	30. Lithuania	486	49. Montenegro	399
12. Belgium	520	31. Latvia	486	50. Indonesia	391
13. Australia	520	32. Spain	480	51. Jordan	384
14. Estonia	515	33. Azerbaijan	476	52. Argentina	381
15. Denmark	513	34. Russian Federation	476	53. Columbia	370
16. Czech Republic	510	35. USA	474	54. Brazil	370
17. Iceland	506	36. Croatia	467	55.Tunisia	365
18. Austria	505	37. Portugal	466	56.Qatar	318
19. Slovenia	504	38. Italy	462	57. Kyrgyzstan	311

Table 1. Mean score in students' proficiency in mathematical literacy in 2006

Students’ mathematical attainments in a given country are also manifested by the percentage of students of a given country placed in higher Levels 5 and 6 and percentage of students placed on Level 1 and below. The graph below presents the percentage distribution of students on levels of mathematical literacy scale in 2006 for selected countries.

Country (mean score)	Levels (L) on the mathematical literacy scale:						
	<L1	L1	L2	L3	L4	L5	L6
Poland (495)	5.7	14.2	24.7	26.2	18.6	8.6	2.0
Hungary (491)	6.7	14.5	25.1	26.5	16.9	7.7	2.6
Spain (480)	8.6	16.1	25.2	26.2	16.8	6.1	1.2
Portugal (466)	12.0	18.7	25.1	24.0	14.4	4.9	0.8
Italy (462)	13.5	19.3	25.5	22.1	13.3	5.0	1.3
Average for OECD countries (498)	7.7	13.5	21.9	24.3	19.1	10.1	3.4
Taiwan (594)	3.6	8.3	14.3	19.4	22.4	20.1	11.8
Finland (548)	1.1	4.8	14.4	27.2	28.1	18.1	6.3
Netherlands (531)	2.4	9.1	18.9	24.3	24.1	15.8	5.4

Table 2. Percentage of students on the levels of mathematical literacy scale in 2006 in selected countries. Source: OECD PISA database 2006.

Mean scores of students in Poland and Hungary indicate that literacy of the assessed students reaches Level 3 of the mathematical literacy scale. Mean scores of students in Spain, Portugal, and Italy reach Level 2 on the scale. In all countries participating in the PDTR Project (Professional Development of Teacher Researchers, the grant of the Program Socrates of the European Commission) a large percentage of students with low mathematical literacy rank at Level 1 and below Level 1, and a small percentage of students have high mathematical literacy rates at Levels 5 and 6.

Mean scores in countries like Taiwan, Finland or Netherlands show that it is possible that almost ¼ of students in total reach proficiency Levels 5 and 6 and fewer than 10 percent of students rank at Level 1 or below.

2. SELECTED PISA TASKS IN RESEARCH OF THE RZESZÓW/KRAKÓW[1] PDTR TEAM

The PISA assessment results provided us with information about the Polish students’ literacy as compared with students in other countries. This information forced a reflection. What factors contribute to such results? How mathematics education can be changed so that students attain high scores? What is the impact of problem selection in the process of mathematics teaching, teachers’ work style, students’ attitude to learning mathematics and the belief that mathematical literacy is useful in one’s life – on students’ skills to apply mathematics in solving problems?

We used the revealed PISA tasks of 2000 and 2003, among others: “Apple trees,” “Continental area,” and “Carpenter” in our classrooms. First, we solved the tasks ourselves and discussed what mathematical activities as defined by Krygowska (1986) and Niss (2002) are developed by these tasks. We agreed that Polish students aged 16-17 possess sufficient mathematical knowledge to solve these tasks; however, the

[1] Ewa Szczerba , middle school in Radecznica, Agnieszka Drąg, middle school in Boguchwała, Grażyna Cyran middle school no. 11 in Rzeszów, Jerzy Migoń middle school in Trzebunia; Katarzyna Sasor-Dyrda School Complex in Łańcut, Katarzyna Radoń high school in Kalwaria Zebrzydowska, Katarzyna Dymek-Nowak high school in Kraków, and Agata Przężak-Bielak high school in Kraków.

formulation of these tasks may be untypical for them. Then, each of the teachers used these tasks in their assessed classrooms at the same time learning how to conduct research as part of the project according to the "teaching-research – diagnostic-test instruction" procedure.[2]

2.1 The "Continent Area" Task (Antarctica) (PISA 2000)

Estimate the area of Antarctica using the map scale. Show your working out and explain how you made your estimate. (You can draw over the map if it helps you with your estimation).

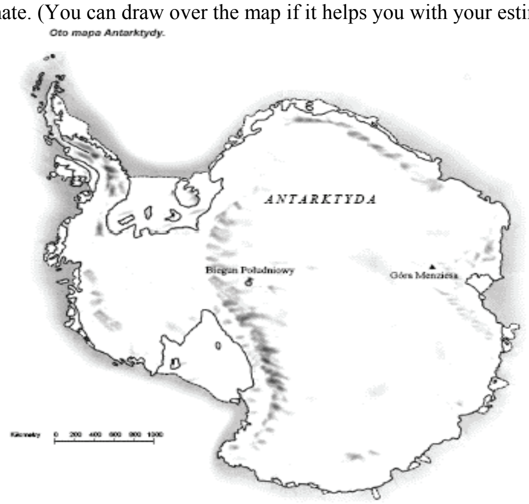

Figure 1. A map of Antarctica

In PISA task typology this task's mathematical content is associated with space and shape; mathematical competencies refer to relationships and situation context is personal. Mathematical activities assessed by this task include: estimating the area of a figure with irregular shape via its approximation to a sum of regular geometric figures; using a map scale to estimate the area of a figure in reality and calculation of length units or area units. In an attempt to employ mathematical activities as defined by Krygowska (1989) we decided that in solving that task we use: analogy observation, schematization, coding and algorithm application. Taking into consideration mathematical competencies defined by Niss (2002), we deal with posing and solving mathematical problems, building mathematical models and communication about a situation that contains mathematical content when solving this task.

PISA Scoring
Full credit - 2 points
These scores are for answers that use a correct method AND give a correct result. The second digit indicates different approaches.
Code 21: Answers which are estimated by drawing a square or rectangle: between 12 000 000 sq km and 18 000 000sq km (units not required)
Code 22: Answers which are estimated by drawing a circle: between 12 000 000 sq km and 18 000 000sq km
Code 23: Answers which are estimated by adding areas of several regular geometric figures: between 12 000 000 sq km and 18 000 000sq km
Code 24: Answers which are estimated by other correct methods: between 12 000 000 sq km and 18 000 000sq km (draws a large rectangle and subtracts the area of the parts outside the map)

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[2] The procedure is described in "Mathematical Tests" by Katarzyna Sasor-Dyrda, Ewa Szczerba, and Grażyna Cyran in the *Handbook of Teaching Research*.

Code 25: Answers which are correct (between 12 000 000 sq km and 18 000 000sq km) but no working out is shown
Partial credit - 1 point
These scores are for answers that use a correct method BUT give an incorrect or incomplete result. The second digit in parentheses indicates different approaches, matching the second digit in parentheses of the full credit scores.
Code 11: Answers which are estimated by drawing a square or rectangle – a correct method but an incorrect or incomplete result: (i) Draws a rectangle and multiplies width by length, but the answer is an overestimate or underestimate (e.g. 18 200 000); (ii) Draws a rectangle and multiplies width by length but the number of zeros is incorrect, (e.g. 4000 x 3500 = 140000); (iii) Draws a rectangle and multiplies width by length but forgets to use the scale to convert to square kilometers (e.g. 12 cm x 15 cm = 180); (iv) Draws a rectangle and states the area is 4000 km x 3500 km. No further working out.
Code 12: Answers which are estimated by drawing a circle – a correct method but an incorrect or incomplete result
Code 13: Answers which are estimated by adding the areas of regular geometric figures – a correct method but an incorrect or incomplete result
Code 14: Answers which are estimated by a correct method – but an incorrect or incomplete result, e.g. draws a large rectangle and subtracts the area of various parts outside the map
No score
Code 01: Calculation of the perimeter instead of the area: e.g. 16 000 km as the scale of 1 000 km would go around the map 16 times.
Code 02: Other incorrect answers: e.g. 16 000 km (no working out is shown, and the result is incorrect.)
Code 99: No answer

Results

This task was solved by third-grade students in middle school and first-grade students in high school. The students were taught by the teachers who were members of the team.

Coding evaluation and estimation method (correct or not)	Percentage (%) of the middle school students No.=89	Percentage (%) of the high school students No.=138
(C21, C11) draws a square or rectangle	30	28
(C22, C12) draws a circle	8	4
(C23, C12) draws and adds the areas of regular geometric figures	18	14
(C24, C14) draws a large rectangle and subtracts the areas of figures	11	7
(C25) correct result, other method (remembers from geography)	0	3
(C01) calculation of perimeter instead of the area	4	4
(C02) other incorrect answers	15	16
(C99) no answers (attempts)	14	24

Table 3. Estimation methods selected by students solving the "Continent area" task

Number of students solving the task	Full credit 2p Correct method and result 12mln km^2-18mln km^2	Partial credit 1p Correct method, errors	No answer or 0 p
Middle school No.=89	16%	52%	32%
High school No.=138	16%	40%	44%

Table 4. The percentage of correct, partially correct and incorrect results in the task by our middle and high school students

Difficulties

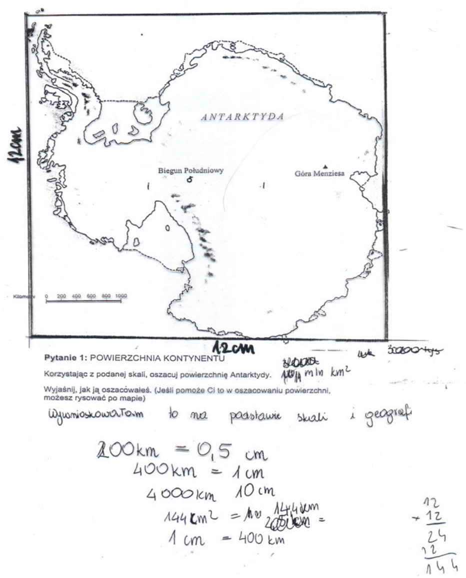
(1) The task was not typical for students for two reasons: very rarely in mathematics class do they encounter an instruction “estimate,” or figures of irregular shapes when asked to calculate the area of figures. Therefore, most likely, such a large percentage of students did not even attempt to solve the problem.

(2) A greater variety of estimation methods and more correct answers were given by middle school students than by high school students. (In high school the subject of the area of figures is not reviewed and most likely students did not remember much from middle school.)

(3) Even a correct estimation method was often realized irrationally due to: (a) drawing to many figures (squares), which required a calculation of many areas; (b) lack of calculation planning and making many operations on large numbers; (c) understanding the instruction “estimate” as “estimate as precisely as you can.”

(4) On both levels there were students who had difficulties with: (a) calculation of the length in scale; (b) calculation of the area of a figure in scale; (c) carrying out correct calculations.

Examples of these difficulties are illustrated by the solutions presented in Figures 2, 3 and 4.



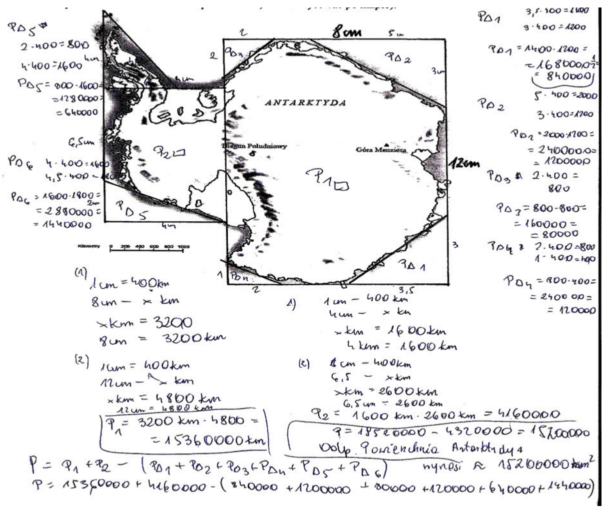
A solution by a 17-year-old student of third grade in middle school.

Estimation methods C11 – draws a square.

Process: using proportions when calculating length units in reality.
Area of the drawn square is 144cm^2
Difficulties with calculating the area in real life 144cm^2 = ???

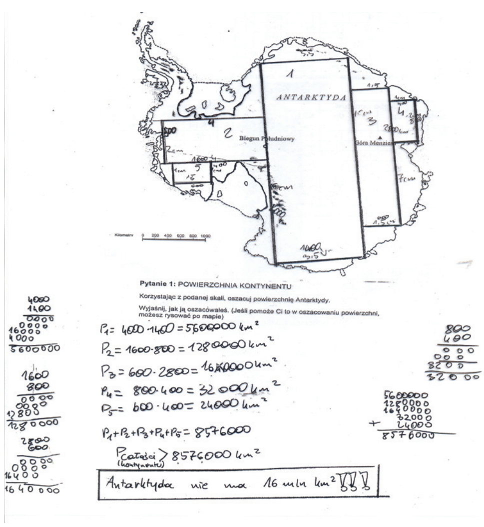
Eventually, the student did not give a result of the estimation.

Figure 2. A solution of the “Continent area” task by a 17-year-old student



A middle school student used estimation method C24; he subtracted the area of triangles from the area of rectangles. Using proportions he calculated the length unit. He subtracted the area P6 while he should have added it. He conducted calculations in an irrational way. He attained a correct result.

Figure 3. A solution of the "Continent area" task by a 16-year-old student



A solution by a third-grade middle school student.

She used estimation method C12 of adding the areas of rectangles.

She made calculation errors in multiplying by the 10th power as a result of these errors the areas of P4 and P5 were 10 times smaller.

The result was not between 12 000 000 and 18 000 000 sq km.

Reflection: Despite a good idea, the correct estimation of the continent area was not possible due to low calculation competency and errors.

Figure 4. A solution of the "Continent area" by a 16-year-old student

2.2 The "Carpenter" Task (PISA 2003)

Question 1: A carpenter has 32 meters of timber and wants to make a border around a garden bed. He is considering the following designs for the garden bed.

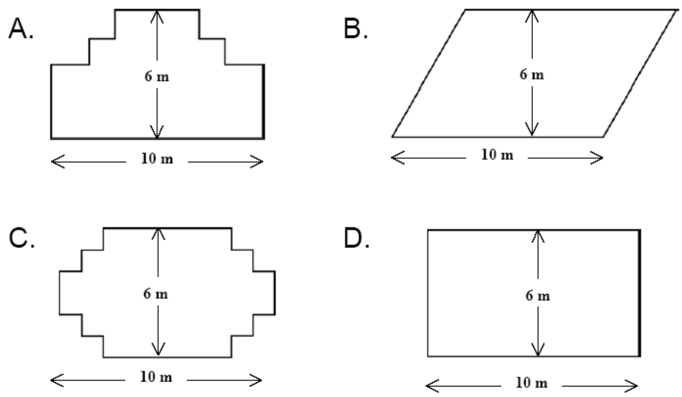


Figure 5. Shapes of the garden beds in the "Carpenter" task

Circle either "Yes" or "No" for each design to indicate whether the garden bed can be made with 32 meters of timber.

Garden bed design	Using this design, can the garden bed be made with 32 meters of timber?
Design A	Yes / No
Design B	Yes / No
Design C	Yes / No
Design D	Yes / No

Table 5. Answer table in the "Carpenter" task

We added an instruction to this task: "justify your choice in each case."

In the PISA task typology the mathematical content of this task refers to space and shape; mathematical competencies refer to connections, and situation context was described as professional. Mathematical literacy as used in solution of this task includes estimation of the perimeter of polygons and their comparison and understanding that there is no relationship between the area and the perimeter of a polygon. In an attempt to define these activities in Krygowska's language we decided that this task gives an opportunity to use analogies and to deduce (justify decisions that were made), while in Niss's terminology this task can be described as a solution of closed tasks, mathematical reasoning and the employment of a mathematical symbolic and formal language.

PISA Scoring "Carpenter"
Full Credit
Code 2: Exactly four correct:
Design A: Yes; Design B: No; Design C: Yes; Design D: Yes
Partial Credit
Code 1: Exactly three correct.
No Credit
Code 0: Two or fewer correct
Code 9: Missing

Results

Number of students solving the task	Full credit 2p	Partial credit 1p	No answer or 0p
Middle school No.=88	44%	28%	28%
High school No.=104	37%	28%	35%

Table 6. The results percentage in the "Carpenter" task

The assessed group of our students attained more correct results (full credit 2 points) than in the 2003 PISA assessment – then only 22 % of Polish students attained full credit, while 30 % of students attained partial credit 1 point. Correct answers for the flower bed in shape A – were chosen by 76 % of students; for shape B – 59 %, for C – 69 % and for D – 92 % of students.

For figures in shapes A and C the following justifications were given. Some students: (i) allocated proportionally lengths of the polygon sides and justified their choice by calculations on concrete numbers *1m+2m+4m+2m+1m=10m* (not always written down); (ii) allocated and marked the side lengths with letters and used algebraic expression, e.g. *a+b+c+b+a=10m d+e+f=6m*; (iii) used geometric transformations (parallel shift or rectangular projection) to justify the equal length of particular segments. Such approaches are presented in Figure 6.

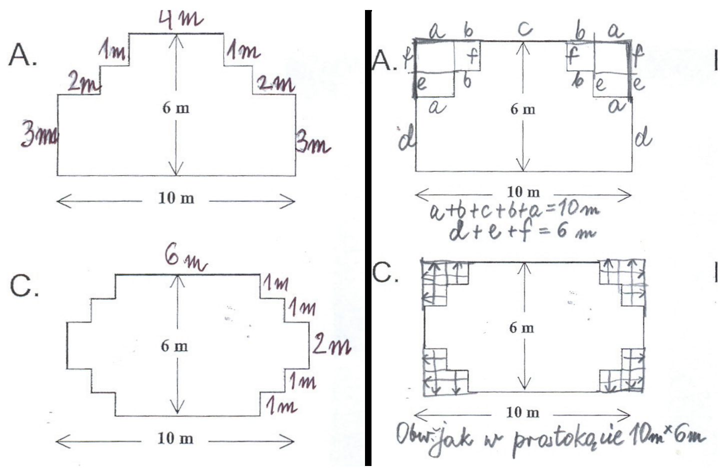


Figure 6. Justification methods for the A and C shapes in the “Carpenter” task

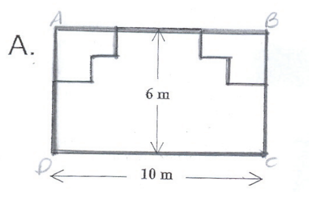


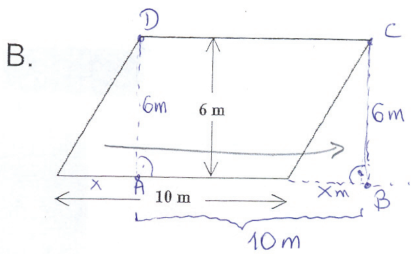
Figure 7. A polygon inscribed into a rectangle

Students inscribed figures of shape A and C into a rectangle (see Figure 7) and drew various conclusions: (a) some stated that perimeter A would be smaller than the perimeter of the rectangle (because A is encompassed by the rectangle) and though they reasoned erroneously they chose a correct answer; (b) others stated that the perimeter will be larger because the length of the ‘cut’ segments is greater than the corresponding parts of the rectangle sides and they chose an incorrect answer; and (c) the remainder demonstrated that the perimeter of figure A and C are equal to the perimeter of the rectangle.

Students had the largest difficulty with the estimation of the perimeter of the parallelogram. The justification that the length of the shorter side of the parallelogram is greater than 6 meters was given in the following manner. Some students: (a) stated that "you can see on a figure that the shorter side is longer than 6m;" (b) used a symbolic expression "b>6" (15 % of students – 15 % SS); (c) drew a right triangle whose longer leg was a height of the parallelogram and used the triangle properties (hypotenuse has longer length than the legs or that the longest side in triangle is opposite the largest angle) (23 % SS); (d) applied the Pythagorean Theorem in order to calculate the length of a side (13% SS); (e) referred to the properties of parallel segments in a parallelogram which is not a rectangle that "the height is the shortest segment (distance) between the parallel segments" (10 % SS); (f) used the fact that "a diagonal of a rectangle is longer than its sides" (having drawn a rectangle whose diagonal was the shorter side of the parallelogram (1% SS); and (g) even gave a non-constructive proof: "If the perimeter of the parallelogram (as shown in a figure) was equal 32 m, then it would have to be a rectangle, and since it is not, then its perimeter must be greater than 32 m. It cannot be smaller since the height in the parallelogram is 6 m and it is the shortest segment between the parallel sides."

Errors that appeared in the justifications might originate from the misapprehension of the concept of the area and the perimeter of a polygon, or in schematic actions in a typical task, like "calculate the area and the perimeter of a given figure." When solving this task, 10 % of students calculated the area of a parallelogram or rectangle, (it happened a few times that when calculating mentally they wrote down 32 instead of 60 as the perimeter).

Another kind of erroneous reasoning by analogy and erroneous conception that "figures with equal areas have equal perimeters" is revealed by the following conversation with the author of such a solution:



We can make a rectangle with dimensions 6m x 10m (rectangle ABCD) out of a parallelogram and analogically to shape D the perimeter will be 32m.

Figure 8. An erroneous solution of the "Carpenter" task

When asked to clarify her reasoning, she answered: "I checked it for the rectangle, it was easy. I only needed to calculate the perimeter. When I saw the parallelogram I associated it with the area, and I obtained a rectangle for which I already knew that there would be enough timber."

Incorrect answers for the rectangle were a result of random choices ("I picked randomly – every other yes, every other no") or of calculating the area instead of the perimeter (60>32).

2.3 The "Apple trees" task (PISA 2000)

A farmer plants apple trees in a square pattern. In order to protect the trees against the wind he plants conifers all around the orchard. Here you see a diagram of this situation where you can see the pattern of apple trees and conifers for any number (n) of rows of apple trees:

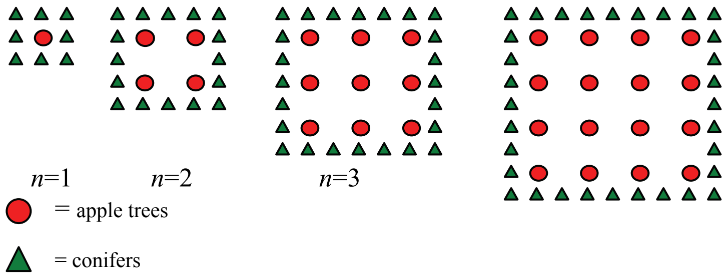


Figure 9. Apple and conifers in an orchard

Question 1. Complete the table:

n	Number of apple trees	Number of conifers
1	1	8
2	4	
3		
4		
5		

Table 7. A table from the "Apple trees" task

Question 2.
There are two formulae you can use to calculate the number of apple trees and the number of conifers for the pattern described above:
Number of apple trees = n^2
Number of conifers = $8n$
Where n is the number of rows of apple trees.
There is a value of n for which the number of apple trees equals the number of conifers. Find the value of n and show your method of working this out.

Question 3.
Suppose the farmer wants to make a much lager orchard with many rows of trees. As the farmer makes the orchard bigger, which increase more quickly; the number of apple trees or the number of conifers? Explain how you found your answer.

In the PISA task typology in terms of mathematical content question 1 refers to observing change and relationships between the values given in a geometric situation; in terms of mathematical competencies it refers to connections, and mathematical situation is educational. Questions 2 and 3 refer to the estimation and approximation of values described by means of algebraic expressions, equations and inequalities or relationships between variables, which require generalizations from a concrete situation to an abstract mathematical model. In Krygowska's terminology the activities used in this situation are associated with observing a structural analogy and recurrence, and a rational usage of a symbolic language. In Niss's terminology the competencies needed in this task include appropriate object representation, mathematical modeling and mathematical reasoning that refer to proposing formal or informal arguments in proving the posed hypotheses.

PISA "Apple trees" scoring
Question 1.

N	Number of apple trees	Number of conifers
1	**1**	**8**
2	**4**	*16*
3	*9*	*24*
4	*16*	*32*
5	*25*	*40*

Table 8. Answers to question 1 in "Apple trees"

Full credit (2 points)
Code 21: All 7 answers are correct.
Partial credit (1 point)
These scores are for only ONE incorrect or missing number in table. Code 11 is for only ONE error in row n=5. Code 12 is for only ONE error in row n= 2 or 3 or 4
Code 11: Answers which give correct numbers in rows n = 2, 3, or 4, but in row n = 5 only ONE answer is incorrect or missing: (i) last answer '40' is incorrect, all other answers are correct; (ii) Answer '25' is incorrect, all other answers are correct
Code 12: Numbers in row n = 5 are correct but ONE answer in rows n = 2, 3, or 4 is incorrect or missing
No credit
These scores are for TWO or more errors or no answer.
Code 01: Correct values in rows n = 2, 3, 4, but BOTH numbers for row n = 5 are incorrect;
Both answer '25' and '40' are incorrect, all other answers are correct.
Code 02: Other incorrect answers
Code 99: No answer

Question 2.
Full credit (1 point)
These scores are for the correct answer $n=8$, using different approaches.
Code 11: Answers which give $n = 8$, with the algebraic method explicitly shown, e.g.
$n^2 = 8n$, $n^2 - 8n = 0$, $n(n - 8) = 0$, $n = 0$ and $n = 8$, so $n = 8$
Code 12: Answers which give $n = 8$, but no clear algebra is presented, or no work shown.
e.g.: $n^2 = 8^2 = 64$, $8n = 8 \cdot 8 = 64$; $n^2 = 8$n. This gives $n = 8$; $8 \cdot 8 = 64$, $n = 8$; $n = 8$; $8 \cdot 8 = 8^2$
Code 13: Answers which give $n = 8$ using other methods e.g. using pattern expansion or drawing
These scores are for the correct answer, n=8 PLUS the answer n = 0, with a given justification.
Code 14: Answers which are similar to those given under score 11 (clear algebraic method) but give both answers $n=8$ AND $n=0$
Code 15: Answer which are similar to those given under score 12 (no clear algebraic method) but give both answers $n=8$ AND $n=0$
No credit (0 points)
Code 00: Other answers, including the answer $n=0$, for example: (a) $n^2 = 8n$ (a repeat of the statement from the question); (b) $n^2 = 8$; (c) $n = 0$. You cannot have the same number because for every apple tree there are 8 conifers.
Code 99: No answer

Question 3:
Full credit (2 points)
Code 21: Answers which are correct (apple trees) AND which give some algebraic explanations based on the formulae n^2 and $8n$; e.g.: (i) Numbers of apple trees = $n \cdot n$ and conifers = $8 \cdot n$. Both formulas have a factor n, but apple trees have another n, which will get larger while factor 8 stays the same. The number of apple trees increases more quickly; (ii) the number of apple trees increases faster because that number is being squared instead of multiplied by 8; (iii) number of apple trees is quadratic. Number of conifers is linear. So apple trees will increase faster. Answers which use graphs to demonstrate that n^2 exceeds $8n$ after $n = 8$.
These scores are for answers when students present algebraic expression based on n^2 and $8n$.
Partial credit (1 point)
Code 11: Answers which are correct (apple trees) AND are based on specific examples or on extending the table: (a) number of apple trees will increase more quickly because if we use the table (question 1) we find that the number of apple trees increases faster than the number of conifers. This

happens especially after the number of apple trees and number of conifers are the same. (b) The table shows that the number of apple trees increases faster.
OR answers which are correct (apple trees) and show SOME evidence that the relationship between n^2 and $8n$ is understood but not so clearly expressed as in Code 21, e.g.: (i) Apple trees after $n > 8$; (ii) After 8 rows the number of apple trees will increase more quickly than conifers; (iii) Conifers until you get to 8 rows then there will be more apple trees.
No credit (0 points)
Code 01: Answers which are correct (apple trees) but give an insufficient or wrong explanation, or no explanation; for example: (a) Apple trees; (b) Apple trees because they populate the inside which is bigger than just the perimeter; (c) Apple trees because they are surrounded by conifers.
Code 02: Other incorrect answers: (a) Conifers; (b) Conifers because for every additional row of apple trees, you need lots of conifers; (c) Conifers. Because for every apple tree there are 8 conifers; (d) I don't know.
Code 99: No answer.

Results

Codes according to evaluation, estimation method correct or not	percentage of middle school students No. =72	percentage of high school students No. =140
Question 1		
(C21) all numbers correct	90	76
(C11) one error for n=5 others correct	0	10
(C01) two errors for n=5, others correct	4	6
(C02) other answers	6	4
(C99) no attempts	0	4
Question 2		
(C11) algebraic method, equation and answer n=8	3	11
(C12) no clear method (equation $n^2 = 8n$ divided by n or guessed number 8 and checked)	58	47
(C13) table expansion, graphs, n=8	3	4
(C14) algebraic method, equation, answers n=0 and n=8	4	0
(C00) other answers	14	24
(C99) no attempts	18	22
Question 3		
(C21) correct answers (number of apple trees) and justification	15	17
(C11) correct answers, justification by examples	17	19
(C01) correct answers, incorrect justification or no justification	42	12
(C02) incorrect answers (number of conifers increases more quickly)	19	46
(C99) no attempts	7	8

Table 9. The "Apple trees" task results

(1) A correct completion of a table in question 1 shows that an overwhelming majority of students noticed a relationship between the number of apple trees in the orchard and the number of conifers in a concrete situation for small numbers. (2) The algebraic formulae given in question 2 describing the number of apple trees and the number of conifers in relationship with n were used by about only 35 % of students (C11 and C14 and almost half of the students who chose C12). Despite the fact that the task content contained information about the existence of n number of apple trees in an orchard, for which the number of apple trees and the number of conifers are equal, almost half of the students were unable to create an equation $8n=n^2$ and solve it. About 25 % of students managed to give a correct answer by guessing number 8 and checking

that $8 \cdot 8 = 8^2$ or by using the table expansion from question 1. (3) It was much more difficult for students to pose a hypothesis in question 3. High school students often posed an incorrect hypothesis (there will be more conifers). About 40 % of middle school students posed a correct hypothesis but gave an incorrect justification by: (a) imprecise description of the observed relationships in the mathematical language e.g. “the number of apple tree will increase more quickly since it is increased by a square of a subsequent number of apple trees in a row;” “the number of conifers increases by 8 each time, while the number of apple trees increases constantly;” (b) usage of examples as general arguments, e.g. $10^2=100$ and $8 \cdot 10 =80$ or $20^2 = 400$ and $8 \cdot 20=160$;

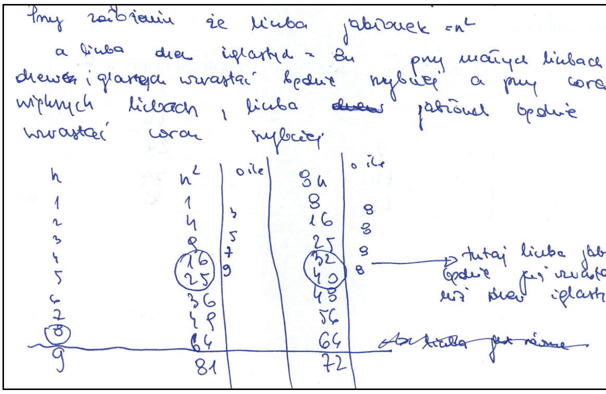
(c) an imprecise interpretation of observed mathematical relationships in reality (confusing the mathematical language with the natural language), e.g.:

> Apple trees will increase more quickly when $n>8$, and the conifers will increase more quickly than apple trees when $n<8$. They will increase equally when he plants 8 of each; The number of conifers will increase faster since they surround the apple trees. They grow on the perimeter;

(d) confusing the biological growth of the trees with the increase of the number of the trees (in incorrect statements by high school students) e.g.:

> conifers – because there are more of them and they grow faster; conifers will increase faster because there will be more trees surrounding the orchard and the apple trees; depending on what we plant first.

There was also another way of understanding question 3: “which number will increase faster when the orchard expands – the number of apple trees or the number of conifers?” In an attempt to answer the question students analyzed when the increase of the apple trees in the orchard will be larger than the increase of conifers and attained an answer that already for n=5. Such reasoning is illustrated by Figure 10.



Assuming that the number of apple trees equals n^2 and the number of conifers is $8n$, with small numbers the number of conifers will increase faster, whereas with larger numbers the number of apple trees will increase faster and faster.
Here (circle) the number of apple trees will increase faster than the number of conifers.

Figure 10. A sample solution of the “Apple trees” task

3. CONCLUSION AND A FEW REFLECTIONS

We were able to view the official PISA assessment results differently after our students solved the PISA tasks. We realized that we can see more clearly in untypical tasks how our students are able (or not) to use their knowledge and mathematical literacy.

The team work in the project enabled us to discuss the difficulties and errors of students from various schools, various levels and various locations, to reflect together on the origins of these difficulties, to make a common attempt to find corrective methods. It was an interesting experience for the teachers because in the majority of Polish schools there is no tradition of systematic cooperation between teachers of the same subject (in small schools with one mathematics teacher, such cooperation is not even possible).

For the majority of teachers the assessment of the results according to the scoring criteria given for each task was an encounter with a new way of assessment. Assessment according to the evaluation code requires a certain discipline in practice. Regardless of very detailed instructions there were doubts about choosing the appropriate code for a given result. We had an opportunity to get acquainted with codes that considered the number of points and kinds of reasoning. We realized that this new assessment method that was created as a result of standardization is not entirely perfect. We are not able to predict all the possible solutions that students can come up with.

An analysis of the "Carpenter" and "Continent area" tasks revealed students' difficulties in employment of their knowledge and skills concerning the perimeter and area of figures. These concepts have been part of school curriculum for years and one would think that such methods had already been created and applied that each student after 8-9 years of mathematics education should be able to use them even in a new, untypical situation. We had an opportunity to see that this is not the case. Another difficulty which appeared in these tasks was an instruction "to estimate." It is not clearly specified what this instruction really means: "estimate approximately," "give more or less how many," "without performing any complicated calculations give a result, which is not much different from an actual (correct) result," or "give a numerical range of the result." A result estimation skill can be a simple and easy didactic strategy used by teachers to motivate students in order to verify the predicted result, or a posed hypothesis. Satisfaction students feel when they manage to predict a result with a high precision motivates them to further education. Result estimation can also be an element of self-control.

An analysis of the "Apple trees" task drew our attention to equations, the usage of letters in recurrence generalizations in order to write down observed regularities and the argumentation with the usage of number properties. Our team's work concentrated on equations. We attempted to understand students' difficulties with equations. We wanted to learn how students comprehend equations, how they solve them, how they interpret the equation results in the context of the task content, what relationship they see between the formulae and equations, between equations and variables, which kinds of equations cause more difficulties. We decided to explore these difficulties while conducting teaching-research in a normal process of mathematics education. We created a series of tasks-tests, which students solved and then we analyzed their results at the team meetings. We were astonished to realize that when students are given two algebraic expressions, almost half of students were unable to create an equation (they calculated on concrete numbers or substituted numbers in algebraic expressions). What is more, when they had an equation they desperately tried to find an algorithm matching the equation (e.g. they tried to transform a square equation into linear equation while making errors in algebraic transformations – by substituting x^2 with $2x$), or they created a new algorithm themselves. There were also students who faced a quadratic equation $x(x-3) = 0$ and immediately stated that they were not taught such equations and they did not even attempt to solve them, when in fact they could try to guess numbers and check or apply the theorem on numbers. More information about this team work can be found in the article "Mathematical Tests" by Sasor-Dyrda, Szczerba, and Cyran (2008). Teachers explored the usage of letters in generalizations and argumentation in individual research.

All in all, the PISA task solving shared by teachers and students and the teaching-research process contributed to numerous passionate discussions about mathematics teaching and learning; evoked a deeply felt need to exchange experiences between teachers, between teachers and students; and, finally, created a new and positive atmosphere in the classroom. It was a unique learning experience for the teachers who turned into teacher-researchers.

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PISA AND PISA-LIKE PROBLEMS

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ABSTRACT
In this survey we shall analyze our investigation on solving PISA and PISA-like problems at four Hungarian high schools. 167 students and 30 teachers-in-training took part in our assessment. We posed original and modified PISA problems to students. With the help of these PISA-like problems we wanted to confirm the supposed beliefs of mathematics teachers and clarify the problem of confusing the concepts of area and perimeter. Also, we assessed student's competency in the application of individual models.

KEY WORDS:
PISA assessment, PISA problems, PISA-like problems, measurement of competencies, application of oriented problems, gender problem

INTRODUCTION

For the teachers participating in the Krygowska PDTR Project, the TR Seminar was the first opportunity to get acquainted with PISA problems, among them the "Carpenter" and "Apple trees." We tried to identify the difficulties that students might have in solving these PISA problems. Our group suggested that we should change the problems to make them more challenging for our students. We wanted to clarify the problem of confusing the concepts of area and perimeter, which students exhibited when solving the "Carpenter" problem. We modified the PISA problem "Apples, too: we changed the values in the table and did not give the formulas. It is worth saying that real life problems or quasi-realistic problems (like PISA problems) have been playing greater role in the last few years in the Hungarian mathematics teaching, parallel to the introduction of the new graduating system.

The investigated classes

We observed and assessed the work of 167 students and 30 teachers-in-training in the following schools: Svetits High School (Debrecen): 40 students of grade 9 (modified "Apple I" and "Carpenter I"); Dóczy High School (Debrecen): 20 students of grade 8, and 20 students of grade 10 (modified "Apple I" and "Carpenter I"); Kossuth High School (Debrecen): 65 students of grade 9 (modified "Apple I" and "Carpenter I"); Hőgyes High School (Hajdúszoboszló): 13 students of grade 7 (modified "Apple II"), 9 students of grade 11 ("Carpenter"); 30 teachers-in-training of upper secondary schools (University of Debrecen).

Svetits High School is a Catholic School without mathematical orientation. In the observed classes there were only girls. Dóczy High School is a Calvinist School. Kossuth High School is the Teacher Training Practice School of Debrecen University. It is a school with a double function: educational duties and teacher training. The school has a good selection of students, a large number of them participate in contests, and 80-85% of them are going to study at universities or colleges after finishing their secondary school studies. This school is one of the best schools in the country. The tested classes were: class 9D with orientation towards mathematics and sciences, class 9E with general interest. Hőgyes High School is situated in a well-known little town near Debrecen, with a famous spa and health centre. The students of this secondary school are good at mathematics. There are classes with mathematical orientation (grades 7-12). For us, it was an opportunity to compare the results of younger and talented students with the results of older and weaker students.

OUR EXPERIENCE IN SOLVING PISA AND PISA-LIKE PROBLEMS

Apple trees (PISA 2000)

Content: Change and relationship. Context: educational. In Question 1 students had to complete the given table, in Question 2 there were connections and integration for problem solving. Question 3 was connected with mathematization, mathematical thinking and generalization.

Apple trees (the original problem)

A farmer plants apple trees in a square pattern. In order to protect the trees against the wind he plants conifers all around the orchard.
Here you see a diagram of this situation where you can see the pattern of apple trees and conifers for any number (n) of rows of apple trees:

$n = 1$ $n = 2$ $n = 3$ $n = 4$

```
X X X     X X X X X     X X X X X X X     X X X X X X X X X
X ● X     X ●   ● X     X ●   ●   ● X     X ●   ●   ●   ● X
X X X     X       X     X           X     X               X
          X ●   ● X     X ●   ●   ● X     X ●   ●   ●   ● X
          X X X X X     X           X     X               X
                        X ●   ●   ● X     X ●   ●   ●   ● X
                        X X X X X X X     X               X
                                          X ●   ●   ●   ● X
                                          X X X X X X X X X
```

X = conifer, ● = apple tree

Question 1 (548 scores, competency class: 2):
Complete the table

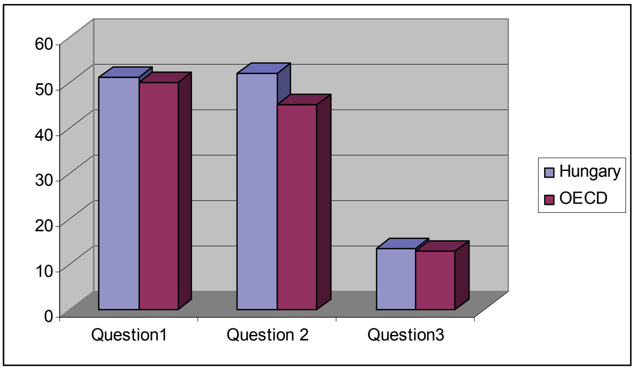
n	Number of apple trees	Number of conifers
1	1	8
2	4	
3		
4		
5		

Question 2 (655 scores, competency class: 2):

There are two formulae you can use to calculate the number of apple trees and the number of conifers for the pattern described opposite:
Number of apple trees = n^2,
number of conifers = 8 n,
where n is the number of rows of apple trees. There is a value of n for which the number of apple trees equals the number of conifers. Find the value of n and show your method of calculating this.

Question 3 (723 scores, competency class: 3):
Suppose the farmer wants to make a much larger orchard with many rows of trees. As the farmer makes the orchard bigger, which will increase more quickly: the number of apple trees or the number of conifers? Explain how you found your answer.

The official results:



Apples I (modified problem)

At first we changed the table. In the original table were the values *1, 2, 3, 4, 5* in succession. In our table we gave more numbers: *1, 2, 3, 4, 5, 7, 9, 15*. These numbers were not consecutive numbers. Some of the students (5%) did not recognize this fact and counted the values in order.

This is the modified table:

n	Number of apple trees	Number of conifers
1	1	8
2	4	
3		
4		
5		
7		
9		
11		
15		

In solving Question a) we expected not only visual solution but calculations too.

We also opened the problem. We asked students to find the two formulae to calculate the number of apple trees and the number of conifers as a function of *n,* where *n* is the number of rows of the apples trees (Question b). We left Question c) and Question d) in the original form of Question 2 and Question 3. Aim of the change in the table of Question 1 was that we wanted to know how students were thinking, what way they followed for greater numbers *n,* whether they worked visually with the help of concrete drawings and counting or they could recognize the hidden rules. We did not give the number $n=8$, for which the number of apple trees equals the number of conifers, but we gave numbers 7 and 9, which closed the case of the equation.

Our hypothesis was that for the younger or weaker students it would be easier to find out the place of the equation with the help of the table. This hypothesis became true, we verified it. We found that the first 7 answers, the original wish, were good, because students could count them in a concrete way from the figures, the bigger values were more difficult to find. It was necessary to conjecture the general form to count the case $n= 15$, but it was easier to find the value n for which the number of apple trees and the number of conifers is equal or to answer question d). 60.5% of students and teachers used the table in their argumentation.

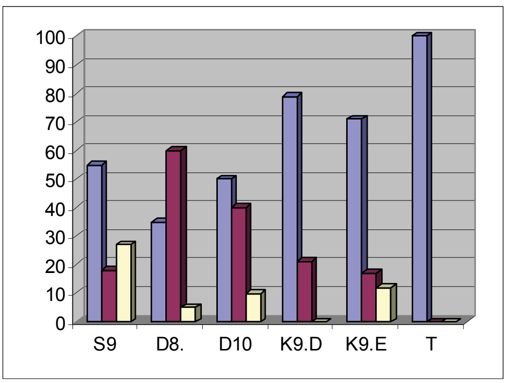
For us the answer to Question 2 in the original "Apple trees" problem was not obvious. We had a problem with "the number n." What is the meaning of "any number n"? Which kind of number is n? At the diagram n=1, 2, 3, 4; we do not find n=0. Should we think that n is necessarily a positive integer; that $n\neq0$? Why do we need to accept that in a garden there are no trees? Is it a garden at all? I did not agree with the answer n=0 nor did a lot of students (80%) and teachers (80%). $n=0$ is only a formal mathematical answer! Only one student wrote as a result n=0.

The wrong answers to question d) were very interesting. Students used everyday life language (65%). These answers were like those in PISA 2000 results. They were affected by the confusion between the concepts of perimeter and area.

About the results
Question a)

Most students had no difficulty in solving this part of the problem. We compared the results in the original and modified tables. The "completing part is wrong" means that the students filled the original table correctly, but they made mistakes in filling the extended table. The "other mistakes" means that they were absent-minded, they failed to notice that after the consecutive numbers 1 to 5 there came number 7 instead of 6, and they used in the whole table the rule which worked for the first five numbers.

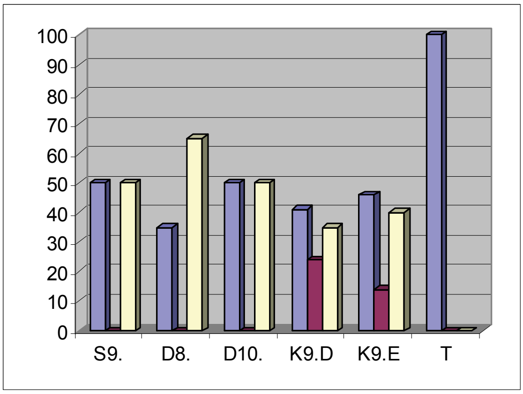
Svetits 9.	Dóczy 8.	Dóczy 10.	Kossuth 9.D	Kossuth 9.E	teachers-in-training	solutions
55%	35%	50%	79%	71%	100%	correct
18%	60%	40%	21%	17%	0%	completing part is wrong
27%	5%	10%	0 %	12%	0%	other mistakes



Question b)

It seemed to be problematic. Half or less than half of the students could find both formulae. It was interesting that the younger students (Dóczy, grade 8) had a problem in finding the formulae. Later we will see that all of the youngest students (Hőgyes, grade 7) with mathematics orientation were successful in solving part b).

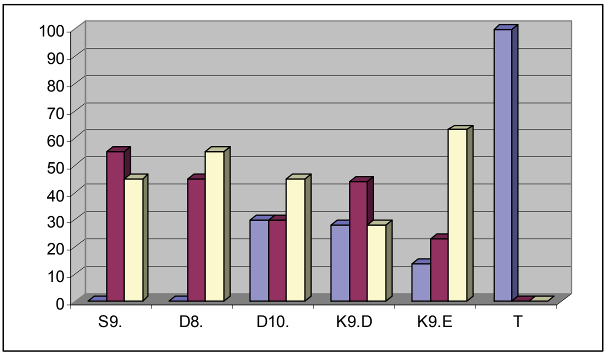
Svetits 9.	Dóczy 8.	Dóczy 10.	Kossuth 9.D	Kossuth 9.E	teachers-in-training	solutions
50%	35%	50%	41%	46%	100%	2 of the formulae are good
0%	0%	0%	24%	14%	0%	1 of the formulae is good
50%	65%	50%	35%	40%	0 %	Formulae are missing, or there are no good formulae



Question c)

Less than one third of the students gave a correct solution to the equation. They chose another correct solution. They compared the two qualities with the help of the table since they could not solve quadratic equations. They will learn it in grade 10. But in grade 10 (Dóczy) 30% of the students chose the two possible ways.

Svetits 9.	Dóczy 8.	Dóczy 10.	Kossuth 9.D	Kossuth 9.E	teachers-in-training	solutions
0%	0%	30%	28%	14%	100%	correct with equation
55%	45%	30%	44%	23%	0%	other correct
45%	55%	45%	28%	63%	0%	incorrect or missing



Question d)

More than a half of the students gave the correct answer. The explanations were acceptable, but sometimes they could not express their ideas precisely. Their argumentation was based upon the values of the modified tables, as we expected it in our hypothesis.

In the correct answers they wrote:

(a) the number of the apple trees increases more quickly as we can see it from the table; (b) the number of apple trees is $n \times n$, the number of conifers is $8 \times n$, and if $n > 8$ then their number will get larger; (c) At first the number of conifers is greater than the number of apples trees, later (after $n = 8$) the number of the apples trees will be greater than the number of the conifers, because the square of a number, bigger than 8, is bigger then eight times this number. For example: If $n = 9$, $9^2 = 81$, but $9 \cdot 8 = 72$ and $81 > 72$.

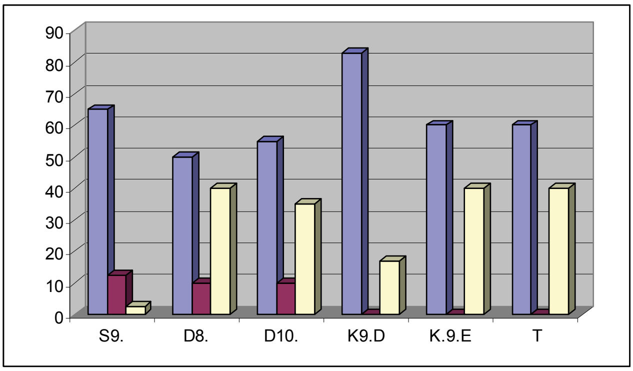
We found correct answers with an insufficient or wrong explanation:

(a) Apple trees because they are surrounded by conifers; (b) Apple trees are planted inside, and the inside is bigger than the perimeter.

We found wrong answers too. Sometime the cause was that the students confused the concepts/terms of area and perimeter. We noticed that in Hungarian the words perimeter (*kerület*) and area (*terület*) differ only in one letter. Their answers were:

(a) The area is bigger than the perimeter; (b) The apple trees are planted scattered. (c) Conifers grow more quickly than the apple trees; (d) We need more conifers for enclosing the apple trees.

Svetits 9.	Dóczy 8.	Dóczy 10.	Kossuth 9.D	Kossuth 9.E	Teacher-students	solutions
65%	50%	55%	83%	60%	60%	correct answers with argumentation
12,5%	10%	10%	0%	0%	0%	no argumentation
2,5%	40%	35%	17%	40%	40%	wrong answers



Selections from the works of the students
1. Grade 9 (Svetits High School): correct answers, use of the table
2. Grade 8 (Dóczy High School): absent-minded

Apples I (modified problem)

In the Hőgyes High School (Hajdúszoboszló) the modified problem of "Apple trees" was a little bit different. It was investigated by their teacher, Éva Balla. Questions 1, and 3 were the same as in the original problem, Question 2 was formulated in the above-mentioned open form with no formulae given. Students' task was to find the formulae for the connection between the number of the apple trees and conifers.

The students were only 13 years old. Our *hypothesis* was that they would solve the problem visually, they would make further diagrams to count the number of the trees, after that, they would start to fill the table and give the observed rules. But in spite of our expectations these students filled the tables with the help of numerical calculations. They realized the rules of the sequences and used it in those cases when the numbers were bigger. Only one student made a drawing for $n = 5$. 100 % of the students could fill the original table. In Question 2 all the students could give the formulae, and 72% solved the equation. 100% gave correct formulae; 1 student gave it in a different way. 72% of them could calculate the value of n in the case when the number of the trees equals the number of conifers. It was interesting how some students guessed the solution of quadratic equation: $n^2 = 8n$, $n \cdot n = 8 \cdot n$, that is, $n = 8$.

Over 54% of the students gave a correct answer to question 3. They observed the difference between consecutive numbers of trees and conifers. We got insufficient answers in 9%, incorrect or no answer in 36.5 % cases. Question 3 was more difficult because at grade 7 they learnt only about linear functions. Their task was to solve a quadratic inequation. In their argumentation they often used the data or their own completed table. Two of the students completed the table with number $n = 8$, too. This problem was complex. The students had to connect different mathematical areas, write and solve a quadratic equation and an equation, to compare the growth of a linear and a quadratic function. We did not expect that about half of them could answer question 3.

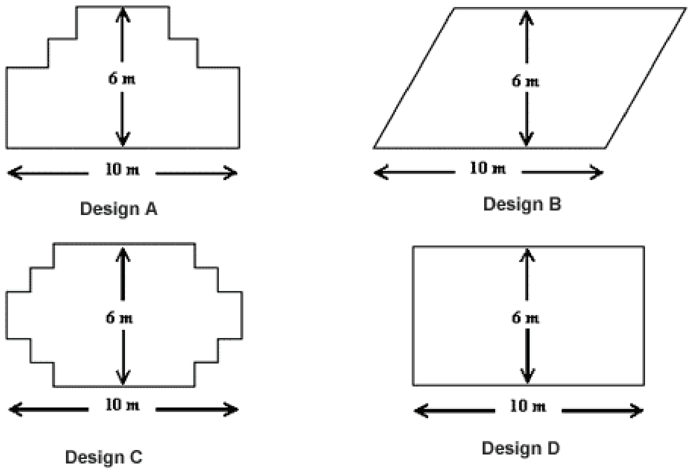
Student 3, Grade 7 (M.O.) Hőgyes High School: another strategy for the counting of apple trees. This student was the only one who made a drawing for $n = 5$, and followed another strategy of counting the numbers of conifers. His rule was: $n \alpha \ (n \cdot 2 - 1) \cdot 4 + 4$. We need to mention that this expression is equal to $8n$. He was one of the students who completed the given table with the number $n = 8$.

Carpenter (PISA 2003)
Content: area, space, and shape. Difficulty: 687 score points. Level: 6. Context: educational. OECD percentage of correct answers: 20.2 %.

Comment*:* It was a complex multiple choice item, a quasi-realistic problem. The students needed the competence to recognize that the shapes A, C and D have the same perimeter. They had to decode the visual information and see similarities and differences. They needed to see that in Design B the border shape was a parallelogram and its perimeter was longer than 32 m*.* But the "distractor" was that its area is equal to the area of the rectangle. This problem fits the connections competency cluster, as the problem was not a routine one.

Carpenter

A carpenter has 32 metres of timber and wants to make a border around a garden bed. He is considering the following designs for the garden bed:



Garden bed design	Using this design, can the garden bed be made with 32 meters of timber?
Design A	Yes / No
Design B	Yes / No
Design C	Yes / No
Design D	Yes / No

Carpenter I (modified problem)

In the modified problem we wanted it to conform to the students' mathematical literacy. We changed the order of the figures A and C. We left the Design D, the shape of the rectangle. Our order was: C, B, A from the original form. We thought that at this level students could work with geometrical models. They could select and compare forms, formulate and precisely communicate their action, their interpretations, interpret their thoughts and their findings. We opened up the problem. We constructed a new question, Question b). The task of the students was to plan a garden bed with 32 meters of timber. So the problem of the carpenter became more realistic. In Hungary a carpenter makes roofs of houses and uses timber for this work. The gardener makes flower beds; he makes their wooden bordering, too, but it is not of timber.

Our hypotheses

(1) Original Design A (modified Design C) is more complicated than the original Design C (modified Design A) because the original Design C is more symmetric; (2) original Design D, the shape of the rectangle, is the easiest one; (3) the task of planning the form of our own garden is more realistic and solvable for the students as a “prefabricated” realistic mathematical task; (4) the cause of the incorrect solution would be misunderstanding and confusing the concepts of perimeter and area.

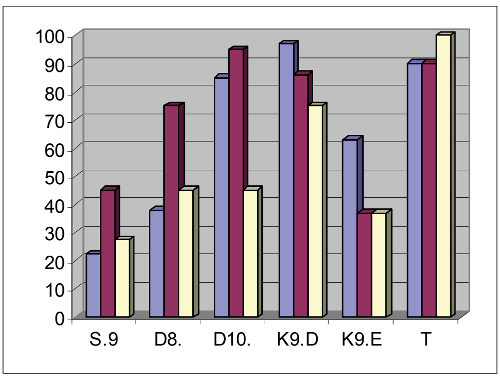
Our experiences

(i) Our first hypothesis did not prove true in all cases; (ii) the second hypothesis proved true. We could see it from the realization of the planning of the garden bed. Almost all of the students chose rectangle or square; (iii) the third hypothesis brought surprising results: there were students, who solved the mathematical problems (Design A, B, C) incorrectly or gave no solutions, but they could make their own and right garden bed plans. Some of their plans were witty. The students were very creative. A student’s explanation was: “It is easy to make the plans. I chose the form of rectangle (the length of its side were: 10 meters and 6 meters), and the form of a triangle (the length of its sides were: 15 meters, 10 meters and 7 meters);” (iv) our fourth hypothesis proved true, too. Even one of the teachers-in-training confused these concepts. She calculated the area of the parallelogram and considered that Design B is convenient, and that the answer was yes! Their method was to transfer a right-angled triangle and proving that the parallelogram has the same area as the rectangle, in which they could transform the Designs A and C.

The results

Part a) students had to reason in a similar way in the cases of Designs A and C, although in some classes (Dóczy grade 10, Kossuth grade 9) students had more problems with Design C. This may suggest that students either did not care enough or misunderstood the question word “which;” they thought that there was only one correct solution, so after finding A to be correct, they did not bother to go on and check the other two designs.

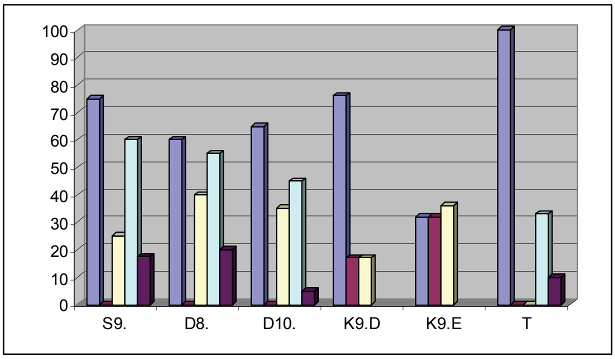
Svetits 9.	Dóczy 8.	Dóczy 10.	Kossuth 9.D	Kossuth 9. E	Teachers-in-training	answers
22.5%	38%	85%	97%	63%	90%	A yes
45 %	75%	95%	86%	37%	90%	B no
27.5%	45%	45%	75%	37%	100%	C yes



Part b): At the Kossuth School a relatively large number of students indicated only one garden bed. I do not consider this to be a sign of being unable to design one more, but rather as a kind of carelessness and indifference on behalf of the students. In most cases they did not give any reasons for their answers. When they did, their explanation or mathematical reasoning was merely giving a formula for the perimeter or extending the sides of the Designs A and C to transform the polygon into a rectangle.

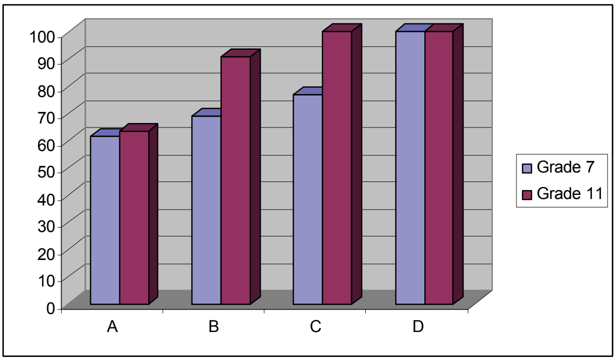
At the other schools, with lower mathematical levels, as Svetits, and Dóczy, their behavior was quite different. They liked this kind of work and they were creative. They drew a lot of different and various forms. We shall show some of their designs.

Svetits 9.	Dóczy 8.	Dóczy 10.	Kossuth 9.D	Kossuth 9.E	teachers-in-training	answers
75%	60%	65%	76%	32%	100%	2 good plans
0%	0%	0%	17%	32%	0%	1 good plan
25%	40%	35%	17%	36%	0%	no plan or bad plan
60%	55%	45%	-	-	33%	rectangular, square
17.5%	20%	5%	-	-	10%	other forms



At the Hőgyes High School the students solved the original problem. We will compare the results of the students of grade 7 and grade 11. We can see that in choosing the correct answer of Designs B and C the older students' results were better.

Hőgyes	A	B	C	D
Grade 7	61.5%	69.2%	76.7%	100%
Grade 11	63.6%	90.9%	100%	100%



In grade 7 class (Hőgyes) there were some students, who had difficulties with interpretation: “How can we enclose a garden with timber? Is the thickness of the timber important? How do the beams fit at the corners?” These practical questions were brought up by boys with good practical sense. They tried to imagine the making of the border. But everyone recognized quickly that the task is – from a mathematical view – to determine the perimeter.

There were 8 students who gave correct answers, 3 students gave wrong answers, and 2 could not answer, they said there were not enough data. The question on Design C was answered correctly; it was easier to determine the perimeter of Design C. They said it was easier because the lengths of the little stair-lines were equal in Design C, but not in A. Since they gave concrete values for the lengths of these lines, this kind of solution was really easier in case C. Among the eleventh-grade students there were only 3 students thinking similarly; they solved the question in C, but only one could solve one in A. One of the seventh-grade students measured the length of each line on the figures, and then enlarged it to the given scale. The others – in both classes – thought the following way: they moved away the little pieces, as one student wrote: “projecting the pushed lines - the perimeter is invariable. So the perimeters of A, C and D are equal.” In this case our first hypothesis was true.

Design C was a parallelogram. In grade 11 everybody knew what to do: they referred to the right angled triangle drawn, recognized that the longest side is more than 6 meters, so the perimeter is more than 32 meters. Design D caused no problem.

To solve the “Carpenter” problem the students needed only elementary geometrical considerations, less syllabus, that is why the lower and the upper grade students solved it in the same or a similar way.

Some opinions from the interviews with students*:*

Design A: (i) I could not answer the question because there were not sufficient data; (ii) I placed the stair-sides outside. I saw that there is no difference between the perimeters of Designs A and C; (iii) The perimeter of Design A is exactly 32 meters.

Design B: (i) Its base is as long as the base of Design A, but the length of the oblique side is greater than 6 meters. The perimeters of C and D are the same, 32 meters; (ii) In this form we cannot surround the garden bed, but if we replace the little right-angled triangle than we get a rectangular form; in this case we could surround it.

Selection from the student works

1. Grade 8 (Dóczy High School): forms of different “stairs” (similar to the given designs)
2. Grade 9 (Svetits High School): forms of different “stairs” (similar to the given designs)
3. Grade 8 (Dóczy High School): only 3 plans are correct
4. Grade 10 (Dóczy High School): two plans, the easiest and a witty one.

SUMMARY

In the 21st century it is necessary, for students too, to know the applications of mathematics. The Hungarian students do not realize the importance and applicability of what they learnt beyond the school walls. They have probably much wider mathematical knowledge, but in new situations they cannot apply it. In a lot of our schools the instruction is traditional. The students reproduce only what they learned. They do not have a close connection with the real world. They are not accustomed to making independent decisions. If they face an unusual problem that does not fit into the category of mathematical problems known to them, they do not know what to do, they are not able to understand and solve the unknown problem.

Our education does not make them ready for real world problems. We should rather say that we do not represent the level that is measured by international PISA tests. The problem is that in many cases students are unable to apply their knowledge for solving everyday problems, or they cannot see the mathematical meaning behind everyday problems. They must be taught how to see mathematics in everyday life; they have to learn how to look for the relationships between mathematics and the problems piling up in front of them.

Teachers have to help students develop these competencies and provide them with guidance. Teachers have to share these new goals with the parents. Schools must be environments that reinforce intelligent student life developing the supports of learning. The teachers’ task is to prepare their students for the challenges. It is important that the assessments are going in that direction. It is necessary to deal with PISA and PISA-like problems. Without knowing this kind of problems students will not be able to solve them. We have to change the students’ attitude, their education and training if we want to increase their motivation and results in solving application-oriented problems. We have to inform the teachers about PISA problems and methods of solving such tests.

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A STUDY ON HOW HUNGARIAN STUDENTS SOLVE PROBLEMS THAT ARE UNUSUAL FOR THEM

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"[The role of teaching of mathematics] is to demonstrate different aspects of mathematics, such as ...[that it is] a tool used in daily life and in a number of occupations."(Hungarian National Core Curriculum)

ABSTRACT
Three PISA problems ("Carpenter," "Walking" and "Growing up") were analyzed from various points of view and thence students' difficulties were hypothesized. Then the problems were solved by Hungarian students aged 12-15, who afterwards were asked to express their opinion on the context and difficulty of the problems. In the article, both solutions and opinions of the students are discussed and contrasted (students were from different age groups).

KEY WORDS:
PISA, context, typical problems, curriculum

INTRODUCTION

Although our National Curriculum says that one of the main aims of teaching mathematics is to show how it can be applied in everyday life, in Hungarian mathematics education the emphasis is often on the abstract and not on the practical aspects of mathematics. In Hungarian secondary mathematics education all students usually have to cope with quite high level mathematics. They have to understand and learn abstract rules and theorems and they have to solve problems using these rules. The problems are usually close-ended. Even if they are in a real life context it is often hard for students to see how it really relates to real life.

In Hungary there is more emphasis on improving the more able students' achievement in mathematics than helping the less able ones to catch up. There are extra lessons and lots of after school clubs available for those who are talented in mathematics and want to practice and learn more about the subject. There is a tradition of mathematics competitions at different levels and our students usually have good results in them.

While some students struggle even with the multiplication table or with solving simple equations they still have to learn higher level mathematics. The question is: Do students with average ability need to learn about abstract mathematics like, for example, logarithm? I think in real life they need logical thinking, basic numeracy skills, problem posing and solving skills, some knowledge of basic geometry, e.g.: measuring length, finding area and perimeter.

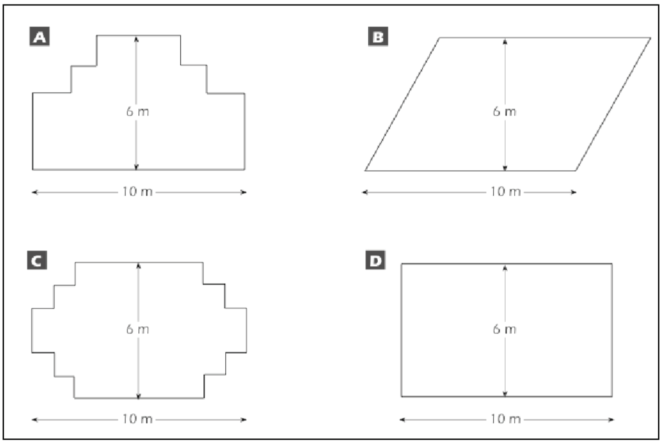
The following example shows teaching the same topic from two different points of view. In Hungary we teach percentages by teaching a formula. Solving percentage problems means that students have to find a missing number with the help of a formula. This way of teaching percentages results in the fact that a lot of students do not understand the concept of percentage at all. On the other hand, in the UK we start teaching percentages wit 0%, 25% and 75% of a number. At this point the majority of students already know that 50% means a half and 25% means a quarter. This step is followed by finding 10% of numbers without using a calculator. It is now easier for the students to find 20%, 5%, 15% ... etc. without a calculator. First, students do not really work with percentages as such. They rather do basic number work like halving numbers or dividing numbers by 10, then doubling them. Doing this simple number work helps to introduce the concept of percentage.

In this study I would like to talk about how Hungarian students aged 12-15 solve mathematical problems that are a bit different from the ones they are used to. The context of the problems is different; the way questions are asked is different. Students were asked to solve and to comment on three PISA problems. I will start with analyzing the problems and then discussing the results and opinion of students. Currently I am teaching at a British secondary school so I can look at the problems from a different point of view, too.

A PRIORI ANALYSIS OF THE "CARPENTER" PROBLEM

The problem:

A carpenter has 32 meters of timber and wants to make a border around a garden bed. He is considering the following designs for the garden bed.



Circle either "Yes" or "No" for each design to indicate whether the garden bed can be made with 32 meters of timber.

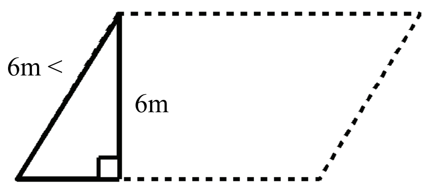
Garden bed design	Using this design, can the garden bed be made with 32 meters of timber?
A	Yes / No
B	Yes / No
C	Yes / No
D	Yes / No

The “Carpenter” problem is an open problem. To solve it students need to use different mathematical competencies. These competencies were defined by Niss as: *mathematical thinking, problem handling, modeling, reasoning, representation, communication, symbol and formalism, aids and tools.* In this problem mathematical thinking, mathematical argumentation, modeling representation and problem solving skills are essential. To be able to make the right decision students need to have a clear understanding of the concept perimeter – all the way around a 2D-shape. Perimeter is introduced in primary school at the age of 10-11. It is usually demonstrated by finding the area of rectangles and squares. In Hungary, students have to learn the formula for the perimeter of a rectangle: $P=2\ (a + b)$ or $P=4a$ for a square, where a and b are the length and the width of the rectangle / sides of the square. Furthermore, to solve the “Carpenter” problem students have to see the connection between the perimeters of the different shapes given in the problems.

A general difficulty with this question can be caused by mixing up the concept of perimeter and area. The Hungarian words *terület* ‘area’ and *kerület* ‘perimeter’ differ only in one letter. Hungarian students often use one instead of the other which shows that they are not clear with the difference between area and perimeter and they do not always realize, which one needs to be used. However, I face the problem again and again with English students as well. In English the two words, *area* and *perimeter*, are clearly different. But students still get confused with these concepts and sometimes they find the perimeter instead of area.

Another difficulty might be that Hungarian students usually have to work out the area or perimeter of ordinary shape e.g. rectangles, squares, triangles and they rarely have to work with shapes like the ones given in the “Carpenter” problem part A, and C. Probably, it is hard for them to recognize that while the area of the shape is changing, the perimeter still can remain same. To help students to solve this problem the order of the shapes could be changed. If they are in order of difficulty that suggests that they need more and more complex thinking to answer the question. The majority of the students would start with part D, as this is a rectangle and they well know that $P=2(a+b)$. The following order would be really helpful for the students:

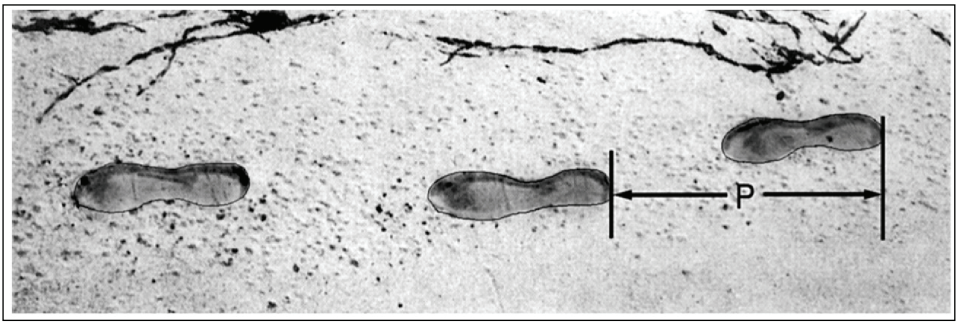
If they see shape D and A next to each other that might realize that their width is the same, and that might help them to see that their perimeter is the same. To see that the perimeter of shape B is bigger than 32 m they need to recognize that cutting along the perpendicular height of the parallelogram we get a right-angled triangle as shown in the figure below. Furthermore, they need to know that in a right-angled triangle the hypotenuse is always the longest side; therefore it must be longer than 6 m, which means that the perimeter of the parallelogram is longer than 32 m. This step requires the knowledge of the properties of right-angled triangles and parallelograms.



A PRIORI ANALYSIS OF THE "WALKING" PROBLEM

The problem:

The picture shows the footprints of a man walking. The pace length P is the distance between the rear of two consecutive footprints.



For men, the formula *$n / P = 140$* gives an approximate relationship between n and P where:
n = number of steps per minute, and P = pace length in meters.

a) Bernard knows his pace length is 0.80 meters. The formula applies to Bernard's walking. Calculate Bernard's walking speed in meters per minute and in kilometers per hour. Show your working out.

b) If the formula applies to Heiko's walking and Heiko takes 70 steps per minute, what is Heiko's pace length?

The "Walking" problem is another open question. To solve it mathematical thinking, modeling, problem posing and solving, representation, symbolic, formal and technical skills are needed. The question consists of two parts. In part a) students have to find the walking speed on the basis of the given formula and in part b) ~~they~~ have to find the pace length.

The first step in solving part a) is to recognize that 0.80 is the value of P in the formula, so all what needs to done is to substitute 0.80 in the formula $\frac{n}{P}=140$ and to work out the value of n, which is the number of steps per minute: $n=140\times 0.80=112$. The next step, which might be hard for some students, is to see that n is not the same as the sought speed. Knowing that Bernard takes 112 steps per minute they have to find the distance he can walk in a minute. For this, students have to notice that the distance can be calculated from the pace length and the number of steps per minute: $112\times 0.80=6$. To see that this is the speed in m/min students have to know the relation between time, distance and speed. The majority of students will stop here. Either because they do not read the question properly and they forget about changing the speed into km/h or because they have difficulty with converting m/min to km/h. However, even if they do not remember that this is how you convert m/min to km/h: 89.6×0.06=5.4, they could use the following reasoning to find the right answer:

1 meter per minute
60 meters per hour
0.06 kilometers per hour

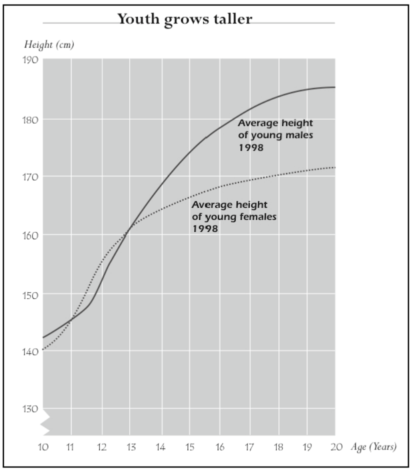
Part b) is less difficult. We have to start with a substitution: n=70, which gives us a simple equation: $\frac{70}{P}=140$. Solving this type of equations is introduced in grade 6 or 7 (students aged 11-13). If students are not confident in solving equations of this type then finding the value of P might be a bit tricky for them as students usually understand division as the number we divide gets smaller by the division. While here we have the opposite. So, they have to realize that we need to divide by a number, which is smaller than one.

Although this problem was put into real-life context it is worth considering how realistic the given formula was. $\frac{n}{P}=140$ means that there is a direct proportional relationship between the pace length and the number of steps per minute i.e. the more steps someone takes in a minute the bigger the pace length is or the other way round, the bigger someone's pace length is the more steps they take in a minute. These, of course, are not necessarily true in real life.

A PRIORI ANALYSIS OF THE "GROWING UP" PROBLEM

The problem:

In 1998 the average height of both young males and females in the Netherlands is represented in this graph.



a) Explain how the graph shows that on average the growth rate for girls slows down after 12 years of age.
b) According to this graph, on average, during which period in their life are females taller than males of the same age?
c) Since 1980 the average height of 20-year-old females has increased by 2.3cm, to 170.6cm. What was the average height of a 20-year-old female in 1980?

The "Growing up" problem is an open problem, too. The competencies needed are the following: mathematical thinking, mathematical argumentation and problem solving skills. It has three parts.

To answer part a) students need to interpret the graphs and translate their meaning into words. They have to understand what the axes represent. They have to see how the curve shows the relationship between age and height and they have to know that the steepness of the curve shows the growth rate. In general, students feel that if the curve gets "flatter" it means that e.g. girls do not grow that fast. However, the majority of the students would find the "explain" part hard because it means that they have to translate their mathematical thinking into language. They have to use mathematical expressions and reasoning. Hungarian students are not used to giving mathematical explanations or to talk about their way of thinking.

In part b) students have to compare the two curves. They have to realize that where one curve is above the other it means that the height at that age is bigger. Basically here they just have to find the right interval where the dashed curve goes above the other curve and read the correct values for age. A mistake could be made if they gave only a single value instead of an interval.

In part c) only a basic calculation needs to be done: 170.6−2.3=168.3 This is a simple primary school task. Students do not even have to pay attention to metric conversion since both measurements are given in cm.

However, the way the question is asked might be confusing. After the first reading students have to stop and think about what they know and what the question wants to know. In reading questions students usually look for keywords. This question has the word *increase* in it, which suggests that they have to add. But, they have to realize that although the date given is 1980, the height mentioned is the height of women now. This might be hard to notice for a lot of students. If they can connect the given height to the right date then it is easy to find out that they have to do the opposite of *increase,* which is subtracting 2.3 from 170.6.

At the age of 12 Hungarian students should have the mathematical knowledge required to solve these 3 questions. However, the way these questions are set up is rather unfamiliar for the majority of them. These questions are not in line with the traditional Hungarian mathematics questions and we hardly ever meet this type in traditional mathematics learning and teaching.

ANALYSIS OF STUDENTS' SOLUTIONS AND OPINIONS

It is always interesting to see how students try to solve problems that are a little bit different from the ones they are used to. The method they use, the notes they make and the opinions they give tells us a lot about what is going on in their heads. In this investigation I worked with three different age groups who had to solve three well known PISA-problems.

The three groups of students were the following: students in grade 7 (aged 12-13), 12 students in grade 8 (aged 13-14) and 33 in grade 9 (aged 14-15). Students aged 12-14 (grade 7 and 8) were from an average primary school, while ninth-grade students were from a selective secondary school. In Hungary children start primary school at the age of 6. Primary schools are usually 8 years long, so students in grade 7 and 8 were at the end of their primary school studies. After finishing primary school they take the so-called entrance exam and enter a secondary school of their choice. Secondary school is 4 years long; students finish it at the age of 18. So the ninth-grade group was in their first year of secondary school.

I was interested not only in the results – whether the students could find the correct solution or not – and in the competency and skills they used, but in the way they tried to solve the problems and in their opinion about the tasks. That is why I asked the students to write down in a few words how they felt about the different problems. Here, I would like to compare and contrast both the opinions of the students from different age groups and their mathematical results.

The “Carpenter” problem

The majority of students thought that this was quite an easy problem and it was clearly understandable what they had to do. Approximately one quarter of the students from all age groups thought that a good idea was needed to find the solution but a few of them said that the problem was a routine task. Only one seventh-grade student wrote that the problem was difficult to solve and its text was not really understandable. None of the eighth-grade students found the text either ambiguous or not understandable and only two of the ninth-year students said that they were not sure what to do.[1] Some students said that you only had to think logically and it was easy to find the right answer.[2]

Half of the seventh-grade students were able to find the right answer for each part. In grade 3 this ratio was 2/3 and in grade 9 it was about 3/4. One seventh-grade student did not even try to solve the problem. In other age groups all students attempted the question.[3]

The methods they used were different. First of all, students had to realize that they have to work out the perimeter of the shapes and they should use the relationship between the shapes. Obviously, the easiest to solve was part D – there is only one incorrect answer from grade 7 and one from grade 9 – as it is a rectangle with 6 m and 10 m long sides. Students only had to calculate the perimeter of the rectangle. The majority of them, especially students from lower age groups, tended to use the earlier mentioned formula for finding the perimeter of the rectangle. In this part they had to use recalling skills, they had to recall the definition of perimeter and carry out basic calculation**.** Although part A and C looked similar more students gave correct answer for part C than for part A – especially in grade 8. Students usually tried to complete the drawings to get a rectangle. This method helped them realize that the perimeters of the shapes in part A and C are equal to the perimeter of the rectangle. However, there were a few examples where students chose the correct method but arrived to the wrong conclusion. The solution of this problem required the application of more complex mathematical thinking and reasoning skills.

Part B seemed less straightforward than the other three. However, especially in grade 7, the number of wrong solutions is not significantly higher than in part A and C. The reason might be that for the younger students part A, B and C were equally challenging. To be able to answer these questions correctly students had to know how to apply their geometrical knowledge and this part required the use of a more advanced technical skill, too. The fact that the younger students had difficulties with solving the three harder parts suggests that their mathematical reasoning skills, modeling skills, geometrical technical skills are not so well-developed yet and it is more difficult for them to apply their knowledge in an unusual situation.

As predicted, a lot of mistakes occurred as a result of mixing up the concept of perimeter and area, and this led students to the wrong conclusion. In teaching, the difference should be emphasized and teachers could use more visual tasks to make the difference clear.

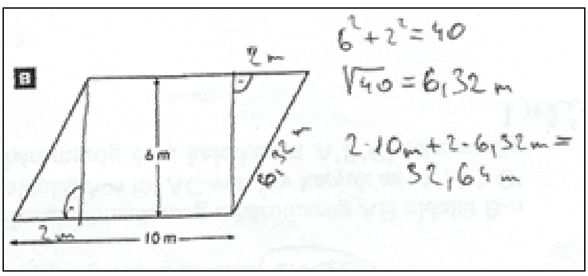


Figure 1. Correct solutions from different age groups

[1] The text of the questions was translated into Hungarian.
[2] Appendix I.
[3] Appendix II.

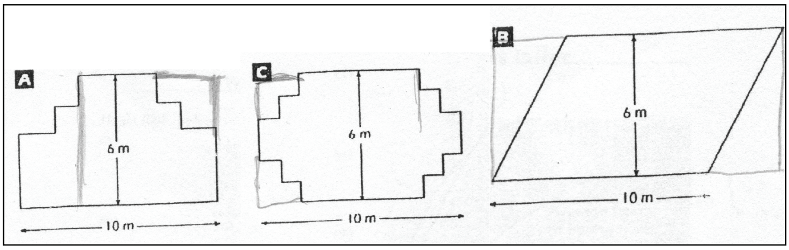


Figure 2. Incorrect solutions from different age groups – using area instead of perimeter

The "Walking" problem

When talking about the "Walking" problem there is a difference between the opinions of the seventh- and eighth-grade students and that of the ninth-grade students. Most of students in grade 7 and 8 consider this problem as a difficult one (67% and 91%). They had difficulties with understanding the text because it was not clear and was often ambiguous for them. None of the seventh-grade students said that this was an easy task and only 16% of students in grade 8 found it easy. They wrote that the walking problem is not a routine task. On the other hand, the ninth-grade students could easily understand the text. They knew exactly what they had to do. Only ¼ of the students thought that the problem was difficult and even if they said so they usually managed to solve it or at least they found the first step. Even fewer ninth-grade students thought that it was hard to understand (13%) or ambiguous (9%).[4]

Some students said that the picture helped them understand the formula and the text of the problem. Others were surprised that they had to calculate the speed not only in km/h but also in m/min. This was a bit unusual for them and many students had difficulty with converting km/h to m/min.

The older students found the task easier and their results were better than that of the younger ones, however, in general the number of good solutions was quite low. In part a), where 17% of students in grade 7, 8% in grade 8 and 38% students in grade 9 could work out the problem[5]. The most typical wrong solution was: $V = 0.8 \times 140 = 112$ m/min (They calculated the pace length correctly, but they thought that the pace length equals the unknown speed.) The percent of incorrect solutions is the following: 44% of students in grade 7, 42% in grade 8 and 62% students in grade 9 made a mistake or followed a wrong plan. Quite a lot of students in grade 7 and 8 did not even try to solve part a), but all of the students in grade 9 had some idea to begin with.

In solving part b) the ninth-grade students were more successful. 84% have found the correct pace length, while in grade 7 only 22% and in grade 8 only 25% had the right solution. Many younger students had an incorrect answer and most of grade 7 did not even solve this part (67%). The seventh- and eighth-grade students' opinions correspond to their result. It means that all students – except for one in grade 7, who said that this is a difficult question – gave wrong answers or had no solution at all. This shows that students are aware of what they are capable of.

Summing up, part b) was easier for all age groups and we have more correct answers here than for part a). The fact that the majority of the seventh-grade students did not deal with the problem suggests that students from this age group have less developed interpretation skills, i.e. it is hard for them to understand complex word problems.

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[4] Appendix I
[5] Appendix II

The “Growing up” problem

The “Growing up” problem was easy, clearly understandable and a routine task for most of the students[6]. Some of them said that it was easy because the answers were on the diagram; they just had to read them from the graph. Only a minority – one student in grade 7, one in grade 8 and two students in grade 9 – said that it was difficult to solve. It is interesting that a lot of students in grade 8 wrote that to solve this task you need a good idea. For the younger students its text was not ambiguous, but one student in grade 8 and three students in grade 9 found that the text is not always straightforward – especially in part a).

The students’ results show that the easiest for them was part b), where students had to define the correct interval. Those who failed to give the right answer wrote only one number and not an interval, but they searched for the answer in the right place. Part c) was also quite easy but in this part there were more incorrect answers. Part a) seems to be the most difficult, especially for the younger students. Only 17% of students in grade 7 gave the correct answer. The eight-grade students were more successful, 58% managed to answer correctly. And 75% students in grade 9 could also work out this problem.[7]

All of the students tried to solve part b) and only a few students had no idea what to do with part a) and c). Furthermore, more than the half the seventh-grade students did not write anything to part a).

As we can see, there is a difference between the achievement and opinion of students in grade 7, 8, and students in grade 9. The latter use their knowledge more confidently and they are aware of their knowledge and skills. The reason for this can be the one- or two-year difference in age and in education, which means more problem solving experience and more mature thinking. Whereas, we have to take into consideration the fact that the younger students were from an average primary school.

CONCLUSION

The aim of this study was to investigate how students can cope with solving problems which are slightly different from the ones they are used to and what they think about them. In theory, they know everything to solve these problems. The interesting thing was to see how they reacted when they first saw them.

The “Carpenter” problem suggests that the difference between area and perimeter should be more emphasized. Teachers have to make sure that students understand what perimeter and area are right from introducing the concepts. Our students are usually good at finding formulas or theorems that can be applied to solve a given problem. But they need to practice rewording either the question or the definition so that it fits to the current problem. They need to practice applying what they know in a different context. For this students need to meet open problems regularly.

The “Growing up” problem shows that although reading information from graphs looks easy it is not always straightforward for students. Therefore, it is important for them to practice it. These types of questions appear more often in exam questions and in textbooks now. Students have to practice understanding questions and they have to make sure that they answer what it asks for. This seems to be a general problem that students do not read questions properly. They just have a quick look, see some numbers, but skip over the important details, which results in them answering a wrong question.

Translating between mathematics or mathematical language and their own

[6] Appendix I.
[7] Appendix II

language is also an issue. Students should practice explaining things, talking about their thinking and changing mathematical information into language. This investigation showed me that students, even those who struggled with finding the right answers, enjoyed doing something different. In my own practice I have to make sure that I provide interesting and challenging questions to my students so they do not lose their interest in the subject. The above study shows that we can make questions challenging just by changing the usual format. I also learned that students appreciate if they can share their views about a task with their teacher.

I mentioned that currently I am a mathematics teacher at a British secondary school, so I have some experience about the types of questions students usually have to solve. Here, their Standard Assessment Tests (SATs) contain a lot of questions which are similar to the three analyzed PISA problems. Students are trained right from the beginning to solve problems like these. One of the aims of teaching mathematics is to teach logical way of thinking. Furthermore, students should be prepared for unexpected problems – unexpected means asking for something they know but applying it in a different context, with different wording... etc. Teachers should equip their students with a wide range of problem solving strategies and make sure that students gain enough experience in choosing the most appropriate strategy in a given problem situation.

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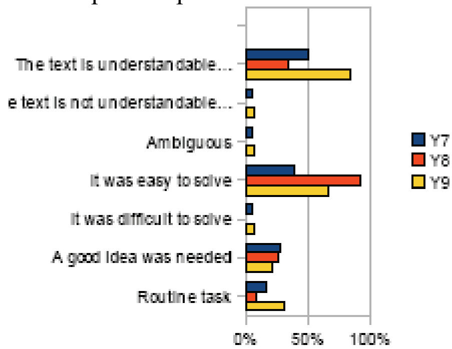
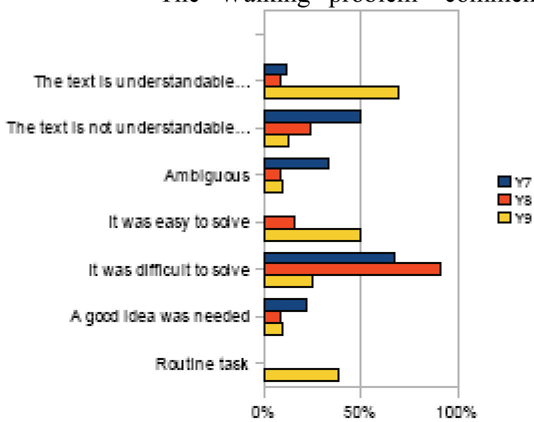
Appendix I.
Students' comments on the three problems

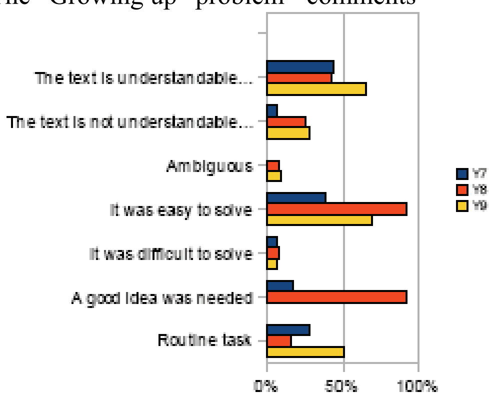
The "Carpenter" problem – comments



The "Walking" problem – comments

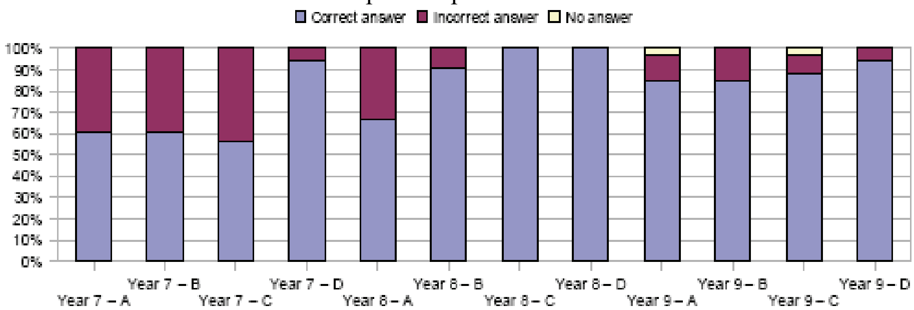


The "Growing up" problem - comments

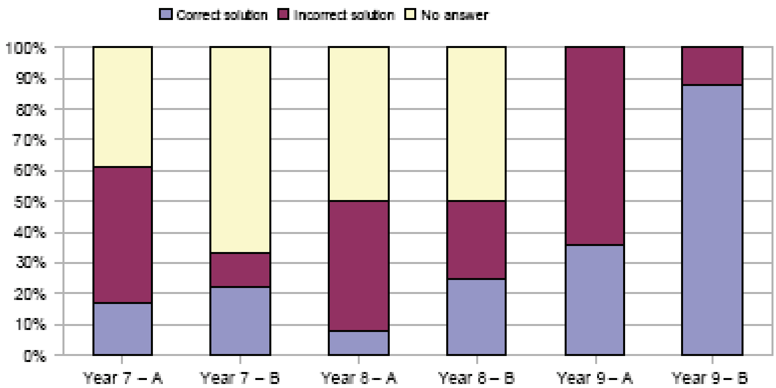


Appendix II
Percentage of correct, incorrect and missing answers

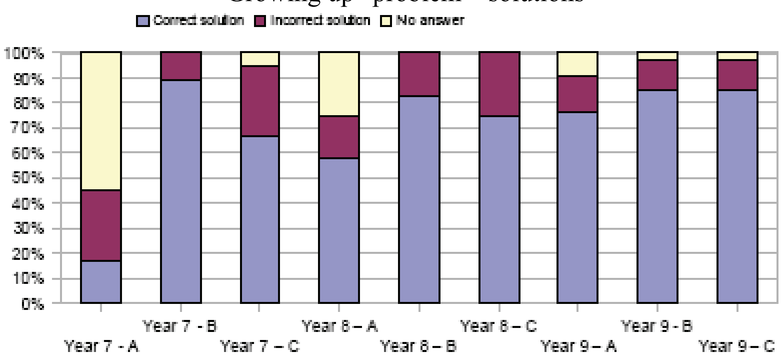
The "Carpenter" problem – solutions



"Walking" problem – solutions



"Growing up" problem – solutions



RAGIO$^{C}{}_{N}$ANDO: A TEACHING EXPERIMENT ON THE SEARCH FOR REGULARITIES THROUGH A COLLECTIVE CONSTRUCTION OF KNOWLEDGE

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ABSTRACT

In this article we give details of the work we carried out between 2006 and 2008 within a wider research project aimed at the introduction of both methodological and curricular innovation in the classroom, as well as at our own professional development through the methods of educational research. The project involved both design and experimental setting of a constructive teaching sequence, based on the identification of regularities in the exploration of numerical and figural sequences. At the same time, the project entailed the actual use of methodology for studying and critically analyzing the enacted classroom-based processes. In the article we describe: the main features of this sequence, some of the experimented activities, students' behaviors with respect to a task drawn from the PISA test and included in the teaching sequence, the general results obtained in the classes. The methodology used to analyze classroom-based processes is then illustrated. Finally, a reflection is made upon the value of the activities, of critical comment and reflection, the awareness we reached and the possible spin-off for our teaching activity.

INTRODUCTION

The work we present here is meant to illustrate how the development of an autonomous and creative form of thinking in students requires a constant spirit of methodological and educational research by teachers, leading them to reflect upon their own professionalism and competencies, for a continuous improvement.

In particular, we present some aspects and results of an innovative teaching experiment, aimed at sixth-grade students and based on the study of sequences for an introduction to generalization and algebraic modeling. Effective teaching and methodological strategies that promoted a critical and research-directed attitude in students are illustrated: these strategies also allowed students to develop divergent thinking, which helped them grasp regularities in an autonomous way, make conjectures and translate them into algebraic language.

THE SEQUENCE

The teaching sequence – jointly developed by trainee teachers-researchers and researchers from the University of Modena and Reggio E. – was named "Ragio$^{c}{}_{n}$ando," a word which combines playing, meant as pleasure in investigation and discovery, and reasoning, meant as pleasure in thinking, reflecting upon and arguing about one's ideas. It was designed and implemented with modalities that point to an improvement and empowerment of students' skills in argumentation and communication on mathematical issues, both in written and in oral form. It was started in the school year 2005/06 and, after initial pilot studies, it was implemented in the school years 2006/07 and 2007/08 with a careful monitoring of the classroom-based teaching and learning processes.

At the basis of the sequence's design were: an accurate study of some problem situations proposed by the PISA test (OCSE 2003) as well as some research papers (Orton & Orton 1994, 1996; Frielander & Tabach, 2001; Malara, 2003; Navarra, 2000; Sasman et al., 1999; Taplin, 1995); a discussion – within the PDTR seminars – about the mathematical value of the examined activities, their consistency with the national curriculum and the modalities to be implemented in the classes.

Our objective was to construct an innovative teaching sequence within the current curricular programs and, at the same time, to make the related assessment consistent with both context and teaching practice related to mathematics. The sequences' contents are not generally included in the traditional mathematics teaching in Italy and this is the reason why teachers had to tackle the topic with the aim to explore it, evaluate its feasibility, study the problems related to its didactical transposition, with the global aim of including it into curricular programs in the future.

General objectives for students are basically set at cognitive and metacognitive level, with social implications, as summarized in the following table:

cognitive level	**metacognitive and social level**
to search for regularities in a sequence of numbers and/or figures	to listen, grasping others' resources
to find a functional link between two sets	to manage the situation in a collective discussion
to formulate and test hypotheses	to develop divergent, critical and creative thinking
to generalize	to argue from one's point of view, reflect upon and make thinking processes explicit
to make abstractions	to favor a constructive dialogical exchange
to understand and use specific language	to acknowledge the importance of a collective construction of knowledge

In the next sections we present the main activities of the sequence and deal with modalities and issues which emerged from the experimental setting: our aim is to stimulate a reflection on the purposes and difficulties of some exploratory contexts we proposed, on their consequences from a pedagogical and disciplinary point of view for students.

THE WORKSHEETS

The activities we selected in the phases of study and discussion were preliminarily analyzed and structured in worksheets with the aim to construct a gradual sequence that would enable students to acquire those competencies promoted by the PISA test, so that they might explore some situations as a preparation to tackle questions taken from the test itself calmly. Some particular numerical sequences were initially proposed (arithmetic progressions), followed by figural sequences, with the aim of comparing the effects of different visualizations in generating a given sequence (some examples are reported in the Appendix). Two particular numerical sequences (one of these was assigned as homework) and two figural ones were proposed to get to the PISA task called "Apple trees" given as final assessment test.

In formulating the worksheets we paid attention to: the context, linguistic aspects in wording questions, numerical aspects and graphical appearance (which was made as much attractive as possible). We were particularly careful to guide the progressive transition from natural language to arithmetical language, by enacting numerical representations that functioned as expressions of the link between the positional index and the related term in a sequence, to end up with a representation in general terms by means of algebraic language.

We then outlined sketches for carrying out mathematical discussions as a support to keep under control some key points of arrival, effective stimulus-questions to prevent students from taking ambiguous or dispersive routes or routes that are little consistent with the main aim to be reached.

The worksheets we outlined led students in the exploration and search for regularities in numerical sequences with an increasingly higher degree of difficulty. In particular, while they are observing the recursive nature of a sequence, students must predict how the sequence continues, thus identifying its generation through the application of the same operator, until the functional link between the positional index and the term of the sequence is made explicit. The analysis of numerical data progressively leads to a relational analysis of the tables containing results from the exploration of the examined cases and later to complete them with the explicit inclusion of the identified relations. Numerical data are always reported in tables that favor the identification and formalization of relations. Students are gradually led to generalization by means of algebraic language. In particular, students are initially guided in the search for a rule that enables them to express a term of the examined sequence not as a function of the previous one, but rather as a function of the first term. The final step is then to lead students to translate the rule that expresses this functional link in formal terms.

Exploration and search for regularities in geometric sequences favors different learning styles and, at the same time, is meant to make students more autonomous in the exploratory activity, since the sequence's terms are not immediately given, but rather they are deduced from techniques for counting the represented elements. Focus is thus on the enactment of visualization skills, and specifically of skills of analysis and decomposition of graphical configurations characterizing the sequence's elements, with the aim of identifying strategies for counting that may be generalized to determine the total number of elements that generate them. Students are later led to reflect upon their own thinking strategies when they are asked to explain how they got to realize the graphical terms in the sequence or rather their counting strategies.

EXPERIMENTAL SETTING AND WORK METHODOLOGY IN THE CLASSROOM

After the pilot studies carried out in the first year of affiliation to the PDTR project, in the following year (2006/07) the teaching sequence was implemented within the curricular programs. This required a careful scheduling of the proposed activities, since the latter were not included in the 'minimum curriculum' (as concerns both contents and aims) that teacher are supposed to complete, in order to offer educational continuity, especially when they are assigned a temporary school.

The teaching and learning methodology was characterized by a strong metacognitive and socio-constructive approach: individual exploratory phases alternated with collective discussions, centered on exchanges and reflections upon both enacted strategies and mistakes made. Work in the classroom was supported and strengthened by a fruitful work at home, which provided a chance for a formative assessment to both teachers and students. The teaching and learning activities were sometimes presented as a challenge or a game, in the attempt to overcome a didactical contract centered on the mere application of rules and mathematical procedures, in which students might get involved if they are not given the chance to become aware of the processes underlying their own and others' rational thinking. We attempted to overcome possible blocks, both cognitive and psychological, that might have discouraged weaker students and prevented them from learning even strictly disciplinary topics.

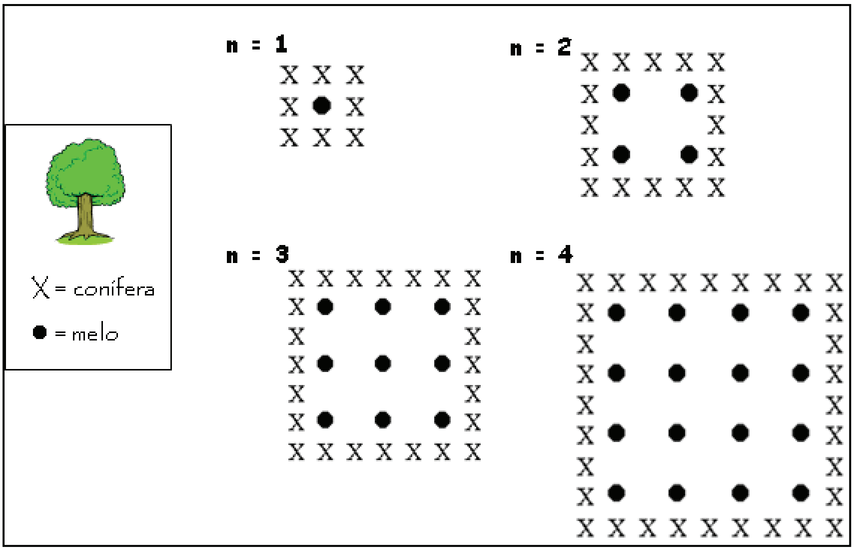
In the second year of the teaching experiment (2007/08) the sequence focused on the study of figural sequences, with the aim of leading students to identify and represent the joint variation of the pair <place-term> in a sequence, to get to the general representation. Students were thus led to compare strategies for constructing the sequences under exam, to reflect upon their own thinking strategies for counting as well as on their efficacy and validity, to predict either the presence or the absence of a certain term in a sequence. In particular, the last proposed activity, taken from Friedlander & Tabach (2001), was meant to make students tackle a more difficult task: given the figural representation of a term of a sequence at a given place, represent immediately preceding and subsequent terms, deduce the underlying regularity and see the sequence in functional terms, making the generating rule explicit and, finally, compare and verify, with an appropriate argumentation, strategies for counting the elements that constitute the terms of the sequence, on the basis of algebraic modeling of the rule.

BEHAVIOURS OF STUDENTS TACKLING THE STUDY OF THE PISA TASK "APPLE TREES"

The theoretical framework of the PISA test assessment centers around four particular content areas: space and shape (including geometrical and spatial problems), change and relations (mathematical representations of change, functional relations and interdependence of variable quantities), quantity (quantitative representation of phenomena, relations and schemes), uncertainty (study of combinatorial, probabilistic and statistical phenomena and related representations).

The "Apple trees" task is placed in the PDTR's key idea "change and relations" as it develops specific skills referred to connections through a coding of identified relations in functional terms. The task's problem situation is centered on the identification of relations between the involved data, by means of open-ended questions: the data are apple trees and conifers scattered around squared fields, with variable side, apple trees being at the center, and conifers along the border, following one single figural scheme (see picture).

The task is clear and split into a series of questions. The first two are quite simple, as they are limited to the study of figural representations of the examined situation; it is with the third question that students are introduced and gradually guided towards generalization and algebraic representation. In particular, they are invited to search for a functional relation for the distribution of apple trees with respect to the field's index and later for another relation in the distribution of conifers. The last part of the task is the most complex: students are asked to investigate on whether there is a case, in which the two identified relations allow them to predict if there is the same number of apple trees and conifers for one same field.

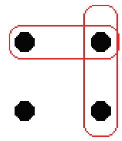


Among the different possible strategies we distinguish relational and recursive ones. In the exploration of the numerousness of apple trees, in the very first cases the simplest strategy is based on counting the single dots and later an additive strategy will be considered. The latter will then be substituted by the multiplicative strategy in the subsequent, more complex cases. Several routes might be followed in reasoning about how to count the distribution of apple trees when n=5. A relational type of argumentation can be produced, by generalizing from previous cases: the field with index 2 has 4 apple trees, the field with index 3 has 9 apple trees, that with index 4 has 16 apple trees and then the field with index 5 has 25 apple trees. A second way might be the recursive one: on the basis of the previous case 4, a row might be added horizontally and a line vertically with 5 apple trees, but taking out one because otherwise the tree at the intersection would be counted twice. A third way might be to add an apple tree for each of the 4 rows horizontally and vertically and add an apple tree to complete the square. Counting conifers implies a subtle analysis of the regularities characterizing the configurations. Recursive strategies for counting seem to be more immediate: one of these is based on the addition of two conifers at each side with respect to the previous configuration. The other strategy is based on the addition of one conifer at each vertex of the square.

There are two relational strategies. One of these is based on the remark that in each configuration, if we exclude the four vertexes in each side, the number of conifers is twice the number of the configuration minus one: in case 2 there are 3 conifers, in case 3 there are 5 conifers, in case 4 there are 7 conifers, so in case 5 there must be 9 and in the general case n conifers are 4(2n-1)+4.

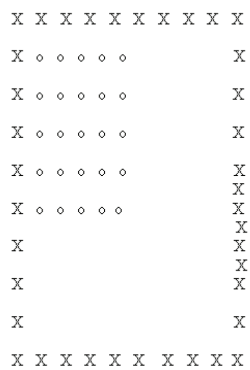
Another strategy stems from the remark that in each configuration the number of conifers is 4 times twice the number of the index: it is enough to go along the perimeter clockwise, starting from top left, and count the conifers in groups of as many trees as the number that follows the index. The comparison between the quantity of apple trees and conifers as n varies is particularly interesting: the square of a number is to be compared to its multiple by 8. If the concepts of square and multiple are clear enough, it is obvious that the multiple by 8 of a number equals the square of that number only when this is exactly 8.

The analysis of answers provided by students in their individual work, shows that in the search for regularities in the apple trees' configuration, the strategies enacted are substantially: a) counting dots; b) counting by row or column and subsequent discovery that *n* corresponds to the number of apple trees in each row (or column); c) counting by reducing to the previous case. As concerns strategy b), some students express themselves with statements like "multiplying it by itself," referred to the configuration number and to the concept of multiplication; some other students use graphical representations like that illustrated above. As concerns strategy c), students suggest that one dot more might be added to each row and column of the previous configuration. The following is an example of an argumentation they provided to support the given numerical result: "It turns out like this because looking at the drawings you always add 1 in each row." Then they add 1 to complete the square. Only one student uses both strategies and writes: "In order to draw the apple trees you must take the number of the figure, for instance 3, and multiply it by itself. Or rather you need to put one column more and one dot more in each column, which is the same as the previous strategy."



When the numerousness of conifers is to be determined, the iconic representation becomes more difficult: some students realize the configuration for n=5, focusing on apple trees only, without considering the relation linking this number with that of conifers. An example of this is given in the representation aside.

For others, the exploration is more systematic and precise than at the very beginning. In particular, some of these students highlight the relation between the conifers distributed along one side of a generic configuration (index *n*) and those situated along the same side in the previous configuration, thus pointing out the need for adding 2 "I did so because in *n*. 1 there are 3 conifers per side, in 2 there are 5 per side, therefore conifers will have to proceed by two."



Only one student remarks that the total number of conifers increases by 8 moving from one configuration to the next and argues by saying "making plus eight" to determine the total number of conifers. Students then get to the generalization, with the aid of tables: some make horizontal and vertical relations explicit and formalize the number of apple trees with $n \times n$, or n^2 and the number of conifers with $n \times 8$. Some difficulties have emerged in the answers to the last question: some students equal the two relations without actually realizing that, others proceed by trial and error. In the justification they either use or not the graphical representation to support their argumentation.

A student only analyses the first cases and, generalizing wrongly, gets to say: "There isn't– a number n for which the number of conifers is equal to the number of apple trees – because there are always more conifers."

Finally, one student shows his divergent mathematical thinking, looking at the case with index 0 and answering the last question "0, because there are no trees to be protected" thus showing a creative use of his own knowledge and skills.

COMMENTS ON RESULTS OBTAINED IN THE CLASSES

Generally speaking, by exploring sequences linked to a certain rule, students are led to acquire a capacity of generalization and representation through algebraic language. These activities, although requiring a lengthy time to be implemented (together with curricular ones), seem to favor the development of divergent, creative and critical thinking, which enables students to tackle strictly disciplinary topics being more serene and open to exchanges with others. The activities we implemented have a metacognitive value: they ask students to make their reasoning explicit, thus encouraging them to reflect upon their own mental procedures, besides favoring a development of argumentative skills and a refinement of specific language. In particular, students in the classes we examined, learn to manage their own participation in collective discussions, refining their listening skills and becoming able to grasp everyone's resources: this favors a constructive dialogic exchange that leads to an aware collective construction of knowledge. The chosen methodology entails everyone's active participation, independently on one's knowledge or disciplinary competencies. The sense of self-efficacy is thus favored and students are likely to overcome their fear of making mistakes.

The teaching experiments we implemented suggest that the explored situation might be a stimulus for students, even if they are carried out sporadically, because they show the actual potential of arithmetic, the richness of problems, numbers' "own life" and nets of mutual relations.

RESEARCH METHODOLOGY AND REFLECTIONS UPON ITS PRODUCTIVENESS

Our research methodology focused on the *analysis of classroom-based processes*, and was supported by audio recordings of collective discussions, their transcription commented by teachers, the enrichment of comments at various levels and the analysis of students' productions, in order to have an overall definition of the Project's educational and disciplinary spin-off. In the next paragraphs, we will reflect upon the productiveness of the chosen methodology, by drawing a distinction between effects on students and effects on teachers.

Students' side

Transcribing collective discussions allowed us to constantly monitor what we planned in order to make the subsequent activities more suitable, to check, in each situation, the actual mastering of both concepts and instruments by students and to follow their cognitive and argumentative evolution in view of the assessment. Audio recording allowed us to carry out an analytical observation and, in particular, to pay higher attention to those relational, communicative and argumentative processes of the class as a group that cannot always be grasped immediately during the activity, but which might nevertheless enable teachers to manage students' answers more effectively and thus modify the teaching sequence accordingly.

The exchanges among teachers who shared the same project was a moment of professional development: it widened the range of possible future projects, of possible, even alternative, answers by students, of possible and not foreseen difficulties they might meet, as they depend on the class context.

Teachers' side

Process-related protocols we edited as teachers were the basis of an exchange between us, our mentors and the research head. These protocols, enriched with comments jointly added by the mentioned subjects, were discussed initially in working sessions with the mentor and later in specific collective sessions (Malara, in this volume). This led to the constitution and sharing of a set of observations, reflections, methodological and operative guidelines that allowed for a widening and deepening of points of view, a wide open exchange not only related to methodological issues, but also to knowledge about the discipline. At the end of the first year of the teaching experiment some meaningful follow-ups on our teaching professionalism emerged. This favored an in-depth reflection upon methodological and disciplinary skills to be refined, as well as on the pedagogical sensitiveness that needs to be used in order to be able to carry out an *authentically* constructive teaching activity.

Our repeated reading and thinking over commented transcripts allowed for a self-evaluation of our own professionalism, a critical metareflection on our own way of managing collective discussions, on our way to send students' suggestions back to the class, to intervene and direct, sometimes categorically, the discussion itself. After this process we got to a higher professional awareness: in particular we became aware of the need to refine our capacity of grasping immediate feedback by students in a meaningful way, always keeping in mind the aims of the route we undertook. We also reached a higher awareness of our own capacity of mediating in teaching situations as well as of the need for a careful control over a clear distinction and coordination of natural language and specific language. The role of teachers and the variety of competencies they have to be able to enact in a collective discussion emerged clearly: teachers should be able to value students' interventions and send them back to the class, must not make explicit judgments, trying to keep a consistency between verbal and non verbal language, must be able to direct the lesson, giving space to the groups' dialectics, must be able to give students the chance to reflect upon ideas, opinions, mistakes and successful results obtained together.

There is a long way to go

This experience led us to a different view on our profession, as open to study, analysis and exchanges, and also instilled the idea that our professional life will continuously develop for the whole productive time.

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APPENDIX

Example of an arithmetic sequence

Continue the sequence

4 11 18 __ __ __ __ __ __

From 4 the rule is ____

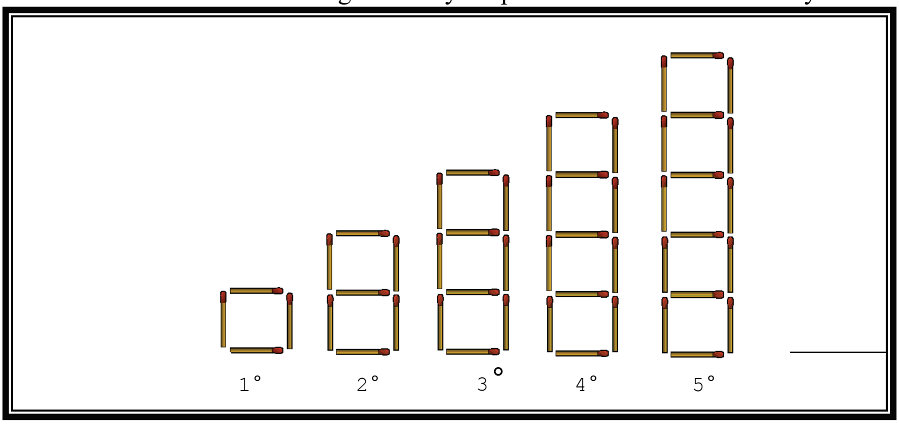
Complete the table

Ranking number	Number	Operation to jump from the first number	Mathematical recipe to build the number
1°	4		
2°	11	4 + ___	4 + (__ x 1)
3°		4 + ___ + ___	
4°			

Example of a figural sequence

We construct with matches!!!!!!

Lorenzo is constructing some skyscrapers with matches in this way:



1. Try to realize the skyscraper that would occupy the sixth place.
2. Explain the procedure that you have adopted in order to realize the sixth skyscraper.

A PISA-LIKE PROBLEM FOR 8-YEAR-OLD CHILDREN: TEACHERS' CHOICES

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ABSTRACT

According to the main cultural aim of the PISA project, mathematics should be used as a tool to model reality and to make inferences. In this work we present an activity based on this cultural idea of mathematics and finalized in the acquisition of the general concept of proportionality in primary schools. In particular, we will analyze the teachers' role in managing activity through the theoretical lens of Bruner's dichotomy between narrative and paradigmatic thought.

INTRODUCTION AND THEORETICAL REMARKS

This paper is about an experimental path on proportionality conducted in a third-grade class. The main aim of the experience was to analyze the teacher's role in promoting students' knowledge reconstruction. Here, we focus our attention on the teacher's choices in planning and conducting the activity and on the theoretical framework that supports them. We start from the strong assumption that "knowledge construction is the result of a process of *resonance* between individual cognition, social culture and reality structures, along cognitive paths efficiently addressed and controlled in their meaning-driven dynamics. It requires, at any level, also *resonance* between various "dimensions" of natural thinking: perception, language, action, representation, planning, interpretation" (Guidoni et al., 2005). So, we focus our attention on children's natural thinking, taking into account just two variables among others: the way children naturally acquire information from the external world and the way they organize such information. With regard to the first variable, we recognize that all children possess a basic network of biological/cultural cognitive strategies through which they become able to see/look at the various sides of reality. In particular, children use mostly their senses to know the external reality so that first knowledge of the properties of the surrounding environment comes from their feelings (Gardner, 1982; Quaglia, 2003). With respect to the strategies devoted to organization of knowledge, we refer to Bruner's hypothesis that

> there are two kinds of cognitive working, two ways of thinking: each of them provides its particular method of organizing the experience and building the reality. These two ways of thinking are irreducible to one another, though complementary. Every attempt to reduce them to just one or to ignore one favoring the other one unavoidably produces the effect of losing sight of richness and variety of thought.[1]

Moreover, Bruner introduces the duality between "paradigmatic" thought and "narrative" thought, the first being

[1] This quotation and the following one are translated by the authors from the Italian version of (Bruner, 1986).

devoted to the search of the causes of general order ... it makes use of procedures able to ensure the referential truthfulness ... its language is regulated by the principles of consistency and of non contradiction ... the creative use of paradigmatic thought produces good theories, rigorous analyses, correct argumentation and empiric discoveries that lean on reasoned hypotheses.

On the contrary, the narrative thought deals with the typical human intentions and actions, and its organization is based on time-space coordinates and on casual links rooted in experience.

THE CLASS EXPERIENCE

The following experimental path has been proposed in a third-grade class to children that for the first time approached a mathematical laboratory.

Firstly, we proposed the problem in Figure 1.

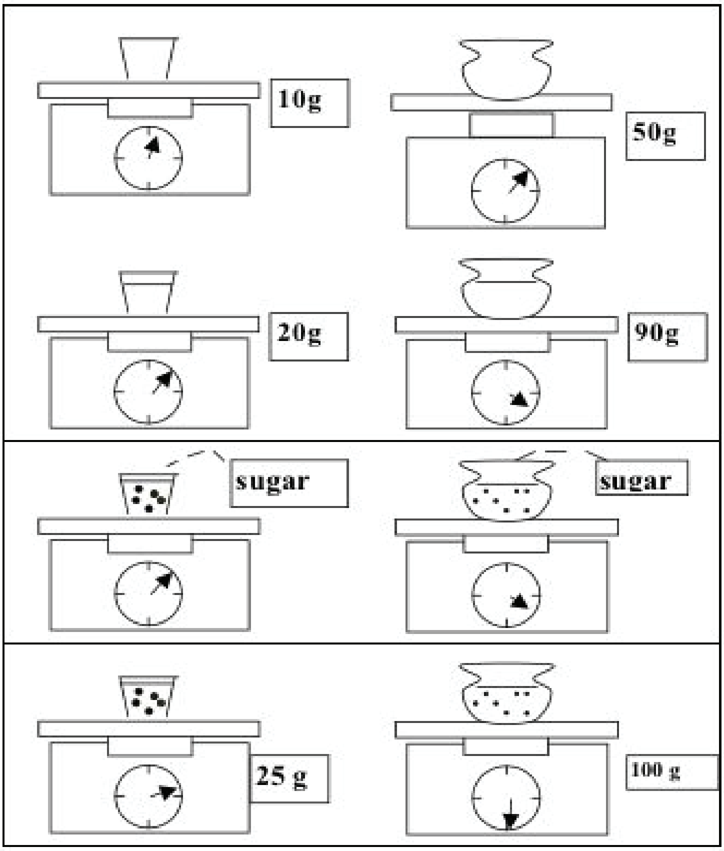

Figure 1.

Is it sweeter: the water in the glass or the one in the carafe? If they're not equally sweet, what can we do to make them equally sweet?

We chose this context for many reasons: (1) it is strongly rooted in everyday experience, so that a lot of pertinent thought-action-wording-representation aspects are actually available to support knowledge construction; (2) it can be approached by means of several strategies, from bodily to formal, crossing through different kinds of representations; (3) it is at the same time complex enough to demand a careful previous individuation of pertinent variables, and simple enough to allow for an exploration not too rigidly guided. The third condition, in our opinion, is the crucial one in bridging narrative to paradigmatic thinking, in both directions.

The children's request of concretely realizing the described experiment comes up immediately, confirming one of our hypotheses: in problem solving situation they trust their perceptive/motor strategies. Then we arranged the experiment. Soon, a difficulty arose in looking for a glass and a carafe corresponding to those in the problem, but this allowed a rich discussion about what is important in order to answer the first question and what can be neglected. We suggest that all these collective reflections promote awareness of the dialectic between narrative and paradigmatic thought.

Stefania: I *see* there's more sugar at the bottom of the carafe, isn't there?
Martina: I want to *mix* it with the teaspoon, so I can *look* at the sugar again and then I'll be able to decide.
Giuliano: Let's try to *color* the sugar so that we can say where there is more sugar.
Joseila: Can I *taste* the water once again?… I did not feel well… From my point of view the water in the glass is sweeter.

We note that children activate perceptual strategies: to see, to taste, to color, to mix. In this phase the mathematical data are used only to realize the two solutions, not as reasoning tools: in their reasoning children completely forget the "quantities" introduced by the problem.

In order to draw children's attention to numerical aspects, we introduced a space visualization of the sugar/water rate. For us, this representation had to act as a semiotic mediator toward a paradigmatic organization of their observations. We supplied children with some blue squares in order to represent water and with some smaller yellow squares to represent sugar. Students proposed their own representations sticking the colored squares to a sheet of paper. Many students distributed the sugar pieces at the bottom of the containers revealing again a perceptual reasoning (Figure 2); others distributed the same pieces in a scattered way. Only 3 students out of 18 produced a representation like that in Figure 3.

Figure 2.

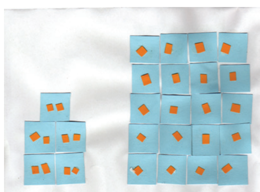

Figure 3.

We want to emphasize the fact that the choice of measure units for sugar and for water was the result of a negotiation in a collective discussion. A further discussion on the features of the various representations allowed us to choose the distribution in Figure 3 as the most effective.

Salvatore: It is clear! The water in the glass is sweeter because in every piece of water there are two pieces of sugar, while in the carafe in every piece of water there is only one piece of sugar.

Salvatore's words "in every piece there is" later shared by the whole class, show that children's attention focuses now on sugar/water rate.

Children's goal was to "discover" the answer, while for us the problematic situation was just a pretext to induce children to use a more effective and technical linguistic tool that could put better linkages in evidence, that is, the *tables*. According to Vygotsky, we believe that language is a social tool that drives the thought: in this sense, a two-column table, being a translation of Salvatore's "natural" words in an iconic/symbolic language, can shift the focus from the particular situation to the more general proportionality relation between two variables.

	pieces of sugar in the glass	pieces of sugar in the carafe
1 piece of water	2	1
2 pieces of water	4	2
3 pieces of water	6	3
4 pieces of water	8	4

Figure 4.

As usual, children accepted teacher's proposal for a new representation very well. They invented many types of tables like that in Figure 4 in order to discover the answer to the second question. This table shows and clarifies the relationship between sugar in the glass and sugar in the carafe, when we have the same quantity of water. At this point the teacher suggested to construct two tables (see Figure 5), in order to underline the relationships between water and sugar in the two containers.

Glass	
units of sugar	units of water
1	2
2	4
3	6
4	8
5	10
6	12
7	14

Carafe	
units of water	units of sugar
1	1
4	4
6	6
8	8
10	10
12	12
14	14
16	16
18	18
20	20

Figure 5.

Through the use of these tables children explored many regularities, for example, the way to move between the two columns by the same factor of multiplication.

Moreover, children's tables were not "closed" at the bottom like in Figure 5. Once they discovered the proportionality factor, they also realized that the tables could go on. So, through a visual perception, they were becoming aware of the generality of the two proportionality relations. The students still needed to see that not always a result obtained by working with numbers can be transposed without restrictions into a real world situation. Therefore, the next step was to try to translate the mathematical knowledge so hardly acquired back to the side of the perceived reality.

Adriana: We know that the two relationships can go on forever, but the tables have an end.
Teacher: So, how can we continue them?
Lucia: I tried to do it in my notebook and I arrived to 144 and I could still continue. It can be continued up to infinity: if you think to glass and sugar, you can't continue because water comes out. But with numbers you can arrive even to thousands!

Finally, using their tables, children tried to solve the last part of the problem:

Lucia: I think you can add sugar into the carafe…precisely twenty pieces of sugar.
Antonio: Yes, but you can also take off the sugar from the glass.
Lucia: No, you can't, because in the glass water is mixed with sugar, so you can't take sugar off. The only allowed action is to add, so you can also add water into the glass.

It is very interesting to note how Lucia activates both kinds of reasoning in looking for a solution to the new question: this is the starting point of the process toward the ability of selecting a way of reasoning on the basis of the actual goal.

Of course, the path did not stop here: the exigency of comparing different answers to the second question lead us to introduce a Cartesian representation. Our aim was to exploit this new tool in order to allow children to use again perceptual strategies for reasoning about sugar concentration at a different and higher level. In fact, reflecting on the picture in Figure 6 they discovered that each line corresponds to a different "sweetness" of the sugar/water solution, that the sweeter is a solution, the closer to the "sugar axis" is the corresponding line, and so on.

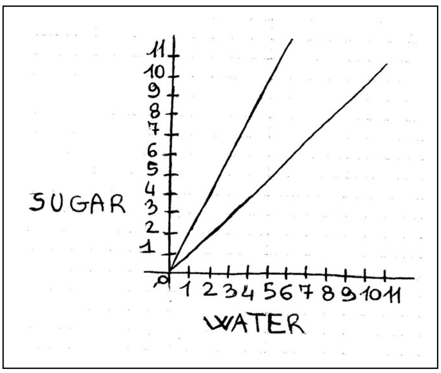

Figure 6.

CONCLUSIONS

The awareness of the complementary roles of paradigmatic and narrative kinds of reasoning is a powerful tool in teacher's hands in promoting knowledge construction. As we have tried to show, it guides both teachers conduct and their interpretation of children's cognitive behavior. This is particularly important if we look at mathematics education in the sense of the PISA project: in fact, using mathematics as a tool to model reality requires awareness of the potentiality and of the limits of a mathematical model. This is the reason why the experimental path we have presented was not imagined as a one-way path from the concrete level to the formal one, but rather as a cyclic path that, starting from the real world problem, goes to the formal level and then back again to the concrete context. In these back and forth dynamics the synergy of two kinds of reasoning plays crucial role.

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…CAN A SEAL SLEEP WITHOUT BREATHING?

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ABSTRACT
The PISA 2003 problem the "Seal" is analyzed and criticized from various points of view: language, information provided, biological and physical realism, solving difficulties. Then students' solutions are quoted, which confirm the criticism.

KEY WORDS:
PISA, translation, cross-disciplinary

THE PISA TEST

The Program for International Student Assessment (PISA) is an internationally standardized assessment administered to 15-year-olds in schools. The PISA test tries to assess how far students near the end of compulsory education have acquired some of the knowledge and skills that are essential in adult life and for full participation in society. Skills and knowledge, such as reading, mathematical and scientific literacy and problem solving are investigated. In particular, "problem solving is an individual's capacity to use cognitive processes to confront and resolve real, cross-disciplinary situations where the solution path is not immediately obvious and where the literacy domains or curricular areas that might be applicable are not within a single domain of mathematics, science or reading" (www.pisa.oecd.org).

THE EXPERIMENTAL CLASS

During the school year 2005/06, I taught in two different secondary schools in two small towns 15 km and 30 km away from a bigger city, Parma (Italy). I submitted my second-grade (secondary school) students to one of the problems from the 2003 PISA test, the "Seal." Both classes were average classes, with some good pupils and some others showing logic difficulties.

THE PROBLEM

The PISA test was first formulated in English and then translated into the other languages. The Italian version is slightly different from the English one. Version that follows is my re-translation from Italian.

> A seal has to breathe even if it is asleep. Martin observed a seal for one hour. At the start of his observation the seal dove to the bottom of the sea and started to sleep. During the following 8 minutes the seal swam to the surface and took a breath. After 3 minutes it was back at the bottom of the sea again. This whole process was a very regular one.
> After one hour the seal was:
> a. at the bottom
> b. on its way up
> c. breathing
> d. on its way down

The text begins with information that a seal has to breathe even if it is asleep in the water. The question that solvers probably ask themselves is this: Is there a living organism that does not need to breathe while asleep? The answer is obviously no, but this useless piece of information makes the readers put to it an importance, which it does not have. It is misleading information.

The original English version says that “At the start of the observation the seal was at the surface and took a breath. It then dove to the bottom of the sea and started to sleep.” This difference with the Italian text gives rise to a slight phase displacement. It looks like in the English version there is a zero time, identified with the seal being at the surface and taking a breath. Taking a breath is like the beginning of the process. The Italian version says that “during the following 8 minutes the seal swam to the surface and took a breath.” The English version says instead that “from the bottom it slowly floated to the surface in 8 minutes and took a breath.” It makes an obvious difference in meaning. The Italian version tells a story of a seal that actively swims to the surface and takes a breath, whereas in the English version the seal passively floats to the surface. The two versions justifiably take to different conclusions. On the one hand, there is a seal that wakes up, swims to the surface and breaths; on the other, there’s a seal that floats to the surface still sleeping.

Beyond these differences in meaning that the translation caused, analysis of the Italian version raises many doubts: (i) How long does the seal take to get to the bottom the first time? (ii) Does the seal start sleeping as soon as it reaches the bottom? (iii) Does the seal sleep on the bottom for some time before going up again? If yes, for how long? (iv) Is the seal asleep or awake when it goes up? (v)When the seal reaches the surface to take breath, does is stay there for a while or does it plunge in immediately?

The Italian text turns out to be unclear and difficult to be interpreted, because the reader does not know how to arrange the various actions the seal does. Moreover, the most important information is not given: seals can sleep both on land and in the water, making the water surface to rock them; they can rest alternating diving (during which they can lay at the bottom) and slow going up to breath. This information is anything but useless and it cannot be known to people who are not seal fans or researchers. This lack of information makes readers guess and, above all, it gives the text poor reliability. The text also looks artificial and unnatural since it does not even say which instruments Martino uses to exactly measure how much time the seal needs to get to the bottom and how he knows when and if the seal falls asleep.

Beside, some terms used in the text do not help readers understand: “during,” “following,” “very regular” are misleading words that lead more to a personal interpretation than to an objective reading.

It’s also important to consider the key answers’ sequence. One of the potential answers gives chance that the seal might stay on the bottom. This answer is misleading for the entire understanding process because it suggests that the seal effectively stays on the bottom for some time. But for how long?

SOLVING WAYS

When the time variable appears in a problem, the attention is necessarily drawn to the process. The problem can be modeled through different solving strategies: (i) 11 minute modeling: the seals needs 3 minutes to reach the bottom and 8 minutes to go up to the surface; the entire cycle lasts 3+8=11 minutes; (ii) 8 minute modeling: at the beginning the seal dives to the bottom and it falls asleep; it seems that all that happened at time zero (T_0) and that the entire process up & down lasts 8 minutes; (iii) modeling with a starting point at the bottom of the sea: at the beginning, the seal dives to the bottom and it looks like this situation (the seal at the bottom of the sea) is observed on T_0; (iv) modeling with a starting point at the surface: the seal needs 3 minutes to get to the bottom of the sea. It is right to think that it needs the same time the first time it dives to the bottom at the beginning of the observation and that the observation T_0 should be when the seal is still at the surface.

Any of the above models take to the B key answer (the seal is on its way up).

ANALYSIS OF SOLUTIONS OF SEVENTH-GRADE STUDENTS (aged 13)

Many students in the process of solving the problem expressed some doubts:

Salma: But how can he see it if it's at the bottom of the sea?
Lucrezia: During the 8 minutes, how long does it take to go up and to breathe?
Barbara: Is 8 minutes after the observation hour or during it? How long does it sleep? How long does it take to go to the bottom the first time?
Alessandra: How can it stay on the bottom in a continuous way? But does it go up by itself or because of the water reaction?
Federica: But how long does it sleep?
Clarissa: But is breathing the same thing as taking a breath?
Valentina: But was it taking a breath at the surface or at the bottom?
Giorgio: But if it takes 8 minutes to go up and 3 to go down, what does it do during that time?
Maria: But when does it sleep?
Chiara: But when and how long does it sleep?
Roberta: But what came after the 8 minutes? The diving?

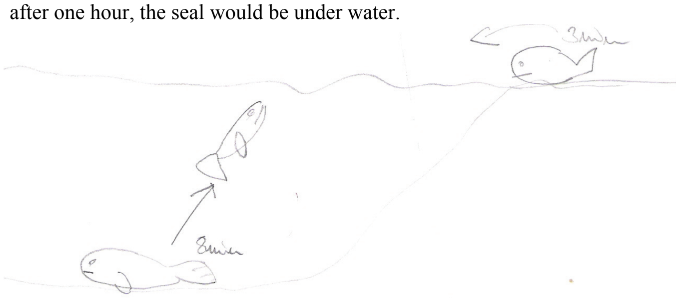
The students tried to solve the problem using different types of modeling, as seen in the following chart:

	No. of students
11-minute modeling	12
8-minute modeling	5
11-minute and 8-minute modeling	2
other modeling	8
answer with no modeling	11
no answer	12

11-minute modeling

Some students solved the problem, more or less correctly, using the 11-minute modeling. All of them incorrectly fixed the starting point at the bottom of the sea, since they did not consider the time needed for the first diving.

Maddalena: To solve the problem would be enough repetitiously adding 8 and 3 minutes till 60 minutes, and then check, which the last number to be added is. In this case it's 8; thus, after one hour, the seal would be under water.



Francesca: The seal is on its way down. If it gets to the bottom in 3 minutes and it goes back up in 3 minutes, it means that it stays at the bottom for 5 minutes. Going up and down is 6 minutes, plus the 5 minutes at the bottom. It makes the same up and down 5 times, 55 minutes total, when the seal is at the bottom. It goes up in 3 minutes and down in 3 minutes. While it's going down, it's 60 minutes. But I'm not sure we need any calculation to solve it.

Francesco: It was at the bottom because at 55 minutes it was breathing, but it stays in water 8 minutes and it takes 3 minutes to go down and 3 minutes to go up. If we count to 60 with 8 and 3, we get to 55, plus 8 it's 63. I then understood that it's at the bottom and that it's there for 2 minutes, at 59th and 60th one.

Anita: The seal goes up and down 5 times in 55 minutes. If we add 8 minutes during which it's under water, we get to 63 minutes. Since it's over 10 minutes, it means that the seal is still at the bottom of the sea. The right answer is then A (at the bottom).

+	3	3	3	3	3	3
8	11	-	-	-	-	-
8	-	22	-	-	-	-
8	-	-	33	-	-	-
8	-	-	-	44	-	-
8	-	-	-	-	55	-
8	-	-	-	-	-	66

Pasquale: The seal is at the bottom because it stays 8 minutes there and then goes up to take a breath and after 3 minutes is at the bottom again and it takes 11 minutes to do so. If every 8 minutes goes up and after 3 minutes is at the bottom again, after 55 minutes it did it 5 times. But it's not one hour yet, there are 5 minutes more. But if it takes 8 minutes to go up, at 60 minutes the seal is still under water.

Federica: It sleeps 8 minutes, goes up and goes back down after 3 minutes. It sleeps 8 minutes, goes up and goes back down after 3 minutes and so on. According to my construction, after one hour the seal is at the bottom because at the hour it's in the 3 minute period it spends at the bottom.

Giorgio: I made a cycle reproducing one hour and all the minutes needed to breathe. Making a calculation, I get to 60 minutes. Deleting 3 minutes to go down and deleting 3 minutes from the 8 it needs to go up and breathe. I obtain 60 minutes. Thus, after one hour, it's breathing.

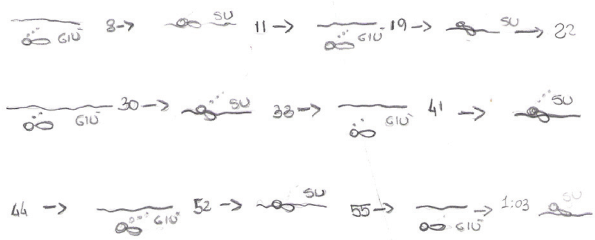
Mattia: The trial lasts 11 minutes. Five times this period makes almost one hour (60 minutes). The text says that the first 8 minutes he dives back up. But there are 5 minutes left to the hour. It's then going up.

Martina: The seal is going up because, counting up to 60 minutes from 11 (8+3=11) we add 8 and then 3. When we get to 60 the seal is still in the 8 minutes it spends going up.

Lucrezia: Taking a breath takes less than a minute. Going up takes 7 or 8 minutes. Doing all the up-and-downs takes 11 minutes. 11 minutes five times is 55 minutes. There are 5 minutes left to the hour. It takes 7 or 8 minutes to go up. During that remaining 5 minutes, it's then going up.

Giulia: The seal needs 3 minutes to go down, 5 minutes stays at the bottom, and it needs 3 minutes to go up. After 8 minutes it goes up. In the 11 minute period it goes down stays there and goes back up. But we don't know how much time it needs to breathe.

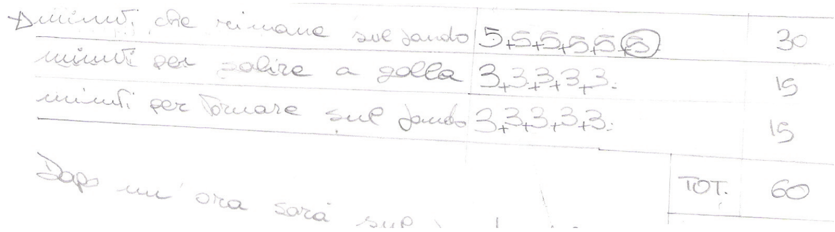
Jessica: The seal is going up because if it's at the bottom and it goes up after 8 minutes, and then it goes down after 3 minutes, we understand that, after one hour, it's going up breathing.



8-minute modeling

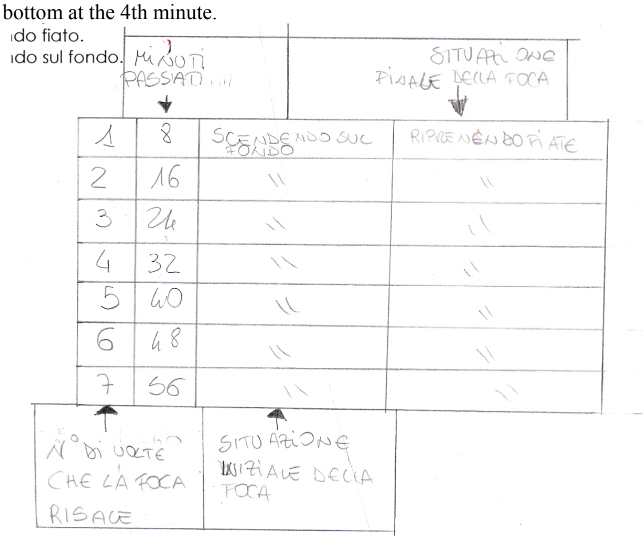
Some students solved, more or less positively, the problem using the 8-minute modeling.

Angelica: The seal stays at the bottom for 8 minutes, it then goes up, breathes. But if it takes 3 minutes to go up, it stays at the bottom only 8-3=5 minutes. After one hour it will be at the bottom.



Giulia: The seal spends 3 minutes going down, 2 minutes at the bottom and 3 going up. After 8 minutes the seal is at the surface but we don't know how much time it needs to breath.

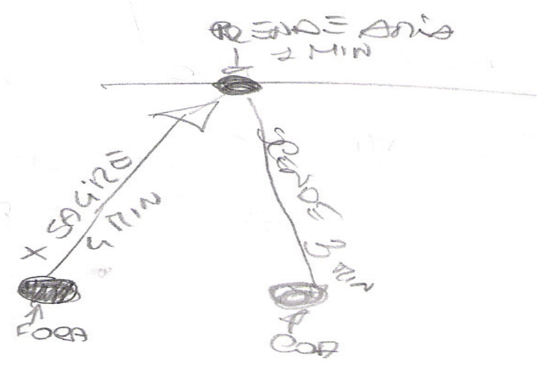
Beatrice: But there are still 4 minutes to the hour. It spends 3 minutes to go down. Thereby, it's at the bottom at the 4th minute.



Marco: 8=going up; 8+8=16 going down; 16+8=24 going up; 24+8=32 going down; 32+8=40 going up; 40+8=48 going down; 48+8=56 going up; 56+8=64 going down. 1 hour=60 minutes. 64-4=60. 1 action=4 minutes=breathing.

Simone: 60=1 hour. 8=minutes spent by the seal to perform its cycle. It is impossible that the seal perform its cycle in 8 minutes, but if one hour is 60 minutes, the seal is at the bottom around the 63rd minute.

Chiara: 4 minutes to go up, 1 minute to breathe and 3 minutes to go down. I couldn't do it very well.



Other modeling

Some students try to solve the problem using modeling different from the above ones.

Barbara: The seal sleeps for one minute, it goes up for 8 minutes and after 3 minutes is back to the bottom. It sleeps for one minute, it goes up for 8 minutes and after 3 minutes is back to the bottom. It sleeps for one minute, it goes up for 8 minutes and after 3 minutes is back to the bottom. It sleeps for one minute, it goes up for 8 minutes and after 3 minutes is back to the bottom. It sleeps for one minute, it goes up for 8 minutes and after 3 minutes is back to the bottom.

Antonietta: It goes up after 8 minutes and breaths, it goes down after 3 minutes. It keeps doing it. 8x7=56+3=1h. After one hour, I think it's going down, but actually I didn't understand much…

Davide: It dives and sleeps: 20 minutes. It slowly goes up and breaths: 8 minutes. It goes back to the bottom: 3 minutes.

Alessandra: I think the seal stays at the bottom 2 minutes because if it goes up every 8 minutes and it takes 3 minutes to go down, it would 3 minutes to go up again: 8-(3+3)=2 min.

Maria: During this hour the seal repeatedly goes up and down. During these 8 minutes the seal goes up and breaths and after 3 minutes it's at the bottom. Then, after one hour it's at the bottom.

Lorenzo: 8+3+8+3+8+3+8+3+8+3=60. The seal is breathing for sure, because last number is a 3 that matches with breathing. But I have some doubts about how to solve the problem.

Paolo: Martino watches the seal for an hour. The seal stays at the bottom for 19 minutes, it goes up in 8 minutes and after 3 minutes is under water again. Then, after one hour, it's going down. And then it repeats it regularly.

Emanuele: 8 and 3 are exactly right for an hour. Thus, it doesn't take a breath or stay at the bottom. It's then going up or down. But since the hour is passed with the 8 minutes, the seal is going up.

Answer with no modeling

Some students try to solve the problem with no modeling, but with other kinds of motivations.

Noemi: Martino could see what the seal was doing because it couldn't stay at the bottom without breathing. It was then taking a breath.

Valentina: I chose answer C "taking a breath" because, even if it breaths while it's sleeping, is under water and thus, after it slept, it has to breath and it has, I think, to go to the surface.

Clarissa: If breathing and taking a breath is the same thing, the correct answer is then "taking a breath" because the problem says that it breaths while sleeping.

Stefano: It's taking a breath because, while sleeping, it's a dead weight and it keeps afloat and takes a breath.

Zorana: The seal is taking a breath because after one hour it's not at the bottom: it was at the bottom after 3 minutes, not after one hour! And it's not going up because it goes up during the following 8 minutes. Moreover, a seal cannot sleep without taking a breath.

Nicolas: It's taking a breath because it goes to the surface to breathe every hour.
Samir: It's going up because after one hour it needs to breath and goes to the surface.
Marta: The seal is a mammal and it's obvious that after 8 minutes it goes up because it cannot breathe under water.
Marina: Solving the problem is not possible because the action is always performed in a regular way and the seal uses all the time to do as many actions as it can.

No answer

Some students face the difficulties in the text comprehension and they cannot find a solution.

Riccardo: 1h □ sleeps □ 8 m □ takes a breath □ 3 m □ goes down. I cannot do it because I cannot understand what happens after one hour. If there was only one hour, it would sleep. But since there are those 3 and 8 minutes, everything is turn up down, because it would be I hour and 11 minutes.
Alessandro: To 8 min – 3 min down = 5. We cannot find out if the seal goes up or down because we don't know how much time it needs to go up. I don't understand what those 8 minutes are.
Federica: If the seal takes 8 minutes to go up and breathe and 3 minutes to go down, it will need 8-3=5 minutes to breath and 3+3=6 minutes to go up and down. The problem doesn't say how long it slept. Without this information, we cannot find a solution.
Giuliano: The text doesn't clearly say what the seal does in those 8 minutes but it says the seal goes to the bottom. It gives importance to the action it's doing.

CONCLUSIONS

The difficult text analysis entwines with psychological aspects and with an inevitable personal resolution. Making the text more clearly readable, adding clue information and deleting misleading and pointless ones could help eliminate difficulties. The subject makes the problem suitable for all ages and grades. Its solution, though, is difficult for everybody. If all the useful and necessary data were given, it would be a suitable problem for the first grade of secondary school.

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THE USE OF OECD/PISA MODEL QUESTIONS IN DIDACTICS: AN INNOVATIVE CLASS EXPERIENCE

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ABSTRACT
Five questions addressing the area/perimeter confusion, similar to the PISA "Carpenter" problem, were posed to students aged 12-13, attending the second grade of lower secondary school. A metacognitive approach to the questions helped students reflect on the proposed tasks, favoring a critical and non-passive approach. In the discussion on students' solutions their dramatic difficulties were shown in reporting their own thinking processes and facing the metacognitive aspects of the questions, aspects they usually considered secondary in mathematics.

KEY WORDS:
Area, perimeter, metacognitive, language

INTRODUCTION

As indicated in many reports, Italian students have poor knowledge of mathematics and approach the subject with evident difficulties (Bolletta, 2005). Although everybody is equally able to *understand* the world through numbers (Lucangeli et al., 2007) some students feel absolutely unable to face mathematics tasks and they develop a negative feeling about this subject. Difficulties connected with mathematics learning are related to many factors and heterogeneous aspects such as basic abilities, practice of strategies, metacognitive abilities, beliefs, environment support, cultural conditioning and willingness to learn (Lucangeli et al., 2007).

According to PISA (Program for International Student Assessment) data, only 7% of Italian students reach the upper levels on the mathematical competency scale, against OECD (Organization for Economic Cooperation and Development) data of 16%, with excellence level attained by 20% in some countries. On average, the mathematics competency scores registered by students from the best countries (Hong Kong, Finland, Korea, Belgium and Netherlands) are 70 points higher than those scored by Italian students. Indeed, for many Italian students mathematics represents the most difficult subject due to the fact that it involves lots of social-cultural factors such as didactic, cognitive and metacognitive abilities, social behaviors, cultural approaches to success, and other. Moreover, while proceeding in this subject, the increase of its complexity associated with mistakes learned during the earlier scholastic years, and bad feeling toward the subject, make many students to definitely lose any interest in the subject. It has been proved that poor mathematics performance observed in adults could be related to the above mentioned causes as well. Furthermore, the higher the mathematics level becomes the less the topics seem related to reality. Because of this apparent lack of connection with the real life many people consider mathematics useless.

For these reasons I decided to "make mathematics a living thing" to Italian students through presenting it in a realistic and successful way, with the aim of having students experience the subject first-hand. Moreover, I aimed at developing in students a right and positive approach to study through transforming the learning experience into a positive one that gives them a measure and a value of their own knowledge.

ADDRESSES AND METHODOLOGY

Questions were proposed to students aged 12-13 who attended the second grade of lower secondary school. I did not know the group very well. It was composed of 28 students before this experience; hence I decided to test students approach to mathematics, in particular to problems, covered during the previous years. The questions proposed to the class group, based on PISA model involved knowledge normally learned at school but applied in different environments. With reference to the theoretical ideas of the PISA project, questions can be sorted into *reproduction* and *connections* clusters. Questions were formulated as open questions and students had to describe their own working out the results (see table below). In this way it was possible to evaluate students' thinking processes, aspect that is lost when the answer has to be chosen on a multiple choice test. No time restrictions were given to avoid excuses related to pressure and anxiety. In these friendly conditions students were mentally relaxed and reflective.

After this first phase of individual work answers were discussed in the classroom and, through the comparison of different strategies used by the students, a constructive and meaningful cognitive conflict spontaneously arose among them. This metacognitive approach to the questions helped students reflect on the proposed task, favoring a critical and non-passive approach.

The teacher's role was to orchestrate students' interventions and to mediate the discussion, without playing down utterances but, on the contrary, with throwing back to the student questions raised by others. The teacher's competence involved not only the specific subject knowledge but also the ability to listen to students and guide their cognitive processes.

Class discussions were recorded in order to enable each student to make a personal evaluation based on individual work and level of participation. Recordings were also used to evaluate the quality of the teacher's work.

THE DIDACTIC PATH

The Italian teaching approach to geometry is based on conceptual aspects, which are built on an abstract level, while in England and in other countries geometry is more based on figural aspects that concern images as sensor representations of objects. Since with the Italian approach many students feel geometry as a subject very distant from reality, through this didactic path I tried to expose students to a more concrete geometry, based on operative problem solving that would integrate figural and conceptual aspects of geometry (Fischbein, 1992). Questions were not posed in a casual way but followed a pre-conceived path and incorporated in the curriculum. In particular, the didactic path on geometry intended to give space to the empiric-inductive study of space and objects, rather than focus on axiomatic and deductive aspects. Specifically, the didactic path used in this study involved the concepts of "isoperimetrical figures"[1] and

[1] Relationship among the polygons of a plane which have the same perimeter

"equidistant figures,"[2] and focused on the conflict between perceptible and cognitive aspects.

Questions were adapted from PISA tests, from tests by the national mathematics Kangaroo competition for non-university students (2007 edition), from NMP English project (Harper ed., 1988) or from the text books. The first question (see table below) evaluated students' thinking processes, events that can not be evaluated with multiple choice test. Using this approach I had the opportunity to discuss with them the importance of the procedure beyond the result in itself. The second question (see table below) focuses on the conflict between perceptible and cognitive aspects of reality. Some of the aspects of this conflict were deeply analyzed with the third question (see table below), which had a strengthening function on the topic. With the fourth question (see table below) we came back to numeric data and to a simple mathematical operation, which, however, was derived from the integration between conceptual and figural aspects of the geometry. On the other hand, the fifth question (see table below) concerned the abilities of exploring possible different solutions of the proposed question. Again the geometric problem involved areas and perimeters; students were asked to discuss their own reasoning. Students' habit of generalizing evidences and thinking approaches was noticed.

Table:

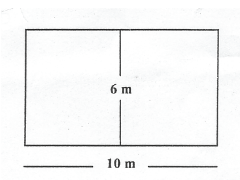
Question 1:

- First version: "a square tablecloth has got an area of 4 m^2; can it cover a rectangular square measuring 180 cm by 150 cm? Explain your answer."
- Second version: "a square tablecloth has got an area of 4 m^2; can it cover a rectangular square measuring 210 cm by 180 cm? Explain your answer."

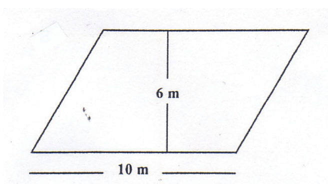
Question 2: "A carpenter has got 32 meters of wooden planks and wants to build a fence. He considers different projects but doesn't know if they are realizable. Help him, showing for each project if it is realizable with the 32 meters of wooden planks available and explain your reasons."

Projects:

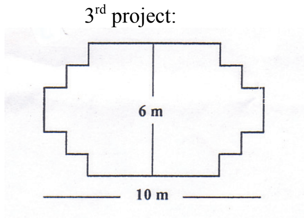
1st project:



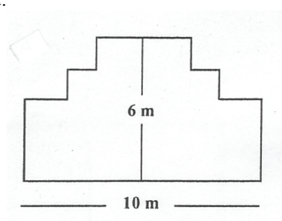
2nd project:



3rd project:



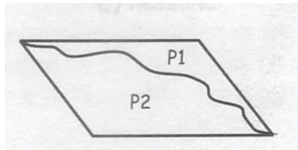
4th project:



[2] Relationship among the polygons of a plane which have the same extent

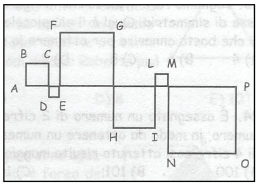
Question 3: “A parallelogram is divided into two parts P1 and P2 as shown in the figure. Which is the right assertion?”

- P1 and P2 have the same area
- P1 and P2 have the same perimeter
- P2 has a smaller area than P1
- P2 has a larger perimeter than P1
- None of the above

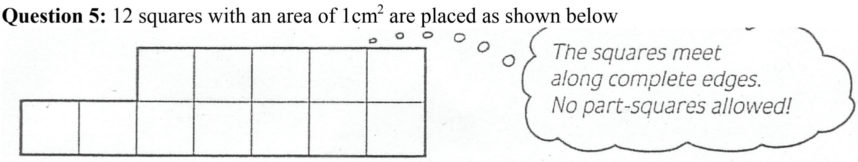


Question 4: “Squares in the figure result of, first, drawing the segment AP, 24 cm long, then a broken line ABC.....OP, which crosses the segment. How long is the broken line ABC.....OP?
Data are not sufficient to answer

- 72 cm
- 96 cm
- 56 cm
- 106 cm



Question 5: 12 squares with an area of 1cm^2 are placed as shown below



- How could you place the 12 squares to obtain a figure with a smaller perimeter?
- How could you place the 12 squares to obtain a figure with a bigger perimeter?
- Try with a different number of squares. What did you observe?
- One of your best friends wants to replace his carpet and cover the floor with square tiles. Assuming that no tile will be broken during the transport, how could he arrange them if he wants to obtain a bigger perimeter? and smaller?

ANALYSIS OF STUDENTS’ PROTOCOLS AND OF EXCERPTS OF CLASSROOM DISCUSSION

Question 1

The problem concerned the comparison between the side of a square tablecloth (2 meters) and the longer side of a table (1.8 meters by 1.5 meters). Almost all students did not compare these lengths but solved the question finding the area of the rectangle (1.8 meters x 1.5 meters = 2.7 m^2) and comparing it with the area of the square (4 m^2). Two thinking approaches are possible: comparing tablecloth with table side length and comparing tablecloth and table areas. Even though both approaches give the same solution, the second procedure is conceptually wrong.

From the analysis of the results I realized that even those who solved the question comparing the table and tablecloth side length thought that it was necessary to compare also the areas. Nobody solved the question comparing only the lengths of tablecloth and table size. These observations indicate the importance of assessing students’ thinking processes through open questions. To better analyze the process beyond the final result it was useful to reformulate the question with different data. In the second version side and area's comparison drives the students to different results and to solve correctly the problem the student has to compare table and tablecloth’s sides only.

Question 2 (Carpenter PISA question)

For many students two polygons "equally divided"[3] have the same area and the same perimeters and, on the contrary, two polygons not "equally divided" do not. Since the question asks to motivate their own conclusion, students had to invent a way to explain their assertions and, doing that, they demonstrated great creativity. Some of them attempted to measure each segment of the polygon, ending in being imprecise; some invented data and tried to work applying useless mathematics formulas; some attempted obstinately to divide not equivalent figures into parts of the same area; some others established relationship in the projects by using letters, numbers or symbols. I will discuss the Carpenter PISA question's protocols in the next paragraph.

Question 3 (Kangaroo question)

Even students on a middle-upper grade level did not give the correct answer. Since they could not use mathematical formulas to calculate the two perimeters including a curved line, they based their answer on their own perceptions, concluding that bigger area always corresponds with bigger perimeter.

Question 4 (Kangaroo question)

Every student found this question exceptionally difficult and they thought they had not enough data to solve the problem. Facing numbers lead them to think: "which is the formula I should apply?", "If I can't find the formula I won't be able to solve the question!" These thinking approaches blocked the students and they were not able to solve the question. Even though previously we had worked on figural aspects of geometry, this approach was completely forgotten when students faced a numbers related problem. According to Mariotti (1992), the harmonization between conceptual and figural aspects of geometry should not to be taken for granted.

Question 5 (NMP question)

Students had many problems in finding different figures with the same perimeters built of a fixed numbers of squares. Generally, students found just one arrangement of squares that satisfied the first and the second part of the question. Few students replayed the third point, and, even understanding that this point asked something different from the first two was a challenge for many of them. Among the students who tried to answer this third point nobody found any regularity and their conclusions were wrong. Similarly, nobody was able to infer the generalization requested in the fourth point.

DISCUSSION

In the discussion on students' solutions I pointed out the dramatic difficulties that student had in reporting their own thinking processes and facing the metacognitive aspects of the questions, aspects they usually consider secondary in mathematics. As far as the Carpenter PISA question was concerned, at first the students commented that the question was easy, probably because it did not include numeric data, while later they saw some difficulties. Everybody tried to describe and defend their own process and conclusion, even thought poor and inexact language was used. Often the arguments were directed just to demonstrate what they intuitively asserted, but most conclusions were

[3] That is 'split' polygons where we can put a 1-1 correspondence between the parts of one polygon and the parts of the other polygon and each pair of parts in correspondence is constituted by congruent polygons.

wrong. This shows that at this age intuitive thinking is significantly conditioned by perceptive aspect of reality.

The language used by students showed a great confusion among different registers of representation (verbal, symbolic and graphic), and they were not able to clearly express their thoughts in any of them. According to Malara (1996), when teaching geometry it is necessary to promote in the students two skills: using different systems of representation and coordinating the tree different registers of representation.

During group discussion that followed the individual thinking phase I was surprised by the obstinacy that some students showed on defending their results. In fact, even though they could be correct from the multiple choice point of view (yes/no), they were not able to justify their conclusions on an effective manner. Clearly, mathematics should not to be seen and experienced as a quiz with final awards and students should get used to give importance to the whole thinking process and not only to the final result.

Concluding, I would like to emphasize how many study-oriented students obtained poor results because of their dependence to the mechanical use of mathematical formulas.

SOME GENERAL CONSIDERATIONS

This didactic path showed students a new approach to mathematics, which at first surprised then fascinated them. Students aged 12-13 need to test and to start knowing themselves, and certainly this kind of approach to mathematics can lead them to discover and love the constructive side of this subject. Some students defined the given questions as mathematical games and they were pleasantly involved in them, in spite of the poor results obtained. This playful experience clearly clashed with their idea of mathematics as a boring and difficult subject.

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PART 3

CONTEXTS, MODELING, DEMODELING

ALWAYS LEARNING...

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ABSTRACT:
This article presents three moments in which students and teacher learnt. These moments took place with eighth-grade students during the teaching of geometry topics with dynamic geometry software. While students built figures and solved investigation activities, the teacher reflected on students' learning and the way he was being influenced.

KEY WORDS:
Exploration, investigation, triangle, tesselation

Since the day I decided to become a teacher, I committed myself to study to be the best teacher I can. It did not occur to me that I would learn so much with my students. This has been happening during my ten years of teaching and, in particular, occurred with a set of students whom I will never forget.

The next three episodes occurred during an educational investigation that I carried out with a group of eighth-grade students, when I taught them an extended geometry unit. For four months, the students solved 26 tasks using dynamic geometry software in a classroom equipped with 14 computers and a multimedia projector. There were essentially three kinds of tasks: explorations, investigations (open problems) and problems (Ponte, 2003). The students worked in pairs. I learnt from all of them, but here I just mention a pair of students, André and José, who had a notable performance in this teaching unit.

André was 13 years old and was very reserved. He was born in Guinea, an African country, and came to Portugal when he was 8 years old. He lived with relatives, since his parents stayed in his homeland. His expressive look, last generation mobile phone and cap were his trade image. Inside the classroom, he behaved well but had some learning difficulties in mathematics. He was a quiet boy who did not intervene much in class. He was slow in doing exercises and just copied what was on the blackboard or in his classmate's notebook. During group work, he hid himself behind the work of his colleagues, because he had some difficulties in speaking Portuguese. In spite of everything, he was successful in school.

André had a great friendship and respect for José. They started working together in grade 7, when André had failing grades in eight subjects that he needed to improve. José started helping him in "supported study"[1] and continued doing so in several other subjects. This partnership allowed André to improve considerably his school results and consequently to pass to grade 8 with no failing grades.

[1] *Estudo Acompanhado*, component of the students' curriculum devoted to study subjects in which they have more difficulties, such as Portuguese, English, Mathematics, etc.

José was a brilliant student in all subjects, except physical education, in spite of practicing several sports. In the remaining subjects usually he had all the answers correct and was very concerned when that did not happen. In his interventions in class, always at a high level, he used a brilliant reasoning and quite an advanced vocabulary for his age. He did not turn down a challenge. José had great expectations about the kind of work I asked them to do: take part in a study, in which students would use software to learn geometry for a considerable period of time.

CONSTRUCTING TRIANGLES

In this exploration task, I intended students to learn how to build isosceles and equilateral triangles. Afterwards, students had to relate to two kinds of triangles' classification, regarding sides and angles. The right triangle construction was also important, because it would be studied in later tasks. The following question led students to relate to all triangles and made all groups experience great difficulties.

> Investigate the relations that exist among triangles: acute, right, obtuse, equilateral, isosceles and scalene. Write down the relations that you encounter.

André and José had no problems with this question. They were the only students who completed totally the investigation, related to the classification of triangles regarding sides and regarding angles (Figure 1).

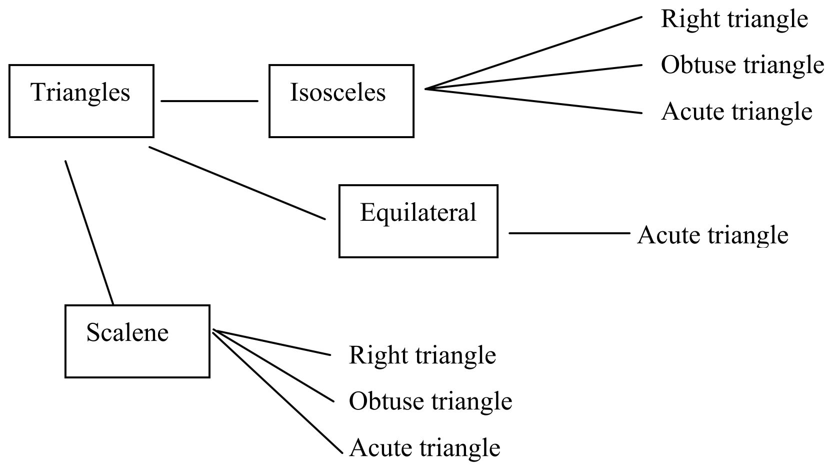

Figure 1. Students' scheme presented to answer the question "Constructing triangles"

This scheme was the result of several conclusions made by André and José summarizing their answers to all the questions of the task. The most debated one was the question that asked students if an equilateral triangle could be a right triangle.

LOCUS PROBLEMS

This task consisted in solving nine geometrical problems involving locus. Students had to build and to relate circles, perpendicular bisectors, triangles, and rectangles. Problem 3 was the following:

In a basketball game Manuel is 4 meters from the ball and Sara is 5 meters from the same ball. Where is the ball?

André and José presented a solution (Figure 2) that started from the position of the ball, and then indicated Manuel and Sara's possible positions; in other words, they inverted the problem simplifying it. If they had marked first the position of Manuel and Sara, the answer to the problem (position of the ball) would depend on the distance the two friends were from each other. In my comment about their work, I proposed to them to try out to solve the problem again, but starting by marking the friends' positions, drawing the respective circles and, afterwards, investigating the several possible answers that could exist. The students accepted the challenge and tried to find all the possible solutions for the problem. They built two circles: one centered in Manuel, with radius 5 cm, and another centered in Sara with radius 4 cm. After, they dragged one of those two circumferences to verify if there was a solution. They synthesized the several possible answers as follows:

(i) If they are more than 9 m away, there is no solution for the problem (the circumferences do not intersect); (ii) if they are in a distance between 1 and 9 meters away, the ball can be in two different locations (the intersection points of the two circumferences); (iii) if they are exactly 1 m away, there is only possible place for the ball (the circumferences are tangents); and (iv) if they are less than 1 m away, there is no solution again (the circles do not intersect).

Student's answer:
3. Manuel can be in a circle with 4 cm radius, while Sara can be in a circle with 5 cm radius.

Solution process:
We marked a point – Ball – then with Transform menu we bid 2 points, on with 4 cm form point "Ball" and another with 5 cm from the same point.
Finally, we constructed two circles with radius 4 and 5 cm, each one of the built points.

Figure 2. Students' initial solution.

Translation tesselation

Before working on this investigation task, the students already solved two tasks related to translations and vectors. In this task I wanted them to study the tesselation using only this geometric transformation. The last question of this task generated a lot of interest, because it led students to a small investigation on polygons that allowed them to make a translation tesselation:

…And if we started initially with other quadrilaterals [the rectangle had been studied in previous questions], could we also make a translation tesselation? And with triangles? Write down your discoveries.

André and José elaborated interesting conjectures, extending their investigation to other polygons such as pentagons and hexagons. After the experiences and discoveries, they tried to find a relation between the number of symmetry axes of a polygon and the possibility or not to make a translation tesselation. At this time the following dialog took place:

Teacher: So, have you answered the last question?
José: Yes! We realized that we can do translation tesselations using only squares or using only rectangles.
Teacher: Why?
José: We manage to do translations and cover the plane.
Teacher: And have you already tried with triangles?
José: We realized that we could cover the all plane with equilateral triangles, but we had to rotate some of them, so it wasn't a translation tesselation.
Teacher: OK! And with rhombus or kites?
José: We could with rhombus, but with kites we could not. I think the reason is somehow related with the number of symmetry axes.
Teacher: Why?
José: The square has four, the rectangle has two, the rhombus has also two and the kite has only one.
Teacher: And can you do a translation tesselation with parallelograms?
José: Yes, because they have no symmetry axes.
Teacher: Then that was the conjecture that you could write?
José: I think that it is possible to make a translation tesselation with any polygon that has an even number of symmetry axes.
Teacher: This conjecture is interesting. We would have to find a proof to see if your conjecture is true, or a counter-example to show that it is false.

Some minutes later José called me to tell that a regular hexagon had six symmetry axes and it was possible to do a translation tesselation with it. Although he did not present a proof, he was convicted that his conjecture was true, since he did not find any counter-example. Their answer was:

> It's impossible to cover the plane with triangles only using Translate [a Geometer's Sketchpad menu]. But with quadrilaterals, the rectangle, the square and the rhombus allow us to do translation tesselation, because they have an even number of symmetry axes. Further more, we can cover the entire sketch with all polygons that have an even number of symmetry axes to cover the plane.

At night, in front of my computer and with the help of Sketchpad, I draw a regular octagon and realized immediately that it was impossible to make a translation tesselation with this polygon. At the beginning of the next class I sat down with the students at a computer and we reviewed the sketch made in the previous class. Then, we built a regular octagon and the students tried to make a translation tesselation with this polygon. They were surprised to verify that it was not possible, although the octagon had an even number of symmetry axes. We talked about the measure of the octagon's internal angle, 135º, not being a divisor of 360º (circle angle), therefore not allowing this figure to tessellate. Students went on to solve other tasks, but it was evident for me the connections that this subject has with the geometry topics taught at grade 9: Rotations and measures of polygons' internal angles.

CONCLUDING REFLECTIONS

These three episodes concern André and José's learning process and my role as the teacher influencing their learning. But also they present situations in which the students influenced my own learning. In the first task reported, "Constructing triangles," the initiative of the students in systematizing their ideas carried them to build an outline that explains and organizes locally various kinds of triangles and the relations among them. This organization allowed them to realize the relations that exist between the sides and angles of triangles and to understand that only one of them, the equilateral, can be acute, and, therefore it has unique characteristics regarding the others. It is a very special triangle and has a relevant role in the study of plane geometry. I learnt that a good question can lead students to develop this systematization ability, so neglected in our teaching. Instead of presenting a theory already systematized of facts and properties, teachers can provide a set of suggestions and explorations to help students learn it in a more consistent way. The type of questioning, written or oral, is decisive to improve students' comprehension (Long, 1992; Menezes, 1999).

The second learning episode presented in this article shows the importance of checking a problem solution that we believe is correct, questioning if it answers the problem completely. But for that it is necessary to give students some time to attempt to find the solution. That leads to the beginning of their comprehension. André and José had more difficulties in solving geometric problems than working in open problems, but I have to point out that the students tried to solve the task again, although it was already graded by me. I learnt with them trying and trying again to solve a problem. They were persistent, like a teacher also has to be. They liked a good challenge!

The last learning episode shows a situation in which the teacher did not manage to explain or to justify any affirmation at the very moment. After a deeper reflection he created insight and found not only a counter-example, but also a possible strategy to initiate the study of a new theme. In open or problematic situations it is possible that students present conjectures whose acceptance or denial is not possible right away. But that also enables learning by teachers, even in themes that they know. Never letting students with wrong mathematical ideas is always one of my main concerns. Sometimes the difficulty is in trying to find the best way to quickly explain, justify or deny what a student in the classroom tells. However, I think that is advantageous not leaving students with the wrong idea, even when the explanation comes in the next day. In this case, the discussion just involved the teacher and a pair of students. However, it could have occurred with an affirmation done by students before all the classmates. In this case it becomes necessary to discuss and, sometimes, discuss again later, with all, the veracity or falsity of the conjectures. The discussions with all the students performed a fundamental role in this learning process, in particular regarding the geometric concepts (Gardiner & Hudson, 1998).

Finally, I would like to state the role that the dynamic geometry software had in the students' learning, as in my own learning. I do not concur with the skepticism showed by a teacher that took part in the study by Hannafin, Burruss and Little (2001), who thought that this kind of software was not significant in students' geometric learning and considered that his students did not learnt in a deep way the topics studied. My students did! They managed to do geometric constructions, investigations and solve problems with this powerful tool. I wonder what the ancient Greek mathematicians would have done if they could use a tool like this.

I believe that the presence of a dynamic geometry environment in the classroom only makes sense if students use it by themselves, preferably since the early years. Therefore, the students need time to use this software, but it is gratifying to see their learning. Teachers have to learn to use this tool and also to learn with their students. To learn as they learn, all seeing geometry in a different way. They were learning, and so was I...

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CASE STUDIES ON DEMODELING ACTIVITIES

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ABSTRACT

In this paper we want to describe some classroom modeling and demodeling activities, starting from the interpretation of a graph of motion in a Cartesian plane. According to a model of cognitive dynamics based on resonance processes (Iannece & Tortora, 2007 and 2008; Iannece et al., 2007; Guidoni et al., 2005), the Cartesian plane is used as a powerful tool of "communication," while the choice of the topic motion depends on the fact that it immediately goes in resonance with an enormous variety of daily life experience of every human being. Finally, we show how our choices favor the development of symbolic mathematical treatment.

INTRODUCTION

The PISA assessment results about mathematics literacy of Italian 15-year-old students gave rise to a wide debate about our students' difficulties to reach the mean level of literacy with respect to the competencies fixed by OECD. Our participation in the 3-year European project PDTR (Professional Development of Teacher-Researchers, www.pdtr.eu) was the opportunity to deeply reflect and wonder about our own professionalism as school teachers of mathematics and physics, and about the possibility to improve our didactic actions: indeed, we often wonder why a lot of our students are not able to tackle and solve a problematic situation, even if they studied the mathematical instruments fit to solve the problem itself. This is a very common error we make as teachers: we expect our students' good performances concerning the relational understanding of a mathematical concept, only on the base of a long-lasting work concerning the attainment of instrumental abilities (according to the definition of *relational* and *instrumental understanding* given by Skemp, 1971). By now, we have learned that a professional change, which aims at enhancing our role of teachers as learning mediators, needs a deep reflection upon our own action. As suggested by Mason (1998), the improvement of students' performances should be pursued through teacher's own professional development.

In this paper we want to describe our efforts to change our teaching practice, in order to improve students' results with respect to PISA items concerning graphs, and their interpretation. We are going to describe some classroom activities concerning modeling and demodeling[1] processes, according to the definition given by Guidoni et al. (2005). In the following section we will report the activity proposed to our students, with a brief account of the theoretical reference frame, which inspired the activity itself, while in the last section we will present some classrooms excerpts, and our reflections on them.

OUR DIDACTIC CHOICES AND METHODOLOGY

A great number of items in PISA (2003) start with a graph, describing various situations, in order to assess the ability to use graphical representations as one of the mathematical instruments to communicate in mathematics, to display relationships, to describe phenomena; shortly, to use and/or to recognize a mathematical model. Here, we refer to a model in the sense given to this word by Iannece & Tortora (2008): "a very complex process, neither deterministic nor one-way, where the formal structures are seen as one of the different correlated ways into which the cognitive reconstruction of external world structures takes form." But what is fundamental, as emphasized by Guidoni et al. (2005), is the ability to merge all the possible representations of a concept (actions, words, graphs, laws, and so on) in a "continuous (even subliminal, in expert situations) shifting from one cognitive dimension to another in a mutual progressive enhancement," or, again in the words of the authors, in alternative actions of modeling and demodeling. It is worthwhile to notice that this is one of the eight competencies indicated by Niss (2003) to define the mathematical literacy for OECD.

The teaching activities we are going to describe are inserted in a long lasting path, as the fruit of a deep study made with our mentors in the PDTR team in Naples, in order to introduce in the mathematical curriculum several modeling and demodeling activities, to improve our students' competencies in this direction. Here, we will describe a demodeling activity, which follows several modeling ones concerning linear functions and direct proportionality (see, e.g., Grasso & Mellone, 2008).

We proposed the following problem:

Observe the Cartesian graph in Figure 1: the two half-lines give information about the movements of two slugs on a wall.

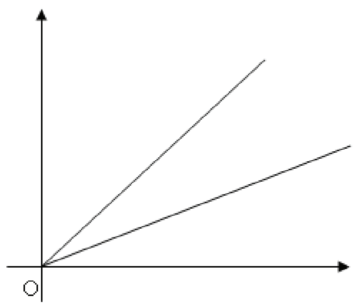

Figure 1.

Try to imagine a story that would describe the movements of the two slugs.
Try to describe their movements in the way you prefer (you can use natural language, symbolic language, numerical tables, etc.)

[1] Interpreting model elements in terms of the "reality modeled" (Niss, 2003)

There are several reasons why we have chosen this particular problem: (1) intentionally, the graph does not contain any information about the “name” and the “surname” of the axes, i.e. the coordinate variables, just to underline the fundamental role of the choice of the variables in interpreting a graph, and to develop in our students the ability to select the most important features of a problematic situation, which can be translated into mathematical variables and/or parameters. For the same reason there is no even a fixed unity of measure, so also this choice has to be made in a suitable way. (2) The concept of *motion* immediately goes in resonance with an enormous variety of daily life experience of every human being, and this fact favors its symbolic mathematical treatment, according to a model of cognitive dynamics based on resonance processes (see Iannece & Tortora, 2007 and 2008; Iannece et al., 2007; Guidoni et al., 2005). (3) As we already said, the problem was proposed after several modeling activities on linear functions and direct proportionality, presented as a natural approach to describe phenomena and not only as a typical mathematical content given by the teacher (see Guidoni, 1985; Guidoni et al., 2003). (5) This problem gave us the possibility to treat mathematics and physics without distinction to a great surprise of students (the first question in both classrooms was: “Teacher, is this math or physics?”).

Of course, all these cognitive aspects are strictly related one to another, and it is not easy to correctly manage the activity in order to catch all the possible didactic opportunities, also because time, at school, rolls by fast! In our classes, this has been partially possible thanks to the choice of a constructive didactic methodology: we greatly encourage discussions in the classroom about the possible solution of a problem, so that all students, alone or as members of small groups, have to formulate a solution hypothesis, and in the meantime have to reflect on the exactness of other students’ proposals. In this way, the role itself of teachers considerably changes: they have to mediate among different positions and, above all, they have to choose the direction the discussion has to take, on the basis of students’ answers. So, we usually ask our students to record the lessons, to write down transcripts, to reflect on the discussions, and to talk about their homework. In this way we have the possibility to evoke answers previously neglected, or to postpone the analysis of didactic aspects not immediately related to the current activity. In the next section we will give punctual account of the strictly disciplinary goals related to the chosen problematic situation.

EXCERPTS FROM THE CLASSROOM ACTIVITIES

As said above, the problem concerning the two slugs was proposed in different classes at different levels. Nevertheless, we observed in students’ answers the same cognitive dynamics with respect to the conceptual knots, that is why we will report classroom’s excerpts without distinguishing among the various groups.

The stories

The students are mainly concerned with the first question, so they try to imagine a story of the two slugs.

Rosario: The slugs had a quarrel, and now each of them is going her own way, so… the Cartesian plane represents the wall and the two half-lines represent the slime of each slug.

Gennaro: It represents a motorcycle race starting from the same point but along two different paths. The arrival is in two different places, since the two half-lines do not intersect.

Mario: Let’s talk about slugs! It is a race between two slugs: they walk along different paths but they arrive at the same point, in fact at the arrival there are different values of x but the same value of y. It is possible to convince oneself looking at the following graph [pointing at a drawing, here Figure 2]:

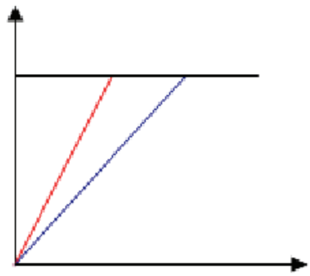

Figure 2

Martina: It seems to me that the journey of the slug moving along the half-line below (let's call it the first slug) goes on for a double time with respect to the journey of the second slug, because of the different inclinations of the two half-lines. I mean: the second slug spends less time than the first to cover the same distance.

Antonella: So, if the graph represents a race between the two slugs, the second slug is the winner.

We can recognize in students' words the classic conceptual knot concerning the study of a motion: it is clear that Gennaro and Rosario are talking about trajectories, Mario is thinking of a particular *x-y* graph, while Martina interprets the graph as a time-space law. So the teacher asks them which variables they choose for the two axes.

Gennaro: The plane represents the wall, so I think *x* and *y*.

Martina: The two half-lines represent two rectilinear uniform motions, i.e. motions where equal distances are covered in equal times, therefore *t* (time) and *s* (space) are the variables on the axes.

T.: So, who is right?

Alessandro: It depends on the story you decide to tell: both of them are right!

Alessandro's answer is just what the teacher was hoping for. Indeed, it is clear that Alessandro becomes aware that each graph is susceptible of several different interpretations, on the base of the choice of the variables. This gives the possibility to the teacher to underline how important is the "semantic" of a Cartesian graph.

T.: Well, can you tell me which is the faster slug?

Gennaro: Perhaps the first, but I am very confused, on an *x-y* graph I only see the direction!

Martina: In fact, the concept of velocity is strictly related to intervals of time, so I think you can give an answer only if it is a *t-s* graph.

We believe that this is one of the crucial points of the activity. Indeed, Gennaro, in spite of the perceptive-type information constituted by the two differently inclined lines, does not forget that he is reading the graph as a trajectory, while Martina is choosing the name and the surname of the two axes in a pragmatic way. We can say that they are getting used to formulating hypotheses and to using them.

Valeria: According to Martina, each point represents a couple time-space… then each ratio space/time represents a velocity!

Alessandro: But the ratio s/t is also the angular coefficient of the half-line…

Martina: Then, if the two slugs had the same speed, the two half-lines would be parallel… as two parallel lines have the same angular coefficient.

Lucia: So, the more the half-line is inclined, the faster the slug goes!

In this dialogue all students become aware that the same object, a graph in a Cartesian plane, is susceptible of both physical and mathematical meaning, and they are trying to connect the two aspects in order to properly interpret the situation. On the other hand, the conclusion of Lucia, who is looking at the graph from a physical point of view, is greatly influenced by the inclinations of the half-lines, the same perceptive-type information previously invoked by Gennaro.

At this point the teacher asks the students to re-interpret the graph as a *t-v* graph, in order to enrich their possible points of view.

Martina: This time, the ratio between velocity and time is constant, as it was the ratio between space and time in the previous case.
Alessandro: In any case, it represents a direct proportionality.
T.: Yes, but now the constant ratio has a different meaning.
Martina: Yes, in this case the inclination represents acceleration!
Valeria: This time it is a uniformly accelerated rectilinear motion.
T.: Are you sure that this graph represents a rectilinear motion?
Gennaro: No, it is certainly rectilinear only in an *x-y* graph, and in this case you cannot say that it is uniform or uniformly accelerated!
Valeria: Sorry, you are right, a straight line, in a *t-s* graph, only means that you take equal number of steps in equal intervals of time.
Martina: It means that the motion is uniform, not necessarily rectilinear!

Once again the students have to reflect on the possible meaning of the graph. This time, they reflect on linear functions as mathematical objects and recall that they always represent a direct proportionality. Their knowledge and competency about this tool has been induced by the teachers by means of previous activities on modeling, as we said before. However, Valeria's words show that a line, or a half-line, is a *strong* perceptive data, and it can generate mistakes. But, in spite of this, we recognize in the subsequent intervention of Valeria, as well as in the words of Gennaro and Martina, the great efforts made by the teacher and the students in order to control the semantic of a graph in a Cartesian plane. Of course, the ability to interpret graphs cannot be reached with only one activity, or only with demodeling activities. As we already said, a continuous shift between modeling and demodeling activities is necessary. So, at the end of the lesson, the teacher decided to give the following task:

One of the two slugs, whose movement was represented by the previous graph, suddenly stops because of a small obstacle. For 10 seconds she looks all around to decide which direction to take, then she turns right. Let's represent on a Cartesian plan a graph fit to describe this story, distinguishing three cases: the slug decides to continue walking a) with the same velocity, b) slower, c) faster.

Figure 3 below shows one of the graphs proposed by students, which represents a correct answer to the question. The correct answers of most students are a proof that they become able to switch between modeling and demodeling activities. It is an important goal for the teacher, as it constitutes a sort of proximal development area for future skills inserted in the yearly curriculum.

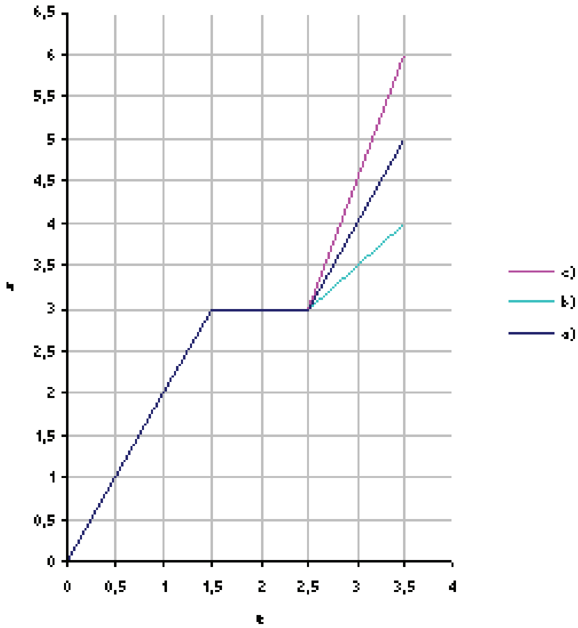

Figure 3.

CONCLUSIONS

The classroom excerpts presented in this paper refer to an activity chosen by the authors to render students competent in mathematical modeling, which consists, in the words of Niss (2003), not only in "performing active modeling in a given context," but also in "decoding existing models, i.e. translating and interpreting model elements in terms of the 'reality modeled'" (what we call demodeling). On the other hand, Guidoni et al. (2003) emphasize the necessity of a continuous switching from modeling to demodeling in order to achieve an effective competency by our students. Therefore, it is clear that an improvement of our students' performances in PISA-like tests requires a deep reflection by teachers themselves on the needed didactic choices when implementing the traditional curriculum, in order to achieve the mathematical competencies required by OCSE. In this paper we have tried to show that in this direction it is possible to re-read usual skills and tools on the base of a renewed awareness of teachers of their potentialities.

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MAKE YOUR OWN PROBLEMS

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ABSTRACT
Starting to design problems based upon a more or less realistic situation will create "problems" in the different meaning of the word. What does it mean to make a "good" context problem? Are all teachers good test designers or learning materials developers, because of their professional training? These are two important questions to be addressed in this contribution to the PISA handbook. Characteristics of context problems that are used by professional test designers, either implicitly or explicitly will be identified. For teachers, the purpose of the test or problem and how this makes a difference in the problem design, is important. For a quiz, teachers may focus on reviewing and using algorithms, whereas in another test, showing mathematical insight is stressed. Finally, we found that almost absent in most text books is how to translate from a problem situation to a mathematical model of the situation, solve the problem within the mathematical model and translate again to decide whether or not the solution found fits the situation. We will pass on some guidelines for teachers when they try to help their students in this respect.

KEY WORDS:
Problem design, context problems, Assessment Pyramid, PISA

INTRODUCTION

The term *mathematical literacy* in the sense as it is used in the OECD PISA[1] study emphasizes the functional use of mathematical knowledge in a variety of different situations. As the PISA 2003 Assessment Framework states

> Mathematical Literacy [is] an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of the individual's life as a constructive, concerned and reflective citizen. (15)

"Concerned and reflective citizen[s]" live in an ever changing society and thus need to adapt to these changes in a creative and flexible way. Their mathematics education needs to focus on these adaptations and therefore it is not enough for students to master definitions, basic skills and the performance of standard procedures, that is, having all kinds of mathematical tools available. Students must also learn how to use those tools in unfamiliar situations as encountered in private life, in further studies as well as in occupational life.

Based upon the work of de Lange (1994, 1999), used to design a National Option for the Third (now Trends) in International Mathematics and Science Study (TIMSS) (Kuiper et al., 2000) the OECD has chosen to describe the cognitive demands for mathematics in three competency clusters, the *reproduction* cluster, the *connections*

[1] PISA Program for International Student Assessment

cluster, and the *reflection* cluster. For use with professional development in assessment design, instead of the description provided by PISA, we offer a pyramid model for these competency clusters that, in the words of a participating teacher, proved to be *sweet, short, and simple* (Figure 1)

Level 1: Reproduction, procedures, concepts, and definitions
Level 2: Connections and integration for problem solving
Level 3: Mathematization, mathematical thinking and reasoning, generalization and insight.

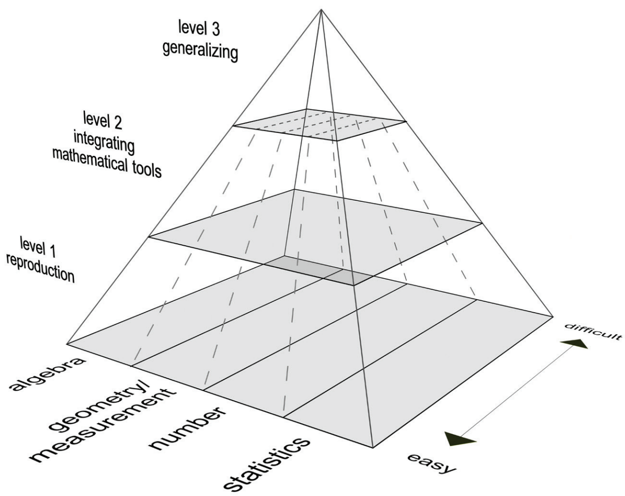


Figure 1. The Assessment Pyramid model

The three parts of the pyramid model indicate the number of problems at the three levels, there is no hierarchy in the competencies. All three of them are important, but since most questions at level 1 are often closed questions whereas questions at level 2 and especially at level 3 are open questions that need more time, in a balanced test as a general rule the ratio below is used: level 1 : level 2 : level 3 = 3 : 2 : 1.

The third dimension in the pyramid, from easy to difficult, shows that a more difficult problem is not necessarily at a higher competency level, it is just more complicated to solve and it may take more steps, albeit at the same level, to do so.

An example of a PISA problem from the reproduction cluster, level 1 (meant for students aged 15) is

Science test. In Mei Lin's school, her science teacher gives tests that are marked out of 100. Mei Lin has an average of 60 marks on her first four science tests. On the fifth test, she got 80 marks.
What is the average of Mei Lin's marks in science after all five tests?
(Answer: 64 marks)

An example from the connections cluster, level 2.

Pizza. A pizzeria serves two round pizzas of the same thickness in different sizes. The smaller one has a diameter of 30 cm and costs 30 zeds. The larger one has a diameter of 40 cm and costs 40 zeds. Which pizza is better value for money? Show your reasoning.
(Answer: The larger pizza, because the surface area increases more rapidly than the prize of the pizza or other correct reasoning.)

And finally an example from the reflection cluster, level 3

Litter. For a homework assignment on the environment, students collected information on the decomposition time of several types of litter that people throw away:

Type of litter	**Decomposition time**
Banana peel	1-3 years
Orange peel	1-3 years
Cardboard boxes	0.5 year
Chewing gum	20-25 years
Newspapers	A few days
Polystyrene cups	Over 100 years

A student thinks of displaying the results in a bar graph. Give *one* reason why a bar graph is unsuitable for displaying these data.
(Sample answers: The difference in lengths of the bars would be too big. The length of the bar for "polystyrene cups" is undetermined. You cannot make a bar for 20-25 years.)

Both in real life and in occupational life it is important to be able to "translate" a problem posed within a context into a mathematical model of the situation. In other words, to find out what you need to do to solve the problem. Let us consider a simple example, PISA 2003, question 1 "Growing up," M150Q01-019, which stated:

Since 1980 the average height of 20-year old females has increased by 2.3 cm, to 170.6 cm. What was the average height of a 20-year-old female in 1980? (Answer: 168.3 cm PISA score 477, level 1, reproduction).

A graph showing the average height of both young males and females in the Netherlands is provided but not really needed to answer the first question, though it might help to see that the average height of 20-year-old females was 170.6 in the year 1998. Students need to translate the problem into the mathematical problem 170.6 – 2.3 = ... (168.3). It is this translation that makes the problem a more difficult one. The subtraction algorithm is one that many students will show mastery of if it is presented as a bare problem. Hardly any mathematics textbook address the importance of making a mathematical model of the problem situation first, next solve the problem within the mathematical model and last decide whether or not the solution found fits the situation. A student who thought of the word *increase* only, and solved 170.6 + 2.3 = 172.9, should look back at the problem situation and the graph and decide that the average height measured in 1998 is the highest so the answer cannot be right.

PROBLEM DESIGN, FINDING GOOD PROBLEMS FOR PROBLEM SOLVING

Many teachers complain that during teacher education, test and problem design are hardly addressed at all. One of the reasons may be that test design is easy, as long as only problems at the reproduction level (level 1) are used for classroom quizzes and tests as is often the case. Just change the numbers in a problem for similar ones, change oranges for apples and change the names of any people mentioned. That's all.

The bath superintendent wants the height of the water in the swimming pool to be exactly 4 meters. How many centimeters is that?

Jessica is filling the swimming pool in her garden. She wants the height of the water to be exactly 3 meters. How many centimeters is that?

What is being assessed here? Does the student know that 1 meter equals 100 centimeters? Then why use a silly context instead of asking:

3 m = ... cm (Fill in the blank).

The context is probably just there to make the problem look more interesting. However, it is not taking students seriously since no student will really think that it is wise to have a swimming pool with a depth of 3 meters (nearly 10 feet for those using different units of measurement) in your garden. Moreover, the unit "centimeters" is too exact to estimate the height of the water in the pool.

Take students seriously. The problem situation must be realistic or at least imaginable, and the numbers used should be as authentic as possible. Always check! Not all mathematical concepts can be posed within a plausible and meaningful context situation. In that case, a bare question is more suitable. Our advice is to start with the context situation and try to pose a variety of (mathematical) questions instead of asking which context situation can be found to pose a question about for example a parabola.

If a problem situation is used to ask mathematical questions, the starting point is the situation and not the mathematical content. The problem situation used should provide a mathematically interesting problem that is worth solving in other words.

The questions asked within a context should be meaningful; designers should ask themselves "Why would anybody want to know?" This does not mean the problem is always interesting for every student, they all have their own interests and preferences; they hate soccer or they know everything there is to know about prehistoric animals. But at least students should be able to imagine that this particular problem is important or interesting for a researcher or important within an occupational situation. We sometimes call a context that is entirely "made-up," not imaginable and uninteresting to almost anybody, a *non-text*. Here is an example from a mathematics textbook:

> For her birthday, Lois got a box of writing paper. She thinks the sheets are too big; they are 20 cm × 30 cm. From both sides of each sheet of paper, she cuts an equally wide strip. After cutting the strips off, she knows the surface area of one sheet of paper is 416 cm^2. Use an equation to find the width of the strips that Lois has thrown away.

It is not easy to find "rich" contexts that enable interesting mathematical questions. Newspapers, scientific journals, objects of art, interesting buildings, utensils, crafts and packaging are important sources. The situation that is found initially, almost always needs to be adapted for use with students.

For less able students, the context should be closer to the student. For these students a scientific context is less suitable.

The language used must be adapted for the age and reading competencies of the students taking the test or using the curriculum materials. This means for example that the text of a newspaper article used as a starting point for a (mathematical) question being posed almost always needs to be changed. Photos and drawings may help explain the situation. Visuals should be added to help students; they are not there for fun or to make a test paper look "nicer." Taking a closer look at the drawing, photo, or diagram must be worthwhile for students, not a waist of time in a test situation or learning situation.

A really rich context makes it possible to pose questions from a variety of mathematical domains, number, algebra, statistics, or geometry. Problem designers should be aware that for most students, even an interesting context becomes boring after a while. Especially if a problem situation does not appeal to students, they may find it very hard to keep motivated. Therefore, a *"rich" context enables mathematical questions from different domains. In general, limit yourself to one mathematical domain, e.g. algebra and use the context another time for a different mathematical domain.*

An article from a Dutch newspaper is summarized below to illustrate the previous characteristics. A lot of information about the contest, the cost per tile, weight per tile, etcetera is already omitted.

> Ivo ten Hove won a contest to design a new tile to be used for pavements. He made one big, more or less H-shaped tile measuring 90 x 90 centimeters and several smaller ones.

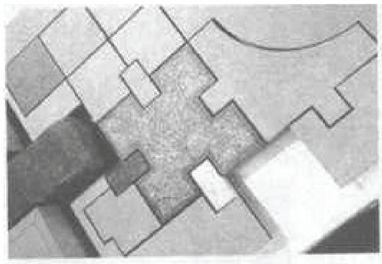


Figure 2. Tiles designed by Ivo ten Hove

Below you see a mathematical model of some of the tiles.

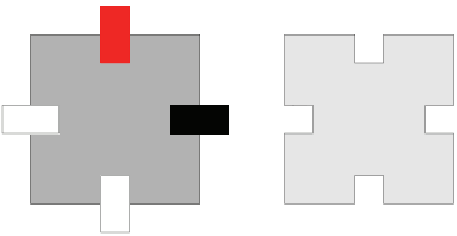


Figure 3. Mathematical model of some tiles

Many interesting questions could be posed within different mathematical domains. As a start, designers might do just that. Of course, the tile is simplified and the questions are posed regarding the mathematical model of one or more tiles. Here are some examples:

> Compute the area of an exactly square tile measuring 90 × 90 centimeters. Converse the answer you gave to square meters.
> One of the requirements for the contest was that a winning tile would not cost over 85 Euro per square meter. Does a square tile of 90 × 90 centimeters which costs 90 Euro, meet the requirements?
> Different square patios can be made using the large tile. We will make a row of growing square patios. This row can be extended to higher numbers.

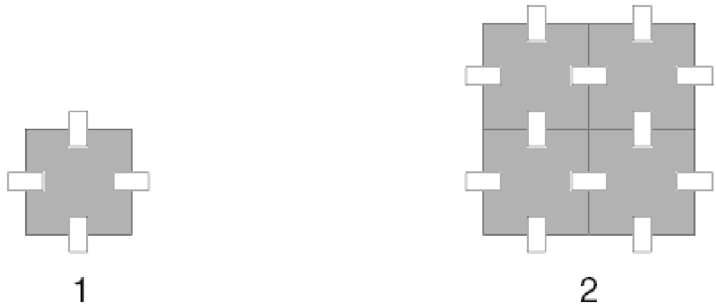


Figure 4. One example of a row of square patios using Ivo's tiles.

How many large tiles are needed for the fourth model?
Is it possible to make a square patio using exactly 210 square tiles?

Here are two formulas to compute the number of large tiles and small rectangular tiles in each model:
$B = n^2$,
where *B* represents the amount of large tiles and *n* represents the number of the model,
$R = 2n^2 + 2n$,
where *R* represents the amount of rectangular tiles and *n* represents the number of the model.
The number of rectangular tiles grows faster than the number of large tiles.
How does that show in the formulas?
Use the formulas to make a new formula that can be used to calculate the total amount (T) of tiles in each model.

Shown above is a limited number of questions that might be posed using the tiles context, teachers using this context found many other interesting questions. For students it would be confusing to go from one mathematical domain to another in one problem and they might get bored after a while trying to keep involved in the same context. If the context is used as a test *problem, designers would choose just one mathematical domain (algebra, geometry, arithmetic) and limit themselves to a relatively small number of questions. The same context may be used again later, or with other students, with different questions.*

The number of questions to be used with one context is advised to be 3 – 5 to keep students motivated. It often takes time for students to get involved in the context situation. This time can be rewarded by first asking a simple question, for example reading data from a graph that is part of the context, substitute a number in a given formula, perform a simple calculation that will clarify the situation, etc.

The first question to be posed within a context situation (if more than one question is posed) should enable students to get "involved" in the context. Usually this will be a simple question at the reproduction level. For a learning situation this recommendation can be "translated" into:

> The basic problem to be solved in the context situation should elicit student's ideas. It should evoke a "global" motif for students to become involved and it should continuously evoke "local" motifs to keep the learning process going, leading to solutions to the problems they are confronted with. (Westra 2008, 96)

And also:

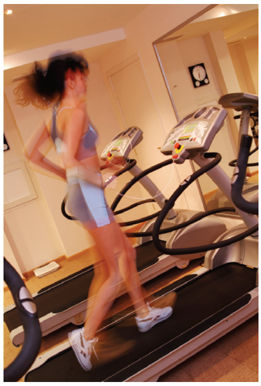
> To support the autonomy of the learner, the learning and teaching strategy should be transparent for students: this implies that at any point it should be clear to the students what learning activities they have to do, when and why. (Westra 2008, 198)

TEST DESIGN, AN EXAMPLE FOR THE CLASSROOM

From our own practice as test designers, educational materials designers and mathematics teachers, we will take an example from the PISA2003 study to make the problem solving process more explicit. The problem may serve as a basis for teaching how to reason mathematically based upon the structure of a formula and think about mathematical modeling of a situation.

The problem situation was derived from a newspaper article. The text is already adapted for use in the classroom. Note that for less able students, we do not provide all information available at once, because the students might feel overwhelmed when having to deal with a lot of information. A photograph is added to clarify the situation for those students who are not exercising regularly. Many sportsmen and women regularly measure their heartbeat while exercising, so we may expect that many students are interested in the context.

Heartbeat



For health reasons people should limit their efforts, for instance during sports, in order not to exceed a certain heartbeat frequency.
For years the relationship between a man's recommended heart rate and his age was described by the following formula:
recommended maximum heart rate = 220 – *age*

As you read the information, many questions come up, some of which may be used with students, in a classroom discussion or in a test, as well: What is a "normal" heart rate for somebody my age?

Discussing this with students in a classroom, doing an experiment might be a good idea. Count your pulse for half a minute. Are there differences among students? Now jump up and down for a minute and count again. How did the heart rate change? Whose recommended maximum heart rate is higher, a student's or their teacher's? Of course it is not necessary to know the teacher's age, in order to answer the previous question, if we assume that the teacher is older than the students.

The formula is a *rule of thumb* and may not apply to everybody. Give an example for somebody the formula may not apply to.

A newborn baby has a heart rate of over 100 beats per minute, but not nearly 220. So the formula is not suited for young children. (Note that students argue informally here about domain and range without using or even knowing the appropriate mathematical terms yet). A top athlete at rest may have a very slow heartbeat, of about 30 beats per minute. It is possible the formula does not apply to top athletes. Obviously,

because the information is given, the formula does not apply to women. Their heart rate is slightly higher than for men. A similar formula for women is
recommended maximum heart rate = 230 – *age*

Suppose you would graph both formulas. (You do not have to do that) What would they look like? How can you be sure?

Note this is a question that is *not* asked, because it does not take students seriously:

Christian found his recommended heart rate to be 180. What is Christian's age?

There are much better ways to find out somebody's age!

Going back now to the newspaper article. Recent research showed that the formula *recommended maximum heart rate* = 220 – *age* should be modified slightly. The new formula is as follows: *recommended maximum heart rate* = 208 – (0.7 × *age*). The article further stated: "A result of using the new formula instead of the old one is that the recommended maximum number of heartbeats per minute for young people decreases slightly and for older people it increases slightly.

From which age onwards does the recommended maximum heart rate increase as a result of the introduction of the new formula? (Show your work)

Note that this question was used in the pilot of the PISA study but proved to be too difficult for the majority of the students taking part. It certainly is a difficult question. The question might become easier if there had been an introductory simple question first, to get involved in the context, to have students draw the two graphs (or give one graph and have them draw the other one) or state the problem in a different way, for example:

For which age does it not make any difference whether the old or the new formula is used?

When answering the question, students may compare tables and check; they can use the graphs, find the intersection point and check, use equations or just guess and check. That is why this is a question from the connections cluster (level 2 in the assessment pyramid), students need to find their own mathematical tools to solve the problem. There are different tools that lead to a right answer, albeit not all equally efficient.

In the newspaper article more information about exercising and checking your heart rate was provided. Research has shown that physical training is most effective when the heartbeat is at 80% of the recommended heart rate.

Change the formula *recommended maximum heart rate* = 208 – (0.7 × *age*) so that it becomes a formula for calculating the heart rate for most effective physical training, expressed in terms of age.

The same question would have been a lot easier if it was posed much earlier in the problem situation that is if it was posed for
recommended maximum heart rate = 220 – *age*, the original formula.

Students' answers may show a variety of equivalent formulas, such as

effective heartbeat = 0.8 × (208 – (0.7 × *age*))
effective heartbeat = 166.4 – (0.56 × *age*)
$y = 166 - 0.6x$

y is the recommended maximum heart rate while exercising effectively and x represents the age in years.

When discussing answers to the last question in class, the relationship between realistic situation and mathematical model of the situation should be addressed. Since the formula is a rough estimate, 166 might be a better rounding for 0.8 × 208 than 166.4. Within the mathematical model of the situation, 166.4 is correct but after finding the answer, we should "translate" back to the situation in which the problem originally was posed and check whether the answer should be adapted according to the situation.

A classroom test should reflect the teaching and learning process before the test was taken. In the test, some questions could be posed about the same context that was discussed in class. The context is a rich one that may lead to many different questions at different levels of competency. For a level 3 problem, however, a teacher may look for another, unfamiliar context, so students can show they are able to use the mathematical content studied previously in this new context as well.

CONCLUSION

Below we will summarize the design characteristics discussed in this chapter. The authors limited themselves in this respect; much more could be said about the problem situation, about the design of an assessment plan for the school year or about the design of a balanced classroom test. For teachers, it may be reassuring to know they do not have to invent the wheel themselves. If you want to design problems at other competency levels than just *reproduction*, start by looking at the work by other designers and adapt their problems for use in your own classroom. And take care, once you start looking around for interesting problem situations it is hard to stop and you may get many problems!

TEST DESIGN CHARACTERISTICS, SUMMARY

1. A full test should be "balanced," which means that the competencies *reproduction*, *connections*, and *reflection* are addressed. (See Dutch pyramid model included).
2. A context used as problem situation should be as authentic as possible. Note that "authentic" may involve fictitious elements such as the name of a country, the currency used, etc.
3. If a problem situation is used to ask mathematical questions, the starting point is the situation and not the mathematical content.
4. The questions asked within a context should be meaningful; designers should ask themselves "Why would anybody want to know?"
5. A "rich" context enables mathematical questions from different domains. In general, limit yourself to one mathematical domain, e.g. algebra and use the context another time for a different mathematical domain.
6. For less able students, the context should be closer to the student. For these students a scientific context is less suitable.
7. The language used must be adapted to the age and competencies of the students taking the test or using the curriculum materials. This means for example that the text of a newspaper article used as a starting point for a (mathematical) question being posed almost always needs to be changed.
8. The first question to be posed within a context situation (if more than one question is posed) should enable students to get "involved" in the context. Usually this will be a simple question at the reproduction level.
9. The number of questions to be used with one context is advised to be 3 – 5 to keep students motivated.
10. Make it clear whether or not the question posed should be solved within the situation or within the *mathematical model* of the situation. (For test purposes often a mathematical model of the situation is already provided).

For a learning situation recommendation 8 can be "translated" into:

The basic problem to be solved in the context situation should elicit student's ideas. It should evoke a "global" motive for students to become involved and it should continuously evoke "local" motives

> to keep the learning process going, leading to solutions to the problems they are confronted with. (Westra, 2008, 96)

Recommendation 10 can be "widened" into:

> To support the autonomy of the learner, the learning and teaching strategy should be transparent for students: this implies that at any point it should be clear to the students what learning activities they have to do, when and why (Westra, 2008, 198)

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"LET'S PRETEND!" ROLE-PLAYING, DRAWING, CLASS DISCUSSION AT THE ROOTS OF MODELING PROCESSES

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ABSTRACT
We report here on a "mathematical workshop" realized in two second-grade classes, whose goal was to enhance children's autonomy and creativity in managing modeling processes. Starting from a challenging problem, a tale about space time and speed, children experienced different solving strategies and means to represent "reality:" words, realistic drawing, role-playing, different kinds of symbols and graphic representations. These strategies were built and compared during class discussions, where children learned to reflect on power and limits of each representation. This paper focuses on the first phase of the workshop, where the "creative interaction" between different representation strategies is particularly evident. We try to analyze the role of this interaction in order to trigger a process towards the construction of mathematical models. Furthermore, we emphasize the role of the teacher as a mediator between children's "natural" ways of reasoning, and scientific thought.

INTRODUCTION

In the following paper we give an account of a mathematical workshop experience in two second-grade classes. The activity starts from a mathematical tale, centered on the conceptual tangle "space-time-speed." The tale is quite complex and rich in suggestions. It can be faced at different levels of mathematical skill and through a multiplicity of representation strategies. Above all, the workshop is meant as an occasion to experience abstraction and modeling processes. Our goal is to strengthen children's autonomy and creativity in managing this kind of processes. Indeed, we think that one of the reasons of the widespread negative attitude towards mathematics can be found in a very common teaching practice: at every school grade, abstract objects are imposed on students, but no one teaches them how to manage abstract processes (Nazzaro, 2005). Usually, people are supposed to gain ability in using complex systems of mathematical representations, but no one cares if these systems have any meaning for them (more: if these meanings correspond to the culturally valid ones). At the same time, the ways by which all individuals (children or adults) represent the world to themselves are not considered, made explicit, and improved at school. That means that people are left at the representation level where they were (Mazzoli, 2000). This way, the risk is that the gap between cultural tools passed at school and individual life experience become wider and wider along the years.

During our courses of mathematics education and workshops with primary school teachers we frequently hear statements like these: “You suggest I should use the Cartesian plane, but it doesn’t fit my way of thinking;” “I know the right words to explain it, but I’ve never really understood it;” “mathematics and me: we never met.” In our opinion such statements represent quite well the school failure, today as well as in the future, as they are so popular among teachers. For this reason our goal, while working with both teachers and children, is to create a “meeting ground” for people and mathematics. The problem is how to identify each student’s resources and how to enhance them through (not to replace them by) mathematical tools (Hawkins, 1974).

In this process, the work on representation is crucial. At first, it is important to make people aware of their resources: what can be obtained putting them in a challenging problem solving situation. The comparison of different representation modalities and of problem solving strategies proposed by peers and with the aid of the resources brought in the group by an expert guide stimulates a metacognitive reflection on everyone's ways of understanding. This can be seen as a first step from natural towards scientific thinking (if we define it, following Vygotsky, 1990, Chapter 6, as characterized by awareness and willingness to use). If people reflect on the power and limits of their own representation strategies, and begin to build some criteria to recognize an effective representation, then mathematical tools can be welcomed: they can be seen as something that helps them solve some problems, overcoming their individual limits. In this way, also the meanings of mathematical representations are rooted in their experience and in their spontaneous way of thinking.

In the workshop that we will describe we worked with a multiplicity of representations of the same situation, using drawings, movement, words, and symbols. For many years the research literature in mathematical education and in cognitive sciences underlined the importance of the use of different representations and languages. This practice gives teachers the possibility to reach different students with different kinds of prevailing intelligence and cognitive style (Gardner, 1983). At the same time, each student can compare different representations of the same mathematical object (Duval, 1995; Ferrari, 2004). In this paper we focus our attention on the latter point of view, for different reasons: from one side, the experience of a multiplicity of representation strategies avoids the risk of confusing a mathematical object and its representation. Furthermore, we have experienced the effectiveness of provisional and non-conventional representations, created by children on the basis of their needs in each phase of the understanding process, and shared by the class community. This practice is crucial also because it helps students to experience the nature of mathematics as a search for patterns and structures. We agree with Hawkins when he says that the definitive formalization of these structures is just a stable product of the mathematical process, but not its essence.

In our workshop a central role is played by the use of the body and of its movements in the space, in order to represent the problem. Obviously this is also due to the kind of the problem proposed: we speak about movement, paths, speed of two characters: a prince and a knight. Attempt to reproduce the characters’ actions is a natural play for the children. Nevertheless, as we will try to show in this paper, this reproduction is neither “a game only,” nor it is by any means easy: the choice of the rules for our representation is a crucial source of reflection on the problem structure, and on the delicate passage between concrete and abstract thinking. Of course, the body movement and its observation play a crucial mediation role, but this kind of representation is tightly intertwined with the others. In our experience, understanding is made possible above all by the interaction among different means of representation. “The Prince and the Knight” workshop is a good context to observe how children solve a problem “with the help of their speech, as well as with their eyes and hands, as Vygotsky would say (we could add: with the help of their feet, too). According to the Russian psychologist, this “unity of action, perception, and language, which produces the inner visual field,” is a basic aspect of human behavior.[1] Didactic mediation is effective, in our opinion, as it works on this unity, instead of hindering it.

THE “PRINCE AND THE KNIGHT” STORY

The following tale has constituted the starting point of many workshops for students of all ages. In this paper we have chosen to focus our attention on the initial phase of the workshop experience in two second-grade classes with the same mathematics teacher (a member of our research group). Due to space limits, we have decided to choose only a fragment of our experience (the first 4-5 sessions of laboratory in every class) and to describe it in a detailed way, but not so detailed as to lose the whole sense of the activity. This paper focuses on the interaction among different strategies and ways of representing “reality.” We try to analyze the role of this interaction in triggering the process of construction of mathematical models. The first representation modality is actually the proposed text: of course, the story, although full of realistic details, is not the reality but just one of its representations based on a system of specific rules. Children are used to these rules, although they would not be able to formulate them explicitly. The following text, translated from Nazzaro (2005), was read aloud to the children before starting the discussion.

> A prince, when he was grown-up, was curious to visit his father’s kingdom. Everyone said that the kingdom was huge, full of woods, lakes, fields and green, scenting grasses. One day he decided to leave with his retinue: knights, servants, carriages, tents and provisions. They walked 50 km. At nightfall they camped: this was the first stop. Next morning the servants left the camp to set out again. Before leaving, the prince called his most trustworthy knight and said to him: “You must go back to the castle to bring me some medical herbs; you have also to bring me news about my parents, while I continue travelling ahead.” Then they said goodbye and parted on their separate ways. Every day the prince covered 50 km and the knight 100 km. The second night everyone stopped, rested and on the third day kept walking. Finally, in the evening the knight caught up with the prince, gave him the medical herbs and told him about his parents. Then they ate and rested. The day after the prince asked the knight to go again to the castle to take jewels for the countesses. So they left in opposite directions. After many days, the knight arrived at the castle and then met the prince for the second time and gave him the jewels. The next morning the prince called the knight and asked him to return to the castle to take the kingdom’s map because he was afraid of going over the frontiers. The knight left once again. Many more days passed and the knight delivered the map to the prince...

THE FIRST CLASS: DISCUSSION AND BODILY ACTION

In the first of the two classes where the story was told, after the reading, the teacher asked the students to repeat the story in their own words during a collective discussion. Everyone paid attention to a different detail. During the discussion the teacher raised the fundamental question of the problematic situation: how can we foresee when the prince and the knight will meet next time? (Gradually, during the workshop sessions, other questions were emerging: after how many days they will meet the other times? Will they meet forever, or there will be a moment in which the knight cannot

[1] (Vygotsky, 1978, 26). Vygotsky's intuitions seem to be confirmed by recent research in neurobiology: the unity among abstract thinking, language, motion and its observation has been observed also at the level of the neural system of our brain (Gallese & Lakoff, 2005).

catch up with the prince any more?) The children let their doubts emerge. In formulating hypotheses, they introduced the word "speed" – not present in the text. This way they showed to have focused on the importance, for solving the problem, of this "complex variable" that had emerged from their reflections on "kilometers and days."

Mattia: It depends on the horse speed: if it is slow it will take longer, if the horse is fast it will take fewer days.

The students, further stimulated by the teacher to foresee the next meeting, spontaneously decide to perform the tale in order to understand "how the things have really gone."

Teacher: But how can we know when they meet?
Giuseppe: Let's pretend!
Mattia: Yes, I play the prince and someone else plays the knight.

The collective discussion went on with reasoning about the modalities of interpreting and dramatizing the tale. The idea of "pretending," typical of children's role plays, took them to building a first model of the situation and to introducing crucial ideas like scaling down.

Marta: Teacher, he goes around and then he arrives here and *we can pretend* that this distance is 50 km.
Alberto: We can pretend that an hour is until here and then here is another hour.
Mattia: But in this way the distances are small, it finishes too soon!

The discussion was lively because the children perceive that modeling is useful, but many things that the children did not want to forgo were left out. For example, the journey's length, or the hard work made by the characters were very important for many children: they represented the very spirit of the story. Anyway, after various hypotheses and trials, the space and time constraints (that is, the classroom walls and the school-time) induced them to agree on the use of suitable measurement units. As Gabriella said, "We have to find an exact method that takes up a little space."

Finally, we decided that one floor tile represents 50 km, and that the running day is marked by a child who counts until ten (the night passes while he counts until four, but soon the children realize that it is not essential: "at night everyone sleeps and therefore the situation does not change"). The two characters move at the same time on two parallel tile rows. This way we could visualize the meeting points: that is when the prince and the knight were at the same distance from the starting point. Later on, after trying the journey of the prince and the knight many more times, Chiara noticed that "the prince is like time, in fact time goes always ahead, it is not like a DVD." In this way she recognized the necessity to think of the time through the spatial metaphor of an infinite path in one single direction. Indeed, this metaphor is widely present in our common language and represents our fundamental way to imagine and to represent time (Capriolo, 1995).

THE SECOND CLASS: DRAWINGS AND THEATRICAL MACHINERY

In the other class the teacher asked children "the problem question" immediately after the reading. Soon, the children tried to answer in the way they usually apply for classic mathematics problems, that is, combining the numbers in the story through arithmetical operations. But this time it was not easy to choose an arithmetical operation. A student, in order to overpass the blockage moment, proposed to draw the problem situation. The teacher passed this proposal to the whole class and re-opened the discussion asking: "How can we draw in order to understand how to answer?"

Of course, the meaning of "drawing" in the original student's proposals was an "illustration of a scene." Actually, this is the only kind of drawing that teachers usually ask children in primary school. On the contrary, this time the teacher raised a different question in order to encourage a global representation of the situation: if we want the representation to help us solve the problem, it would be better if it shows all the important data and the relations among them. Then, the children directed their search towards two aspects. The first problem they discussed was the need for a strategy to represent the two directions of the knight's motion, in contrast with the prince who "goes always ahead." They proposed characters' portraits turned in two opposite directions, left and right oriented arrows, even overlapping tunnels. After that they tried to solve the difficulty of a global glance at the story. For example, Erika proposed:

> Maybe everyone can draw one day and one night and then we attach the drawings one after the other, by a sticky tape. So we make a very long horizontal line.

The second meeting started with looking at the drawings made by each child the previous time. The students did not succeed in joining the drawings in only one temporal sequence. Moreover, single drawings did not help us make any forecast. Therefore, there was a blockage moment. So, the teacher informed the children about the idea proposed by the students of the other class: to represent the situation by dramatization. The children enthusiastically agreed with this suggestion. At first, the idea of a "performance" recalled their recent experience at a theatre. Therefore, they made a lot of proposals, very complicated to realize: theatrical machines, rolling carpets, scenarios made by screens with a projection of a running landscape, to simulate the movement of the characters (as Alice said: "The scenes move and we pretend to walk").

Although these proposals were too complicated and apparently not suitable they hid some fundamental reflections on the problem of space-time representation. At first children noticed that there was neither space nor time enough to reproduce the experience of the prince just as he did it. We must simulate and reduce it. Proposals like Alice's showed the children's "Copernican" awareness of the relations between the idea of movement and the reference system: if we move our body, or move the reference system, we get the same result. Later, another refined remark emerged when Lisa asked: "Perhaps we have to use two screens to see the movement of the two characters, because it is true that they move at the same time, but the prince and the knight are not attached one to the other."[2]

This discussion about "theatrical techniques" went on until the teacher thought that it is better "to come back to earth," in order to avoid getting lost. So she proposed to start moving and to represent the journey by means available at the present moment. Since this passage, the representation process was analogous to the other class, even if more "realistic." When a child found a solution, the requirements of the stage fiction were always present: children were emotionally involved in the prince's adventure, as much as in the intellectual challenge to detect the encounter moments. In fact for children it was important to act as they were really delivering the medical herbs as well as to notice, with enthusiasm, that their method worked and that the third evening the encounter between the prince and the knight really would occur. This continuous game within and outside mathematics, within and outside the story promoted progressive levels of abstraction. It could be imagined as a spiral-like process rather than a linear progression.

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[2] In our opinion, this is a very deep reflection, since it seems to make explicit a problem frequently observed also in adult learning. In our long experience of primary school teacher training we noticed that many adults have strong difficulties to represent motions of different bodies on the same Cartesian plane, although they understand its usefulness to make comparisons.

Mariano: Yes, we can make a huge coil by the numbers and then we can color it with green like a meadow. We can do a giant ruler!

Rita: But there is already the ruler, it is the floor tiles.

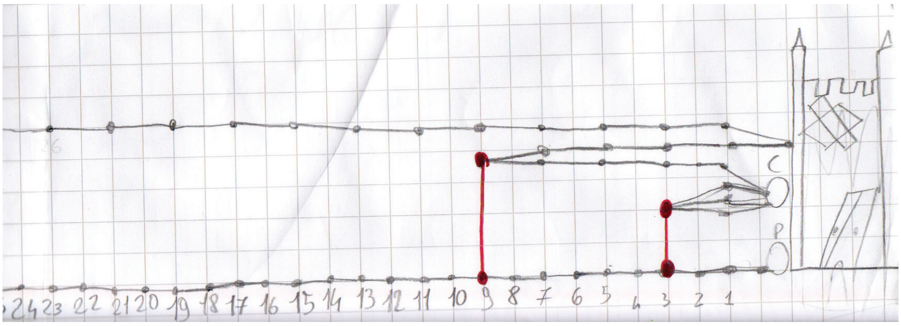
NEW GRAPHIC REPRESENTATIONS

In both classes the representation by body movement was a powerful instrument: it gave us a first idea of the problem shape, and allowed us to find the first days of meeting (they meet on 3rd, 9th, 27th day, etc.). The body representation caused us to reject the first hypotheses advanced by children (based on additive series). That was the reason for their being more anxious to understand the end of the story – if there is an end! But there came a moment, in which we were too tired to go ahead and back. We were tired because no evidence of our activity remained. We had to do it all over again: at the beginning of every workshop session, when we made a mistake, or if a new person entered and wanted to understand our activity. Moreover we couldn’t make so many steps in the corridor of the school. Meanwhile someone played with the calculator and made a sensational discovery: maybe we have to always multiply by three! But the numbers are too big, we can not verify them walking!

Thomas: To use the body is a sure method, but now it is the moment to stop and to work with our minds.

Probably this was the right moment to go back to graphic representations or try them for the first time, in the case of the second class. We believe that the dramatization influenced the graphic representation in a strong way: the latter is a kind of representation where only useful elements are present. Moreover, this experience helped the children to construct criteria for recognizing an effective graphic representation: the bodily representation worked in that it allowed them to find solutions that otherwise we would not have seen. But it was not easy to transfer all on the paper. For example, Alessio asked: “Yes, I have the squared sheet like tiles, but how can I draw the prince’s and the knight’s moving?” Genny tried to solve the problem by moving both his hands on the sheet with a pen in each hand at the same time: this way he could represent the movements of the two characters leaving a trace on the paper. During the play, we left some signs on the floor, in order to mark the encounter and departure points. These signs were now a source of inspiration for our graphic representations. Gradually, drawings became less realistic and more essential. They became richer in lines and notches, numbers and symbols, colored arrows for each of the two characters. But there was still the difficulty to represent time passing by. During the performance children already used a second dimension (time): this dimension was embodied, for example, by the sequence of the days written on the blackboard, or by the children’s voice announcing the new day. Someone tried to transfer on the sheet the bi-dimensionality (the two related variables) through two parallel lines (a space line and a time line), which they tried to connect with thinner vertical lines. Other children reproduced the second dimension with voice: while they marked the characters’ movements on the number line, they whispered counting the days.

Now it seemed that the road was opened to gradually introduce the Cartesian plan. This useful instrument could be used to solve our problems in a “resonant” way with the children’s ways of understanding. But this was the goal of the second phase of our workshop and the hypothesis to be verified in the next step of our research.



CONCLUSION

When reflecting on this experience, an aspect emerges very strongly: the cognitive power of the expression "let's pretend." This expression, proposed by the children and typical of their role plays, represents during the workshop a bridge between their ways of exploring the world, and of reasoning and mathematical thinking. "Let's pretend" means many things for the children. On one hand, this idea fulfills their need to become subject-protagonists of the situation in order to explore it by themselves. The opportunity of putting oneself in characters' shoes promotes emotional involvement, therefore motivation and ability to activate solution strategies.[3] At the same time it stimulates the involvement of children's bodies, with many important consequences on their mental representations.

First of all, it is only through the movement of their body in space and time that children realize the necessity of a dynamic model of the explored phenomenon. This way, they grasp that it is better to reproduce an experience "in a shorter and simpler way," in order to reflect upon it. Doing this, they can observe it globally, they can manipulate it more and more times, with a measure of control that is impossible in the real world. This function is common to children's games and to mathematical models. These two ways of thinking proceed according to different rules and cannot overlap (Vygotsky, 1978). However, in our experience, they have some points in common. Through the connection with their experiences of spontaneous play games children accept the abstraction from reality required to start the modeling process: if *we are pretending*, then I can be a prince, or a chair can be a castle. Then we can accept if we need that a long and hard journey is like a single tile, or like a small mark on the paper sheet.

As we said before, the idea of "playing the situation" promotes the bodily representation. This allows children to see the situation in a clearer way and gives them the occasion to experiment an effective model that allows them to make forecasts and to verify them. However, a bodily representation suffers from some limitations, like evidenced in Section 5. The experience of these limits originates the need for a more powerful and abstract model that allows them to see new things and enhances comprehension. In our activity, different representations – drawings, symbols, body movement, and speech – develop simultaneously. For instance, the limits of the graphic or verbal representations open the road to dramatization that favors the production of more effective graphic representations; discussion, verbal formulation of hypotheses and reflection on the experience precede, accompany and follow the construction of other representations; speech allows the construction of any kind of representations and

[3] For the crucial role of emotions in rational cognitive processes, see Damasio, (1995).

indicates their strengths and weaknesses. This search for more and more effective representations has the secondary effect of stimulating reflection on the explored phenomenon: gradually, during our discussions, the conceptual tangle space-time-speed starts to disentangle. The students recognize variables, notice their relationships, and compare their behaviors.

Our previous remarks can give the impression of a linear reading of our experience, like a straight path from complexity and confusion of “reality” towards a more powerful and clarifying abstractions. Actually, things went in a different way. In the “pretending” game students were also deeply involved in emotional and pragmatic aspects. They discussed on and on about the fatigue of the young prince far away from home, the reactions of the knight that had go back to the castle many times, the danger of a frontier war. These facts were a powerful sources of motivation, but they also triggered many reflections and questions on the problem structure. In this sense we can say that, in our experience, narrative thinking and mathematical formalization reinforce each other (Bruner, 1996). Also in some conflictual moments, the opposition produces new ideas. This does not happen spontaneously, but through a careful didactic mediation, aimed at integrating scientific thinking with the various forms of natural thinking.

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AN IDEA ABOUT FACILITATING STRUCTURAL UNDERSTANDING OF TRIGONOMETRIC FUNCTIONS

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ABSTRACT
Trigonometric functions, cosine, sine, tangent etc. should be from the very beginning taught as functions of the arc length measured in radians. This seems difficult for many teachers and textbook authors. So that they begin with proportions in right triangles and at the end of a long road, they arrive at the trigonometric functions of real arguments. Most students then remain with the first concept image they grasped at the beginning of that road. That image is imprinted *in their minds and is difficult to change. In this paper, I propose another way, aimed at the structural understanding of the topic. I propose to use a simple tool, called* trigonometric stamp *which makes the arc length a natural argument. As pilot studies confirm, it is advisable to begin then with cosine rather than sine graph, because cosine graph is symmetric and gives better start to* syntonic *structural understanding in the sense of Papert.*

KEY WORDS:
Cosine, sine, imprinting, trigonometric stamp

In Polish schools initiation to trigonometry traditionally takes place at the age of 14-15. After the change in the school system, this age group of students is in middle school (gymnasium). It was first introduced for right triangles as proportions: sine, cosine, and tangent, with related tables, and with visual graphical representation, usually for positive angles. Then, a year or two later a new definition of angle was introduced using arc length and full graphic representation of sine, cosine, and tan as functions on the real line. Recently the topic was cancelled in middle school, and is treated only in secondary schools at the age of 17-18.

The problem was discussed several times at our PDTR seminar, and some critical remarks emerged. The traditional approach was time consuming and caused difficulties when students accustomed to degrees had to switch to radians. The detailed list of trigonometric identities that were difficult to remember came in traditional approach before the graphical representation was available. Anyway the problem was posed, if there exists an easier way of initiation to trigonometry and of course aiming directly at structural understanding rather than formal (Krygowska 1977, 1979; Mostowski, 2003).

We found that there is one such direct approach sketched in the literature (Steinhaus, 1989). Steinhaus proposed a direct way of obtaining sine curve by rolling a sheet of paper into a cylindrical roll, cutting it across with sharp knife and then spreading the paper on the table. The edge appeared then as familiar wavy shape, exactly like the sine line.

This approach, proposed by Steinhaus, was not practical for schools. First of all the sharp knife is rather to be avoided in the classroom, and also, the resulting line which theoretically should be neat and accurate, usually came out as irregular. And the idea was abandoned. Another idea of cutting across a plastic pipe and trying to sketch the sine line with a pencil or a marker was also unsatisfactory.

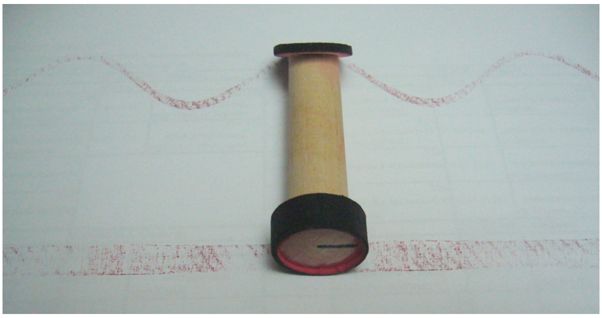

Figure 1. Notice the radius, marked on the circle

My idea was to make a wooden stamp as in Figure 1, a wooden peg with a wooden circle attached on one end and an elliptic shape attached on the other end. Using a stamp pad so that both ends of the stamp leave trace when the stamp is rolled on paper, we get a general rough view of the wavy sine line. The initial play with the stamp is to be followed by a usual use of a graphic calculator to connect wavy line with precise graph of sine or cosine.

We tried it first in some individual cases, not in the whole classroom. The results were in some sense unexpected. The wave line given by the stamp came first, and then we tried to fit it into a coordinate system. It was natural, in such case, for students to fit it so, that it was vertically symmetric. As a consequence, it was not the sine line but the *cosine* that came first to mind, and the value was given by the projection of a radius that was marked on the circle (Figure 2). So unlike the usual order of things, the cosine function came first and sine function followed.

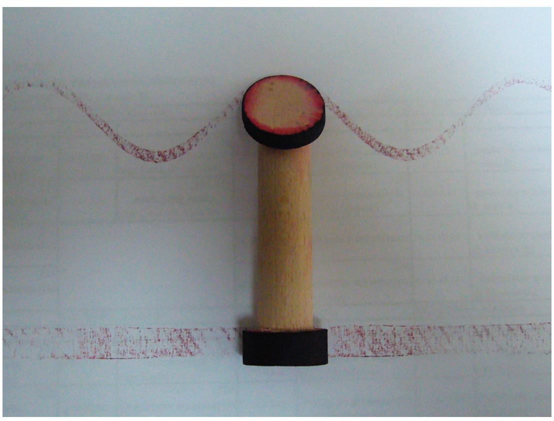

Figure 2. The wavy line traced on the paper

Then, it turned out that for students such a sequence is easier to follow than the traditional one, because it was easier to interpret and remember the vertically symmetric graph of the cosine, then the sine graph, which is not vertically symmetric. The stamp was used as a manipulative tool for understanding the situation in the *holistic* way. The graphic calculator added the numerical accuracy.

Another surprise was that it was easy to explain why we should measure the angle by the arc length of the circle, i.e. in radians, and it was natural to make the circle the unit circle, i.e. the circle of radius one, in the coordinate system. Motivation for it, for the students who were used to degrees is not so obvious (Mostowski, 2005). The relation (cosα, sinα) for the points on the unit circle followed, which was another way of defining cosine and sine functions: as the coordinates of points on the unit circle.

One of the principles of didactical designing was Krygowska's Metaphor, stating that we should "extract concrete actions for the students from the definitions, proofs and theorems." This principle is in general difficult to implement, and it often requires special apparatus or special manipulatives, not always at hand.

After some pilot studies, I think that the trigonometric stamp is one such concrete manipulative for arranging didactically meaningful situations, giving the concrete *action field* aiming at *structural understanding* of an important topic. The initial play with the stamp and related *language games* (Wittgenstein, 2004) should be followed of course with work using a graphic calculator (Figure 4) or a computer. I cannot make any decisive statements about this new approach, but I think it is worth exploring and further trials are pending.

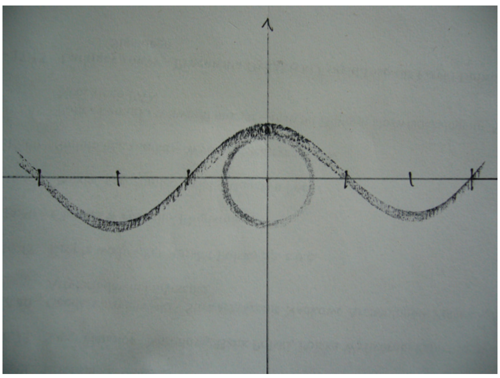

Figure 3. Graph of the *y=cos x* is vertically symmetric

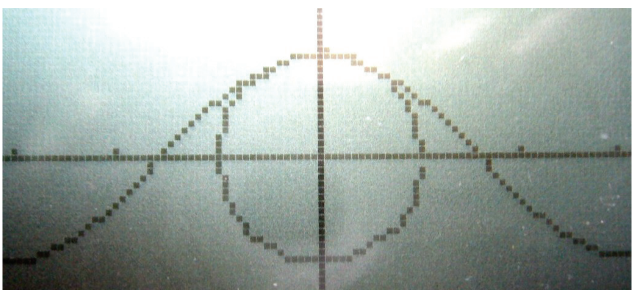

Figure 4. Using graphic calculator gives numerical accuracy

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APPROACHING LINEAR OPTIMIZATION PROBLEMS: AN EXPERIENCE IN UPPER SECONDARY SCHOOL

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ABSTRACT
In this article I will analyze my experience in class modeling activities and evaluating relevant competencies acquired by students' efforts. In the final reflections I give suggestions for teachers who wish to reproduce this ambitious project.

KEY WORDS:
Optimization problems, algebraic modeling, Cartesian graphical representation, coordination of representative registers, students' difficulties

INTRODUCTION

In recent years we have seen an increase of the use of mathematics, whether in its theoretic use to various scientific disciplines, or in contexts of everyday life. Parallel to this demand of subject increasing knowledge, at the same time research grows to find better ways to motivate students to study mathematics. Today mathematics can be considered quite rightly a fundamental element of the cognitive process, which is indispensable for disciplines such as, for example, physics, chemistry, biology, engineering, medicine and economy.

By mathematical modeling we call the proceeding that starts from a problem originating within one of them, through its representation with the use of mathematical language (equations, inequalities and their graphical representations), analysis of the representation, discovery of different methods of numeric simulation fit to approximate them, and at last the implementation of these methods with the use of computers. Modeling and its applications to everyday life help the mathematics teacher to motivate students and catalyze their attention, to involve them better than in a traditional lesson. Moreover, as it happens frequently to a lot of problems as, for example, the one related to field of economics, the mathematical models, with a number of variables help to obtain much more quantitative information than one you could get with just a quantitative analysis. It happens to theories, which advance problems on phenomena that are not independent from each other that require maximizing certain quantities with limited resources. Such problems, concerning very interdependent systems with changing constraints, are crucial to foresee the answer to important questions, the way of acting rationally.

As for tools, the new mathematics curriculum shown at the *XXII Convegno UMI-CIIM del 2001* states: "To solve a problem it will be necessary to use up your own resources, fighting in the open, exploring knowledge to find the one useful for the goal, developing new knowledge, changing the way to use them, going in deep, distinguishing between important and redundant data, finding, in case of lack, data necessary for controlling the solution process referring to the goal and to the validity of the result." The ministerial program states moreover that the scientific-area objectives are to understand the scientific importance of the experimental method (quantitative observation, formulation of hypotheses, mathematical modeling, experimental expectations and experimental checks). All that is possible through a study on relations and functions, in particular collections of elementary functions, and their graphics seen as modeling tools. Moreover, interesting examples of linear modeling and optimization can be found in the books *Matematica 2003* and *Matematica 2004*, where you can find innovative didactic suggestions suitable for secondary school, and useful for the new curriculum (Anichini et al., 2003, 2004).

So, it is easy to understand why in the PDTR project I decided to work on modeling as useful tool to amplify the competencies mentioned above. This type of problems can develop and test students' competencies and reflection (PISA 2002, 2003): students when solving a problem have to come back several times from the model context (linear function and algebraic and graphic representation) to the problem context. It is just this work of interpreting and reinterpreting in different representative registers and theoretical scope that aims to develop and strengthen competencies of highest level, which involve also the concept of function as a mathematical object.

Feasibility of the introduction of problems of this type into different secondary schools is disputable, though, above all because certain type of schools programs are sometimes very strict and there is lack of time and didactic tools necessary for deeper work on linear programming problems. In fact, in Italian text books devoted to *liceo scientifico* it is almost impossible to find this problematic except for classical textbooks e.g. by Prodi (1975, 79) or Speranza & Rossi Dall'Acqua (1979).

This article is divided in three parts: the first one is an introduction to modeling and its power in the mathematical teaching, in the second one I will show more significant problems. Finally, in the third part I will briefly describe my students and discuss some final reflections about my work, also suggestions for teachers, who want to reproduce this project in their school.

PROBLEMS

The topic that I have worked on includes a class of problems, which require finding maximum or minimum points of a linear function of two variables in a limited plane domain determined by a system of two-variable inequalities. It does not explicitly occupy part of the secondary school curriculum, except perhaps for one technical school oriented to economy. The topic is a simple but essential application of algebra to the real world and is also a good training field for the algebraic calculus, analytic geometry, and mathematical modeling. Obviously, problems of this type are simpler than those in the real life (at least the number of variables is smaller); however, they are significant from the theoretic and application point of view. The context of these problems offers the possibility to verify students' ability (i) to formulate a formal algebraic problem, (ii) to coordinate different representatives' registers: Cartesian representation of linear functions and analytic geometry, and (iii) in competencies concerning the concept of function and their development.

My work on modeling began in the past year. At the beginning it mainly consisted in collecting problems and analyzing them with respect to difficulties in text comprehension, easiness, attractiveness of the context, mathematical difficulties, and required competencies. But the problems, which you usually find in textbooks have, we can say, a scholastic taste, even if they are different in different schools; moreover, it seems they are a mass-product: in fact, when you have solved one, others are easy. So I needed new problems, with different problematic situations, not concerning situations of everyday life, but above all that were not analogous. My first experience was problematic. I carried it out in another school with a different ambient, which did not allow me to freely do what I prepared previously. Innovation and choices based on literature sometimes crash when confronted with everyday work with difficult classes, where the innovated teaching fades into the background because of the lack of students' motivation. Due to this experience I began my project this year by reconsidering the old problems, studying and creating new ones, but above all by working on the means of students' motivation and awaking. The goal of the first phase was to create situations able to involve students through their everyday curiosity.

In the a priory analysis we had to work not only on possible difficulties, errors, misconceptions, obstacles, stressed in the didactic research, but also to choose, modify or create problems directed towards students' preferences and help to launch debates involving essential knowledge, thus to expose its value or to institutionalize it.

The following problems are the ones proposed and then discussed in the class. I should add that students needed to be prepared for them through exercises to be done at home: to find the range of variability of a parameter so that an equation has solutions, to understand the meaning of the intersection point of two lines when solving linear systems, and to reinterpret the solution point in the problem context.

Problem 1

A farmer grows little cherry tomatoes and potatoes in a 4.200 m^2 field, having an amount of 6.600 litters of water everyday to irrigate. For producing 1q of tomatoes, earning € 50, he needs 40 m^2 of field and 100 l of water each day. For producing 1q of potatoes, earning € 60, he needs 100 m^2 of field and 20 l of water per day. How many tomatoes and potatoes should the farmer produce to obtain the maximum earning?

The difficulties connected with this problem are: (1) to recognize the unknown values (x = tomatoes [quintal], y = potatoes [quintal]); (2) to find the constraints, partially hidden (water constraint: $100x+20y \leq 6.600$, field constraint: $40x+ 60y \leq 4.200$ m^2, constraint of sign: $x \geq 0$, $y \geq 0$); and to (3) to find the objective function (the earnings to be maximized $z = 50x + 50y$).

Problem 2

Foods	Price (cents)	Calories (×100)	Protein (g)
Chocolates	*200*	*2*	*9*
Chocolate milk	*300*	*3*	*3*

Helen is a thin teenager that loves a boy who likes shapely girls, so she decide to create a fattening and a bit dangerous diet composed only of chocolates and chocolate milk, but she has a limited budget. Let's help Helen! Try to calculate the minimum cost of the daily diet that holds the following restrictions: minimum amount of 1200 calories and 27 grams of protein, six chocolates at most. Necessary data are in the table above.

The difficulties linked to this problem are: (i) to determine the unknown values *(x* = quantity of milk *y* = quantity of chocolates); (ii) to find the constraints, even partially hidden (calories constraint: $200x+300y \geq 1200$, protein constraint: $9x+3y \geq 27$, non-negativity constraints for food $0 \leq x \leq 6$; $y \geq 0$); (iii) to find the objective function: the cost to minimize $s=2x+3y$, (100 cents = 1 Euro); (iv) to notice that the cost attains the minimum value not only at one point (x, y) but at infinity many points inside a segment; that from these points we have to select those with integer coordinates; (v) to know how to modify the problem so that the question of maximum cost might have a numerical answer.

Problem 3

Is subscribing advantageous? And if you have an eight-year-old brother? Would it be advantageous for him? Analyze the situation and represent algebraically the cost as a function of the number of entries in each case (adult or child). Analyze the situation using graphs.

Entry
Full…………………………. € 8
Reduced……………………€ 5
(until 10 years)
Subscribers
Full……………………………€ 4
Reduced……………………€ 2
Subscription € 20 (validity from 15/6 to 10/9)

The difficulties linked to this problem are: (i) to determine the unknown values (y is the entry cost and x is the number of entries) and to represent algebraically the costs in both cases; (ii) to interpret formulas in terms of functions and represent them graphically as lines $y=8x$ and $y=20+4x$; (iii) to understand that the subscription is convenient only if $8x > 20 + 4x$; (iv) to find the value $x=5$ and to be able to recognize it as the point of indifference for the two cases; (v) to know how to modify the problem so that the question of maximum cost might have a numerical answer.

Problem 4 – First part

A family, composed of two adults and three children, has a budget of 200 Euro for three months. They try to maximize the number of the adult and children entrances, let all the adults and all the children enter the swimming pool maximum times. Analyze the situation in terms of functions of the number of entries. (Suggestion: use x as the number of adults' entrances, and y as the children's ones.)

The difficulties linked to this problem are: (i) to divide the problem in two different sub-problems: "subscriber case," "single entries case" to be analyzed separately; paying attention to the subscriber's case; (ii) to find the constraints (price constraint: $2(20+4x)+3(20+2y) < 200$, months constraint: $x+y < 5\times 90$ (at the most everyone, that is five persons, can go to the swimming pool everyday for three months, that is 90 days), adult and children number constraint: $x\geq 2$ $y\geq 3$); (iii) to find the objective function: the expenditure (to maximize) $s=4x+2y$ and graphically explain the constraints (you can observe that the subscription of 20€ disappears, because it is embodied in the parameter, but when you need to interpret the expenses constraints it reappears).

Problem 5

Using a pattern calculate what the lowest price is, according to the number of the monthly entrances.

Single entry…………………................... *€ 6,50*
Carnet of 10 entrances…………….....…...*€ 52,00*

The main difficulties linked to this problem are: (i) to express the unknown values (x number of swimming pool entries, y the price); (ii) to draw and to recognize step functions; (iii) to choose the right modeling among several possible through comparing the two types of rate and stressing the indifference points; (iv) to recognize the winner strategy as a combination of single entrances and a carnet.

At first a sheet was given to the students with a series of problems. I expected collective solving and discussion on that. From the students' point of view the beginning of this class was quite strange: the first phase was spent in silence because they were frightened by the recorder (used by the teacher), believing that it was there to immortalize their own failure; this time of little shyness passed and they continued with a joyous confusion climate because they were in a different place of work: the school computer laboratory. After this phase things changed for the better and the students began to talk to each other in a polite manner. The students liked the new subject very much. The conversations were quite lively, above all when they had to explain to another one their strategy or when, at the end of a discussion about a problem, they had to find one or more solutions. There even was a strange competition to succeed. I did not solve their problems; my role was to repeat in a better and correct way their ideas, to summarize their discussions, and sometimes to pose questions, as a spectator. I embodied a student who did not understand and frequently I played the game to follow their train of thought in order to show them the contradictions. I put myself at the level of my students and I accepted, using the time that the ministry program offers, questions on questions, even the strangest ones. The student solutions appeared always quickly and even wrong solutions or strategies were debated among students. They never saw them as a defeat. All this was due not only to the good class climate but also to the fact that the class seemed quite closed at the beginning.

SOME RESULTS

I worked in a third-grade class, with a strong mathematics program (PNI), of a secondary school named Liceo Scientifico Wiligelmo in Modena. It was composed of 27 students, 10 girls and 17 boys, and their level was a medium one. Three students came from another class, because of a break-up in the second year, and another one was a grade repeater. Moreover, there were some issues related to immaturity (mainly boys) and low self-esteem (mostly girls). At the end of the past year teachers judged eight students not ready for the third year mathematics class, but they already provided for filling the gaps. I did not know the student before because I am a substitute teacher for one year maximum. I ascertained myself just at the beginning they were not familiar with the "mathematics dialogue:" they found it difficult to argument, to reason out loud, and at the beginning they seemed to prefer frontal lessons. Because my first attempts at "dialogued lesson" seemed more of a monologue, all students regardless their level of preparation remained silent.

I thought it would be interesting for students to know the history of linear programming. It might help them understand the birth of a concept and see that mathematics develops continuously. I started with a series of problems to solve with the use of algebra, namely of equations of the first and second degree and systems of two linear equations with two unknowns. After this we analyzed the meaning of a graphic as a representation and possible aid or solution of a problem situation where linear functions act. This choice was made to carry the student through a step by step discovery of linear programming. Then I presented problems to choose from where a quantity (for example expense) depending on several variables (the objective function) had to be optimized i.e. become minimum or maximum. The variables on their part had to fulfill constraints, that is, conditions that come from reality or from the particular type of the problem.

It is necessary remember that we can talk of linear programming only in these cases: (i) all the constraints that determine the acceptable solution domain are expressed by first degree inequalities; (ii) the function that is to be optimized, called objective function, is a linear function; (iii) all variables at least non negative.

The pre-requirements necessary for this activity have already been completed before the beginning of the project, either as a biennium review or as a part of the third grade program (to be able to solve an equation or a system of equations of the first degree, to know how to represent a linear equation on the Cartesian plane, to understand, even superficially, the concept of function). We can consider among the pre-requirements some knowledge of analytic geometry, in particular on canonic equations of lines, and the dependence of lines $y = mx$ and $y = mx + q$ on m as a parameter.

When I started, my students were just working on the modules: Cartesian plane, line, circumference, using these arguments to solve particular equations and inequalities of the first and second degree, with absolute values and roots, either from the algebraic or graphic point of view. I taught them how to find the solution of a system of inequalities starting from drawing of the boundary, then using a couple of coordinates of a point to determine the right semi plane to choose. I suggested to hatch the boundary-lines if they do not belong to the solution area and to color the right piece of plane, or to dot it if the values belong to naturals. Initial linear programming problems involving only two variables can be represented effectively on the Cartesian plane and solution can be found with simple geometric deductions. I stressed that the unknown values of the problem modeled by inequalities had to be distinguished from the unknowns of an equation. Paying attention to these details is not pedantry. In fact, the unknowns are here not just numbers to be identified, as they were till now.

Since manual graphing takes much time and the lines are not always so “good,” I mainly used the computer room with Derive. The use of this software was expedient to save time, but also to obtain correct, clear and visually meaningful graphics thanks to using colors.

During my activities on modeling I intended to teach the students to: (i) distinguish in the problem data, unknowns (variables), and constraints to work on; (ii) to give names to variables and to express algebraically the connections between them; (iii) to find the variability field of the unknowns; (iv) to choose among different graphical representations the right one, adequate to optimization criteria; and (v) to explain, in order to obtain the solution of the problem, the results of the syntactic treatment of the algebraic formulas and of the related geometric representations.

Discussions were always lively. The students confronted their ideas, showing a good interpretative ability of the algebraic form, either in the Cartesian plane or in the real problem. Yet to assess the individual learning I decided to give at the end of the course these problems to solve in writing (class work):

Problem 1 for the class work

Chiara joins a charity event, selling cakes; the proceeds of the sales will be given to the parish church of her town. She is not a good cook, so having to prepare two types of cakes she buys what she needs at the supermarket, that is, cake mix and milk. With the money she has got she is able to buy not less than 1200g of cake mix and at least 9l of milk. Each pancake needs 200g of cake mix and 3l of milk; each chocolate cake needs 300g of cake mix and 1l of milk. Knowing that every pancake will be sold at 20€ and every chocolate cake will be sold at 30€, how many cakes will be sold of the two types? Which is the minimum income?

The parish church forced Chiara to sell not more than 6 cakes, because another parishioner will prepare the same type of cake for the charity event.

a) Draw on a Cartesian plane the lines which represent constraints and minimum income, choosing the right unit of measure;
b) Represent algebraically the relationship between: quantity of mixture, amount of milk, numbers of cakes of the two types;
c) Represent algebraically the constraints on the numbers of pancakes or chocolate cakes or in general of each type;

d) Represent the income function;
e) Explain carefully what the income function that comes from your formalization represents;
f) Show on the Cartesian plane, using colors, the "feasible region" or distribution area of variables and explain which characteristics have points belonging to it.
g) How can you find points (x, y) where the income is minimum?
h) How can you calculate the minimum income?

Problem 2 for the class work

A farmer was to plant two different products on 20 hectare of soil. For type A the seed costs 300€ per hectare and for type B it costs 500€ per hectare. Besides, product A needs 20 hours of specialized work by hectare, paid 15€ per hour; product B needs 30 hours of work paid 10€ per hour. Government restrictions limit seeding product A at 15 hectare, but they do not limit it for B. If the income given by product A was 1,500€ per hectare, and the one of product B was 1,300€ per hectare, how the 20 hectare should be seeded to obtain the maximum revenue? Which value represents the maximum net income?

Draw the constraints and the line that represent the maximum net income choosing the right measurement. (Suggestion: substitute with x and y the areas of the cultivated fields A and B expressed in hectare. Remember that you can have the net income subtracting the expenses from the gross income.)

	Problem A	Problem B
Represent on a Cartesian plane the constrains and the line of minimum income	60%	70%
Represent algebraically the link between components	60%	60%
Represent algebraically the constraints	70%	70%
Represent objective function	70%	60%
Show the "feasible region" on the Cartesian plane using colors (no one has dotted the area)	50%	60%
Diligence about graphical representation, in general	40%	50%

Percentage table of right answers

Explain what the objective function represents on a Cartesian plan	30%
Explain which characteristics the points belonging to the "feasible region" have	40%
Explain how to find the feasible region point to be optimized to solve the problem	60%
Argumentation, in general	40%

Percentage table of good answers to metacognitive questions in the first problem

I have chosen to assign this task with different questions and different difficulties because I wanted to give students an opportunity to try to solve at least one of the two problems: the first had a metacognitive question to sound out the verification knowledge applied and the argumentation abilities used during the reasoning; it provided a comparison, with not only a single solution, but with a couple of solutions. The second problem offered an operative suggestion, but it had a more difficult objective function to represent. Both the problems had not the real numbers as application field, but the natural ones, which would highlight any interpretative gap linking the mathematical solution to the problem.

The first problem was not absolutely the easiest one for the students: it was not simple to find what the unknowns were. In spite of great work done, paths taken by some students were almost shocking. They put constraints as *2hgx+3ly* without noticing that they summed hectogram with liters, that it were different magnitudes, and without asking themselves what the two quantities represented. Those who solved the problem finding the right solution did not answer adequately. The algebraic procedure was correct, but too concise and argumentation poor. What surprised me was that this attitude was common even among better students. When I asked them for explanations they said that the answers were obvious from the graph. Instead, I had better results from weak students. Maybe for "strong" students the exercise consisting in wording the information of the Cartesian graphic was considered useless, because for them the tool spoke in a clear way. Almost all the students used tables as solution strategy. I want to stress that this was not suggested or mediated by the teacher; it came out spontaneously.

The second problem was the less difficult one; perhaps it was due to the hint. Some difficulties came in search of the net income function. Almost all the students wrote correctly the constraints. Here only a small number of students continued to use the grid strategy, because they felt no more need.

Only two students were unable to solve the problem, and to find the unknowns.

What I noticed assigning this task was that the majority of students, as you can see from our table, did not have great difficulties putting both problems in formulas and algebraic representation. But inversely, answering the metacognitive questions was difficult. This could mean that the students had a scarce capability to think on their own mental process. During the phase of class discussion the recorded spontaneous students' reactions about the given problems were noteworthy, but all this phase was done orally, not in writing. All the collective analysis involved the algebraic code comprehension and the cognitive aspect related to the comprehension of the "game rules." The students understood these rules, but they thought it was useless to write down on paper what was clear. So even if the participation in class was good and the students' comments were positive, either on the project in general or the problems discussed, all this partially disappeared in the first class work.

It is not an easy task to assess students in a short term. In fact, it could be important to give this type of problems some time. It would give the possibility to evaluate how much of the acquired knowledge was applied significantly. In school programs, time should be assigned for passing again through some important activities done during the learning process. This is what mainly helps creating real competencies in every subject. So after some months when we did not work any more on this type of problems I decided to give another class work, more essential, but more diversified, which I do not discuss here. In this second class work the verbal interpretation of the graphics was scarcely done; it seems the cause for it was laziness in explaining orally the meaning of the graphical and algebraic representation implications.

However, I can say that the class just at the beginning of the year seemed not to be much inclined to dialogue, maybe because of the temperament, but undoubtedly they were not accustomed to it. As the language teacher confirmed, the students had trouble exhibiting their thoughts freely, and frequently they aimed to be too short and concise in reasoning. The work done has produced positive results at the communication and motivation level; however, even if things got better and the way was right, the results have not been reached completely on the level of written communication. What surprised me was that when they found a difficult problem they aimed at using the grid as a tabular representation: a personal strategy to overcome difficulties! The students anyway showed they were growing up, developing their strategies, creating their tools: the first step to become adults in mathematics. About this I could say I am satisfied.

Reconsidering my entire project, at the beginning, creating problems suitable for our scope seemed to be the more complex phase of work. But very soon I realized that the greatest difficulties were on the evaluation, that is, how to reconcile the evaluations obtained by the students on the discussion level with the one of the individual written texts.

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HOW TO TEACH THE TRIGONOMETRIC FUNCTIONS?

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ABSTRACT
The Hungarian mathematics teaching in higher secondary grades is rather formal, abstract. The science of mathematics is the main factor to influence this approach. This style of teaching is hard for average and below average students. To change this approach I started using knowledge from the outside world. This study describes a small experiment in teaching the trigonometric functions.

KEY WORDS:
Realistic mathematics, trigonometry, concept image

INTRODUCTION

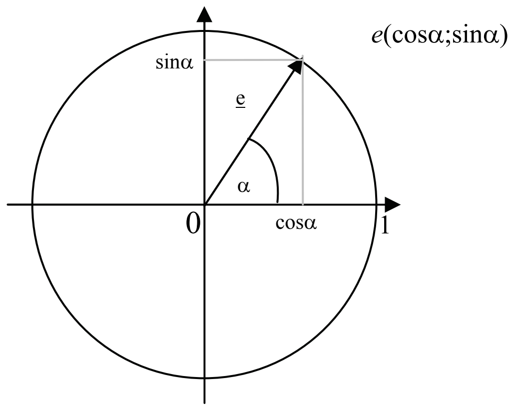
After visiting a lot of Hungarian secondary schools, Laurinda Brown (Bristol University) formulated her opinion about Hungarian mathematics education in the following way: "You, in Hungary, are teaching mathematics, we, in England, children." This statement expresses clearly that in Hungary the science of mathematics stands in the center of mathematics teaching. Due to long traditions, the abstract, formal side is preferred; the learning problems of average and below average students are no topic of discussions. To put it simply, the Hungarian mathematics education is elite oriented; fostering talented students is the most important task. But most of our students are not talented in mathematics. In my secondary school in Kecskemét we have students interested first of all in humanities, languages.

THE STUDENTS

There are 32 students in my class, but only 20 of them attend the bilingual class, so they learn mathematics in 2 groups – in English and in Hungarian. In these classes the language and the culture of the target language are the most important objectives, so the students prefer to learn art subjects. Therefore, as a science subject teacher, I must face great challenges to make them like my subject. How to motivate such students to learn mathematics? This is my favorite question from the beginning of my teacher carrier. Participating in the PDTR project as a future teacher-researcher I changed my formal style of teaching. First of all, the discussions of PISA problems convinced me about the necessity of change. In this paper I will analyze two lessons on the sine function.

In Hungary, we introduce the trigonometric functions in grade 10. As the first step, we introduce them in right-angled triangles using similarity. Having done this, we generalize the trigonometric functions for arbitrary angles. I will restrict myself to the teaching of sine function. It has a long tradition in Hungary to introduce the concept of sine with the help of unit vector in the following way:

Sin α means the second coordinate of a unit vector *e*, with direction angle α in the vector-coordinate system *i, j*. The direction angle of vector *e* means the measure of the angle of the rotation which brings the vector *i* into the vector *e*.



This definition causes a lot of difficulties for average ability students. I have chosen another definition. I will report it in this paper.

THEORETICAL BACKGROUND

The PDTR project helped me to give a theoretical basis to my teaching; until now I considered only the pure mathematical viewpoints. For the teaching of the concept of function the following PISA competencies are relevant: *modeling skill, representations skill, symbolic, formal and technical skill, communication skill.*

Representations

Bruner introduced the notion of enactive (material), iconic (visual) and symbolic representations. According to him,

> any domain of knowledge (or any problem within that domain of knowledge) can be represented in three ways: by a set of *actions* appropriate for achieving a certain result (*enactive representation)*; by a set of *summary image or graphics* that stand for a concept without defining it fully (*iconic representation)*; and by a set of symbolic or logical propositions drawn from a symbolic system that is governed by rules or laws for forming and transforming propositions (*symbolic representation)* (Bruner, 1966).

Due to Hungarian traditions, until now the use of symbolic representations was dominant in my teaching. Of course, the graphical representations were used too, but the material was neglected. At the introduction of the concept of sine function I used the Ferris-wheel as an enactive representation. For the graph of the sine function (visual representation) we used a heated fork and a smoked glass.

Concept image

Vinner and Tall introduced this notion. They call to the concept name associated concrete examples, pictures (graphs, tables), experiences, inner connections to other concepts (networks), impressions, emotions as the concept image. They state that a rich concept image helps students mobilize their knowledge about the concept after a long time, too. With the use of Ferris-wheel we wanted to enrich the concept image of sine function.

Narrative psychology refers to a viewpoint within psychology, which is interesting in the "storied nature" of human conduct. Human beings deal with experience by constructing stories and listening to the stories of others. Psychologists studying narratives are challenged by the notion that human activity and experience are filled with "meaning" and that stories, rather than logical arguments or lawful information are the vehicle, by which that meaning is communicated. This dichotomy is expressed by Bruner as the distinction between the "paradigmatic" and the "narrative" forms of thought which, he claims, are both fundamental and irreducible one to the other (Bruner, 1966). Focusing on the narrative forms of thought as Sarbin proposes that the "narrative" becomes a root metaphor for psychology to replace the mechanistic and organic metaphor which shaped so much theory and research in the discipline over the past century. His fundamental concepts in connection with cognition are summarized in the table below (Sarbin, 1986).

The mode of cognition	**Narrative**	**Paradigmatic, theoretical**
Organization	time, sequences, actions	timeless, categorical, subordinating
Textual	story, deliberate	description
Correspondence	Intended	hierarchy-relations
Idea	special, episodes	impersonal, validity
Embedded	contextual, personal, social	context free tendency

In our cases the story with "Ferris-wheel" and "My running with the heated fork" gave a meaning for the mathematical concepts.

The most recent discoveries about the nature of mind focus on the *embodiment of mind*. The detailed nature of our bodies, our brains and our everyday functioning in the world structure human concepts and human reason. This includes mathematical concepts and mathematical reason. The *cognitive unconscious* plays a significant role as most thoughts are unconscious – not repressed in the Freudian sense but simply inaccessible to direct conscious introspection. We can not look directly at our conceptual systems and at our low-level thought process. This includes most mathematical thoughts. Not only mathematical but also *metaphorical thoughts* are relevant in this theory. For the most part, human beings conceptualize abstract concepts in concrete terms, using ideas and modes of reasoning grounded in the sensory-motor system. The mechanism by which the abstract is comprehended in terms of the concrete is called c*onceptual metaphor*. Mathematical thought also makes use of conceptual metaphor, as when we conceptualize numbers as points on a line (Lakoff & Nunez, 2000). These ideas convinced me that I am allowed to use concrete, material objects in my mathematics teaching.

Introductory problem for the sine function

We investigated a Ferris-wheel. We concentrated on the motion of a cabin, which was at the beginning of our observation in a middle position that meant its suspension point was lying on the same horizontal level as the rotation axis. We considered this horizontal line as a baseline at the rotation of the observed cabin. We were interested in the height of the suspension point of the cabin relative to the baseline. This height can be positive (above the line) vs. negative (under the line). Let the radius of the Ferris-wheel be 12.5 meters and the time of one whole rotation of the wheel 30 seconds. We determined the position of the cabin at different times. For example: at *t = 5* sec the rotation angle realized by our cabin was 60 degrees.

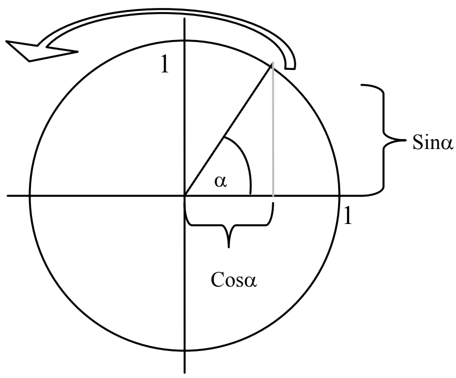
$h = 12{,}5 \cdot \sin 60^0$

For t = 10 sec the rotation angle was 120 degrees. The height would be $h = 12{,}5 \cdot \sin\left(180^0 - 120^0\right)$

We determined the position of the cabin in the cases of the third and fourth quadrant in a similar way.

It was important to notice that students could solve these problems with their knowledge about the right-angled triangles. After these exercises we focused on the unit circle (radius 1) with its center in the origin in the Descartes coordinate system. On the circumference of this unit circle there was a moving point. The position of this moving point could be determined with the help of the rotation angle (relating to the positive part of the *x* axis).

Sine α meant the second coordinate of this moving point at the rotation angle α.



To help my students see and memorize the connection between sin and cos of α and 180°-α I always illustrate this with a book. If we try to close it then the angle of elevation creates an acute angle and an obtuse one, but the virtual “height” is the same. As the example is touchable, they can understand that sinα = sin(180°-α).

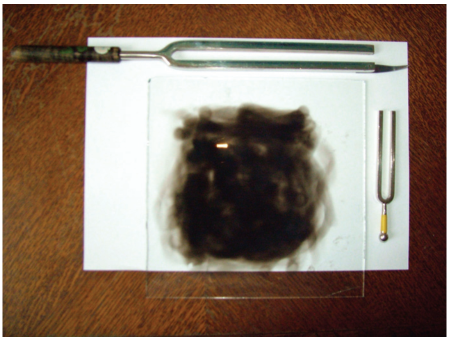
But if we open it we can do it in two different directions: (i) anti-clockwise direction ⇒ acute angle α; (ii) clockwise direction ⇒ reflex angle = 180°-α

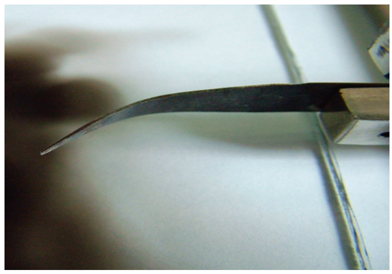
That helps them memorize that cosα = - cos(180°-α).

The Ferris-wheel model helped my students remember this definition. This definition is equivalent to the vector definition, because the coordinates of a position vector are equal to the coordinates of the endpoint of this position vector. Why do we use the “vector definition” in trigonometry in Hungary? The only argument usually is this: because the vectors are very important in modern mathematics and we shall teach mathematics in the style of modern mathematics.

Graph of the sine function

I got the following idea from a physics teacher: Use smoked glass and this…(He gave me the greatest pitchfork I have ever seen, with a special needle at its end.)

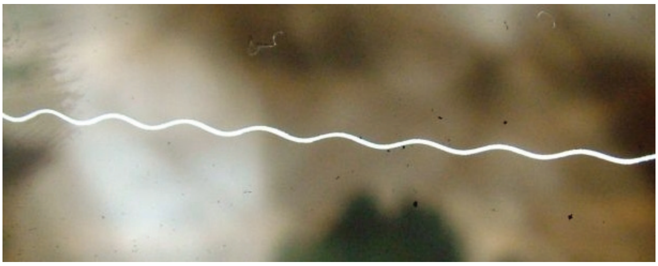




First, let us see an experiment. If we hit the fork it will resonate. On the basis of physics we know that both arms of the fork will oscillate (and give a sound in our ears). The pitch level depends on the lengths of the arms and the distance between them. We know it oscillates; therefore, we need its movement path. That is why we need the smoked glass. If we move the needle of the oscillating pitchfork along the glass and then look through the glass we can observe a beautiful sine wave (with slight attenuation).

At the beginning of the lesson I showed the students a small pitchfork (like the one we use with a choir). I made it resonate and touch the table to hear its sound (the international pitch “A”). I ask the students: “Do you have any ideas why I brought this to the classroom?” I asked them to write down as the main title of the lesson “Piquancy of Sine.” They knew that this was a mathematics lesson but they trusted me. I hit the fork again and said: “Hmm… It’s too small.” And I took out the large one. With this introduction the students were very eager and curious. I think this is the point every teacher wants to reach in a class.

The morale was perfect, “research” flashed them up (that is, the aim to be over the stimulus-threshold). I created curves with the fork on the glass and showed it around the class. After this experiment they got the opportunity to create the function graph on the smoked glass, and I was surprised that they wanted to reach split-hair accuracy level.

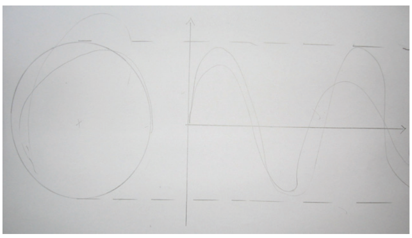


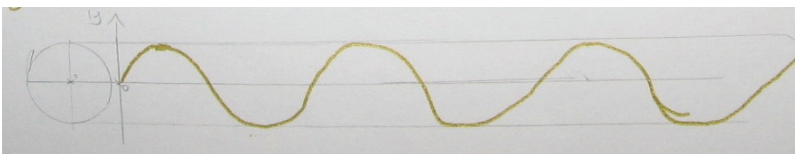
I saw that I could not stop at this moment. I asked one of the class-mates to create an enormous sin(x) function across the blackboard. Naturally, the question came from the students: “What is the connection between the rotation of the cabin and the oscillation of the fork?” To explain it we made another experiment.

I took out my other device prepared for the lesson, which was a simple spring with a small piece of weight at the end of it. Then, by playing with the weight, I moved it up and down and asked where the sine function could be in this case. Most of them found out that if I moved it with the appropriate velocity in front of the blackboard the movement of the weight wrote down the picture of the sine function. With the help of vibrating movements it created some connections between mathematics and physics. A question from one of the students came immediately: “That’s why we see the sine curve after resuscitation in the Emergency room? As our heart does a vibrating movement, too?”

I demonstrated to students that if we projected the rotation of a fixed point on the wheel perpendicularly on the wall, we could observe the connection between the circular motion and the oscillation.

In the last part of the lesson only one thing was left: students had to create the most beautiful graph of the sine function working in pairs. One student holding a pen had to move it along the unit circle; at the same time his/her partner simulating the time-axis had to draw a sheet of paper under the pen.



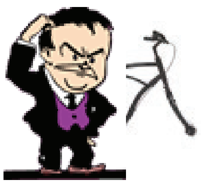


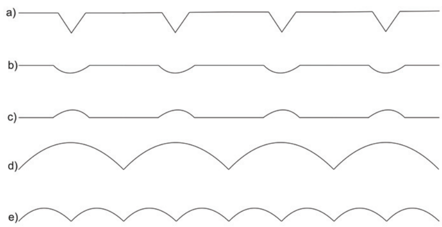
Some of the students asked more questions: “Mr Takács, please! And what do we get if we don’t go on a circular line? If I draw a triangle, a square or a hexagon with my pencil?” That was the turning point of the lesson since the student, whose mind opened already, not only paid attention to what was going on, but also construed theories and problems for himself. That is what I call *different aspects of mathematical thinking.*

Unfortunately, we have been fixed to some theories and definitions and we forgot to leave this "fixation." Has there ever been a mathematics teacher who asked just for fun which functions would come out?

I built up next starting point with this question, which deviated from the syllabus, but I could not leave this problem at that. I brought the following exercise into my next lesson:

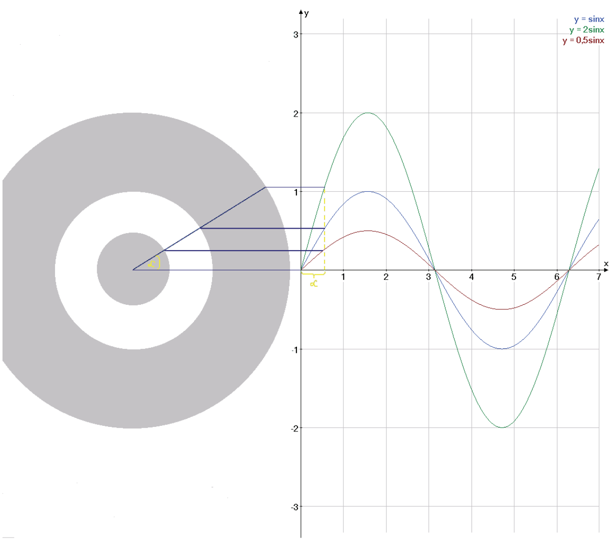
> How might the wheels of a bicycle look like if I don't want to feel the holes in the road (the length of the road equals with one turn of the wheel).



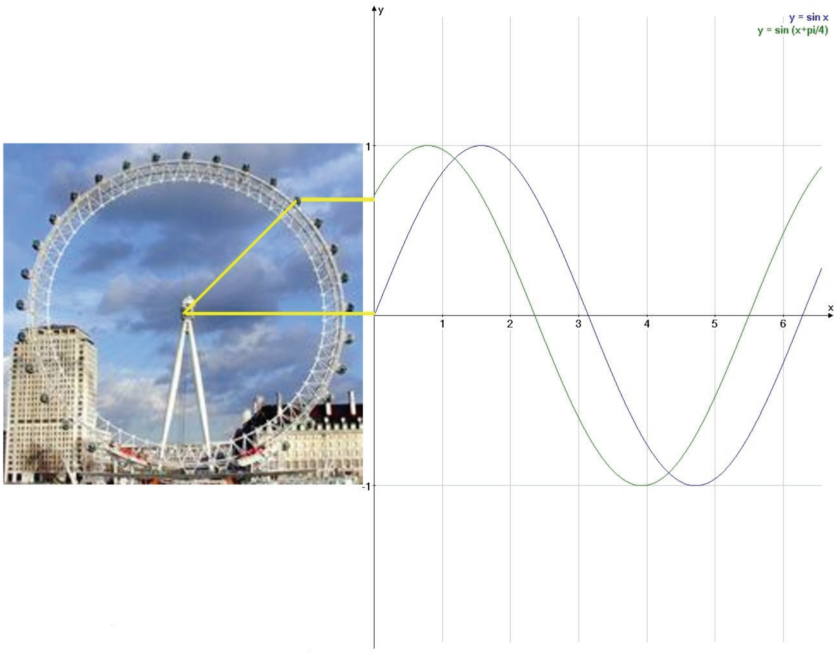


Physical meaning of the transformations of the sine function

The second problem I presented was the transformations of sine function. There are a lot of misconceptions in students' heads; most of them learn mechanically what they shall do in different cases. (Translation along the axis, stretching the graph etc.) If we need to explain the transformations of the trigonometric functions *sin(2x)*, $sin(x+\frac{\pi}{4})$ or $sin(2x+\frac{\pi}{4})$ the path of the cabin also helps us. The multiplication outside "sin" ($2sin(x)$, $\frac{1}{2}$ $sin(x)$) has the physical meaning of in- or decrease the radius of the wheel. The "reflection in the x axis" (e.g. -sin(x)) equals with opposite direction of the rotation of the cabin.



Shifting the function left and right with an angle (e.g. $sin(x+\frac{\pi}{4})$) means that we need to draw that cabin, which is already on the way (and not lying on the "ground"), back to the beginning. The physical meaning is: the cabin was *π/4* sec earlier in the start position, so we need to translate the graph of the function sine x to the left by π/4 units. At the function $sin(x-\pi/4)$ our cabin has a delay, so the graph of the function *sin x* will be translated along the x axis to the right by *π/4* units.



Multiplying inside by a number (e.g. $sin(2x)$ or $sin(\frac{1}{2}x)$) corresponds with changing the speed of the wheel: accelerate it twice when we multiply by 2, so the cabin goes two times around until the "old wheel" turns once, so we will get two periods of function sine during the same time.

The hardest task for my students was to explain how to draw the graph of the function $sin(2x+\frac{\pi}{6})$ or $sin(2x-\frac{\pi}{6})$. At *sin 2x* the argument *2x* means that the wheel has double speed than in original case. With this double speed the cabin needs only half time to be on the position where the cabin was in the beginning case. So it needs in $(\pi/6):2 = \pi/12$ sec to be on the position where sin *2x* was.

SOME FEEDBACK OF THE EXPERIMENT

The trigonometric functions will come back the following year in grade 11. However, I have tested this method only once, but it shed light on the essence of the matter: it was enough to say a word and the whole process of the drawing and later the analysis of the functions flashed on them. The main difference came up in the case of inequalities when they had to solve this problem: $\sin(x) > \frac{1}{2}$.The main difference was in the use of devices: those who learn mathematics in Hungarian tried to draw a unit circle and, with one exception, they didn't find the other angle so wrote down that $\alpha > \frac{\pi}{6} + k \cdot 2\pi$ was the solution. Those who created the function looked for "Where is sin(x) function above $\frac{1}{2}$?" and based on this geometrical view they knew immediately that there are 2 angels with the value of $\frac{1}{2}$. But I was really surprised that they could tell me that one of the angels must have been between 0 and $\frac{\pi}{2}$ and the other between $\frac{\pi}{2}$ and π . So – as I used the symmetry of the function – I did not have to explain that the other angle was $\pi - \alpha$. There was only one step from this to understand $\sin(x)^2 > \frac{1}{2}$ and $|\sin(x)| > \frac{1}{2}$. The students also noticed the connection between sine and cosine functions very soon when $\sin(x) = \cos(x - \frac{\pi}{4})$ came up. When we pulled it with $\frac{\pi}{2}$, it was more "edible" for the students.

Sharing ideas with colleagues

I was rather disappointed in sharing my experiences with my colleges. They disagreed with me, e.g.: "They *just* need *to learn* it, while it is a simple unit-circle." I put asked back: "Should they not understand it?"

Why should everything be about mathematics in mathematics lesson? The task should be understandable not only for the TOP TEN. Roughly speaking, teachers want to be successful in mathematics lesson; I am disappointed when I can see only three arms in the air when I asked a question. Mathematics should not be imposed onto students, however, according to the syllabus we have time for pure mathematics only. But we should shed light on connections with other areas. Mathematicians say that if people are good at mathematics they are also good at physics and vice versa, because they have found the connection. But they are not, if we do not take a step, we do not build up knowledge but just fill the student's memory.

Watch what you do! That could be the core of all the types of learning. But we need to provide the experience of observation and build up the first lookout tower for students. It seems to me that it will be long, hard work to change the opinions and habits of Hungarian mathematics teachers.

In this lesson I dared to "waste" one hour of the curriculum. Many students felt the need to try those experiments. With some luck, they will remember this "research" lesson about oscillatory movement and sine function for ever.

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INTERPRETATION OF GRAPHS BY STUDENTS IN GRADE 7 AND 8

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ABSTRACT

This essay reports the results of an experiment on interpretation of graphs connected with everyday situations, which was conducted in two middle school classes. This experimentation was born inside the PDTR project, which aims at the didactic innovation in the classes consisting in the development of an authentic mathematical thinking in students and, on the other hand, at teachers' professional refinement. The main aims of this work were to put into effect didactic practices suitable for the international contest in the viewpoint of PISA test, that is, to lead students to acquire competencies induced from the test itself, giving an the same time stimuli to their involvement. The work dwells above all on the problems faced by the teacher in the construction of the didactic path and on the difficulties students experienced in the learning process. This has brought the teacher to a critical reflection on her own concepts about mathematics and its teaching and to a research to improve her own way of conduct in the class.

INTRODUCTION

The activity object of this work concerns interpretation of graphs representing real-life situations. It was based on a careful analysis of the results of PISA test (OCSE, 2003). Problem situations were designed jointly with other teacher-researchers in training. Differentiated questions gave an opportunity to confront students with situations different than the descriptive statistics – the standard area where graphs are used at school.

The author worked with students in grade 7 and 8 of middle school. Three different worksheets were administered (see the appendix). In the first worksheet ("The Parachutist") a problem situation was used explained in the NMP project (Harper, 1987); for the second ("Postal rates") relevant problem from PISA test was used, a little bit modified on the language level to make the comprehension easier; for the third worksheet ("Play station") we got inspiration by modifying the contest from an activity presented in the PDTR seminars.

Testing was carried out in a seventh- and eighth-grade class of a middle school. The decision to submit the same worksheets to students of different classes was suggested by the willingness to find out how important the different level of maturity and knowledge were and how much they could influence the results and students' involvement. It took about 12 hours. Students accepted the requirement of motivating and substantiating their answers.

The activity allowed us to point out strong link between mathematics and everyday life, which students usually cannot catch at school. It stressed the importance of communicating with others, telling one's own ideas and justifying choices. That is what is generally difficult for our students. This activity also had an effect on my teaching profession and it let me think about my topic knowledge (that would become better and better) and about the importance of an a priori analysis of problem situations proposed in the classes and of possible strategies put in action by the students.

AIMS

The main aim of this activity was of double nature: on the one hand, it proposed to build up a course aimed at fostering in students the increment (or the birth) of an interest for mathematics, considered not just in its theoretical aspect but also as something useful in applications; on the other hand, the aim was to promote in students the consciousness of how necessary and important was the development of their communicative abilities, both written and oral, in their lives. Moreover, one of the aims was to educate students' attitude towards problematic situations unusual for them, making them learn how to answer simply, trust their intuition, but also teaching them to reflect and build up a solving strategy, coordinating the various kinds of information, according to the registers they belong to. Under a more general point of view there were several objectives of interdisciplinary nature: (i) capacity to read graphs; (ii) capacity to compare different graphs and get useful information for the resolution of further problems; (iii) capacity to deduce in a correct way their own answers; as well as metadisciplinary aims: (a) talking to schoolmates and the teacher; and (b) understanding the value of the collective construction of one's knowledge through comparing it with peers under the guidance of the teacher.

THE WORKSHEETS

It was interesting to see how students would react to the worksheets. Regarding worksheet 1 ("The Parachutist") we expected students to be able to answer correctly to the first three questions, while the last two would create problems. We thought not all of them would be able to connect the information expressed by the chart and the graph, and that many, simply considering the fall of a body in the air (for example, a leaf or a feather) would make mistakes without comparing the chart to the graph. Regarding worksheet 2 ("Postal rates") on the contrary, we thought it would be interesting to observe how students would interpret a chart connected to data gathered in groups, something unusual for them, and also how their answer could be influenced by previous knowledge about graphs. Regarding worksheet 3 ("Play station") it was interesting to observe whether students would answer the questions proceeding by trials and mistakes or calculation strategies, and whether they would determine a unit of measure to draw the graph in order to check what they had found.

METODOLOGY

The first two worksheets with the problem situations were proposed in two different sessions to students working in pairs (heterogeneous pairs properly formed by the teacher) for an hour, with the request to write down their strategies of thought, while the third and last were proposed as a test to be solved independently. We then analyzed the works with the students in a collective discussion, without showing solutions or proposing corrections (the instrument used was an overhead projector). This choice was made to avoid a situation when students would be too much influenced by the teacher's remarks but, on the contrary, to have them freely discuss in group about their solutions in order to reach a self-correction. Students were able to compare their ideas, to listen to and reflect about answers and motivations given by other schoolmates, to express their doubts and difficulties, and to find out their mistakes.

The group discussion phases were recorded. The following written transposition of group discussion by the teacher was a fundamental instrument for her since it enabled her to get evidence on class dynamics and difficulties of the students (she noticed that they talked most of all on the motivation of the answers they had given). In particular, this enabled her to realize what her attitude was while managing the discussions.

STUDENTS' BEHAVIORS AND DIFFICULTIES

Sometimes students' behaviors and difficulties turned out to be coherent with the expectations, sometimes they surprised the teacher who was obliged to reconsider her ideas and to reflect about possible motivations regarding the answers she received. Below, we will analyze the difficulties encountered by students dealing with each worksheet, and report essential parts of the group discussion.

Worksheet 1 "The Parachutist"

As foreseen, all students in grade 7 and 8 answered correctly the first three questions of the worksheet while mistakes occurred in the answers to the last two questions where they were asked to choose the right graph and to motivate their choice. Actually, while in the third class 18 students out of 22 answered correctly, in the second class only 4 out of 20 did. The majority of students who made mistakes chose graph A, as it was in previous studies (Malara & Iaderosa, 2001). Moreover, despite the expectations of the teacher few students checked the data in the chart controlling the graph, which would have favored the correct choice. Many did not consider to analyze the text of the problem, in particular, about the meaning of "free fall" they indicated graph A as the most representative of the relation (fall time – distance from the ground) connecting it to the undulation phenomenon in air of a body under the wind action. Below are (in chart 1) some significant parts of the class discussion about the observations made.

Chart 1

Class discussion extract, which represents various opinions that emerged. It relates to the choice of a modeling graph of the relation proposed in situation 1.
Teacher: How did you proceed to choose one of the graphs and why?
Student M: We chose B because the jump had been made without opening the parachute, maybe A could be the right choice if he had already opened it because he could meet air currents; in my opinion B was the best because when he jumps from the plane he hadn't opened the parachute yet.
Teacher: But what do you understand when you read "free fall"?
Student M: With "free fall" I understand that he still has to open the parachute.
Student E: It's graph B because it represents a speed variation, because in the first part of the graph the line is more horizontal, while after it's steeper, actually during the first 5 seconds it falls for 125 meters, while in the last it falls for 875 meters.
Student F: On the contrary, Miss, I chose graph A because it looked more realistic but I didn't check it was in free fall and he hadn't opened the parachute
Teacher: If I show you how a pen and a sheet fall down, which one best represents the fall of the parachutist?
In group the students answer: the pen
Teacher: Then, why couldn't it be graph C? What if I told you the right one is C?
Student E: It can't be because it represents a constant speed
Student M: Because while falling speed increases, then it's B.
Teacher: But also in C
Student G: But in C it starts immediately fast with a constant speed while if we look at the chart it doesn't have always the same speed, speed varies.
Student Ga: I meant the same thing
Student Gi: No, it's just that in C you can see it starts fast immediately falling from the plane, whereas in B you can't.

Worksheet 2 “Postal rates”

Contrary to the expectations, the majority of students in both classes answered correctly. The ones who did wrong, did so because they chose the graph connecting it to models they got from previous school experiences or because they could not interpret the chart correctly. We confirmed what we had foreseen in the a priori analysis, i.e. that the data in the chart, not giving interval values, represented a serious difficulty of interpretation for the students. We report in Chart 2 some parts of the discussion which is representative of the various opinions.

Chart 2

Discussion extract, which represents different opinions that emerged in the class in the choice of the graph, most adherent to the data chart in situation 2.

Teacher: What kind of reasoning have you made to choose the graph?

Student F: Well, at once we excluded the first graph at the top on the left, because it seems to us it was the less representative of the chart situation, then we compared the other three graphs and the measures of the two graphs on the bottom, which are the same of the chart situation, and we have noticed that the margins between a weight and the other weren’t respected because, if, for instance, we took a space between 2001g and 3001g it was clear that the price rose, while in the chart it remained constant. So, for this reason and following a process of elimination we chose the graph at the top on the right, as further acknowledgment we traced vertical lines, which ran on x-axis, to form little rectangles we then compared with the chart.

Teacher: Has anyone reasoned in a way different from F’s? For example, you, M, what kind of reasoning have you made?

Student M: I was wrong

Teacher: Why do you think you were wrong?

Student M: I chose graph C at the bottom on the left, because it seemed to us more right and furthermore we didn’t consider the weights in the chart, but only the fare and it seemed to us that the points corresponded.

Teacher: And when should you compare the weights?

Student M: I didn’t understand that from 501 to 1000 g the fare was the same.

Student E: Beside the fact that I didn’t consider the constant fare and I chose one of the points and then it didn’t seem possible that it was the graph at the top on the right because I had never seen such a graph .

Student G: On the contrary, we chose the last one at the bottom on the right because, as E said, we had never seen a graph with lines instead of dots; then, since we had already excluded the first at the top on the left, we chose the last one because we thought it was the right one.

Worksheet 3 “Play station”

Confirming previous hypothesis, this paper was the most difficult for the student, also because in this case they worked alone. The students’ approaches were mostly inadequate to the expectations. The teachers expected answers to the questions based upon comparison strategies between data and graphs, which implied calculation procedures. The students, on the contrary, chose different ways without taking into consideration the possibility to connect chart and graph, and without making use of appropriate units of measure. Many did not care about reading, which variables were written on the axis, some chose the graph which starts from the origin (the last at the bottom on the right) without considering that in the problematic situation it is excluded that a game has 0 cost.

Chart 3

Discussion extract, which represents various opinions that emerged in the class about the choice of the graph which is most adherent to the data given in situation 3.

Teacher: Which graph would you exclude first and why?

Student Mi: I remember I chose A immediately because it had no point.

Teacher: Everybody excluded A first?

Student F: I eliminated graph D first because, as the problem suggested (he points to the origin of the axis) at this point Luca had no reward.

Student G: On the contrary, I secondarily eliminated B because it doesn't start from point O, but the sellings are immediately high, while in the other graphs we start from a smaller selling to reach a bigger selling.
Student S: I chose C.
Teacher: Why?
Student S: Because it is the most correct, according to the text.
Student F: I Chose D, on the contrary, because it didn't look possible to me there could be a reward if the game cost 0 Euro.
Student E: I chose D too, at the beginning, but then, thinking about it, the reward can't be 0 but 50 cents, so the point must be off O, then it's C.
Teacher: Why not B?
Student: Yes, but B is wrong because looking at the cost of the games over 10 Euro and those of the games under 10 Euro it's clear they're different, that is, for a cost under 10 Euro it remains constant while in B it increases a lot.
…
Teacher: When Andrea agreed with Luca he didn't talk about a 10 Euro price, what can Andrea tell Luca about it, according to you?
Student M: I answered he gives him 7.5% that is a half way between 5% and 10%, and maybe also 50 cents.
Teacher: Who thought, on the contrary, that the reward might be 50 cents plus 5% and 10%?
(None of them took into consideration these two possibilities or made calculations giving arbitrary values, they found different percentages, particularly 7.5%. To let the class know, which might be Andrea's answer, a student has been called to the blackboard and he calculated 5% plus 50 cents out of 10 Euro (1 euro) and 10% of 10 Euro (1 Euro). The calculations let the students understand that Andrea's answer should have been: It didn't matter because it was the same thing.)

TEACHER'S REFLECTIONS ABOUT THE EXPERIENCE

The activity proved stimulating for the students who participated with a lot of enthusiasm and cooperated with schoolmates and teacher. For the students it was a moment of personal reflection and self-criticism. They noticed how difficult it is for them to express themselves: they often have many ideas but, not being able to explicitly express them, they are tacit audience in discussions. The group discussion made lessons more stimulating for them, attracting also those who usually get distracted more easily, who, on the contrary, participated actively giving their own contributions.

This has been a new and stimulating didactic experience also for the teacher, useful for her formation. It has anyway created problems of various nature: (1) to include this activity, with its time and spaces not always quantifiable, in a curricular planning, which has to be linked to the school planning; (2) construction of a new didactic program, and in particular the deep analysis about the possible attitudes and difficulties of students; (3) managing of group discussion in class; (4) evaluation of students.

The introduction of this experimental segment in the class planning involved a tiring negotiation with the colleagues of the parallel classes, who were not involved in the experience, and a careful restructuring of the whole year work plan in order to make the work fruitful and to enable it to get a certain consolidation and to consider possible effects.

As far as research is concerned, it required time for the planning and a confrontation with teachers and researchers. We thought long about how to structure the worksheets and, anyway, we noticed later that some of our choices were not optimal; for example, the decision to add point P in some graphs of the third worksheet, in order to take the students to a comparison with numeric data, turned out to be distracting because it influenced too much of their behavior in excluding some graphs. Apart from the psychological difficulty to accept the use of a tape recorder (which especially at the beginning inhibited also the teacher), it was difficult to lead the discussion, to know how to intervene with students in order to help them to find the solutions, which words to use not to create ambiguity or forced answers. Listening again to the recordings of the discussions we realized how much lack of experience of the teacher conditioned the development of the discussion. For example, it was not possible to limit the remarks of the best students, as well as to help them try to express themselves at their best, with a simple language; in a way it could be comprehensible for the weaker student, in order to favor their participation, too. Another negative aspect was evaluation of students, since often the motivations to their answers were given with an approximate language, sometimes less comprehensible, or even in contrast with the given answer. The answers to the extemporary questions, the remarks in discussions, individual and in-pair work – all this was hard to assess.

This activity and didactic methodology enabled the teacher to grow professionally and convinced her more about the present necessity to change something in the teaching of mathematics in order to improve its comprehension and to promote a stronger motivation in the majority of students. For instance, she understood, professionally, how significant her role is in a group discussion, for many reasons: (a) she should be able to give the right value to students' remarks; (b) she should not give solutions or express opinions; (c) she should leave space for the discussion, keeping on being a mediator; (d) she should give everybody the chance to express their ideas or reflections fostering the development of the expositive capacities and improvement in the use of the specific language; (e) she should help students going beyond their fear of making mistakes or doing wrong, underlining the importance of the sharing of mistakes as a point to start from in order to reach conscious construction of collective knowledge.

The participation in the project made me become aware of the necessity of a better mastery of the three different kinds of knowledge, interwoven and professionally indispensable: disciplinary knowledge, which enables me to find the crucial knots of the various disciplines and to adapt learning context to the evolving conditions of those who learn; knowledge, which concerns the learning processes enabling to interpret the difficulties and to plan actions aimed at strengthening and motivating students; and didactic knowledge, which considers the teacher as an active mediator between thought and actions of the students.

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http://www.invalsi.it/ric-int/pisa2006/sito/

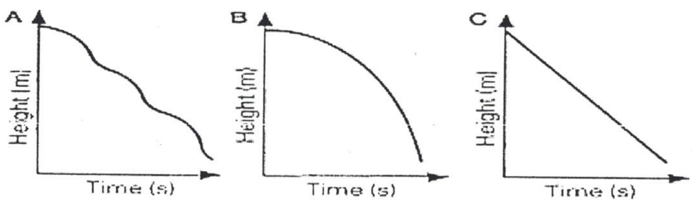
APPENDIX

Worksheet 1 – “The Parachutist”

The following table refers to a parachutist’s free falling jump from an aeroplane:

Time (s)	0	5	10	15	20
Height (m)	3000	2875	2500	1875	1000

a) How high is the plane from the ground at the moment of the jump?
b) How many meters lower was the parachutist after the first 5 seconds?
c) How many meters did he cover in the last 5 seconds?
d) One of these graphs describes the drop. Which one?
e) Explain why you didn’t choose the other two.



Worksheet 2 – “Postal Rates”

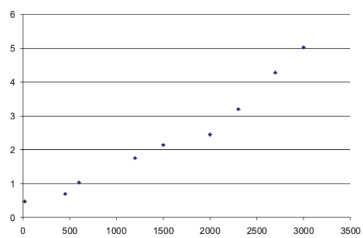
Postal charges are based on the weight of the items (to the nearest gram). In a certain town postal charges are those shown in the table below:

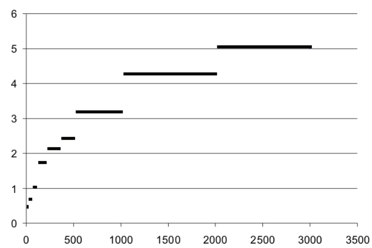
Weight (to the nearest gram)	Charge
Up to 20g	0.46 Euro
21g-50g	0.69 Euro
51g-100g	1.02 Euro
101g-200g	1.75 Euro
201g-350g	2.13 Euro
351g-500g	2.44 Euro
501g-1000g	3.20 Euro
2001g-3000g	5.03 Euro

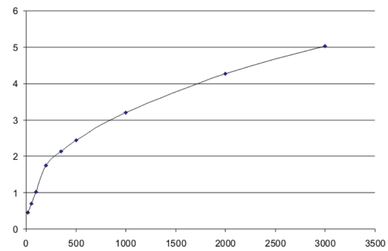
Look at it with attention.

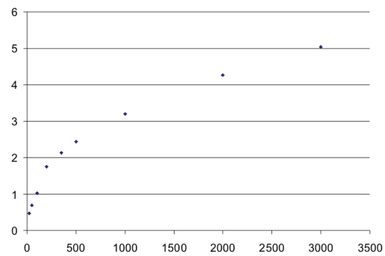
a) Which one of the following graphs is the best representation of the postal charges in that town? Give evidence to your choice.
b) Write also why you didn’t choose one of the other graphs

(The horizontal axis represents the weight in grams, the vertical axis the charge in Euro)



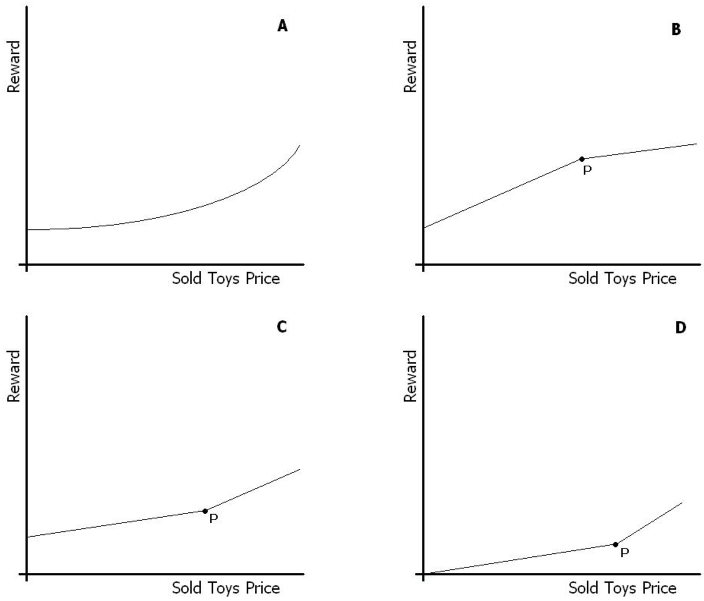






Worksheet 3 – “Play station”

Andrea, who is very fond of play stations, decides to sell some of his video-games in order to buy the new play station. He asks Luca, his schoolmate, for help, and he promises him for every game sold under 10 Euro 0.50 Euro plus 5% of the price of the sold game. For every game sold over 10 Euro he promises him 10% of the price of the sold game. Which one of the following graphs show this situation more realistically?



a) Which graph would you exclude first? Why?
b) In your opinion, can one make hypothesis about the abscissa of point P? What price of a game could it correspond to?
c) The right graph is … Explain why and explain also why you didn’t choose one of the other three.
d) In Andrea’s choice Luca is doubtful…: if a game costs 10 Euro how should he behave?

In your opinion, what could Andrea answer?

PART 4

EXPERIMENTS ON LEARNING AND USING ALGEBRA

AN EXPERIMENTAL STUDY TO PROMOTE ALGEBRAIC AND GRAPHICAL REPRESENTATIONS OF FUNCTIONAL RELATIONSHIPS

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ABSTRACT
Our research concerned designing and experimentation of a didactic path based on the theoretical frame of our ArAl project, which also takes into account the competencies of the PISA test. The path is aimed at the acquisition of mathematical knowledge on linear function and related competencies by middle school students. Classroom activities started from a set of realistic situations and developed towards the construction of meanings about variables, constant, and linear functions. They also influenced the development of logical thinking and reasoning, argumentation and communication through observations and interpretative hypothesis by the class, using specific terms and explaining computations and complex relations with words or in writing. Students' attention was focused on the "mathematical processes," which involved the encoding/decoding and the interpretation of familiar and non-familiar representations of mathematical objects, choice and switching between different forms. Students learned how to collect data; manage the translation from "reality" to mathematical structures; work with or validate a model; use symbols and formal language to describe relationships between variables and constants. In research the metacognitive and metalinguistic aspects are prevalent. Teachers are essentially involved in the planning of classroom activity and in the analysis of conceptual nodes and difficulties that students can encounter, whereas students are stimulated to acquire the capacity of control over their cognitive processes and the adaptation of mental models due the social interactions between fellows. Summing up, students were generally able to recognize variables and to identify linear functions; difficulties appeared when they tried to solve some situations ("Port Said" and "T-shirts") characterized by a high logical complexity and the presence of two different linear functions.

INTRODUCTION

OECD/PISA examines the capacities of students to analyze, to reason and communicate mathematical ideas effectively as they pose, formulate, solve and interpret problems in a variety of situations. Student who are to engage in solving a problem from "reality" have to take on activities such as: (1) identifying the relevant variables with respect to a problem situated in reality; (2) representing the problem in a different way, including organizing it according to mathematical concepts and making appropriate assumptions; (3) finding patterns and relationships; and (4) translating the problem to a mathematical model. At the same time they will build classical PISA competencies, such as: *thinking and reasoning*, *argumentation*, *communication*, *modeling*, *representation*, and *use of symbolic and formal language*.

Italian students had negative results on OECD/PISA tests due to traditional teaching of algebra, where the study of rules is generally privileged, as if it could precede the understanding of meanings. Problem solving activities have episodic nature and, in middle school, the subject "linear functions" is approached as a tool for modeling simple physical phenomena rather than rooted in the modern vision of arbitrary functional correspondence.

Our study was inspired by the OECD/PISA philosophy and based in the ArAl Project (Malara & Navarra, 2003), located in the early algebra theoretical framework. It was aimed at the acquisition of mathematical knowledge on linear function and related competencies by middle school students. Also, it was an approach to 1-to-1 functions together with their inverses (as in Malara, 2006).

THE LESSON PATHWAY AND THE WORK METHODOLOGY

The lesson pathway (Annex 1) was elaborated by using ArAl Project (U9) and NMP Project (Harper, 1987) materials. We decided to (1) explore complex open situations; (2) research for binary correspondence laws; (3) put into the field and coordinate multiple representative registers.

The didactic methodology involved these phases: (a) handing out and reading a worksheet, and identifying the objectives; (b) exploration and production of protocols in the workbook; (c) writing individual argumentation; (d) classroom discussion of protocols and individual points of view; (e) search for a general and shared solution of the problem; (f) recording shared conclusions in the workbooks.

In this research the metacognitive and metalinguistic aspects were prevalent; the teacher was involved in a precise planning and analysis of the conceptual nodes and difficulties that students could encounter, whereas students were stimulated to acquire the capacity of control over their cognitive processes and the adaptation of mental models due the social interactions between fellows. All the sessions were recorded and analyzed "a posteriori" through joint reflection with the mentor (R. Nasi) and project coordinator (N.A. Malara).

ANALYSIS OF PROTOCOLS AND CLASSROOM DISCUSSION

We focused our attention on two activities: "Sticks and pebbles" (Annex 2), proposed in the middle of the pathway, and "Think it through…" (Annex 3), proposed as a final test.

The "Sticks and pebbles" problem

In the "Sticks and pebbles" problem students had to: 1) identify the variables; 2) collect and organize data; 3) study patterns and research relationships; 4) represent the relationships in verbal, algebraic and graphic languages; 5) verify the relationships for specific values of the variables; and 6) work with and validate a model.

A student (S.Z.) writes on the blackboard:

No. Pebbles	No. Sticks	Exploration
2	2	(2 – 1) x 2 = 2
4	6	(4 – 1) x 2 = 6
7	12	(7 – 1) x 2 = 12
10	18	(10 – 1) x 2 = 18

No. of Sticks = (No. of Pebbles–1) ×2

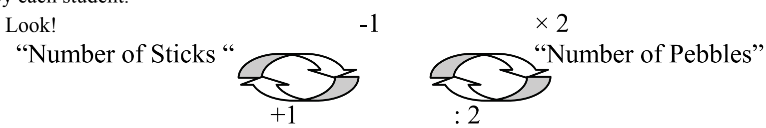
At this point it is fundamental that students know that a number can be represented in various ways and that non canonical representation can help in identifying the relationship between the variables. In the class, this work strategy was very common because students, since the first activities, had the opportunity to appreciate the transparency of the arithmetic expression as the generation process of a rule.

Then I translated the rule for our friend Brioshi.[1]
I put b = No. of Sticks and s = No. of Pebbles   The rule is: *b = (s – 1) x 2*
[The student makes a naive use of the "letter" as a label, Student S. writes on the blackboard and says]: I found a new rule

No. Sticks	No. Pebbles	Exploration
2	2	(2 : 1) + 1 = 2
6	4	(6 : 2) + 1 = 4
12	7	(12 :2) + 1 = 7
18	10	(18 : 2) + 1 = 10

Number of Pebbles = (Number of sticks : 2) + 1

Teacher: What can you observe?
Student N.: Now the subject of the sentence is the number of pebbles.
You can observe that the relationship depends on the interpretation of the problematic situation given by each student.
Student V.: Look!



There are "direct relation" and "inverse relation." The student described the direct and inverse relations by using the arrow representation and he realized the inverse operators swap.

Student J.:   I found a different rule *Number of Sticks = Number of Pebbles x 2 – 2*.
Student C.: I think it's the same as Silvia's rule, because if I put Number of Pebbles = 3, I obtain the same result.

Number of Sticks = (Number of Pebbles – 1) x 2 = (3 1) x 2 = 4
Number of Sticks =   Number of Pebbles x 2 – 2 = 3 x 2 – 2 = 6 – 2 = 4

Teacher: It's not enough to verify the relationships for a single value of the variable.
Student C.: I tried also with Number of Pebbles = 4.

Number of Sticks = (Number of Pebbles – 1) × 2 = (4 − 1) × 2 = 6
Number of Sticks = Number of Pebbles x 2 – 2 = 4 x 2 – 2 = 8 – 2 = 6

It was a necessary arithmetic check of the equivalence.

Teacher: Do you remember what we said about the distributive law?

The teacher did not want to block the flux of the collective discussion, but he suggested activities that may favor identification of equivalence between different representations of the same relationship and lead to understanding the power of syntactic transformation

Student J.: I found this rule …. The difference between the double of the number of sticks and the number of pebbles is equal to 2.

---

[1] Brioshi is a "virtual" Japanese student, who does not speak the Italian language but knows how to express himself with correct mathematical language. He likes to exchange e-mail messages with other class groups. Brioshi is a powerful educational mediator that helps to explain why a symbolic language has to be used and which rules it has to abide to (Malara & Navarra, 2001).

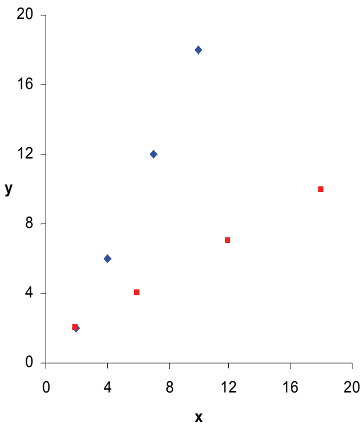
(Number of Sticks × 2) – Number of Pebbles = 2

We constructed an environment able to stimulate an autonomous elaboration of a new language in which rules may be gradually located, within the constraints of a didactic contract that tolerates initial moments of syntactical "promiscuity," named "algebraic babbling."

Teacher: Did somebody find other rules?
Student S. M.: I wrote all the rules for our friend Brioshi …
b = (s – 1) x 2
s = (b : 2) + 1
2 x s – b = 2

S. M. showed a spontaneous passage to the algebraic language as the symbols for variables in play were used.

Teacher: All right! Can somebody represent the relations in a graph?
[Student N. draws Cartesian plot on the blackboard]:



The red points represent the direct relation: *b = (s – 1) × 2*, whereas the blue ones stay for the inverse relation s = *(b : 2) + 1*
The two graphs are *symmetric*; there is a point common for both relationships. It represents a pile with 2 sticks and 2 pebbles.

This step was very difficult and some students had to draw two separate graphs for direct and inverse relationships. The Cartesian plot of the direct and inverse relationships points out the conflict between semantics and syntax. Conventionality of the reference system was noticed: the horizontal axis is the locus of the independent variable values, whereas the vertical axis is the locus of the dependent variable values. The swap involves only the role of the variables in relation to the studied situation and it generates geometrical symmetry.

Teacher: What about the Section B?
Student A.: Pile C is wrong because if I use the rule *b = (s – 1) x 2*, the number of sticks should be *b = (3 -1) x 2 = 2 x 2 = 4.* A stick is missing**.**
Student S.Z.: Pile A has no sticks. If I use the rule *b = (s – 1) x 2*, it will be *b = (1 – 1) x 2 = 0.*

The student chose the direct relationship and checked it by "particularization" for a specific value of the independent variable.

Student J.: I used the graph of the rule to solve the problem; I look for the point (1; …) …….
Teacher: What did you find?
Student J.: I found a point with coordinates (1; 0) that means a pile with a pebble and no sticks.

In this case, the student chose to solve the problem by using the graphic representation of the relationship and interpreted the couple of coordinates in terms of a pile of sticks and pebbles.

Student S.M.: Meg removed a stick from pile C, so now I have only 2 sticks. I must find the number of pebbles in the new pile. I use the inverse relation: *s = (b : 2) + 1* and I calculate: *s = (2 : 2) + 1 = 1 + 1 = 2* She should take a pebble off.

The student identified which variable was in play and she used the inverse relationship to calculate its value. The lessons pathway went on with open situations involving multiple variables, manipulation of algebraic sentences, generation of new relationships and problem solving activities.

The "Think it through ..." problem

In the "Think it through…" problem a set of geometric patterns, graphs and verbal or algebraic sentences were given. Students had to analyze them in order to identify the relationships they represent. Then they had to match them as different representations of the same relationship.

Students can encounter difficulties in 1) interpretation of the text that is synthetic and abstract for them; 2) seeking for the relationship; 3) interpretation of the graph; 4) coordination of the iconic, verbal, algebraic, and graphic representations.

Teacher: Have a look to the section "Rules and letters." In your opinion, what is the meaning of the letter "m"?

Student S.: Maybe "m" means "number of white tiles," because there is a geometric pattern where the number of white tiles is always equal to four. This regularity is represented in the graph (5). In the "Think it through…." problem the letter "n" means "number of black tiles," because in the geometric pattern (E) there is always a black tile. So the rule could be: $n = 1$ …..

Students identify the meaning of the letters by using and interpreting a different representation of the relationship. This behavior requires a complete knowledge of the conceptual nodes about linear functions and good linguistic competencies.

Teacher: Can you find the correct match for graph (3)?

The teacher decided to focus attention of the class group on this point because the graph did not represent a well-known linear functional relationship.

Student I.: It is very difficult. I am sure that it represents geometric pattern (B).

Student M. writes on the blackboard:

No. Black Tiles	No. White Tiles	Exploration
1	0	(1 – 1) x 1 = 0
2	2	(2 – 1) x 2 = 2
3	6	(3 – 1) x 3 = 6
4	12	(4 – 1) x 4 = 12
5	20	(5 – 1) x 5 = 20

She says: The rule could be: *No. White Tiles = No. Black Tiles × (No. of Black Tiles -1).*

The student used non-canonical representations of the numbers to find the relationship between the variables and she applied it to data collected from the graph.

Student J.: …. and, using the letters, the rule would be: $m = n \times (n - 1)$.

Student C.: In my opinion graph (1) represents these data ….. and it matches pattern (F).

No. Black Tiles	No. White Tiles	Exploration
1	2	1 + 1 = 2
2	3	2 + 1 = 3
3	4	3 + 1 = 4
4	5	…….
…..	….	……..
6	7	6 + 1 = 7

The rule is $m = n + 1$, but I'm not able to say this rule in spoken language.

Student M. Z.: I think it could be: No. White tiles – No. Black tiles = 1, because the regularity concerns the difference between the variables, that is always equal to one.

M.Z. recognized the implicit form of the rule, whereas C. observed only the additive relationship.

CONCLUSIONS

The activities were well accepted by the seventh-grade class, maybe because young students are less conditioned by errors and stereotypes. Thus, they can be led to a collective construction of new meanings through the practice of reflections and hypotheses and the "murky" language use. Some of the major difficulties that young students have to face with algebra are represented by the need to understand: 1) why a symbolic language has to be used; 2) which rules does the symbolic language have to abide to; and 3) the difference between solving and representing a problem.

The exploration phase is important because it contributes to questioning the wide-spread conception of mathematics as something complete that must be accepted passively. Arguing round a problem means becoming "knowledge producers" and finding out that there might exist several correct strategies – are expression of completely different mental processes. The generalization process is mediated by the non canonical representations of mathematical objects and the analogical thinking. Then the students learn to paraphrase different formal representations of the law.

These facts are in favor of relational teaching and learning of arithmetic and an early-algebra approach. The teacher's role in the classroom has changed because she is an expert leader who stimulates the discussion, corrects mistakes if necessary and checks verbalization of shared conclusions. She has to acquire that "local flexibility," which enables her to follow the flux of thoughts emerging from the class group, to grasp and develop the potentialities and to insert them into the working context.

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APPENDIX 1. Lesson Pathway

Activities	Objectives
1. “Masked party and pass-words.” 1A. “A new point of view …..” 2. “Graffiti” 2A “An interesting clue”	**Phase 1** Identification of variables and constants Direct and inverse relations Coordination of verbal, algebraic, graphic and arrow representations
 3. “Sticks and pebbles ” 4. “Christmas’ Trees” 5. “A secret code”	**Phase 2** Identification of variables and constants Direct and inverse complex relations Verbal, algebraic and graphic representations Identification of cases not fitting the rule
 6. “The magic cards” 7. “Port Said” 8. “T- shirts”	**Phase 3** Multiple variables and complex relationships Check of equivalence between different forms of the same relationship Generation of new relationships by algebraic manipulation Problem solving
 9. “Think it through…..”	**Final Test** Coordination between verbal, algebraic and graphic representations

TEACHING EXPERIMENT WITH ALGEBRAIC PROOFS

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ABSTRACT
In this article I present an experiment carried out in a Hungarian school. During this experiment a worksheet was used that was compiled by the Italian teacher, Annalisa Cusi. First, I describe the teaching of algebra and arithmetic in Hungarian secondary schools, then the class participating in the experiment, and finally the experiences of facing questions sometimes unusual for the students. One of the main aims of the PDTR project is getting thorough information about various methods of teaching mathematics, exchanging ideas and opinions on the subject. We believe the following experiment to be an adequate part of this.

KEY WORDS:
Algebra, proofs, modeling, generalization, problem solving

TEACHING ALGEBRA IN HUNGARY

The teaching of mathematics in Hungarian secondary schools is basically determined by two factors: the local curricula of the school based on the so-called frame curricula and the requirements of the final examinations. In 2005 the system as well as the requirements of the final exams changed. A two-level system was introduced: most students take the intermediate level final exam; however, those who wish to go on to higher education can also choose to take the advanced level. New topics were included in the requirements that have never been taught before (e.g. combinatorics, graphs, probability calculus, and statistics). Consequently, both the time devoted to algebra and its requirements have been reduced. It can still be stated that about 30% of the mathematics education is spent dealing with topics on algebra and arithmetic.

The greatest emphasis is laid on working with algebraic expressions, solving various linear and quadratic equations, inequalities and simultaneous equations. In grade 7 the systematic teaching of algebra begins. Then the students get gradually used to using letters and carrying out operations with expressions including letters in mathematics. They spend much time on special identities (e.g. commutativity, associativity, distributivity; the factorized form of $(a \pm b)^2$, $a^2 - b^2$, $(a \pm b)^3$, $a^3 \pm b^3$, identities of powers) and on their application (e.g. factorization methods: factorization-out, grouping). In the earlier system of final exams, the operations with rather complicated algebraic fractions had an important role, now the students have to apply the identities (simplification, multiplication, division, collection of terms) only to simple algebraic fractions. Compare the problems below – one from the earlier and one from the present system:

1. Carry out the operations below:

$$\frac{x+3}{x+1}-\frac{2x-1}{x-1}-\frac{x-3}{x^2-1} \quad (|x|\neq 1)$$

[From the final exam in 1996]

2. For the real numbers a and b it holds that $\frac{a^2-b^2}{a-b}=20$. What is the value of $a + b$?

[From the final exam in 2006]

While the requirements have been reduced for some topics, others have gained more and more importance. For instance, expressing a variable from formulae in physics or chemistry, or solving practical, life-like problems using equations. In addition to the algorithms of solving equations, it is more and more important to teach students how to form equations for practical word problems, how to find a mathematical model for a given problem. According to the plans of the Ministry of Education, in a few years' time about 50% of the problems at the intermediate level should be practical; this proportion is already over 30% today, too. Obviously, one of its aims is to improve the performance of Hungarian students in different international (e.g. PISA) tests.

I wish to mention another tool, with which the related authorities try to get the schools to give more and more emphasis to competence-based education. Every year since 2003 (except for 2005), national competence tests have been performed in grades 6, 8 and 10, where student abilities are measured in mathematics and reading with questions similar to the PISA test. Since 2008, the results of all the schools have been available for the public in the internet. I believe that due to this, in educational institutions in the following years more and more emphasis will be laid on practical word problems.

SOME THEORETICAL BACKGROUND

Kieran (1996) has identified three components of algebraic activity: (1) *generational activities*, which involve: generating expressions and equations – the objects of algebra; for example, equations representing quantitative problem situations expressions of generality in geometric patterns or numerical sequences, and expressions of the rules governing numerical relationships; (2) *transformational rule-based activities*, for example, factorizing, manipulating and simplifying algebraic expressions and solving equations. These activities are predominantly concerned with equivalence, form and preservation of essence; and (3) *global, meta-level activities*, for example: awareness of the mathematical structure, awareness of constraints of problem situations, justifying, proving and predicting, and problem-solving. These activities are not exclusive to algebra.

In our experiment the third component was mainly in the center – justifying, proving – of course some expressions of generality and transformational activities were necessary to find a conjecture.

PISA COMPETENCIES

The following PISA competencies were relevant in our experiment: mathematical thinking skill, mathematical argumentation skill, formal and technical skill, communication skill. David Tall speaks about three worlds of mathematics: *conceptual-embodied, proceptual-symbolic, formal-axiomatic.* Arithmetic and algebra belong to the second world.

An *elementary procept* is the amalgam of three components: a *process* which produces a mathematical object, and a *symbol* which is used to represent either process or object. A *procept* consists of a collection of elementary procepts, which have the same object. The expression $3 + 2x$ means two different things, the process of adding together 3 and 2 times x and the product of that process.

To be effective in algebra students need to see the algebraic expressions as procepts, that is, both calculation processes and concepts or objects that can be manipulated with. In our experiment, for example, $2n - 1$ represents an odd number, that is, a concept (mathematical object) but it can also be seen as a result of a process: natural number n is multiplied by two and from the product 1 is subtracted.

Our opinion is that the exercises in the experimental material are good examples for practice both meanings of symbolic expressions.

PRESENTATION OF THE EXAMINED GROUP

In the experiment, 14 of my students from grade 10 at Szent Imre Catholic Secondary Grammar School in Nyíregyháza took part. This is was the second year that I thought them 5 lessons a week, at the advanced level. The students who applied for the advanced level course from the three classes in grade 10 form my group; and the rest have 3 mathematics lessons a week. Since these students took an entrance test for the advanced level, they are more motivated than their fellow students.

The abilities of the group are most objectively described by the above mentioned competency test, as my students answered test questions from 2006. The national average score in secondary grammar schools was 556 points, 25% was 489 points, 75% was 620 points, 95% was 727 points. The average score of the 14 students was 648 points, the deviation 123 points. The results are indicated in the table below:

between 489-556 points (25-50%)	between 556-620 points (50-75%)	between 620-727 points (75-95%)	over 727 points (95-100%)
4 students Lenke, Beáta, Erzsébet, Feri	4 students Anita, Kati, Brigitta, Viktor	2 students Ákos, Zoltán	4 students Edina, Lehel, Péter, Mihály

These data also support my personal impression: on average, the group consists of students with better abilities than the national average in secondary grammar schools, but the deviation is high, the group is heterogeneous. A few students do not intend to participate in the advanced level course next year, due to their poor results. For Anita written tasks seem to be more difficult than the oral ones; she has slight symptoms of dyslexia. There are 5-6 students who regularly participate in mathematics competitions. One of them (Edina) is regularly among the best students of the county, and another student (Mihály) has outstanding results in national competitions besides the county ones (last year he ranked 2^{nd} in the national final). Since the abilities of Mihály are much better than those of the others, I asked him to solve the problems individually, independently from the others. As expected, he finished the test questions two lessons sooner that the others. He needed four lessons to do this, while the others six.

The students are motivated, their attitude towards learning is positive. They participated in the experiment with pleasure. In general, they needed 15-20 minutes to solve the test questions that I planned for a whole lesson; only the worksheet compiled for lesson 5 took up all the 45 minutes. During this lesson, the students worked in pairs, other lessons involved individual work. Individual work was always followed by a class discussion. I think the difficulty of the test questions well suited this group; it was a real, however, not insurmountable challenge for them. The solutions gave the students a feeling of success.

The initial test was taken on March 13th, and the solutions were discussed in the same lesson. The next five lessons were held between March 28th and April 1st.

The test problems by Annalisa Cusi are listed in the Appendix.

RESULTS OF THE INITIAL TEST

The question of the domain was noticed by one student (Mihály). He indicated the domain in parentheses where the wording of the problem made it obvious that the variables stand for integers, and he did not write anything where the variables could stand for any real number. For example: “The cube of a number - k^3; An even number - $2k\ (k \in \mathbb{Z})$” For the others it was natural that the variables represent integers. Therefore, during the discussion we agreed that thereafter the variables stand for integers even without specifying it.

The Hungarian textbooks do not use the concepts “consecutive” or “the antecedent of a number,” instead the expression “greater or smaller adjacent number” is common. This was not problematic; some applied these concepts as soon as in question 2 – where algebraic expressions had to be translated into verbal language.

In addition to Mihály, other three students (Edina, Péter, Lehel) solved task 1 without mistakes. One student (Anita) completely misunderstood it; she tried to rephrase the sentences in other words and did not use algebraic expressions. The others solved the problem with minor mistakes. The following mistakes were instructive: (a) Three students (Erzsébet, Kati and Ákos) did not know what to do with the tasks about even numbers. The discussion made them understand the necessary notations. This is indicated by the fact that in the following lessons they were able to correctly use similar expressions. First, Feri could not solve the problem properly. He translated “an even number” into the form $\frac{k}{2} \in \mathbb{Z}$, then he crossed it out and wrote $2k$. (b) Viktor and Lenke used the absolute value to represent the opposite of an even number: “$-(2|n|)$ and $|n \cdot 2|$”

Task 2 was the opposite of task 1; here propositions had to be translated into verbal language. Except for Anita, Erzsébet, Ákos and Lehel, everybody could solve the task by giving adequate verbalizations. Having given false answers to task 1, Anita could not solve this problem either; she translated only the first expression correctly (to the others she did not give answers): “$4k+1$ - we multiply a natural number by 4 and add 1 to it.” The content of the solutions of Erzsébet and Ákos were correct, still, they were unable to separate themselves form the algebraic expression completely and used its letters in the translation: “$4k+1$ - we multiply k by four and add 1 to it; $2 \cdot 4k$ - four times a number k multiplied by 2;…”

Lehel correctly translated the last two expressions; he created an unusual but correct statement from the first three ones:

$4k+1$ - giving 1 as a remainder after division by four; $2 \cdot 4k$ - the number is divisible by eight; $(k+2) \cdot 3$ - the number is divisible by three; k^2-1 - the antecedent of the square of a number; $(k+1)^2$ - the antecedent of the consecutive of a number

Task 3 turned out to be one of the easiest ones; everybody solved it correctly. Task 4 was more difficult to understand. Before the solution I was afraid that some students would only mention the letter *k* that the expressions have in common. This task received the largest number of different solutions. Four of them (Edina, Brigitta, Mihály and Péter) realized that each expression is divisible by 3. Lehel marked the number 3 in each expression, but failed to give a verbal explanation. Feri misinterpreted the task and tried to find identical expressions; and he did: $(h+k)\cdot 3$ and $3h+3k$. Looking for the feature in common, Ákos and Viktor found that k and $3k$ are present in each expression, but neither of them examined the substitution values. Erzsébet, Beáta and Anita did not give any answers.

Task 5 was again easy to solve, the students had already been familiar with this type of exercises, and everybody worked well.

ACTIVITIES OF TRANSLATION

The first part in the second and third worksheets includes tasks that asked students to translate statements into algebraic language. After the initial test, the majority of the activities of translation were solved correctly also by students who had misunderstood or not been able to solve the previous ones.

Task 1 in the second lesson brought a new challenge for them, the algebraic form of two- and three-digit numbers. In the previous two years they had seen word problems that required this knowledge, so it was surprising for me that only four students knew how to represent expressions with several digits according to unit values (e.g. two-digit numbers in the form $10a+b$ (Edina, Lenke, Brigitta and Mihály). Two of them (Péter and Ákos) represented two- and three-digit numbers as follows: $\overline{ab}$ and $\overline{abc}$. Feri also used a similar representation, but without a line above.

The solutions of task 2 were good, even brackets were used correctly (e.g. the square of a number was represented either by $(2k)^2$ or by $4k^2$). It was only Edina who was a bit forgetful here and interpreted the sentence "Subtract an odd number from its antecedent!" as follows: $2k+2-(2k+1)$.

As a result of revision, in the test questions for the third lesson most of the students already knew how to apply the correct representation of two- and three-digit numbers. At this point, it was the remainder and its treatment that was new. The students had already seen such tasks about a year before; therefore it was surprising to me how many of them could not remember these. The following mistakes were made: (a) Ákos and Brigitta represented a number divisible both by 4 and by 3 as $\frac{n}{3\cdot 4}$ or as $\frac{4n\cdot 3n}{4\cdot 3}$; a number such that the remainder of the division of it by 5 is 3 as $\frac{n}{5}+3$ or as $\frac{5n+3}{5}$; and the multiples of 5 and of 7 were represented correctly; (b) Ákos, Viktor and Anita represented a number, which is not divisible by 3 only as, $3n+1$.

ANALYSIS OF VARIABLE EXPRESSIONS

The second part in the second and third worksheets included tasks related to analyzing variable expressions. Task 1 in the second worksheet was solved correctly by everyone but Anita and Lenke. Task 2, however, was only solved by four students (Lenke, Erzsébet, Edina and Mihály). It can be noticed that, although the others solved the tasks about parity correctly, they could not give the features of the variable for the expressions divisible by 3 and by 6. One of the typical mistakes was that the expression $3b$ was not considered to be divisible by 3 for $b=0$ (Kati, Feri). The other mistake involved stating that the expression $3b$ is divisible by 6 if and only if b is divisible both by 2 and by 3 (Anita, Brigitta and Ákos).

The students could correctly determine the parity of the expressions in the tasks in the third worksheet, but some of them made mistakes when they examined divisibility by 4 and by 9 (Ákos, Brigitta, Anita, Viktor, Zoltán, Lenke).

ANALYSIS OF STATEMENTS

Task 3 in the worksheets for the second and third lessons was also concerned with the analysis of statements, and so was the worksheet for the fourth lesson.

Task 3 in the second worksheet involved the analysis of statements related to distributivity. Two students even put down the formal representation of the statements before deciding if they were true or false. Only two students could not solve the tasks correctly (Anita and Ákos), although Ákos gave the formal representation of the statement and still considered it true: $a3 \cdot b3 = (a \cdot b)3$.

For the statements to be analyzed in the third worksheet several students could correctly form the expressions and draw proper conclusions from them. Mistakes were made at the third statement (Viktor, Zoltán, Feri, Lenke). In the class discussion it turned out that the complex text was difficult for the students; e.g. they had to think over the parity of the consecutive even number of an even number.

The worksheet for the fourth lesson focused entirely on the analysis of statements. The class discussion revealed that the statements 1, 4 and 5 were strange to them. Both the condition and the consequence of statements 4 and 5 were obviously true, so, although it was clear for them that the statement is true, they were disturbed by the lack of a real conclusion. Only Zoltán found statement 5 to be false; the others held both the statement 4 and 5 true.

The students can see two interpretations of statement 1 ("The sum of three odd numbers is divisible by 3") in the Hungarian teaching of mathematics. When analyzing statements like statement 1, primary school textbooks expect three answers from the students: a) certainly true, b) might be true, c) impossible. Secondary school textbooks offer two alternatives for the evaluation of such statements: true or false, i.e. if there is one counter example from the domain, for which the statement does not hold, then it is considered false, otherwise true. This distinction is not clear for the students, because they find tasks of the first type in other subjects, too (e.g. in biology, geography). Due to this, some students (Lehel, Feri, Brigitta, Beáta, Ákos) argued that the statement is true since there are three odd numbers, the sum of which is divisible by three. Furthermore, they even proved it formally that there are infinitely many instances like this:
$(2k+1)+(2k+3)+(2k+5)=6k+9=3(2k+3)$.

The students rephrased statement 1 during the class discussion so that the solution should be unambiguous: "The sum of any three odd numbers is divisible by three" or "The sum of three odd numbers is always divisible by three."

In this worksheet the students were asked not only to find out which of the statements is true of false, but also to justify their answers. It was interesting to see that some students (Péter, Mihály, Viktor, Zoltán) found it obvious that it was sufficient to justify the falsity of a statement by a counter example, while true statements needed some general proof. Besides this, some of them (Anita, Lenke) used only an example for the true statements, too. This mistake can later also be observed when proofs are required.

Some students (Lenke, Lehel, Kati) applied the algebraic identities related to the sum and difference of two terms incorrectly during a correct justification. For instance the justification of the statement "If *n* is odd, the expression n^2+1 represents an odd number" was the following:

Lehel's proof:

True, because if *n* is odd, then $n=2k+1$. $(2k+1)^2=4k^2+1$ is odd. Odd+1=even.

Lenke's proof:

$n = 2m - 1$

$(2m-1)^2 + 1 = 4m^2 - 1 + 1 = 4m^2$

The students' proofs generally consisted of verbal explanation. Some (Feri, Lehel, Brigitta) intended to use algebraic expressions in addition to the verbal explanation for the justification, but the algebraic proof related only to special cases. The reason for the error was that consecutive numbers were considered in the justification of a statement. If it was about an odd number, then they considered three consecutive odd numbers and adequately represented it formally. If two even numbers were mentioned, then they examined two consecutive even numbers. Therefore, their answer was correct, but the justification was not:

Feri's justification:

If the sum of two numbers is odd, then also their product is odd. The statement is false, the sum of two numbers is odd if one of them is even and the other is odd, so: $2k \cdot (2k+1) = 4k^2 + 2k$ according to this the product is even, thus the statement is false.

Lehel's justification:

If the sum of two even numbers is even, then also their product is even. True, because the sum and product of two even numbers are always even. $2k \cdot (2k+2) = 4k^2 + 4 \Rightarrow$ even.

FORMULATION AND ANALYSIS OF CONJECTURE, FIRST PROOFS

Most time (a whole 45-minute lesson) was taken up by the fifth worksheet. The students worked in pairs only during this lesson. The pairs were formed on the basis of friendship, I asked them to choose partners with whom they work with pleasure. The following pairs were created: Viktor – Kati; Péter – Zoltán; Beáta – Lehel; Feri – Brigitta; Anita – Lenke; Edina – Erzsébet (Ákos was absent, Mihály worked individually). I decided on the pair-work because due to the heterogeneous group I had expected some students not to be able to solve these tasks alone, but they had a better chance of benefiting from the pair-work, and also the better students understand the solutions deeper while explaining them to their pairs.

The task was unusual for them; they saw such open problems only rarely. Moreover, they seemed to be uncertain whether to judge the proof adequate or incomplete. After solving task 1 correctly, Edina asked for my opinion about whether the statement is suitable and the proof sufficient.

In task 1, several different conjectures were formed for the difference of the square of two consecutive numbers: an odd number; the sum of the two consecutive numbers; the number 1 less than twice the original number. Everybody had a correct conjecture, most proofs were as follows: $n^2 - (n-1)^2 = n^2 - (n^2 - 2n + 1) = 2n - 1 = n + (n-1)$. It could be seen that following the discussion of the previous worksheet, the students did not make the typical mistake when squaring the sum $\left((a+b)^2 = a^2 + b^2\right)$. There were two instructive proofs:

Anita and Lenke's conjecture*:*

$n^2 - (n-1)^2 \qquad n \in N$

$n^2 - n^2 + 2n - 1$

$2n - 1$

This is an odd number.

Proof:

$n = 3$

$n^2 - (n-1)^2$

$3^2 - (3-1)^2$

$9 - 4 \underline{\underline{= 5}}$

$2n - 1 = 2 \cdot 3 - 1 = 6 - 1 \underline{\underline{= 5}}$

The two numbers are equal.

Kati and Viktor's proof (they had similar proofs for the other statements, too):

$n^2 - (n-1)^2 = n + (n-1)$

$n^2 - (n^2 - 2n + 1) = n + (n-1)$

$n^2 - n^2 + 2n - 1 = n + (n-1)$

$2n - 1 = 2n - 1$

During the class discussion the students soon realized that in the first case it was not correct to prove by offering examples. In the case of the second proof, however, they could not understand why I found it incomplete; all of them thought the solution to be correct. The students and I had the dialogue below:

Teacher: What do you think about the proof of Kati and Viktor?
Viktor: Perfect!
[laughing]
Teacher: Viktor says it is perfect.
Péter: They proved the line at the top.
[In the top line on the blackboard they could see:]
$n^2 - (n-1)^2 = n^2 - (n^2 - 2n + 1) = 2n - 1 = n + (n-1)$.)
Viktor: There must be something wrong if the teacher does not like it …
Péter: …also the logic is good.
Teacher: What is my problem with this?
Edina: It is a bit too complicated.
[In the following three minutes they give further ideas to correct the proof.]
Viktor: What kind of mistake are we looking for? Is something missing? Or did I miscalculate something?
Several students: It is correct! The calculation is correct!
Teacher: What do you want to prove? This: $n^2 - (n-1)^2 = n + (n-1)$. How can you start out? Also from this. This is dangerous. You start out from the statement you want to prove and get an obviously true statement.
Kati: If the order were the opposite, would it be correct?
Teacher: It would. The last line is clearly true. From this the last but one line follows, then the one above it and finally the statement to be proved. So we derive the statement to be proved from a true statement.
Ákos: And why is the first proof not correct?
Feri: Because we do not know if the statement is true, and we still assume it to be.
Viktor: What if we assumed that the difference of two square numbers is not equal to the sum of the two numbers and we proved this? This means that it would be an indirect proof.
Péter: Then this would have led to contradiction, so the original statement is true.
Teacher: This way it would be correct. I'll show you why it's not always good to start out from the statement to be proved. Let the statement be this: $-1 = 1$. The proof is as follows: Let's assume that $-1 = 1$.
Edina: …and we square this …
Teacher: Yes. Then we get $1 = 1$. This is an obviously true statement, so we are ready with this, $-1 = 1$ holds. Do you see what my problem is with the proof of Viktor and Kati? But it also happens that this is a good way to start. We have seen a proof this year, too, in which we started out from the statement to be proved. Does anyone remember it?
[The students were turning the pages in the textbook for a while to find the proof wanted.]
Edina: The relation between the arithmetic and the geometric mean?

Teacher: Yes, this is what I had in mind. (We quickly looked through the proof.) In the end we get an obviously true statement. Do you remember what else there was in the end? One sentence. What was it?

Edina, Feri and Péter interrupting one another: Taking equivalent steps we got an obviously true statement, hence the statement to be proved is also true.

Teacher: This is what I missed from the proof of Kati and Viktor. If we write this at the end of the proof then it will really be complete. At the end of the solution of a similar final exam task the following sentence could be found: "each step of the derivation is reversible, therefore the statement is also true.

Everybody realized that in task 2 the sum of three consecutive numbers was divisible by three. Many of them could state that this sum was the treble of the number in the middle. The algebraic justification was unproblematic. The generalization, however, turned out to be more difficult, because part a) already included the unknown *n*, so it was not clear for them what else they had to generalize. Some observed that the statement held for not only the sum of three terms but of any number of terms. A few of them wrote down the sum of only an odd number of consecutive numbers in a closed form.

Task 3 proved to be the most difficult one, three pairs (Lehel – Beáta, Viktor – Kati, Anita – Lenke) did not realize that $n^3 - n$ can be factorized into the product of three consecutive numbers, but they could factorize-out *n* $\left(n^3 - n = n\left(n^2 - 1\right)\right)$.

Task 4 did not cause any difficulty for the students. It is worth mentioning that they could phrase the conjectures after studying many specific cases.

In the task sheet for the sixth lesson the students had to analyze proofs. During this lesson they again worked individually. The students had not yet seen such tasks during their studies. After the task sheets and discussions of the previous lessons I expected this task sheet to be completely unproblematic. Still, it turned out that some students were still incapable of differentiating between the correct and the incorrect proofs. The proofs of Alice and Filippo were judged to be correct by Mihály, Péter, Erzsébet, Edina, Viktor, Zoltán, Ákos, Lehel, Pista, Kati, Brigitta, and incorrect by the other three students. Péter noted that Filippo's algebraic proof was nicer and clearer than Alice's proof. Lenke went further and said that in her opinion Alice's proof was bad because it was in verbal language.

Based on Anita's and Lenke's earlier proof I expected that it would be clear for everybody that Giovanni's proof using examples was incorrect. It was clear to Péter, Lenke, Beáta, Viktor, Mihály, Edina, Lehel, Pista and Kati, but not to the other five students.

Lenke's proof was also deceptive. Péter, Ákos, Viktor judged it correct, Beáta, Ákos, Zoltán did not answer.

RESULTS OF THE FINAL TEST AND WHAT THE EXPERIMENT TAUGHT ME

In the final test my students had to construct two statements and two proofs. Both tasks were solved correctly by Mihály, Edina, Zoltán, Beáta, Kati, Lenke and Brigitta. Task one was correctly proved by Viktor, Lehel, Ákos and Zoltán. Lehel and Viktor, due to miscalculation, did not accept the conjecture in the second task; they found a "counter-example." Both proofs of Anita and Erzsébet were false; however, Anita was able to translate the text into algebraic language in task one, but she made mistakes during the rearrangement

$$\frac{\left(\frac{(n-1)+n+(n+1)+(n+2)+(n+3)+(n+4)}{3}\right)-3}{2} = \frac{\left(\frac{6n+9}{3}\right)-3}{2} = \frac{2n+9-3}{2} = \frac{2n+6}{2} = n+6$$

The second proof of Péter was correct; in the first one he could not properly transcribe the text into the language of algebra.

The most important outcomes of the teaching experiment

I attach great importance to becoming familiar with the ideas on mathematics teaching in other countries. Otherwise education might become closed and certain emphases remain in the background. Our present experiment drew attention to the significance of algebraic proofs, to the judgment of logical statements, as well as to the fact that much more activities of "translation" are needed in both directions – translating a verbal situation into the language of algebra, the possible meanings of algebraic expressions.

I often believe that if we discuss something with students once, then it will be clear for everybody from then on. During the experiment it turned out that they tend to forget not only about concepts discussed several months before, but also information learned in the previous lesson. The essential, frequent elements of the curriculum (for example the formula for the square of a sum) must be repeated several times. According to my experience, other approaches (e.g. geometric representations) can also contribute to the consolidation and the effective application of algebraic concepts.

The analysis of proofs was very instructive. In the future, I am going to strive to present and analyze correct and incorrect solutions, because the knowledge of students is stable enough if they are able to recognize and correct mistakes. A good opportunity to do this could be discussing the solutions following a test, or stressing the applied proving strategies (e.g. arguing in inverse direction).

The most important benefit for me is the reflection to my own teaching. I think it is this field that I have managed to develop most in the PDTR so far.

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APPENDIX:

Initial test

1) Translate the following statements into algebraic language:

The double of a number
The square of a number
The triple of a number
The cube of a number
An even number
The double of an even number
The triple of an even number
The opposite of an even number
The product of two consecutive numbers
The consecutive of a number
Two consecutive even numbers
The square of the consecutive of a number
The consecutive of the square of a number
The antecedent of a number

2) Translate these propositions into verbal language:

$4k+1$ $2 \cdot 4$ $k(k+2) \cdot 3$ $k^2-1,$ $(k+1)^2$

3) Which of the following expressions are equivalent to $8k$?

$2 \cdot 4k$ $4 \cdot 2k$ $6+2k$ $(5+3)k$

4) What do the numbers represented by the following expressions have in common?

$3 \cdot 5k$ $(h+k) \cdot 3$ $3h+3k$ $9k$ $\frac{6k}{2}$

5) Complete the following statements:

If $u=v+3$ and $v=1$, then $u=$………
If $a+b=43$, then $a+b+2=$…………
If $e+f=8$, then $e+f+g=$……………
If $n-246=762$, then $n-247=$…………

First lesson

Discussion about the test

Second lesson

Activities of translation

1) Translate the following statements into algebraic language:

An odd number
The double of an odd number
The triple of an odd number
The square of an odd number
The cube of an odd number
The opposite of an odd number
The consecutive of an odd number
Two consecutive odd numbers
The antecedent of an even number
The antecedent of an odd number
A two-digit number
A three-digit number

The square of a two-digit number
The cube of a three-digit number
2) Translate the following procedures into algebraic language:
Add 3 to the square of an even number.
Subtract 1 to the cube of an odd number.
Subtract an odd number from its antecedent.
Add to the cube of a number the square of its antecedent.
3) Are the equalities expressed by the following statements true?
The sum of the triple of two numbers is equal to the triple of the sum of the same numbers.
The product of the triple of two numbers is equal to the triple of the product of the same numbers.

Analysis of variable expressions
1) *k* is a natural number:
k+3 is multiple of 3 if k ...
k+3 è even if k ...
2) *b* is a natural number:
3b is multiple of 3 if b …
3b is even if b …
3b is multiple of 6 if b …
3b is odd if b …

Third lesson
Activities of translation
1) Translate the following statements into algebraic language:
A number divisible by 7
A multiple of 5
A number divisible both by 4 and by 3
A number such that the reminder of the division of it by 5 is 3
A number which is not divisible by 2
A number which is not divisible by 3
2) Translate the following procedures into algebraic language:
Add 5 to the consecutive of the square of a multiple of 5
Subtract 2 to a three-digit number
Multiply the consecutive of a number divisible by 7 by a two-digit number
3) Are the equalities expressed by the following statements true?
The difference between the square of a number and the same number is equal to the product between the same number and its antecedent.
The sum between the cube of a number and the square of the same number is equal to the product between the same number and its consecutive.
If we add 3 to the double of the antecedent of a number, we find the consecutive even number of an even number.
4) *a* is a natural number:
a^2 is even
a^2 is odd
a^2 is a multiple of 4
a^2 is not divisible by 9

Fourth lesson
Analysis of statements
Find out which of the following statements are true and which are false, justifying your answers.
1) *The sum of three odd numbers is divisible by 3*
2) *The product of two even numbers is never divisible by 7*
3) *If n is odd, the expression n^2+1 represents an even number*
4) *If the sum of two even numbers is even, then also their product is even*
5) *If the product of two even numbers is even, then also their sum is even*
6) *If the sum of two numbers is odd, then also their product is odd*
*7****)*** *If the product of two numbers is odd, then their sum is even*

Fifth lesson
Formulation of conjectures and first proofs
1) Consider a natural number. Find out the difference between its square and the square of its antecedent. What kind of regularities can you observe? Try to prove what you state.
2) a) Consider a natural number. Find out the sum between this number and its two consecutive numbers. What kind of regularities can you observe? Try to prove what you state.
b) Do you think it is possible to generalize this conjecture?
3) Consider a natural number. Find out the difference between its cube and the same number. What kind of regularities can you observe? Try to prove what you state.
4) a) "Write down a two digit number. Write down the number that you get when you invert the digits. Write down the difference between the two numbers (the greater minus the lesser). Repeat this procedure with other two digit numbers. What kind of regularity can you observe? Try to prove what you state.
b) What happens if we consider the sum of the tow numbers instead of their difference? Try to prove what you state.

Sixth lesson
Analysis of proofs
Consider the following statement: *If a natural number n is even, then n^2+16 is divisible by 4.*
Analyze the following proofs of the statement given by four students. Are they correct? Why?

<u>Alice's Proof:</u>

If n is even, then n is divisible by 2. So it contains a factor 2. If we square n, this factor becomes 4.
So the square of n is divisible by 4. 16 is divisible by 4 too.
So we can collect 4 as a factor from n^2 and from 16, so their sum will be divisible by 4!

<u>Giovanni's Proof</u>:

If n is 2, then $n^2+16=4+16=20$.
If n is 4, then $n^2+16=16+16=32$.
If n is 6, then $n^2+16=36+16=52$.
If n is 8, then $n^2+16=64+16=80$.
So the statement is true!

Filippo's proof:

$$n=2k$$
$$\text{so } n^2=(2k)^2=4k^2$$
$$\text{so } n^2+16=4k^2+16=4(k^2+4)$$
So the statement is true!

Elena's proof:

$$n^2+16=4x$$
$$\text{so } n^2=4x-16$$
$$\text{so } n=\frac{4(x-4)}{n}$$
$$\text{so } n=4\cdot\frac{x-4}{n}$$
so n is divisible by 4!

Seventh lesson
Proofs

1) Consider a natural number. Add to this number its four consecutive numbers and its antecedent. Divide this sum by 3. Subtract 3. Divide the result by 2. What do you find?

2) Try to do the same starting from another number. What regularity can you observe? Try to prove what you assert.

3) Consider two three-digit numbers. For example, 153 and 285.

a) Calculate their product	153·285=43605
b) Subtract the second number from 1000	1000-285=715
c) Subtract 1 from the first number	153-1=152
d) Multiply these last two numbers	715·152=108680
e) Sum the results of the operation a) and d)	43605+108680=152285

The final result is the number which can be obtained putting the previous of the first number near to the second number. Does it happen starting from other three-digit numbers? Can you justify this regularity? (Suggestion: indicate the two initial numbers with x and y).

AND WHEN A STUDENT BRINGS A RULE TO THE CLASSROOM?

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ABSTRACT
The purpose of this paper is to illustrate how the rule of three was included in the students' repertoire in a sixth-grade classroom, during a teaching experiment. Teaching this rule was not on the teacher's agenda. However, it emerged from a strategy of a student when solving a task. This created a dilemma to the teacher: should the rule be integrated into the classroom repertoire or not?

KEYWORDS:
Rule of three, direct proportionality, classroom dynamics.

As teachers, we know all too well that when students come to school and to us, they already have a considerable background of tools including knowledge, strategies, and techniques. However, it is not always easy to deal with this for teachers, for the rest of students, and for those in charge of monitoring students in their learning. This episode came up during a teaching experiment related to the mathematics topic of direct proportionality, in a sixth-grade class (Costa, 2007).

The class was taught by the same teacher in the previous school year (grade 5). It had a classroom dynamics which valued students' involvement and included moments of sharing different strategies. Quite often, students (starting from proposed problems) were induced to devise their own strategies which were discussed afterwards and deepened in the larger group, with the teacher's support (one of the authors of this paper).

The unexpected happened!

In the scope of the teaching experiment, an initial test was applied to identify the strategies of students and their performance in solving problems involving proportionality prior to the formal teaching of this subject. In this test, several different problems were proposed. Some were missing value problems (questions in which three values are given and the fourth is asked for), others involved numerical comparisons (questions in which all the four values are given and students were asked to relate them) and yet in others students had to check the existence of direct proportionality.

In one of the missing value problems, Carla, one of the students in the class, used the cross product to solve it. Many questions arose: Where did she learn it? Why did she use it in this problem? Did she know how to use it meaningfully? This was more than a legitimate doubt, as her use of such strategy in this particular situation was misguided! (Figure 1);

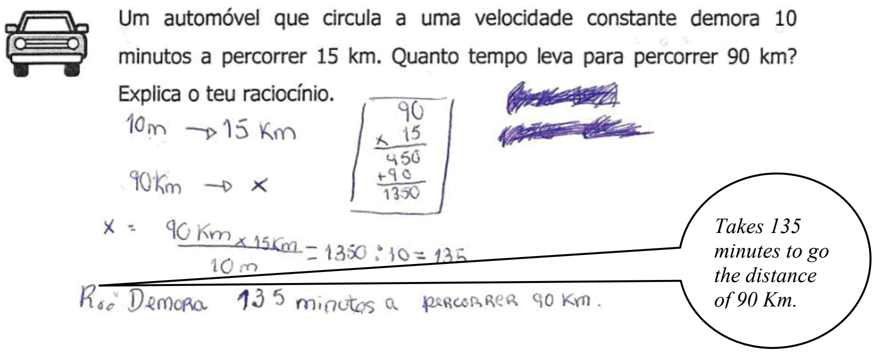

Figure 1. Carla's solution of the automobile question of the pre-test

The teacher was surprised, as Carla attended grade 6 for the first time, and she was not taught such procedure in grade 5. However, she decided not to question her at this time in order to maintain the classroom dynamics. Instead, the teacher suggested a task to the students, to be done individually or in pairs, and later the students' strategies would be presented and the difficulties explored as they came up.

On the following class, during the discussion of the strategies in the class conducted by the teacher's questioning, an arm rose in the air signaling that someone had a different strategy to present. Carla headed for the blackboard, as others did before, and presented the rule of three as her problem solving strategy.

The rule of three entered the classroom. What to do now? Toss it off? When a student is a carrier of some sort of information – extra-curricular knowledge or, at least, knowledge foreign to the classroom – should this be integrated into the working dynamics? Should its discussion be encouraged or, on the contrary, put aside because it does not matter at that particular moment, even though it might be interesting later, in that very same class or in the next week? And what to do, if contrary to the teacher's intent, the presented strategy ends up being adopted by most students?

With Carla at the blackboard, some students immediately showed surprise, as they considered this strategy to be very "fast" and "simple" to use. The teacher decided to strengthen that perception and completed the explanation given by Carla, exploring some properties and relating it to the fundamental property of the proportions. Hence, the teacher chose, as she had done with other strategies, not to put it aside but to integrate it in students' knowledge.

From then on...

This strategy became very appealing to several students, who begun using it successfully in many problems. From then on, this procedure was as valued by the teacher as the others formerly presented (using tables, consider unit rates, perform successive additions, etc.). Nevertheless, this strategy seemed to be the preferred one by students. This was verified in the final evaluation, when many students used this procedure, always successfully. In the final interview, the investigator (the other author of this paper) acknowledged that all the students correctly used the rule of three. The students indicated that they used it because it was faster.

Teacher: Did you understand this task?

Carla: Yes... The purpose is to dilute a glass and a half of juice syrup in 9 of water… Oh, I can use the rule of three! If… Is it a glass plus a half?

Teacher: Yes, a glass and a half is a glass plus a half.

Carla: A glass and a half... Ok… One and a half of syrup… 9 of water, 3 of syrup… *x*. 3 glasses of syrup times 9 glasses of water then divided by one and a half of syrup equals 27 divided by [uses pocket calculator] may I? Equals 18… 18 glasses of water.

Teacher: Do you think there's another way? If you didn't know the rule of three wouldn't you be able to solve it?

Carla: I would, but I don't use the others often...

Teacher: Ok, but write down the answer…

Carla: Oh, I could make a table... But it would be more complicated and it doesn't fit in here and I like it more this way… I have more… I don't know… I'm more certain!

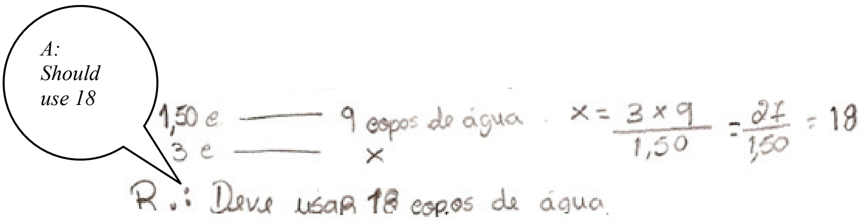

Figure 2. Use of the rule of three in a missing value problem during the interview

The students also showed a big tendency for its use when dealing with repeated calculations (Figure 3). The same thing happened, with several students, in the problem dealing with calculating the necessary quantities for each ingredient of a certain recipe:

6. Na hora do lanche, Ricardo lembrou-se de uma receita que a avó fazia sempre. Foi ver os ingredientes, para pedir à mãe para o fazer…

200 g de açúcar;
250 g de farinha;
4 ovos;
50 g de manteiga

A mãe quis fazer um bolo maior e resolveu usar os 6 ovos que tinha em casa. Indica as quantidades dos outros ingredientes que vai ter de usar, para que o bolo tenha o mesmo sabor.

At tea-time, Richard remembered a cake recipe that is grandmother used to do quite often. He sorted out the ingredients to ask his mother to do it…

200 g sugar;
250g flour;
4 eggs;
50g butter

His mother wanted to do a bigger cake, so she decided to use the 6 eggs she had in the fridge! So that the cake tastes the same, calculate the amounts of the other ingredients.

Marta: Well, if she was... We divide this [quantity of each ingredient] by 4 and we know how much it was for each egg. 50 g divided by 4 is 12.5; 250 divided by 4 is 62.5 g; 200 g divided by 4 was 50 grams… Well, now we could do… If she had six eggs then [pause, looks to initial data] … Oh! If we didn't do this we could use the rule of three.

Teacher: You're already using another strategy...

Marta: Well, then we would do 6 times 50 g divided by 4 equals 75 g. [she does the mental calculation and uses the pocket calculator to check the result] We already know that this one here is flower, then the same thing with the other ingredients… Which is equal to 12000 divided by 4 equals 300 g of sugar and then only the butter is missing… 4 eggs, 50 g if there were 6 eggs we should use 6 times 50 divided by 4. We should use 375 g of flower, 300 g of sugar and 75 g of butter.

Teacher: I'm only going to ask you one thing... Why did you abandon this strategy? Was it wrong?

Marta: No, it's only that it's easier to do it this way, if I did it the other way it would take longer.

Teacher : Other strategies?

Marta: The proportion, the table...

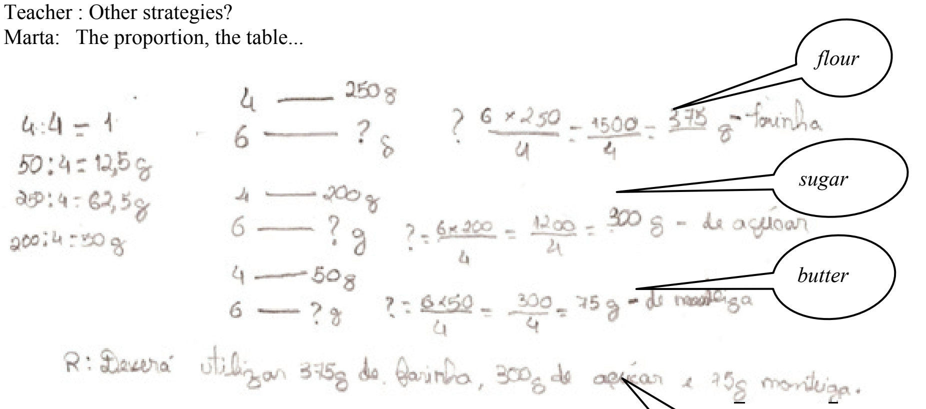

Figure 3. Rule of three in a repetitive calculation

It was exactly regarding a question asked on the interview (Figure 4) that Carla stated that she knew this rule even before she was in grade 5:

Carla: Because I attend the ATL ("Free Time Activities") … Well, like, I mean... It's a way I do it… Because I couldn't remember but back at the ATL they told me that there was this rule… I could remember but I knew how it was done and, like, I started to use it.

Teacher: Did you know it prior to the ATL?

Carla: Yes, I did! With the teacher of the... From the 1 -4 [grades]

Teacher :And did you talk about this rule?

Carla – Yes, we used it, but not quite often... Because they [the problems] weren't that hard!

Fig. 4. One of the moments during the interview when the student used the rule of three.

CONCLUDING REMARKS

The rule of three is, quite possibly, a strategy used throughout life by the most of those who learned it, for instance when trying to figure out how many litters of fuel a car has spent at the end of a trip, when choosing between two sales promotions of the sort of take *x* and pay for *y*, when calculating a currency exchange or converting measurement units. Even though it relies on a numerical relationship, this rule is possibly one of the most visible aspects, along with others, of mathematics in the context of reality. However, it is not always used with understanding, serving as a "magic formula" to find the answer to a problem in which, starting off from three values, we need to find the fourth. Hence, this is a powerful problem solving tool involving direct proportionality situations.

Teaching the rule of three was not on the teachers' agenda. However, the culture of the classroom open to students' communication about their strategies, allowed this tool to reach the entire class. The teacher had made into a habit for students to comment these subjects in open discussions. Usually, the teacher's starting point were the works that students presented to discuss the different problem solving strategies and the representations that students reached.

The strategy adopted by the teacher was to facilitate the integration of the rule into students' knowledge relating it to the fundamental property of proportions. As such, instead of putting aside this tool brought in by Carla, the option was to establish connections with previous knowledge, thus trying to create conditions for its understanding by all students.

On the other hand, it seems to us rather important to bring students to learn how to distinguish situations in which there is direct proportion from those in which it does not occur. It is necessary to consider both situations in a balanced way, as the rule of three is only applicable to situations involving direct proportion.

Hence, when confronted with both kinds of situations, students should be able to recognize which strategies they can use, including personal problem solving strategies. That is, the important thing here is that students, when put before a problem (involving proportion or not) can apply an efficient strategy, formal or informal, with understanding.

It is important that those who know the rule understand that they can not use it always when they are given three values and the fourth is asked for. Besides, if the rule is applicable, they have to know how to use it properly, relating the appropriate values. For that, a good understanding of the multiplication relationships and the fundamental property of proportions are key elements. To ignore them and to only apply the rule can only lead to a superficial knowledge, allowing students to solve routine exercises, but not problems slightly different from the usual ones.

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EQUIVALENCE OF ALGEBRAIC EXPRESSIONS

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ABSTRACT
This article focuses on the writing of the general term of sequences of numbers when an unexpected difficulty arises: the equivalence of two expressions with variables. This situation happened in a teaching experiment as part of an investigation about the development of students' algebraic thinking. Data were collected through written reports by students and audio recording of the lessons. It appears that students can identify regularities, make generalizations and express them in algebraic language but they show nevertheless difficulties in leaving the arithmetic conception in favor of the algebraic conception. The results show that it is necessary to do significant and ongoing work with situations involving the equivalence of algebraic expressions.

KEYWORDS:
Sequences, algebraic expressions, equivalence, algebraic thinking

Working in mathematics throughout school years leads the students to leave arithmetic and gain confidence in algebraic thinking. However, many students aged 15-16 have difficulty with understanding the meaning of letters and their usefulness. Following this thought, and the awareness that students feel more confident in working with numeric expressions and in using numbers to reason, I prepared the task "Number grid" for eighth-grade students. This is an exploration/investigation task based on a square grid with numbers from 0 to 99 (Figure 1) and has four questions.

This task is the first task of a teaching unit aimed at making students feel the need to work with letters, and then working with letters in operations with polynomials and solving first or higher degree equations with and without denominators or parenthesis (Pesquita, 2007). This experiment focused on writing of the general term of the sequence 0, 5, 10, 15, 20, 25 … Students were able to establish generalizations and get two algebraic expressions. However, they had difficulty in realizing the equivalence of those two expressions.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

Figure 1. Number grid from 0 to 99

THE FORMATION RULE

The lesson was divided in two parts – the students began to solve the task working in small groups, and later there was a discussion with the whole class. The first question asked students to identify sequences of numbers and indicate their formation rule. In the final discussion all the eleven sequences of numbers that students found were written on the board. Eight of these sequences were based on a similar formation rule that involved a number of multiple or successive additions, beginning at 0. Another sequence began at 9, always adding 9. Another was 7, 16, 25, 34, 43, 52, 61, 70 …, which was based on the diagonal of the grid that starts at 7, adding always 9. One sequence had a formation rule quite different, using addition and subtraction: 29, 28, 38, 37, 47, 46, 56, 55, 65, 64, 74, 73, 83, 82, 92, 91 … The formation rule was -1, +10, -1, +10 …

Then, I introduced the designations “order” and “term” used in sequences of numbers. After that, based on the sequence 0, 5, 10, 15, 20, 25 …, the following table was built with the help of students:

Order	Term
1	0
2	5
3	10
4	15

My concern was to make all the students understand the formation rule, and how the sequence was formed. After some terms of the sequence were written on the board, there was the following dialogue:

Teacher: What is the 10^{th} term of the sequence?
Margarida: 50.
Teacher: Really? Is it 50?
[Some students say yes and others say no]
Madalena: It is 45.
Teacher: 45… Why?
Madalena: Because in multiplication table the first is 0.
Jacinto: And then?
Marta: The multiplication table doesn’t start from 0.
Madalena: The first is 0.
Teacher: Yes… And???
[Some students say that it is 50 and others say is 45]
Teacher: There are two possibilities: some say it is 50 and some say it is 45. Let us see…
Beatriz: Teacher, on 10 it is 5×10, 50. On 10 it is 50. But as the first is 0, it must be 45.
Marta: That’s right, because of 0.
Teacher: I understand what she said. Did you all understand what she said? Ricardo, what do you say?
Ricardo: 45.
João: It is 45, Teacher.
Aniceto: I think it is 50…
Margarida: It is 45.
Aniceto: It is not.
Teacher: Calm down, calm…
…
Margarida: It is the multiplication table of 5, but it begins at 0. So it is always one less. Instead of being 50 it is 45.

In this dialogue we can see that students were divided, some said 45 and others 50. Some were fixed by the multiplication table of 5 so they said that the 10^{th} term must be 50. Other students realized that there was a variation on the multiplication table of 5, so they said "[this] multiplication table does not start at 0." Margarida's intervention is the one that ended this discussion when she drew attention to the fact that this sequence was based on the multiplication table of 5 with the previous multiple.

THE EQUIVALENCE OF ALGEBRAIC EXPRESSIONS

The following dialogue shows the reasoning that took place to find the general term of the sequence:

Teacher:	Now we must find the general term. What if instead of 10 it was *n*?
Alexandre:	What is that?
Teacher:	*n* in the following sense: if you want the term of order 1000, how would you get it?
Beatriz:	I had to do the computation!
Alexandre:	5×1000
Margarida:	$n \times 5 - 5$
Teacher:	Yes, it is right. Alexandre, it was $5 \times 1000 - 5$. And in this case 1000 was my *n*.
Rodrigo:	It is 995.
…	
Teacher:	The general term is how I can find the general, the generation…
Madalena:	$\times 5$, -5.
Teacher:	$\times 5$, -5?
Madalena:	We have a number; we multiply it by 5 and then subtract 5.
Teacher:	So how is it? The general term is always the term with *n*. So what is the general term of this?
Madalena:	$\times 5$, -5
Teacher:	It is $5 \times n - 5$.

Getting to the expression of the general term was relatively easy. The intervention of Madalena was essential, by referring to the others that she considered a number, then multiplied it by 5 and then subtracted 5. However, the expression of the general term had already been referred by Margarida.

After the students found the expression of the general term, I asked them if they could simplify the expression $5 \times n - 5$. They immediately gave me wrong answers such as $5n$, $4n$ and *n*. After the students realized that these responses were incorrect, replacing them with values, one of them noticed another way to indicate the general expression of the sequence, as the following episode shows:

Rodrigo:	It is possible to make it 5×0, 5×1
Teacher:	And then?
Rodrigo:	It is always following that.
Teacher:	What is the general term if I use it? I will take his idea. You were saying that the 1st term is 0, and it is 5×0. And then it is 5×1, 5×2, 5×3 … [Ricardo is saying that while I'm writing on the board] Now tell me, in 100th term, how much it is?
Ricardo:	It is 99.
Teacher:	And in order *n*?
Ricardo:	$5 \times n - 1$
Madalena:	$5 \times n - 5$
Teacher:	Calm down… We will combine this… We forget what we know so far… You are saying to me that it is $5 \times n - 1$?
Marta:	No. It is 5 times, open parenthesis, *n* subtract one.
Teacher:	Marta told me something else: $5 \times n - 1$.
Madalena:	I can not understand…
Teacher:	Let's go and see, if we can understand why, when I say this [$5 \times n - 5$] is the same as $5 \times n - 1$. Madalena: we did the first reasoning and it was not wrong.
Madalena:	No…
Teacher:	Here, it is not 1, it is 1-1, that is 0. Here, it is not 2, it is 2-1, that is 1. [I pointed to the board]
Madalena:	Yes.

Teacher: Here, it is not 3 is 3-1, here it is not 100, it is…
Margarida: It is 100-1.
Teacher: It is not *n*.
Margarida: It is $n-1$.
Madalena: I have already understood it.
Teacher: When we did this [first reasoning], we did it right. We did 5×1 and then took 5 from the multiplicative table, until we came to $5\times n$ and subtract 5. Now the question is how do you know that $5n-5$ is $5(n-1)$?
Margarida: It is right.
Teacher: Or, is anything wrong?
Margarida and Filipa: It is right.
Teacher: Why?
Margarida: Because it is right on both ways.
Madalena: Because *n* can not be *n,* when it is the previous to *n*.
Teacher: Madalena, this is *n*. $n-1$ means that is the one before *n*.
Madalena: Oh… I do not know how to explain…
Teacher: How do we ensure that this equivalent to this?
Beatriz: We can not, because we do not know what *n* is.
Margarida: We did it both ways, so it has to give the same…
Teacher: So can I guarantee that $5\times n-5$ is the same as $5\times(n-1)$?
Margarida: Yes.
Teacher: We have made the first reasoning and it gave [$5\times n-5$]. We did the second one and it gave this [$5\times(n-1)$] and we have not made any mistake… With the knowledge that you have until now, can you justify it?
Aniceto: I… If *n* is 1000 it gives 5×999, that is 4995 and on the other is $5n$ that gives 5000, less 5 it is 4995.
Teacher: For 1000 it is right.
Filipa: For 2, it is so, too.
Margarida: It is OK for all the numbers.
Teacher: It is, for all the numbers…

Ricardo's reasoning to find the general expression of the sequence was correct. However, for Magdalena it was difficult to follow him since she was stuck at the expression $5\times n-5$. After overcoming this situation, Marta indicated that "It is 5 times, open parenthesis, *n* subtract one." Then, I asked students how they could ensure equivalence of the expressions $5\times n-5$ and $5\times(n-1)$. Beatriz said that "We can not because we do not know what is *n*", suggesting that she needed to concretize *n*. Then, I insisted on equivalence of expressions. Aniceto said that he could guarantee it because it was correct for 1000. Filipa added that it was also correct for 2. It appears that students felt the need to give concrete meaning to the letter, although, they were able to understand it as representing a general number.

My question: "With the knowledge that you have until now, can you justify it?" was intended to check, if the students were able to concentrate on the reasoning involving a generalized number and could verify the equivalence of those two expressions. However no one was able to make it! In this case, students, despite recognizing the meaning of letters in this situation, remained attached to the arithmetic strategies and habits. They showed difficulties in manipulating symbols and in interpreting and using mathematical symbols, important aspects of algebraic thinking (Arcavi, 1994).

CONCLUSION

This task took place with students in grade 8. I taught them in the previous year. In grade 7 they had contact with letters, including simplifying algebraic expressions and solving equations. This task aimed to make them work with letters once more and promote their understanding of their use. I wanted students to be able to abandon the reasoning based on numbers (arithmetic conception) and gain confidence in working with letters (algebraic conception).

This work was intended to indicate the general term of the sequence. I wanted students to understand the formation rule and to be able to indicate a term of a distant order. And I wanted them to get progressively familiar with the algebraic symbolism. The general expression only emerged gradually. They came across two expressions $5\times n-5$ and $5\times(n-1)$. With the formulation of questions and several interactions between the students and me, they were capable of communicating mathematically their ideas and to build a more formal reasoning. This episode shows that it is possible to generate opportunities for generalizing and systematically expressing that generality as suggested by Kaput and Blanton (2005).

Despite that fact that the students previously worked several times with the distributive property, they were blocked because the expression included a letter. The change from arithmetic to algebraic thinking modifies many things and makes students no longer use what they had learnt before. Thus, it seems to me that it is essential that students in grades 7-9 acquire skills in situations involving the use of letters and that the work is done out with understanding. The existence of two expressions for the same sequence is an opportunity for learning about the equivalence of expressions. Students were able to recognize the use of letters to represent concrete numbers, but their answers were still stuck in arithmetic. They substituted concrete values and checked, if the result was the same. This shows that it is necessary to give students a gradual and frequent contact with reasoning that appeals to the abstract nature of mathematics in a way that they will develop their ability to think algebraically.

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TEACHING LINEAR OPTIMIZATION AT SECONDARY LEVEL

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ABSTRACT
Linear optimization is not included in the Hungarian mathematics in the secondary school; however, the knowledge needed for it is required there. Partly as a result of PISA tests, the applications of mathematics have been emphasized in Hungary, too. I carried out an attempt of teaching linear optimization at a secondary school in Karcag, Hungary. In the present document I summarize my experience.

KEY WORDS:
Modeling, optimization, problem solving

INTRODUCTION

In Hungary, the teaching of mathematics traditionally centers around mathematics as a science. Having attended several mathematics classes, Laurinda Brown (Bristol University) gave her opinion about our way of teaching mathematics as follows: “You, in Hungary, teach mathematics, we, in England, teach children!” As a consequence of the rather poor Hungarian results in PISA tests, the applications of mathematics and mathematical modeling have gradually been highlighted in the teaching of mathematics in Hungary, as well. Participating in the PDTR project seminars I also started to consider what I could change in my way of teaching. My students mostly revolted at solving mechanic algebraic problems. “Why should we solve such problems? What is the point in so much practice?” This was when I decided to try to discuss with my students the topic of linear optimization, which needs a lot of algebraic knowledge.

KNOWLEDGE AND COMPETENCE NEEDED FOR LINEAR OPTIMIZATION

The new National Curriculum emphasizes the application of acquired mathematical concepts in practice and in other subjects. Besides interpreting and analyzing word problems, representative diagrams and modeling are also held essential. In the local curriculum of our school the requirements of each topic are stated for each school year. The competencies listed for the requirements of the final exam include the students’ ability to see the mathematical problem involved in a given text, to create a model, to carry out calculations based on the model, and to interpret the results obtained.

The competencies related to the topic of functions include functional representation of simple relations of everyday life. Students have to be able to draw conclusions from the analysis of these functions in connection with the expected course of real events. Among the competencies concerning geometry we can find planar and spatial orientation, and the ability to prepare an appropriate diagram, to represent the possible set in the coordinate system. According to the specific final exam requirements, solving simple linear and quadratic inequalities, simple simultaneous inequalities with one unknown are necessary at the intermediate level. At the advanced level solving of more complex problems is required. Graphing linear functions, as well as normalized direction vectors are essential requirements even at the intermediate level.

Linear optimization belongs to the periphery of mathematics teaching at the secondary level. Earlier, curricula did not require this topic, hence most teachers ignored it due to lack of time. They did not even include it as an additional material, although all textbooks include one or two problems related to it.

THEORETICAL BASES

The necessity of teaching this topic is implied by the fact that *all the PISA-related competencies are developed through it*: thinking and reasoning, mathematical argumentation, modeling, problem posing and solving, representation, communication, using symbolic, formal and technical languages and operations. Most application tasks are complex problems.

In my course, the Pólya phases of solving a problem were applied: (i) interpretation and analysis of a problem; (ii) preparation of a solving plan; (iii) execution of the solving plan; and (iv) looking back, reflection.

In the solving process of problems related to linear optimization metacognitive components also were in the foreground: (a) planning; (b) overview of the solving process, checking; (c) looking back, reflection.

I considered the *realistic teaching of mathematics* as a model, from which a lot can be adopted and learnt. The aims of are: (1) social goal: to become an intelligent citizen (mathematical literacy); (2) economical goal: to prepare for the workplace and for future education; (3) scientific goal: to understand mathematics as a discipline

Finally, I was keeping in mind the following key findings of the PDTR seminars and lectures. (1) Students come to the classroom with preconceptions about how the world works. If their initial understanding is not engaged, they may fail to grasp the new concepts and information that are taught, or they may learn them for purposes of a test, but revert to their preconceptions outside the classroom; (2) to develop competence in an area of inquiry, students must have a deep foundation of knowledge, understand facts and ideas in the context of conceptual framework, and organize knowledge in such a way that it facilitate retrieval and application; and (3) a metacognitive approach to instruction can help students to learn to take control of their own learning by defining learning goals and monitoring their progress in achieving them.

I drew from it some practical principles: (1) teachers must identify and eliminate the preexisting understandings that their students bring with them; (2) teachers must teach the subject matter in depth, providing many examples, in which the same concept is involved, and a firm foundation of factual knowledge; (3) the teaching of metacognitive skills should be integrated into the curriculum in a variety of subject areas. (National Academy, 2003)

THE PROJECT

Research questions

(i) Is it realistic, the idea to teach linear optimization to 15-year-old students? (ii) How many lessons are needed at least for the safe acquisition of the material? (iii) Are the students able to effectively apply the concepts of algebra and functions in problems of real life situations? (iv) What conditions must be satisfied for the teaching of linear optimization to be effective in secondary schools?

Hypothesis

Students can effectively acquire the material when the necessary concepts and procedures are carefully formed, practical problems integrated, and the computer application opportunities made use of.

The project was carried out in class 10B at Gábor Áron Secondary Grammar School, Technical School of Medicine and Youth Hostel in Karcag during the autumn of 2005. Classes were held in one or two afternoons a week. The work began with the participation of 13 students, from among which only 10 completed the course with a final test due to illness or other timing problems. The students were accepted to this grammar school class without an entrance test. They learned English at an advanced level; however, the whole class with 36 members had mathematics classes 3 times a week. The 13 students were all volunteers, having very or quite good grades, one of them had medium abilities. They had successful results at county-level competitions, but there were no outstanding students among them.

The scheme of the project

Lesson 1: Pre-test
Lessons 2-3: Preparation phase
Lessons 4-9: Development phase
Lesson 10: Final test

Applied research methods

A technician video recorded almost every lesson. The knowledge of the students was tested with a pre-test at the beginning and with a final test at the end of the course. In addition to the video recording, consultations with the students as well as an analysis of students' documents also contributed to getting as precise an insight into their learning process as possible.

The detailed way of solving a problem

Below I demonstrate the steps of solving a given problem. After refreshing the necessary knowledge, the problem was approached; then the problem was solved together by the class members, with a lot of teacher's aid and explanation. It was the uniqueness of the topic that made me choose joint work this time; these students are used to working together anyway. I applied the teaching style dominating the Hungarian teaching of mathematics: first the new topic is introduced and explained by the teacher, then students solve problems based on the sample presented.

The solution phases of linear optimization problems: (1) interpretation, collection of data; (2) translation into the language of mathematics; (3) modeling; (4) acquisition of the solution technique; (5) finding the object function; (6) finding the possible range, (7) parallel translation of the object function; (8) construction of level lines; (9) finding the optimum point; and (10) calculation of the optimum point(s) and value(s).

Problem

In a pottery, jars are manufactured in two ways: A and B. Whichever way the product is made, it spends a certain time in the molding, burning and painting machines. The time (in hours) needed for the production is given in the following table:

	A	B
Moulding	0.08	0.05
Burning	0.1	0.2
Painting	0.1	0.12

There are 2 molding, 4 burning and 3 painting workers. Everybody is allowed to work at most 40 hours a week. The jar produced by method A costs 4.50€ each, one made by method B brings 7€ profit for the pottery.
How many pieces should be produced of types A and B if they want to gain the maximal profit? How much is this profit?

This problem introduced the new topic. Since it turned out to be difficult for the students, I asked them to read the problem out several times, which enhanced understanding and finding the essentialities. I tried to get the students to collect the important data and relations from the text. The questions were asked: What did we read about? How many types of products are produced? How many production phases do the products go through? Explain what the value 0.2 stand for in the table. What does a 40-hour-long working week mean? How many hours does a worker work a day in such a week? How many days does a worker work a week?

Solution

An interpretation of the problem is the first phase of teaching linear optimization

Teacher: Do we know how many pieces should be produced of types A and B?
Csilla: We do not. Let us denote the number of products A by x, that of products B by y.
Teacher: What profit does 1 piece of product A bring?
Students [immediate answer]: 4.5 €.
Teacher: What profit does 1 piece of product B bring?
Péter: 7 Euro.
Teacher: What is the profit of x pieces of product A and y pieces of product B? What profit can be reached in total? What determines profit?
Statement: $4.5x + 7y$.
Teacher: Let us name this object function, and this function has two variables! What should this object function be like?
Erzsi: It should be as big as possible.
Anna: It should be maximal, very big.

We agreed to call this maximization. We stated the aim that the values of *x* and *y* should be chosen to give a maximal value for *K(x; y)*, and this should be denoted by: *K(x; y) = 4.5x + 7y→max*.

Teacher: How many working hours are available at each place?
Dénes: Since there are two molding workers, this means 80 working hours a week; the four burning workers can work 160 working hours; while the three painting workers are allowed to work 120 working hours a week.

The simultaneous inequalities were set up after answering the questions below:

Teacher: (1) If 1 piece of product A is made in 0.08 hours in the molding machine, then how much time do *x* pieces of this need? (2) If 1 piece of product B is made in 0.05 hours in the same place, then how much time do *y* pieces of this need? How much time is available for molding in total?

Analogously, the questions were discussed in relation to each production phase. Thus the following simultaneous inequalities were put down:

$0.08x + 0.05y \le 80$,

$0.1x + 0.2y \le 160$,

$0.1x + 0.12y \le 120$,

$x \ge 0$, $y \ge 0$.

Teacher: Let us rearrange the simultaneous inequalities to the standard form and graph the set of possible solutions!

At this point I was still rather invasive with much teacher aid and guidance.

The work of Anett, Sára, Annamari:

$y \le -1.6 \cdot x + 1600$

$y \le -0.5 \cdot x + 800$

$y \le \frac{-10 \cdot x}{12} + 1000$

$x \ge 0$

$y \ge 0$

The rearrangement of the first three inequalities above in *y* was not perfect, it was really difficult to rearrange an inequality with fractions, but the new form made the graphing of linear functions definitely easier. Such sum-division type operations must be practiced a lot as to make sure that students divide not only one of the terms, and they should be able to decide when to represent a fraction in decimal and when in an ordinary simplified form.

Teacher: Conditions $x \ge 0$ and $y \ge 0$ are called non-negativity conditions, which means that the values of *x* and *y* can only be positive integers.

Renáta: The reason for this is that no negative number of pieces, or half pieces can be produced either of type *A* or of type *B*.

Teacher: How could the points be indicated in the plane for which all conditions hold?

Anna stated that the common part of the inequalities should be indicated. When graphing the inequalities we first examined what value the function takes for $x = 0$, and where the zero of the function is.

Teacher: Let us find points in the possible range and determine the profit there! At point (200; 300) the profit is $4.5 \cdot 200 + 7 \cdot 300 = 3000$; at point (600; 400) it is $4.5 \cdot 600 + 7 \cdot 400 = 5500$ Euro.

Teacher: How could the maximal profit be found? Can we choose the coordinates of the point at random?

My question was provoking, and I even suggested the point (1000; 600). They stated that it was impossible, because it was not included in the possible range.

Teacher: What interval could the maximal profit belong to?

The students realized that it was only the points in the possible range that can be solutions. I suggested trying to graph the object function, too.

Let us make it theoretically equal to 0. We get the line with the equation

$y = -\frac{9}{14}x$,

which is a black linear function passing through the origin; it is called function of direct proportionality. After this we examined specific points, for each of which we calculated the profit. Such a point was e.g. point (0 ; 0). In this case *x=0* and *y=0* number of pieces are produced, the profit is 0 Euro. We can step further along the axis *x*. The point (400; 0) was selected and the profit $4.5 \cdot 400 + 7 \cdot 0 = 1800$ Euro was determined.

In this case the parallel translated object function passing through (400;0) with the slope $-\frac{9}{14}$ intersects axis *y* at point (0; 257.12). The intersection of a given function with axis *y* was already practiced before, and it caused no difficulties. The fact was surprising for us that for point (0; 257.12) the profit was also approximately 1800. We found it interesting to calculate the substitution value also for *x=100* and to determine the profit there. A new point was chosen on axis *x*: (700; 0). Lajos: "In this case the translated object function passing through point (700; 0) with the slope $-\frac{9}{14}$ intersects axis y at point (0; 450).

As a result of joint calculation we found that point (300; 257.14) also lay on this line. Here the profit was bigger than in the earlier cases, it was 3,150 Euro.
Similarly, point (1400 ; 0) was also examined.

Éva: The translated object function passing through point (1400; 0) with the slope $-\frac{9}{14}$ intersects axis y at point (0; 900). Point (600; 500) is also a point of this line.

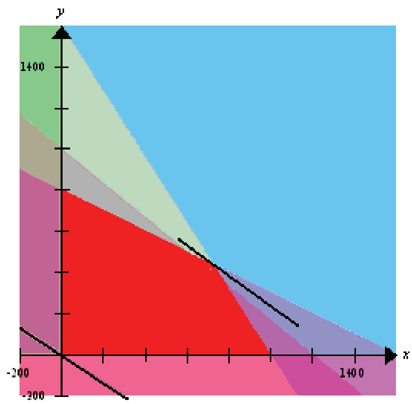
Significant observations were made at this point, namely that the same profit belonged to points that lay on a given line, and also that the lines with different profits would be parallel. The students guessed which two lines intersected at the point giving maximum value, to which the parallel line of the object function should be translated. This point was the ideal solution. This point belonged to the possible range, and the line of the object function did not have a value above or under it that would belong to the possible range.

$$y = 800 - 0.5x$$
$$y = 1000 - \frac{5x}{6}$$

Kata: These two lines are:

$$1000 - \frac{5x}{6} = 800 - 0.5x\,.$$

Solving this linear equation with one unknown gave us the two coordinates of the point, i.e. $x = 600,\ y = 500$, so we got the profit by substitution: $4.5 \cdot 600 + 7 \cdot 500 = 6200$ Euro.



The optimal solution was: 600 pieces should be produced of type A, and 500 pieces of type B, the optimal profit was 6,200 €.

CONCLUSIONS

It was the interpretation of the text, its translation into the language of mathematics and creating a mathematical model that proved to be the most difficult task. To decide what to denote by a new variable even turned out to be problematic. It was also difficult to create the simultaneous inequalities. One of the crucial inventions was the concept of the *object function*. It was difficult to realize that using the coordinates of the points, concrete values can be calculated for the object function; and also to see that the same profit belongs to points that lie on a given line, and that lines of different profits are parallel. Determining and finding the object function, as well as creating its parallel translation were also problematic. It is often challenging to select *the point of the possible range,* through which the object function passes, that result in the maximum profit. On the basis of interviews I drew the conclusion that finding the object function is much easier for them than that of the simultaneous inequalities. Even the good students forgot about the conditions on non-negativity and those relating to positive integers.

The steps of the above solution were discussed at the PDTR seminars in detail. Several fellow teachers, although accepting the strong teacher guidance in the introduction phase, suggested encouraging students to do more individual work and to elicit more student ideas. It was a strange feeling later to see and hear how much I was in the foreground, how I wanted to make students accept my ideas and argumentation. In what followed, the joint class work was gradually replaced by group-work and then by individual work.

In the development phase 5 problems were solved jointly, demanding more and more from students each time. During lesson 8 a further problem was to be solved in groups of 3. At the end of the lesson each group member had to comment on some part of the problem. During lesson 9, 3 different problems were solved by the same groups as in the previous lesson, with only one student explaining the solution at the end. Lesson 9 included problems of different difficulty, so the group-work was graded.

I repeated the project also with another group of students. This time I provided more opportunities for students to present their ideas and argumentation; I tried to stay in the background. This was not always easy, since I have already got used to being the leading and dominating figure in the classroom. Nevertheless, I started changing my style.

The application of computers and interactive boards could further improve understanding, and these facilities are available at our school, as well as some software applicable to represent simultaneous equations and inequalities, and even surfaces. I wish to make use of these facilities in the future.

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THE USE OF CONCRETE AND VISUAL REPRESENTATION IN TEACHING EARLY ALGEBRA IN ORDER TO AVOID BASIC DIFFICULTIES

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ABSTRACT
In this paper I present a method of teaching early algebra by using concrete visual representations. I explain one example dealing with the rule of removing parentheses followed by a minus sign. Testing of this method took place in March 2004, during 24 lessons in grade 7 (12-year-old students) in the Serbian minority school in Budapest in Hungary. I was inspired to seek for alternative ways of teaching because of the special needs of my students – their lack of ability to give verbal explanations of their way of thinking and the reasons for acting in a certain way. I was trying to achieve the development of students' algebraic thinking, which should enable them to understand and use algebra easily. Furthermore, this ability could be used later to cope better with the problems of the contemporary world. Students' positive results encourage me and indicate that using real objects has beneficial effect on the learning process of algebra for this particular age group.

KEY WORDS:
Representations, early algebra, procedural vs. conceptual knowledge

INTRODUCTION

In my teaching of mathematics in a bilingual primary and secondary school I face such a problem that my students, regardless of age, make an amazing number of errors when they apply the knowledge of elementary algebra. I collected and analyzed the mistakes made in and comments made about the tasks they usually have difficulties doing. A possible explanation for these elementary mistakes can be that they consider the algebraic symbols and formulas to be "letters" rather than expressions of general relationships between numbers or quantities. For them, algebra is a frightening world of symbols, which they manipulate using senseless rules they have learned by heart. I suspect that the roots of this misleading way of thinking could originate from the period of their education when they encountered algebra for the first time. Unfortunately, there is a dramatic leap in the textbooks from arithmetic to algebra (without gradual transfer).

Trying to help my students to overcome these difficulties I asked myself the following question: Can teaching method of early algebra be developed so that students make links between mathematical formalism and familiar situations? This would enable them to build up their algebraic notions by using their abilities, experience and the knowledge already acquired.

I decided to extend the period of teaching using concrete manipulative tools and to combine such use with more abstract methods. Besides having a beneficial effect on algebra learning this method enriches students' conception and perspective of mathematics. It will show them that many different every-day activities can be expressed by the same algebraic expression. The second reason for my decision is the special position of my students: I teach in a bilingual school in Budapest (Hungary), which implements the official Hungarian curriculum with an additional aim: to teach fluent Serbian language on a high level. A lot of our students do not have the appropriate command of Serbian language or do not speak it at all so they attend this school in order to learn it. On the other hand, students coming from Serbia do not speak Hungarian. Consequently, in the same class we can find students with very different level of knowledge of these two languages, so we have to use both languages during lessons.

In order to have better communication and understanding with my students I use gestures and body language extensively. I spontaneously point to different objects and use things to clarify issues manually and verbally. The role of language is complex: explaining activities, thoughts and ideas on the appropriate level of representation, but also to bridge over the different levels of representations. Using manipulative tools serves the purpose of continuation of tradition already existing on lower levels of elementary school and students are used to approach this kind of activity seriously. By manipulating objects one can mobilize and use the actual vocabulary already accepted, familiar and connected with every-day life situations.

For this reason I formulated the following research question: Can the use of concrete and visual representations help students build up the basic knowledge of algebra in the complexity of bilingual schools as well as develop specific way of thinking, which is typical in algebra?

BASIC NOTIONS AND THEORETICAL BACKGROUND

Concept development and representations

According to Bruner (1966) there are three different levels of representations of any problem (within a domain of knowledge): (1) *enactive level* means representations by a set of actions appropriate for achieving a certain result; (2) *iconic level* means representations by a set of summary images or graphics that stand for a concept without defining it fully; and (3) *symbolic level* means representations by a set of symbolic on logical propositions drawn from a symbolic system that is governed by rules or laws for forming and transforming propositions.

Algebra teaching usually happens on the symbolic level. Our aim was to use the enactive and visual level longer than it is used traditionally in Hungary. Skemp (1978) distinguished instrumental and relational understanding: (1) instrumental understanding means that a student knows how *to do thing*; and (2) relational understanding *relates knowing how to do things with knowing why to do them*.

Students with only instrumental kind of knowledge have big difficulties to find alternative ways of doing things and can be easily disturbed by outside factors. Student with relational understanding have a possibility to find new solutions independently by creating always new and new conceptual structures (*schemas*). The way of building new structures is self-rewarding and students experience satisfaction because they can develop more and more complex schemas, so their self-confidence builds up. Besides already mentioned, students find also self-motivation in their studies and gaining new knowledge.

Instead of a formal definition, we use, according to Lakoff (1987), the method of *conceptual embodiment*: "the idea that the property of certain categories is a consequence of the nature of human biological capacities and of the experience of functioning in a physical and social environment." The contexts for conceptual embodiment of an algebraic expression that we used in our experiment were "candy stories" and "paper strips."

Gray and Tall (1994) introduced the notion of *procept* as an idea generated by looking at a symbol, such as 3+2, both as a process (of addition) and a concept (of sum). It was extended to include different symbols and different processes that give rise to the same mental object in the mind of a particular individual. Thus 1+4, 2+3, 3+2, 4+1 can all represent the same procept involved in composing arithmetic processes that give 5. An *elementary procept* is the amalgam of a *process*, (which produces) a mathematical *object* and a *symbol,* which is used to represent either process or object. A *procept* consists of a collection of elementary precepts, which have the same object.

We consider a procept not to be just a simple compression of different aspects, but also a flexible way to see and interpret an algebraic expression in two directions: command for concrete processing and symbolic recognition of the expression (as a collection of representations). All of these symbols are considered to represent by a student the same object, though obtained through different processes. But it can be decomposed and recomposed in a flexible manner.

In order to describe the total cognitive structure that is associated with a concept, which includes all the mental pictures and associated properties and processes, Tall and Vinner (1981) introduced the term *concept image*. The concept image is something non-verbal associated in our mind with the concept name.

Some algebraic aspects of thinking

Mathematics is an abstract science and algebra is the most abstract part of it. In the didactical literature the *problem solving, functional, generalization, language, and historical approach* to school algebra are described by Bednarz et al. (1996), Usiskin (1988), and Drouhard (2001). In educational practice we consider the same algebraic problem from different approaches.

According to Mason and Sutherland (2002) we can find *algebraic aspects of thinking* while detecting and expressing a structure: "Imbuing every lesson with algebraic thinking, with expressing generality and particularizing generalities, with conjecturing and reasoning, is vital to successful experiences with algebra." Students can use algebraic thinking for better orientation to and participation in business, economy and political life. The modern world and trends involve these kinds of mathematical mental processes more then ever.

The main difficulties in early algebra learning can appear in developing an understanding of the *meaning* and *use* of algebraic symbols and expressions. There are different sources from which the algebraic expression can take its meaning. One of them is *formal:* an algebraic expression is meaningful if it can be derived from the set of axioms. In contrast with formal we use the term *referential* meaning: (1) *An algebraic expression can represent a relationship between numbers in general – we call it* numerical meaning; (2) *An algebraic expression can recall a relationship among quantities in some situations – we call it* situational meaning.

The numerical meaning of the identity $a + (b + c) = a + b + c$ is the following "rule:" When we add sum of two numbers to an individual number we would get the same result as if we added first number to the same individual number and after that the second number to it. With concrete numbers we show the usefulness of this rule:
In case $a = 15, b = 6, c = 4$ we calculate in the form 15 + (6+ 4), but in case of $a = 13, b = 7$ and $c = 5$ we use $13 + 7 + 5$. If students understand the referential meanings of expressions they can make good explanations of expressions and identities.

The same identity $a + (b + c) = a + b + c$ can be associated with a situational meaning, for example with the following story:

> Ana had *a* apples and she got *b* from her sister and *c* from her brother at the same time. Mary had *a* apples. She got *b* from her sister and later she got *c* from her brother.

They have the same number of apples, but the ways of their calculations are different. Appropriate stories can help students to understand expressions and identities.

Some special difficulties connected with early algebra

Most of the difficulties of my students relate to the generalization and recognition of structure, the meaning of algebraic symbols and expressions, manipulation of symbolic forms. The students manipulate the sides of an equality asymmetrically, therefore we have to deal not only with the four given equalities, but also with the equalities obtained through changing of the sides. One of the typical mistakes that my students make is creating *malrules*. As for Matz (1982) students create malrules by generalizing signs of operations and creating prototype rules from which they construct new rules:

The correct rule is:	$a + (b + c)$	=	$a + b + c$
Prototype rule is:	$a \square (b \square c)$	=	$a \square b \square c$
malrules are:	$a - (b + c)$	=	$a - b + c$
	$a - (b - c)$	=	$a - b - c$

There are some difficulties caused by the properties of Serbian language. For example we use two different terms for expressing addition of objects (*dodati*) and addition in algebra (*sabrati*). Fortunately, in arithmetic we use both terms, therefore it is used as mediator between addition of objects and addition in algebra. Manipulating the objects, students write down the arithmetic expression. However, they express it verbally using both terms for addition and after generalization they write down and express verbally the algebraic expression.

STRIP-ARITHMETIC, THE COMBINATION OF REPRESENTATIONS

Following the Hungarian curriculum for grade 7 (13-year-old students) I applied my strip-arithmetic method in March 2004 during 24 lessons of algebra. The class consisted of 8 students (3 girls and 5 boys). This small number of students enabled me to have individual approach to every student, so I could deal with their individual problems. All of them were bilingual; however, 3 of them expressed themselves better in Hungarian, whereas the rest preferred Serbian.

At the beginning students took both subject pretest and motivation test and were tested at the end of the school year (measuring the some categories). This gave me the insight to the development of their knowledge and motivation level.

1. The hypotheses

Motivated by the question mentioned in the introduction and by my personal 5-year teaching experience in Serbian school in Budapest I could put forward the following hypotheses: The use of concrete and visual representations can help students to build up the basic knowledge of algebra in the complexity of bilingual schools, as well as specific way of thinking, typical for algebra.

2. The mathematical aim of the case study

I am going to describe 3 lessons in order to demonstrate the method by explaining the following identities:

$$a + (b + c) = a + b + c$$
$$a + (b - c) = a + b - c$$
$$a - (b + c) = a - b - c$$
$$a - (b - c) = a - b + c$$

This is the first time we develop the students' skill of generalization, which can be expressed in the language of algebra (associativity of addition, minus sign in front of parentheses, realizing and expressing structures).

3. Pretest

Before conducting these 3 lessons I wanted to sum up the learning preconditions using the pretest. The task connected to the goal of these lessons was:

(1) Without any calculation, associate with the expression on the left side an expression on the right side, which *has the same value*:

	185 – 58 – 9
	58 – 185 + 9
185 – (58 – 9)	185 + 58 – 9
	185 – 58 + 9
	185 + 58 + 9

(2) Write a story which will support your choice.

(3) Using mathematical signs write down the rule.

In order to avoid mental calculation I set three-digit number in the numerical expressions.

Nobody solved this task correctly. 5 of 7 students chose the first expression on the right side (malrule). Two of them connected the expression on the left side with two different ones on the right side (1st and 3rd), without knowing that the result is uniquely determined. Here is one of their five similar stories:

Peter had 185 forints. He gave 58 to Joca and 9 to Milan. How many forints has he still got?

The students gave referential meaning only to the expression on the right side, whereas the expression on the left is not mentioned at all. They cannot control their own solution with regard to "the same value." Two of the girls have wrongly interpreted the rule about minus sign in front of the parentheses and one of them wrote the following equality (without any story):

$n - (x - y) = n - x - y.$

4. Lessons

First, we manipulated with objects, collected experiences, and drew conclusions. The students felt safe using manipulative tools. The order to handle identities was from the simplest towards the more complex ones. We recalled the basic technique of mental addition using exercises like: (a) Generate the number 8 as a sum of two numbers. [We are looking for the processes (1 + 7, 2 + 6, 3 + 5, 4 + 4), when the concept (8) is given.]; (b) Add 9 to 8. [They suggested to solve it by splitting 8 into 1 + 7.]; (c) Compare these numbers without counting: 892 + 171 and 892; 892 – 171 and 892 [This is a preparation for equations, too.]; and (d) Which number can replace the box in the equality 11 + 12 = □ + 8? [Without understanding the meaning of the equal sign one can write the value of the sum on the left side into the box.]

a) The identity *a + (b + c) = a + b + c*

Concrete manipulation with objects: Enactive level of representation

Each positive whole number is represented by discrete sets of real objects. Cardinality of these sets is the given number. To add two numbers means to build the union of two sets and to take its cardinality as the sum. Each student got 36 candies (12 of 3 different kinds: toffis, mint and fruit ones) and they had to construct the sets and their relations corresponding to the arithmetical expression 4 + (6 + 3) written on the board. During this activity a procept of "sum" developed through alternative perception of an arithmetical expression as a process on the one hand, and as a concept on the other:

Students *represented* the numbers 4, 6 and 3 by creating 3 groups of 3 different kinds of candies: (i) They consider (6 + 3) as a *process* and added the group of 3 candies to the group of 6 candies; (ii) The result of the addition (6 + 3) was a *concept* (9).

The steps (i) and (ii) form together an *elementary procept.* The new *process* 4 + (6 + 3) means that we add to the group of 4 candies the group of (6 + 3) candies. The result of this complex process is the concept 4 + (6 + 3), which denotes the number of candies in newly created group. One of the possibilities of *formulating the situational meaning* of the arithmetical expression 4 + (6 + 3) is: Pera had 4 toffis and he got 6 fruit candies and 3 mint candies at the same time. Now he has 4 + (6 + 3) candies. The situational meaning of the parentheses is contained in the words: "At the same time."

During the lesson I demanded that the description story should be written using only separate numbers of the original groups of candies. In spite of the instruction given, 2 of the students used number 9. In their stories students did not refer to the specific time of getting the candies, so by asking questions I urged them to be more specific in their formulation.

It is very important to establish mutual connections between a symbol in the expression with the appropriate word or words in the story. We practiced this through discussions.

Mira had 4 candies and she got 6 from her sister. What should she do in order to have the same number of candies as Pera. Write down your work.

Having seen 4 + 6 as a process and as a concept, they now had to compare the numbers of Pera's and Mira's candies:

$$4 + (6 + 3) = 4 + 6 \square \text{ O},$$

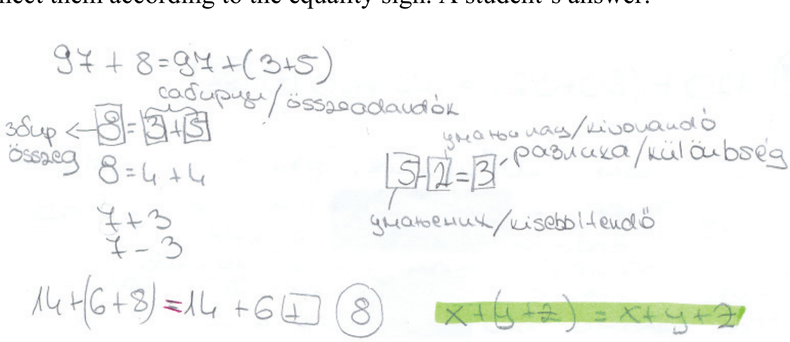
where ☐ means an operation, O means a number of candies. The sign "=" is the rule "get the same number of candies on both sides (numerical meaning of the equality sign).

The situational solution was: Mira should get some (O) candies, ☐ in an addition, O means 3. The basis of this solution was the same number of Pera's and Mira's candies.

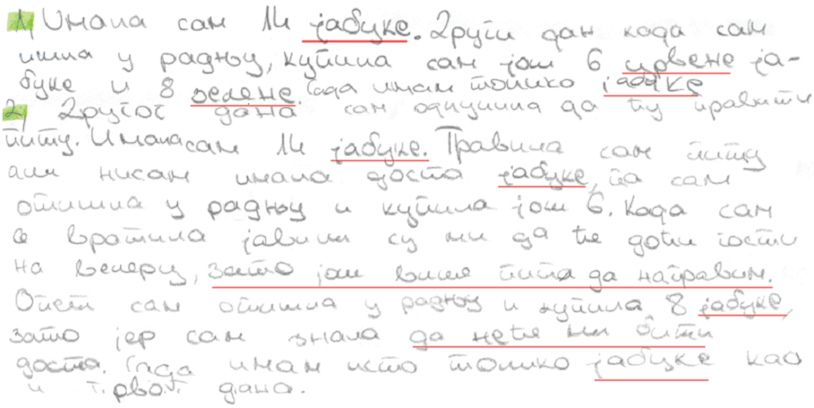
The next exercise originated from the numerical meaning and asked for a situational explanation:

Replace the "boxes" and write an appropriate story: 14 + (6 + 8) = 14 + 6 ☐ O.

Three students wrote the stories for the left and for the right side but they did not connect them according to the equality sign. A student's answer:



A worksheet with correct comments in two languages (“=” is signed with red color).


1. Имала сам 14 јабуке. Други дан када сам ишла у радњу, купила сам још 6 црвене јабуке и 8 зелене када имам толико јабуке.
2. Другог дана сам одлучила да ћу правити питу. Имала сам 14 јабуке. Правила сам питу али нисам имала доста јабуке, па сам отишла у радњу и купила још 6. Када сам се вратила јавили су ми да ће доћи гости на вечеру, зашто још више пита да направим. Опет сам отишла у радњу и купила 8 јабуке, зашто јер сам знала да неће ми бити доста. Сада имам исто толико јабуке као и први дана.

The student wrote a story explaining the equality. Here is a translation of her story with language mistakes, which are underlined in original text:

> I had 14 apples. Next day when I went shopping I bought 6 red apples and 8 green apples. Now I have this much apple.
> Next day I decided to make pie and I had 14 apple. I made a pie but had no enough apples, so off I went and bought 6 more. When I came back rings the telephone: guests coming to dinner! Must make more pie! Again going to the shop and bought 8 more apples. I know I had no enough. Now I have as much apple like the first day.

This work shows that she makes mistakes by using the language (with cases - *jabuke* instead of *jabuka*, incorrect order of words, incomplete sentences); her mathematical understanding of the problem is obvious and can not be denied.

The next step was *using mental objects* instead of real ones: the students faced large numbers in tasks. Replace the “boxes”: 120 + (80 + 300) = 120 + 80 ☐ O.

Arithmetic of strips

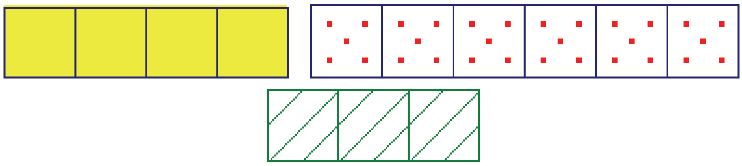
After dealing with discrete objects we used paper strips in order to repeat the tasks above and to have a possibility to generalize the operations for “symbolic” quantity (the length of the strip without concrete measured value). We used graph paper strips in different colors. We agreed that 1 column of squares represented the measurement unit; we did not care for the width of the stripe. Different given numbers were represented by strips in different colors. Within one task we did not change the color used for the same number.

The addition of two numbers was represented by connection of two strips; the new (extended) strip represented the sum of them (concept). In order to connect the strips we used white tape where we wrote the signs of operations (in order to keep in mind the process). For an operation in parentheses we used smaller operational sign and the strips on the both sides of this sign represent the expression in the parenthesis.

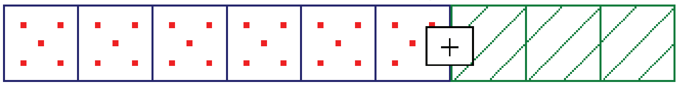
> Replace the “boxes”: 4 + (6 + 3) = 4 + 6 ☐ O. Use the paper strips for representing the expressions.

The students made the following steps:

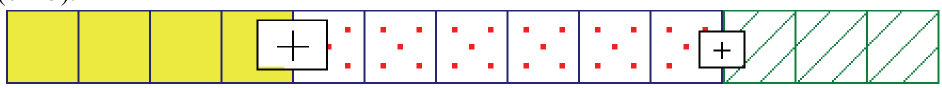
Step 1: Representations of numbers: 4, 6 and 3 by cutting popper strips of 4, 6 and 3 units.



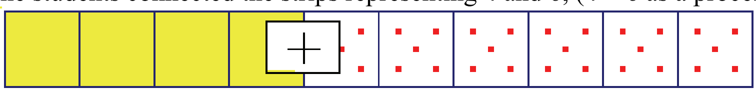
Step 2: The students connected the strips representing 6 and 3, (6 + 3 as a process):



Step 3: The students connected the representation of the number 4 with the representation of the expression (6 + 3) using the result as a concept in the process 4 plus (6 + 3).



Step 4: The students connected the strips representing 4 and 6, (4 + 6 as a process):



Step 5: The students compared the concept 4 + (6 + 3) and 4 + 6 by looking at the length of the strips. To the shorter strip they had to add the strip of the length of 3 units.

Conclusion of Step 1- Step 5: ☐ means an addition, O means 3.

In order *to recognize the symmetric property* of the equal sign we dealt with the analogical task 4 + 6 + 3 = 4 + (6 ☐ O). Next we used 2, 5, 7, 8 or 9 instead of 3, the students discovered the general conclusion: O means always the second number in the parentheses (formal meaning). The role of O (and of each number) can be replaced by a strip (meta-symbolic) and we can rich the formal using a + (b + c) = a + b ☐ c, (three types of adding in process). The formal meaning of ☐ can be confirmed after examining all the three identities.

b) Dealing with the identity *a + (b – c) = a + b – c*
Starting again with the enactive level of representation: using marbles

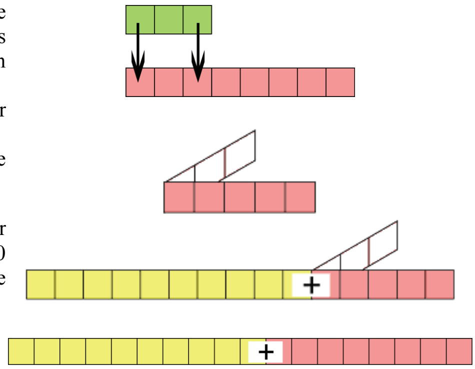
Playing with marbles students gave referential meaning to these expressions: 10 + (8 – 3) and 10 + 8. We added to the first group of 10 marbles another group, which is by 3 smaller than 8, and exactly 8 to the other group of 10 marbles. In the equation 10 + (8 – 3) = 10 + 8 ☐ O the operation ☐ meant a subtraction and O could be replaced by 3. Referential meaning of 10 + (8 – 3) = 10 + 8 – 3 can be: Pera had 10 marbles and he got another group which was by 3 smaller than a group of 8 and Mira had also 10 marbles and she firstly got 8 and after that another 3 marbles. Now they had the same number of marbles.

Using stripes

The same task was represented by strips. After the representations of the numbers 10, 8 and 3 by stripes we defined the “strip subtraction” (for the cases, when the minuend was bigger than the subtrahend):

- We put the subtrahend on the minuend to the left or to the right end and stick it on it.
- We folded back the double-layered part. The difference was represented by the remained strip.

We constructed 8 – 3 (process) and use the result for the process of addition of 10 and (8 – 3). The sum 10 + (8 – 3) has to be compared with the result of the (known) addition 10+8.



The representation of 10+8 is by 3 units longer than the representation of 10 + (8 – 3). Dealing with similar tasks (we use 1, 2, 4, 5, 6, 7 instead of 3, the students discover the conclusion: $a + (b – c) = a + b + c$.

c) Dealing with the identity *a – (b + c) = a – b – c*

Using multicolor sticks

The task was: Replace the “boxes” 15 – (5 + 9) = 15 – 5 □ O.

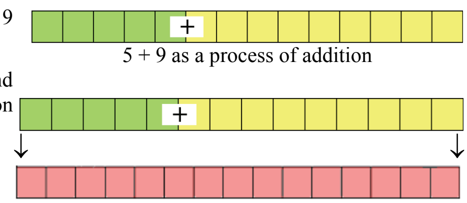
Playing with multicolor sticks students gave referential meaning to the expressions 15 – (5 + 9) and 15 – 5 (always using the same colors for the same numbers). In the first case they took away number of sticks bigger by 9 than in the second case. Comparing the numbers of sticks after doing this it was obvious that 9 sticks should be taken away from the second group of sticks. The answer was correct: 15 – (5 + 9) = 15 – 5 – 9.

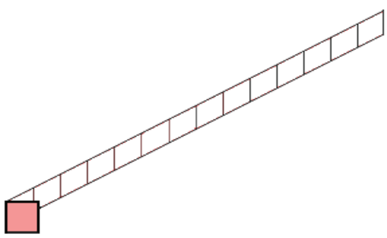
The *referential meaning* of the written can be: Pera had 15 sticks yesterday and he simultaneously gave 5 to Jovan and 9 to Marko, today he has 15 sticks also and he gives 5 to Jovan and latter 9 to Marko, so now he has the same number of sticks as yesterday.

Using paper strips

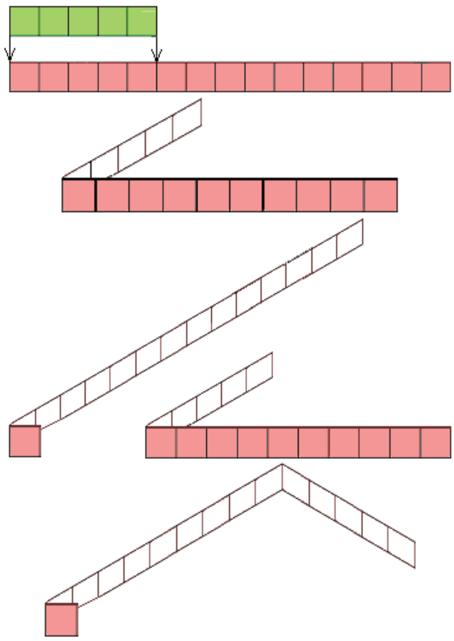
The students made the representations of 15, 5 and 9 and joined two strips and stick the “+” sign over.

The students used the result 5 + 9 as a concept and constructed 15 – (5 + 9) as a process of subtraction (sticking and folding back of the appropriate strips).





The students constructed 15 – 5 as a process of subtraction (sticking and folding back the appropriate strips).



We compared the concepts of 15 – (5 + 9) and 15 – 5 and gave the conclusion:

15 – (5 + 9) is shorter by 9 than the stripe 15-5. In order to get the same length we stuck and folded back a part of length 9 from the strip representing 15-5.

The experiences with strips should be transferred into the world of normal arithmetic:

$$15 - (5 + \mathbf{9}) = 15 - 5 \ \square\ \mathbf{O}.$$

We obtained that the equality was true if we replaced O by 9. The role of □ was determined by the *operation regarding to both* of values included in parentheses and by the operation in the parentheses.

$$15 - (5 + \mathbf{9}) = 15 - 5 \ \square\ \mathbf{O}.$$

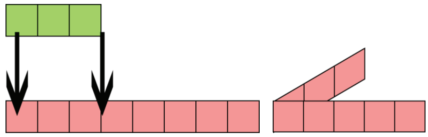
The sum in the parenthesis means that the members have to be manipulated on the same way (*spojiti*= join both of them). Now we have a complex subtrahend (the sum of 5 and 9). The subtraction means *saviti* = folding, therefore we have to fold 5 *and* 9, and this means, that we made the operations (15 – 5) – 9. In this case □ means a subtraction (one has to subtract the second summand, too).

The students had no problem in the stage of generalization and were able to get the identity which follows: *a – (b + c) = a – b – c.*

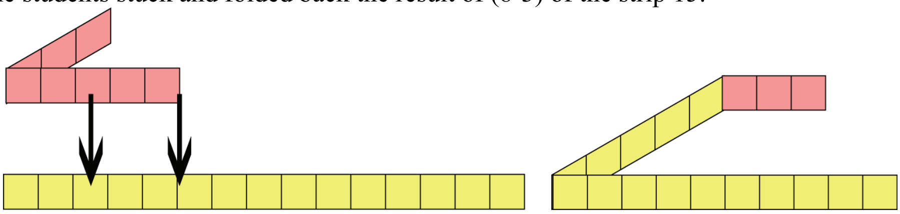
d) Dealing with the identity *a – (b – c) = a – b + c*
Using candies

The students solved the task 15 – (8 – 3) = 15 – 8 □ O using candies and giving referential meaning to the expressions: Pera and Jovan had 15 – 15 candies on the table. Pera wanted to take 8 candies, but 3 of them fell back. Jovan took 8 candies and …. "=" means that they still have the same number of candies. Jovan should to put 3 candies back (+).

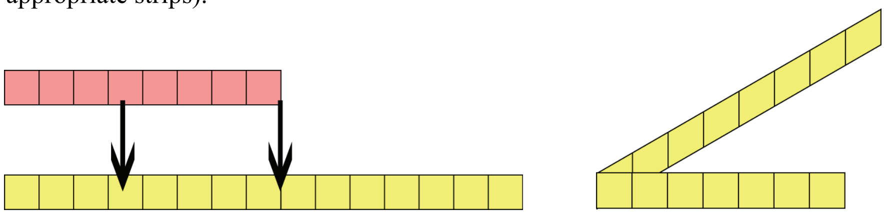
Using paper strips
The students mad the representations of 15, 8 and 3. They stuck and folded back 3 of the strip 8).



The students stuck and folded back the result of (8-3) of the strip 15.



The students constructed 15 – 8 as a process of subtraction (sticking and folding back of the appropriate strips).



We compared the concepts of 15 – (8 – 3) and 15 – 8 and give the conclusion: 15 – (8 – 3) is by 3 longer than the strip 15 – 8. In order to get the same length we join a strip of length 3 to the strip representing 15-8.
This means in the terms of the arithmetic: the equality is in 15 – (8 – **3**) = 15 – 8 □ **O** true if we replace O by 9.
The role of □ is determined by the operation "fold back the part, which was already folded," "put back 3."

15 – (8 – **3**) = 15 – 8 □ **O**.

The difference in the parenthesis means that the members have to be manipulated in different ways. The complex subtrahend (the difference of 8 and 3): folding (*saviti*) 8 and joining 3. In this case □ means addition.

The students had no problem in the stage of generalization and were able to get the identity which follows: $a - (b - c) = a - b + c$.

CONCLUSION

The use of objects, stories and strips (and the cross-connection to the algebraic description) could *promote the ability* of constructing referential meaning of algebraic expressions as well as ability of generalization, perceiving structures, and the creation of procept. As one of the techniques to eliminate the application of malrules created by a student is to try out whether the rule is applicable in arithmetic or real situation (method of control). The systematical considering of all logical possibilities of these identities should cut down creating of malrules (method of mathematics).

It was obvious that students did the tasks gladly. They approached them as a kind of game, were also interested in their classmates' stories and indicated to one another noticed imperfections. Manipulation with objects helped them understand the structures of algebraic expressions and also facilitated correction of their own understanding. Finally, the complex activity during the mathematics lessons had positive effects on the functional *use of Serbian language*.

Paper stripes can also be used as representation means at the beginning of teaching equations.

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