Perceiving the concept of limit by secondary school pupils

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ABSTRACT: *This paper is based on extensive research carried out on secondary school pupils and students commencing their university studies. The basic purpose of the research was to find out the conceptions connected with the limit of a function the pupils and students had.*

.1 INTRODUCTION

In this paper I present the concept images related to limits formed in the secondary schools. Following Tall and Vinner (1981, p. 152), I understand the notion of "concept image" as the cognitive structure containing all kinds of associations and conceptions related to the concept (also related to their properties and theorems), including intuitions, elements of formal understanding, established patterns, procedures applied in different situations and operational strategies. Elements of a concept image need not be all conscious and some may be at variance with one another. I assume that learning, understanding, applying and developing mathematical concepts further involves the creation of this kind of structure in the mind (see, for example, Noddings, 1990; Piaget, 1977; Steffe and Gale, 1995).

Aspects of understanding limits by secondary school pupils were investigated by some authors. See, for example, Davis and Vinner, 1986; Schwarzenberger and Tall, 1978; Sierpinska, 1987; Tall and Vinner, 1981; Tall, 1996.

This paper is based on the research conducted at the secondary schools in Kielce and Warsaw (Poland) and at the Pedagogical University of Kielce in the years 1993 – 1995 and 1998 – 2002. The research involved 358 secondary school pupils and 154 students commencing their university studies in mathematics.

The purpose of my research was to investigate the understanding of the concept of limit formed in the school. It was particularly directed at revealing the intuitions, conceptions and associations linked to this concept. It was significant to determine to what degree they correspond with the meaning of the concept of limit. It was also important to determine the degree of efficiency of particular conceptions, the possible sources of their formation as well as to establish the awareness of the relations holding between them.

To achieve the aforementioned objectives several research instruments were used. They included an analysis of written tests, observation of discussions among groups of pupils and students and interview (one on one conversation).

In the research some expanded sets of selected problems – simple but a bit not quite standard were used (see Przeniosło, 2000, pp. 12-22). They enabled to examine the perception of the concept of limit in concrete situations and at the same time allowed to draw more general conclusions and not only those referring to the cases considered (the way in which the interviews were held served the same purpose). The problems concerned the limits of appropriately selected functions (including sequences) defined by various means (for example, by a formula consisting of two or more parts, a graph or with the limit expressed by infinite decimal). Some problems were mutually reverse (a formula given – complete the formula; a graph given – complete the graph). The problems were selected so as to refer to various situations connected with the limit of a function (for example, at the point not belonging to the domain of a function, at the isolated point, at plus and minus infinity; the limit of a sequence in $x_0 \in N$).

.2 REVEALED IMAGES

The analysis of the research enabled to distinguish a few images of the limit of a function formed in a secondary school. These images were named with regard to their key elements. The images were distinguished basing on the following idea: neighbourhoods, graph approaching, values approaching, being defined at x_0, limit equals to the value, schemes. In order to show the range of each image the percentage of the examined persons applying it was given.

.2.1 Neighbourhoods (1%)

Not many examined persons revealed correct conceptions based on neighbourhoods. These conceptions turned out to be most efficient when considering the associations connected with the limit of a function. With reference to the limit of a function at infinity (also of a sequence) the most frequent conception consisted in considering whether for each neighbourhood or strip around the straight line determined by a candidate for the limit there exists a point, such that starting from it all the function values belong to this neighbourhood. For the limit of a function at $x_0 \in R$ the conceptions based on neighbourhoods were connected with the awareness of the fact that x_0 is the accumulation point of the domain and that for each neighbourhood of the number *g* being the limit there exists such a neighbourhood of x_0 (except x_0), that for all arguments included therein the function values belong to this neighbourhood of *g*.

There also appeared not quite correct conceptions connected with neighbourhoods. In the case of a sequence it was the conviction that the necessary condition for the existence of the limit (equal to *g*) – when considering the strips of an every width around a straight line $y = g$ – is that: “if the terms start falling into the strip then no successive ones can fall out of it”. One more example is the conviction that the sufficient condition is the existence of “one neighbourhood or one strip around the straight line $y = g$ such that the terms contained in it approach *g*”. It should be adding that the interviewers did not used the phrases: ‘a necessary condition’ and ‘a sufficient condition’. The sources of the formation of these conceptions could have been a method of teaching the notion of limit – in the first case discussing monotonic sequences only. The second conception could have been influenced by the graphic illustrations showing that the limit of a sequence equals *g* as quite frequently on graphs in a text-books only one neighbourhood was marked. Such visual presentation could contribute to the formation of a deliberate association even if some necessary explanations were provided. It could have been the case when the explanation contained the phrase “for an arbitrary” neighbourhood or ε, as this phrase is frequently used in definitions. Quite often this phrase is subconsciously understood as “for one arbitrarily chosen”. Owing to mechanical acquisition of the definition the phrase can be comprehended in this way.

Such diversity of meaning of the phrase “arbitrary neighbourhood” could have effected the conception connected with the limit of a function at x_0. It involves inversion of an order of matching the neighbourhoods of the number being a limit and neighbourhoods of point x_0 (without x_0). The reason for such conviction could be the order of considering the arguments and values formed with reference to mapping. The source of ‘degeneration’ could also be associations connected with determining whether x_0 was accumulation point of the domain.

.2.2 Approaching of a graph (35%)

The key element of the images of the limit of a function created by the majority of examined persons consisted of associations connected with the observation of approaching of the graph points. In the case of $x_0 \in R$ the conceptions were used to determine right- and left-hand limits while referring to necessary and sufficient condition connected with one-sided limits allowing existence of the limit at x_0. The characteristic feature was the conviction that both one-sided limits can exist, in other words that the function must have arguments smaller and larger than x_0 (predominantly close and close to x_0, arbitrarily close to x_0).

Some examined persons determined one-sided limits, 'moving' on the graph, starting with "its ends nearest to x_0". They defined the limit, for example left-hand one, as the second coordinate of the point which they reached moving on the graph from the left-hand side of x_0. In order to treat such a number as a left-hand limit the majority of them wanted to reach the point of the first coordinate of x_0 (although x_0 did not have to belong to the domain).

A totally different association connected with observation of approaching of the graph points consisted in the treatment of the limit of a function as its global property. It was revealed through an inclination to approach x_0 from the whole domain, 'moving' on the graph without separating a pencil from the sheet of paper, starting with the "graph's ends" and reaching point $(x_0, f(x_0))$ from both sides or "an empty circle for x_0". This sort of association was probably caused by intuitive conviction concerning existence of the limit at $x_0 \in R$ only for continuous functions in some particular sets. This could only be an random, proper or improper interval whose accumulation point but not the end was x_0, or the sum of such two open intervals if x_0 was their common end.

The various associations connected with the limit of a sequence based on the observation of the approaching of graph points, that is, the terms of a sequence have been revealed. The demand that the terms approach number g – the limit of a sequence, starting from a certain point was closest to the notion of limit. More often, however it was expected that the points (n, a_n) would approach the asymptote $y = g$ or would be located on it from a certain point (similar associations were observed for limits of other functions in infinity). The conviction that finitely many terms of a sequence can "jump out" and not be closer and closer to the straight line $y = g$ also seemed constructive. Examinees said that these terms can be rejected. However, some of them thought that only several, hundreds or a few thousand terms could "act differently" but obviously not a billion or sextillion as this would be "too many". It was often connected with understanding of infinity and perceiving all such numbers as "unimaginably" therefore infinitely large.

What is interesting the majority of the examinees revealed the conviction that the limit of a sequence can exist at $n_0 \in N$. It was sometimes connected with the observation concerning the existence of "a right-hand and left-hand limit". That must have been a degenerating consequence of conceptions about the approaching of the graph points and the belief in the necessity of 'moving' exclusively on the set of natural numbers. As the limit of a sequence (a_n) the pupils and students showed its value only at such point n_0, for which terms – points (n, a_n), from both sides approached, "were closer and closer" to (n_0, a_{n_0}). However, they usually were not able to determine how many points must approach (n_0, a_{n_0}). Some thought a few, tens or hundreds were enough while other demanded that all the points approach it. In the case when $n_0 = 1$ examinees took into consideration only the terms on one side of (n_0, a_{n_0}). It can be interesting to note that some of them applied the same way of thinking to the last term of a finite sequence. Some examinees did not relate the conviction about existence of the limit of a sequence at $n_0 \in N$ to conceptions connected with the approaching of graph points so explicitly. For them such limit of a sequence was equal to its value "by definition". The phrase "by definition" did not mean for those using it that something results from the definition but it was a piece of their own definition of a limit.

Some examinees revealed also the conviction that "by definition" the limit of a function at an isolated point of domain exists and is equal to its value.

The images of the limit of a function formed by the pupils and students using the aforesaid associations turned out to be insufficiently operative.

.2.3 Approaching of values (11%)

More operative conceptions were formed by relating the limit of a function with the approaching of its values. The most advanced conception was based on considering whether for the arguments close, increasingly close and finally "infinitely close" to $x_0 \in R$ in which was determined the limit, the values approach, that is are less and less different from a certain number and whether this difference becomes arbitrarily small. For the majority of examinees, however, the same behaviour of values was expected for the arguments on both sides of x_0.

Some examinees despite observing the approaching of arguments and values demanded that for a function to have the limit at $x_0 \in R$ there must exist in intuitive sense continuous "bits" of a graph "unimaginably small", usually for arguments from both sides of x_0. Others perceived the limit of a function as its global property.

Just as in the case connected with the image "Approaching of a graph" some pupils and students thought the limit of a sequence exists at $n_0 \in N$. With reference to the limits of sequences the approaching of values from a certain index was considered. Likewise the limits of other functions in plus or minus infinity were perceived.

The most fully developed conceptions connected with the approaching of values turned out quite efficient as they allowed to solve numerous problems.

.2.4 Function must be defined at x_0 (23%)

A lot of persons thought that a necessary condition for the limit of a function to exist at the point is the belonging of this point to the domain. Its sources may have been the very name "the limit of a function at the point", frequent dealing with the functions having the aforesaid property or some associations connected with continuity.

The conceptions some examinees had were directly connected with continuity understood intuitively as a possibility of drawing the graph without separating a pencil from a sheet of paper, obviously for the whole domain containing x_0. Such continuity was – according to them – necessary for the limit to exist at x_0. To determine a limit the examinees usually 'moved' on the graph beginning from its ends with the aim of reaching $(x_0, f(x_0))$. Hence, they considered the value at x_0 as the limit of a function at this point and continuity as a necessary and sufficient condition for its existence. Therefore, it seems evident that they identified the concept of limit with the concept of continuity at this point subconsciously.

Other elements of these images of a limit were similar to those considered in two preceding sections, especially to those concerning the approaching of the graph. They also comprised the elements of images which would be discussed in the two succeeding sections.

.2.5 Limit equals to the value (13%)

The basic element discovered in these images of a limit was the conviction that the belonging of x_0 to the domain (when $x_0 \in R$) was a necessary and sufficient condition for the limit to exist at x_0. Furthermore, the examinees pointed at the function value as the limit at this point. They often applied a well remembered statement: "the limit of a function at the point is equal to the function value at this point". The aforesaid property was used so frequently that the fact that it was applicable to continuous functions and accumulation points of their domains was forgotten. Consequently, the pupils and students extended the scope of its application.

Association of the limit of a function at x_0 with the function value at this point was the essence of the image for some other examinees. The belonging of x_0 to the domain was, however – according to them – the sufficient but not necessary condition for the limit to exist at x_0. For x_0 not belonging to the domain they examined the existence and equality of one-sided limits determined, for example, by the approaching of graph points. They did not consider a possibility of applying the aforementioned condition to the function defined at x_0 but instinctively pointed at the value of the function as its limit. Their attitude to sequences and points $x_0 \in N$ was analogous. However, when $x_0 = +\infty$ or was minus infinity for other functions they were interested in the approaching of the graph points again. The reasons for such associations to evoke were similar to those discussed above and concerning identification of the function value with its limit. However, their authors perceived one more variant of the limit when function was not defined at x_0.

.2.6 Schemes (17%)

Obviously, algorithms applied in different situations are an important and constructive element of each notion's image. Their role, however, is constructive only if their application does not obstruct conceptual thinking. Unfortunately, disadvantages of using algorithms were observed in lots of situations. Quite frequently the strategy of solving problems adopted by pupils and students consisted in searching for the most suitable scheme. In addition, during the research a group of examinees for whom schemes were of particular significance was observed. The images of concepts they created almost exclusively in the form of sets of procedures applied in different situations. They processed new information into such schemes and only then linked it with the image. The procedures they used were both correct and incorrect, often specific and contradictory.

The way of thinking the pupils and students adopted may be best explained by a few examples. In the case of two-part formulas some examinees reduced the condition connected with the equality of one-sided limits to the form: "if there is a brace I calculate the limit from one formula and then from the next one, if the limits are the same the function has the limit but if they are different the limit does not exist". This condition was applied to sequences, other functions, points $x_0 \in R$ and those being plus or minus infinity. Sometimes a similar condition was used in the case of the function defined by several formulas.

Looking at graphs, some examinees could see – as they said – "known" functions and on the basis of the information about those functions they determined their limits. As known they considered both elementary functions and those which were thought to possess a certain property. The latter may be exemplified by identifying the limit equal to 1 for x approaching $+\infty$ with a curve monotonically and infinitely approaching the asymptote $y = 1$, or covering it from a certain point. The examinees thought so although these were not the graphs of such functions. So what they remembered were only approximate shapes of graphs in the form of schematic pictures. They were not able to interpret the figures from which "it was not possible to tell which function the graph represented".

The images of the limit of a function represented as a set of procedures to be applied in different situations turned out to be insufficiently operative not only in the case of problem situations but also in the case of simple, a bit non-standard ones. Obviously, the pupils and students could consider the problems only in the context of the well-known schemes.

.3 FINAL REMARKS

The images of the limit of a function formed by the examined persons contained a lot of elements at variance with the key one, contradictory to one another and far removed from the notion of limit. It should be stressed that only some pupils and students knew the definition of the limit of a function and even fewer examinees could apply it while a majority constructed their own definitions as a 'conglomeration' of particular cases.

To sum up it may be concluded that the development of images of the limit of a function revealed by the examined pupils and students was not sufficient. It is, however as follows from the analysed writings comparable with the one observed in other countries.

Apparently, assimilation of the notion of limit carries so many difficulties, obstacles and possibilities of degeneration 'hidden' in its very 'nature' that the organisation of the learning process undoubtedly requires taking special steps. It would be possible for pupils to overcome difficulties if they were given an opportunity to discover the sense of the concept through the process of creating the associations increasingly closer to its meaning (see Brousseau, 1997, p. 125). Appropriate methods concerning introducing the notion of limit as a problem-situation solving may be helpful here. To determine them a very detailed research on associations, conceptions and intuitions is necessary. Basing on my research I have made attempts to work out such methods (see Przeniosło, 2000, pp. 153-176; 2002, pp. 193-204).

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