MATHEMATICA V

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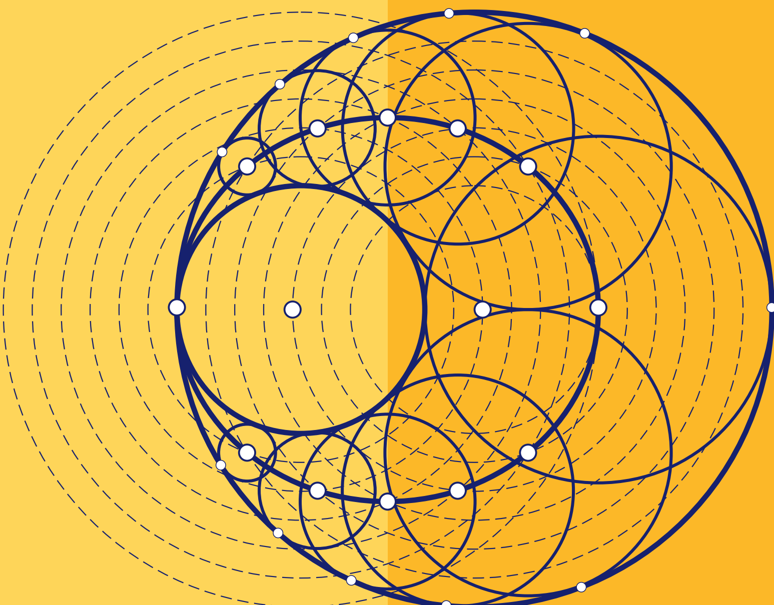

Scientific Issues

MATHEMATICA V

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PREFACE

This volume of 18 texts presents the authors, who took part in the Czech-Polish-Slovak Mathematical Conference, which was held at the Faculty of Education of the Catholic University in Ružomberok, June 3 – 5, 2015. The main goal of the conference was international exchange between researchers in mathematics and mathematics education. The primary support for the conference was the participation of the PhD students from the PhD School in mathematics education of the Palacký University in Olomouc, University of Ostrava, Czech Republic, and other researchers and teachers from universities in Visegrad countries. The participants presented their research results, and possibilities of future cooperative research.

The primary objective of *MATHEMATICA V* is to present results some of these PhD students and experienced researchers and teachers. It is nowadays important exchange between these groups of researchers, because PhD students bring new methods and experienced researchers bring experience and good practices for the teaching. New findings in pure mathematics can also support mathematics education. We hope that potential readers (teachers and researchers in mathematics and mathematics education) of this publication can find many inspirations for their educational and research work.

Martin Billich

The influence of interactive materials on students' study results

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Abstract. The article describes the influence of supporting materials on university students and their study results. Five hypotheses that were tested on first-year students in the course of Basic Mathematics will be introduced.

Keywords: Interactive materials, study results, hypoteses.

Classification: C70, F90.

1 Introduction

When teaching, every teacher should respect so called *didactic principles* (sometimes called *principles of teaching-learning process*). These principles are general suggestions or rules which help education process to be as much effective and efficient as possible and should be followed not only in teaching but also in creating all supportive materials. Following principles are the most presented: *principle of plasticity, consciousness and activity, adequacy, orderliness and feedback.*

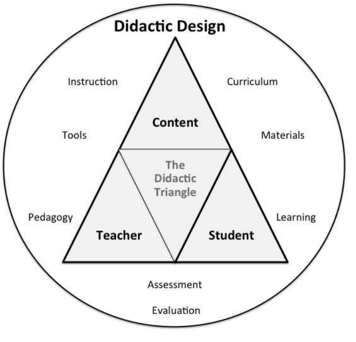

Figure 1: Didactic design.

2 Illustrating

Our brain receives 87 % of information via sight, 9 % via hearing and only 4 % are received through other senses (touch, smell and taste). In education process, verbal communication is used the most, but for teachers, it is also very important to try to illustrate as much information as possible. It is especially important in Mathematics, because students meet such a large ammount of abstract terms which they have to process.

Petty [1] shows five main advantages of illustrating:

- drawing attention;
- change;

- support of understanding;
- easier remembering;
- showing interest of the teacher.

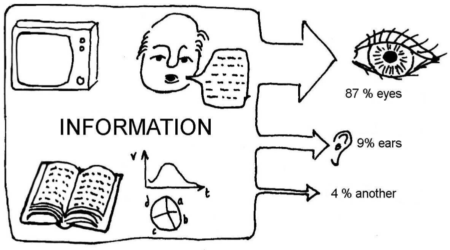

Figure 2: Some description of the figure.

3 Interactive materials

Within the scope of testing our hypotheses, our student Kateřina Hrabálková created a set of interactive materials which were designed in GeoGebra software and which should help other students to understand discused subject matter. Theoretical background and principles of working with these materials were introduced during the lesson and all needed materials were available for students during the particular lesson as well as after its end.

4 A questionnaire survey

A questionnaire survey was done in one of thirteen lessons of the course of *Basic Mathematics*, this lesson was focused on vector algebra - vectors, operations with vectors, linear combination of vectors, dot product and cross product.

There are usually two groups of students of this course and the difference between the particular lessons in these two groups was based on the way how students got to know given materials related to discussed subject matter. Wednesday's group was not allowed to use laptops and tablet computers which meant that this group got to know the materials only passively via the projector and the screen. Thursday's group was allowed to use laptops and tablet computers which meant that approximately 75 % of students could use these materials immediately and set in it its own parametres.

Students of both groups were given questionnaires at the end of the lesson. The aim was to find out their relationship to the lesson and used materials as well as their gained knowledge. The questionnaire contained 10 closed questions (including 7 question based on the discussed theory). In total, 23 questionnaires were distributed in Wednesday's group and 35 in Thursday's group. Rate of return was 100 %.

Several conclusions can be drawn from all the answers. It can be stated that that students like lesson in which GeoGebra is used. 81 % of respondents gave the positive answer. Moreover, 55 % of respondents think that this lesson is more interesting than previous lessons. Anyway, this is the question in which both groups differ the most. The answer "more important" was chosen by 70 % of students in

Wednesday's group but only by 46 % of students in Thursday's group. Students also think that interactive materials can help them to understand discussed subject matter, because this answer was chosen by 90 % of respondents.

It can be also stated, from the answers to questions focused on the theory, that students more understand subject matter that is taught using GeoGebra, because only 36 % of all respondents answered correctly the answer to the question including topic from the previous lesson, which is approximately twice less than an average number of correct answers to the questions focused on the topic discussed in this particular lesson.

It can be also assumed that it is more rewarding for students if the subject matter is taught using interactive materials, because more students answered questions connected with this subject matter correctly in comparison to questions connected with topic which was taught using static pictures.

5 Testing of hypotheses

Five hypotheses were set in total. They were tested using data from the questionnaires or didactic test.

5.1 Hypothesis 1

H_1: Students achieve more correct answers to questions which are connected with the subject matter that is taught using interactive materials than to questions connected with the subject matter that is taught using static pictures.

Answers to questions about the theory connected with the currently discussed subject matter were used for testing this hypothesis.

Count of answers	Kind of study materials		**In total**
	Static image	Interactive materials	
Correct answers	107	132	**239**
Incorrect answers	67	42	**109**
In total	**174**	**174**	**348**

Table 1: The Pivot Table: the actual rate (Hypothesis no. 1)

Count of answers	Kind of study materials		**In total**
	Static image	Interactive materials	
Correct answers	119,5000	119,5000	**239**
Incorrect answers	54,5000	54,5000	**109**
In total	**174**	**174**	**348**

Table 2: The Pivot Table: the expected rate (Hypothesis no. 1)

The number of degrees of freedom is 1, the value of tested criterion is $\chi^2 = 8,3490$ and the critical value is $\chi^2_{(0,95)} = 3,8415$.

Since $8,3490 > 3,8415$, we reject zero hypothesis and accept alternative hypothesis. It is true that the number of students' correct or incorrect aswers is dependent on the kind of used supporting material.

5.2 Hypothesis 2

H_2: Students who enjoy lesson taught using GeoGebra software achieve more correct answers to questions related to the discussed subject matter than students who do not enjoy it.

Answers to the questions about the theory related to currently discussed subject matter were used for testing this hypothesis as well as answers to the question about enjoying the lesson.

Count of answers	Relationship of students to the lesson		**In total**
	Enjoying the lesson	Not enjoying the lesson	
Correct answers	208	31	**239**
Incorrect answers	74	35	**109**
In total	**282**	**66**	**348**

Table 3: The Pivot Table: the actual rate (Hypothesis no. 2)

Count of answers	Relationship of students to the lesson		**In total**
	Enjoyng the lesson	Not enjoying the lesson	
Correct answers	193,6724	45,3276	**239**
Incorrect answers	88,3276	20,6724	**109**
In total	**282**	**66**	**348**

Table 4: The Pivot Table: the expected rate (Hypothesis no. 2)

The number of degrees of freedom is 1, the value of tested criterion is $\chi^2 = 17,8429$ and the critical value is $\chi^2_{(0,95)} = 3,8415$.

Since $17,8429 > 3,8415$, we reject zero hypothesis and accept alternative hypothesis. The number of students' correct answers is dependent on the fact if they enjoy the lesson or not (students who do are probably more intrinsically motivated).

5.3 Hypothesis 3

H_3: Students who use materials created in GeoGebra software to prepare for the test achieve more correct anwers in it than students who prepare without using these materials.

Data from the didactic test were used for testing this hypothesis, especially answers to the question if students used supporting materials uploaded on the LMS Moodle, as well as answers to the questions related to the theory about discussed subject matter, which were represented by these materials.

Count of answers	Kind of preparation		In total
	Using GeoGebra	Not using GeoGebra	
Correct answers	411	76	**487**
Incorrect answers	139	24	**163**
In total	**550**	**100**	**650**

Table 5: The Pivot Table: the actual rate (Hypothesis no. 3)

Count of answers	Kind of preparation		In total
	Using GeoGebra	Not using GeoGebra	
Correct answers	412,0769	74,9231	**487**
Incorrect answers	137,9231	25,0769	**163**
In total	**550**	**100**	**650**

Table 6: The Pivot Table: the expected rate (Hypothesis no. 3)

The number of degrees of freedom is 1, the value of tested criterion is $\chi^2 = 0,0729$ and the critical value is $\chi^2_{(0,95)} = 3,8415$.

Since $0,0729 < 3,8415$, we accept zero hypothesis. The number of correct answers in the test is not dependent on the fact if students use materials created for preparation for the test in GeoGebra or not.

5.4 Hypothesis 4

H_4: Students achieve more correct answers in tested subject matter for which preparation they use interactive materials than in tested subject matter for which preparation they use static images.

Only answers from students who used supporting materials on LMS Moodle were used for testing this hypothesis.

Count of answers	Kind of supporting material		In total
	Static images	Interactive materials	
Correct answers	103	306	**409**
Incorrect answers	62	79	**141**
In total	**165**	**385**	**550**

Table 7: The Pivot Table: the actual rate (Hypothesis no. 4)

The number of the degrees of freedom is 1, the value of tested criterion is $\chi^2 = 17,6252$ and the critical value is $\chi^2_{(0,95)} = 3,8415$.

Since $17,6252 > 3,8415$, we reject zero hypothesis and accept alternative hypothesis. The number of correct answers in the test is dependent on the kind of supporting material which student uses for preparation.

Count of answers	Kind of supporting material		In total
	Static images	Interacitve materials	
Correct answers	122,7000	286,3000	**409**
Incorrect answers	42,3000	98,7000	**141**
In total	**165**	**385**	**550**

Table 8: The Pivot Table: the expected rate (Hypothesis no. 4)

5.5 Hypothesis 5

H_5: The more time students prepare for the didactic test, the more correct answers they achieve.

Data from answers to the questions related to the theory were used for testing this hypothesis as well as answers to the question how much time preparation of students took.

Count of answers	Amount of time for preparation for the test			In total
	Less than an hour	From one to two hours	More than two hours	
Correct answers	87	122	224	**433**
Incorrect answers	33	98	66	**197**
In total	**120**	**220**	**290**	**630**

Table 9: The Pivot Table: the actual rate (Hypothesis no. 5)

Count of answers	Amount of time for preparation for the test			In total
	Less than an hour	From one to two hours	More than two hours	
Correct answers	82,4762	151,2063	199,3175	**433**
Incorrect answers	37,5238	68,7937	90,6825	**197**
In total	**120**	**220**	**290**	**630**

Table 10: The Pivot Table: the expected rate (Hypothesis no. 5)

The number of degrees of freedom is 2, the value of tested criterion is $\chi^2 = 28,6093$ and the critical value is $\chi^2_{(0,95)} = 5,991$.

Since $28,6093 > 5,9915$, we reject zero hypothesis and accept alternative hypothesis. Amount of time for preparation for the test influences the number of correct answers that student achieves.[1]

[1] Apart from these five hypotheses, one more hypothesis was set as well - *Lesson taught using GeoGebra software is more interesting in comparison to other lessons for students who are not allowed to use their own laptops and tablet computers than for students who are allowed to use*

6 Conclusion

On the basis of the four accepted hypothesis, it can be stated that the number of correct or incorrect answers of students is dependent on the fact if the discussed subject matter is taught using interacitve materials or static images. It was also found that the number is also dependent on the fact if students enojoy the particuar lesson or not. Hypothesis stating that students who use interactive materials created in GeoGebra software for preparation achieve more correct answers was rejected. On the other hand, hypothesis that students achieve more correct anwers in tested subject matter for which preparation they use interactive materials than in the tested subject matter for which preparation they use static images was accepted. It was also found that the amount of time for preparation to the test influences the number of correct answers, so it is true that the more time students prepare for the test, the more correct answers they achieve.

Acknowledgements

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them. Data to this hypothesis were shown in the picture, but the rate in every possible pair of nominal values was not greater than 5, this hypothesis was not tested and processed.

Some possibilities for using mobile learning in mathematics

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Abstract. At the moment, several projects in Slovakia are taking place, that are focused on using tablets and smartphones in education. Interconnecting new educational forms within mobile learning brings new possibilities to teaching mathematics. This kind of teaching can motivate students and benefit in active learning, as well as raise effectiveness. This article focuses on some of the possibilities of using tablets to improve the effectiveness of teaching mathematics.

Keywords: mobile learning, digital technology in education, augmented reality

Classification: D40, U50, U70.

1 Introduction

We live our everyday lives in an environment, where we cannot imagine a day without a computer, the Internet, a cell phone and other ICT products. Elementary and secondary school pupils were born into this world and it's natural to them. Fortunately, this natural environment is already reaching into schools. According to the European Commission/ICT cluster, 2010 there exists an increasing discrepancy between the possibilities of using ICT at home and in school, therefore schools should support the development of modern technical environment, thus connecting their experience with these devices at home with school and prepare them for real life situations [1].

Digital technology offers teachers a possibility to make use of new educational methods, e.g. the constructivist approach, controlled search, workshop method or peer instruction method. Digital technologies are very suitable for project teaching, too. Teachers can make use of blended learning, flipped classroom method, etc. Last but not least, the computers are used for electronic testing when knowledge of the pupils is measured.

From a didactic point of view, we can use these resources in different teaching methods:

- method of controlled discovering,
- project education (this is another rising form of mathematics teaching, which is gaining new possibilities in the digital environment. Students use mostly cooperative learning, which is enabled by m-learning),
- peer instruction method (in this case, the tablet is used for voting),
- "The flipped classroom" method, etc.

From an educational point of view, teaching with a tablet is appropriate for both individual and collective work.

We can find several approaches to define mobile learning in professional literature:

Mobile learning refers to the use of mobile or wireless devices for the purpose of learning while on the move. Typical examples of the devices used for mobile learning include cell phones, smartphones, palmtops, and handheld computers; tablet PCs, laptops, and personal media players can also fall within this scope [6].

Peters viewed mobile learning as a useful component of the flexible learning model. In 2003, Brown summarized several definitions and terms and identified mobile learning as "an extension of e-learning" [7].

Taylor [4] has defined mobile learning as "learning mediated by mobile devices, or mobility of learners (regardless of their devices), or mobility of content/resources in the sense that it can be accessed from anywhere".

One-to-one learning with a mobile device falls into the same category of mobile learning in which learners use a mobile device (e.g., iPads, iPods, netbooks, laptops, cell phones, or other mobile devices) with Internet access to engage in learning activities. Many school districts may restrict the access to classroom use for fear of damage, lost, or misuse [8].

Mike Sharples [9] has made a good summary on different views of defining mobile learning.

Current perspectives on mobile learning generally fall into the following four broad categories:

- Technocentric. This perspective dominates the literature. Here mobile learning is viewed as learning using a mobile device, such as a PDA, mobile phone, iPod, PlayStation Portable etc.
- Relationship to e-learning. This perspective characterises mobile learning as an extension of e-learning. These definitions are often are all-inclusive and do not help in characterising the unique nature of mobile learning. What is needed is clarity: in agreement with Traxler [10], the technocentric/e-learning based definitions only seek to place "mobile learning somewhere on e-learning's spectrum of portability".
- Augmenting formal education. In the mobile learning literature, formal education is often characterised as face-to-face teaching, or more specifically, as a stereotypical lecture. However, it is not at all clear that this perspective is wholly correct. Forms of distance education (for example, distance correspondence) have existed for over 100 years, leading to the questions regarding the place of mobile learning in relation to all forms of "traditional" learning, not only the classroom.
- Learner-centered. Any sort of learning that happens when the learner is not at a fixed, predetermined location, or learning that happens when the learner takes advantage of learning opportunities offered by mobile technologies [9].

In the following, m-learning means teaching with tablets, smartphones...

2 M-learning - the current situation in Slovak schools

At the present, nearly all schools make use of digital technologies in education. Most schools are equipped with computers, interactive whiteboards (often many) and teachers have access to notebooks with projectors. However, few schools have interactive whiteboards in every classroom, there they cannot use it all the time. Most of the students own a smartphone and still more of them own a tablet. Despite this fact, education that incorporates mobile learning is spreading slowly.

There are many projects, domestic and foreign, which are focusing on pilot introduction to m-learning to schools. In Slovakia, there are currently 2 such projects that can be found at the following links: http://www.skolanadotyk.sk and http://www.digiskola.sk. The former one, "school by touch", was an initiative from private companies, which supplied 10 schools with interactive whiteboards and tablets for every pupil in a classroom. Besides equipping schools with technology, teacher trainings were carried out as well and an empirical research trying to determine the effectiveness of teaching using these technologies. Teachers were creating demonstrational materials suitable for m-learning, students created and shared videos from classes, visit http://www.skolanadotyk.sk/materialy.html. On a conference that followed the project's end, the research team stated that in all subjects except mathematics, significant results were gathered that point out the effectiveness of such education. We presume, the case of mathematics classes was caused by insufficiently digitally literate teachers of mathematics. The latter project started in 2014. The Ministry of Education of the Slovak Republic, using EU funds, bought 5680 interactive whiteboards, 5680 notebooks, 2686 color printers, 20000 tablets and 1000 wifi routers for schools all around Slovakia. At the moment, teacher training is under way regarding the use of tablets. In the next phase, the ministry wants to create digital content, as well. Besides these projects, there exist several individual initiatives by mathematics teachers, which experiment with tablets and smartphones for teaching mathematics during classes.

At the moment, teachers of mathematics in Slovakia use tablets and smartphones in several ways:

- Pupils are solving e-tests, electronic worksheets for exercising
- Giving access to electronic study materials
- Solving activities, which could not have been utilized without the functions of mobile devices (project teaching, constructivist teaching using a camera, GPS, etc.)
- Smartphone as a voting device
- Using some applications for mobile devices

Some of the above mentioned activities can be done on a PC as well, but for teachers as well as for students it's easier to use a tablet or a smartphone, than to move to a PC classroom.

In the next part, we present a few examples:

1. Tablets - alternative for eBook readers. This means they can be used for studying from school books in electronic forms. Some mathematics school books are available in electronic form in Slovakia.

   The portal eAktovka (http://www.eaktovka.sk/) gives access to digital school books for students in elementary and secondary schools. These are available for free for all that register on this website. Among else, students can also access other internet sources in text, image, audio or video form.

   Many mathematics teachers use the portal http://www.zborovna.sk. It's a web portal, created for exchanging information among teachers, parents and students. It's available for a small fee, which is covered by the school for every of its teachers and students. The portal is very popular and it is being used mostly for sharing PowerPoint presentations regarding a given topic. For this reason, we can classify this portal as a source for multimedial and explanatory materials suitable for m-learning as well. The disadvantage is, that these materials do not undergo a reviewing process and small context errors occur or in the method of explanation.

2. E-tests on tablets. In this part, by "e-test", we mean that e-test is an electronic, interactive material based on a system of questions and the search for answers, created not only for evaluation, but also for achieving educational goals (therefore it can server as an aid for innovative teaching methods). E-tests can not only be used in classrooms equipped with computers and laptops but they are getting more commonly used on tablets, too. One of the most popular e-test making software among teachers in Slovakia, which is available for free, is HotPotatoes, visit http://hotpot.uvic.ca/index.php. Multiple choice tests, crosswords and other methods are great for reviewing study materials, using interactive whiteboards, notebooks, tablets or smartphones.
3. Tablets and smartphones can serve as voting devices (instead of clickers). For this purpose, teachers mostly use the software SmartNotebook14, electronic tests in LMS Moodle or Google Docs Sheets. During mathematics classes, we can use tablets for e-tests not just to measure the students' knowledge, but also for exercising the study materials with immediate feedback. The advantage of this is, that every student can progress with their own pace and gets immediate feedback. In more carefully prepared e-tests, the student can get help, or view some pre-solved examples. In such organized class, the teachers can focus on less advantaged students. From a methodical point of view, teachers use HotPotatoes mainly for creating study materials in modern and attractive forms. Currently, there are many portals and recently even applications for smartphones, which were designed for digital education of mathematics via m-learning:

   - website of PaedDr. Katarina Polacikova
     http://www.supermatematika.wbl.sk
   - website of RNDr. Martha Megyesi
     http://megym.wbl.sk
   - tests on the website of the University in Trnava
     http://vcv.truni.sk/tests.php.

   Some of the applications that are being used during mathematics classes on some schools in Slovakia. Within the project "DIGISKOLA", the following applications are recommended for smartphones and tablets:

   - GeoGebra app
     https://play.google.com/store/apps/details?id=org.geogebra suitable for elementary and secondary school mathematics.
   - Some applications server as a substitute for graphical calculators:
   - Graphing calculator app
     https://play.google.com/store/apps/details?id=com.herbertlaw
   - Graphing calculator Mathlab
     https://play.google.com/store/apps/details?id=us.mathlab.android
   - Fraction Calculator by Mathlab
     https://play.google.com/store/apps/details?id=us.mathlab.android.frac

- WolframAlpha

Furthermore, these applications contain formulas and instructions:

- Pocket geometry
  https://play.google.com/store/apps/details?id=sk.halmi.geometryad
- Math formulary
  https://play.google.com/store/apps/details?id=com.anjokes.apps.math.en

The last category are brain teasers/logical puzzles:

- Logical
  https://play.google.com/store/apps/details?id=com.chilled.brainteasers

There are applications as well, that allow teachers create their own e-tests, such as

- Socrative
  https://play.google.com/store/apps/details?id=com.socrative.teacher

3 Augmented Reality

Augmented reality (AR) ... "provides a simple and immediate user interface to an electronically enhanced physical world" [8]. The basic principle of augmented reality (AR) is to superimpose digital information directly upon a user's sensory perception [3], rather than replacing it with a synthetic environment as VR systems do. "AR has the potential to become the leading user interface metaphor for situated computing" [8]. According to Azuma [2], AR must have the following three characteristics: 1. Combines real and virtual, 2. Interactive in real time, 3. Registered in 3D. All these requirements are often fulfiled by mobile devices and one of the prominent applications seems to be education.

Two representative applications are Construct3D and Codex. Construct3D, a 3D geometric construction tool specifically designed for mathematics and geometry education [4], was successfully tested with real high school students. The main goal is to improve spatial abilities and maximize transfer of learning. Teachers and students can play different roles and see the 3D scene. However, the tool is restricted to the working place of Studierstube device. The Graphics Codex by Morgan McGuire [5], contains 225 cross-referenced equation and diagram entries, rich linking to external documentation and full citations for primary sources and textbooks. However, this purely mobile (iOS) and web oriented solution is limited for computer graphics. Both limitations – hardware dependency and graphics focus – can be overcome by follow-up of EmatikPlus project. Augmented reality seems to be supported by novel sensors and taking the functionality and methodology from Construct3D with adapting, shortening, interlinking the presentation according to Graphics Codex offers a promising innovation. Moreover, an original way how to discover innovative educational methods was tested in two international competitions, MATHeatre and MATHFactor, which targeted student ages 9-18 and teachers Le Math, visit http://www.le-math.eu/index.php?id=15. The evaluation is in progress.

4 Conclusion

The quick rise of DT and their entrance to the lives of students, brings forth many questions about their effective use. Experience shows, that pedagogic research in the field of theory of teaching mathematics must go this way. The use of mobile technologies in teaching mathematics proves itself to be very effective and is attractive and motivating for students. The use can be applied from elementary schools to Universities. At the same time, a strong need occurs for good quality e-materials and e-tests in mathematics for m-learning. According to our opinion, it's of the best interest to use free and well-made software, such as GeoGebra, HotPotatoes and free applications from the Google Play store for the Android operating system. Even universities educating future teachers of mathematics should be addressing this problem more intensively. We believe, that by the use of innovative methods of teaching in a digital school, we can stop the decrease of popularity of mathematics on schools.

Acknowledgements

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Geometric terminology: How do teachers see it?

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Abstract. What is the current attitude of secondary school teachers to geometric terminology? Are you primarily interested in answering the following questions: Which basic symbols of plane geometry do teachers prefer? Which geometrical definitions are used in their practical teaching? Are they willig to accept any potencial changes of terminology?
In addition to this, which sources are used for their maths lesson's preparation.

Keywords: geometry, terminology, symbol, definition, teacher, secondary school.

Classification: G10.

1 Introduction

The main focus of educational research conducted by the authors was to identify the means of geometric terminology used by secondary school teachers in their practice. This mapping of the situation is important at the time when the changes of the Czech terminology, used for several decades without modification, are being prepared. It is also good to know the extent to which teachers are willing to accept any changes.

In 2013 The Terminology Commission for School Mathematics was established as a part of The Union of Czech Mathematicians and Physicists. Its aim is to update the terminology in mathematics. In the same year on the basis of these efforts, we started research which finds out the current status of basic geometric concepts for secondary school pupils, using the method of questionnaire survey. This was the content of the first three surveys.

This fourth questionnaire survey is focused on secondary school teachers, particularly for grammar and specialized secondary schools. The interviewed teacher expressed the opinion on a number of issues relating to the symbols used in geometry as well as on the way of using definitions of selected geometric concepts in the practice. Tasks 11 and 12 are open, the others are multiplied choice tasks.

2 Main results

2.1 Administration.

Questionnaire survey was performed in March and April 2015.
Total respondents: 174 teachers in secondary schools
Female/Male ratio: 1.44 (55 % female, 38 % male)
Education of teachers: Science education/Pedagogical education = 4.6
(78 % science education, 17 % pedagogical education)

Practice duration: How long do they teach?

practice duration [years]	
1-5	10 %
5-10	10 %
10-15	8 %
15-20	16 %
20-25	16 %
25-30	**18 %**
30-35	15 %
over 35	6 %

The average value of practice duration for the sample of 174 respondents is 22 years (median is 23 years). Half of the respondents have teaching experience ranging from 15 to 30 years.

2.2 The presentation of the tasks

The whole survey consists of 17 mathematical tasks. The task processing was done for the sample of 174 respondents. All the tables containing percentage value refers to the number of these respondents.

2.2.1 What symbol do you prefer?

Task 1: What symbol for **line segment** do you prefer?
Task 2: What symbol for **straight line** do you prefer?
Task 3: What symbol for **half line** do you prefer?
Task 4: What symbol for **plane** do you prefer?
Task 5: What symbol for **half plane** do you prefer?
Task 6: What symbol for **convex angle** do you prefer?

Task 1 Line segment		Task 2 Straight line		Task 3 Half line		Task 4 Plane		Task 5 Half plane	
AB	**75%**	$\leftrightarrow AB$	**59%**	$\mapsto AB$	**60%**	$\leftrightarrow ABC$	**53%**	$\mapsto ABC$	**59%**
$\overset{\longmapsto\!\!\dashv}{AB}$	20%	$\overleftrightarrow{AB}$	40%	$\overset{\longmapsto}{AB}$	21%	$\overleftrightarrow{ABC}$	30%	$\overrightarrow{ABC}$	19%
$\overline{AB}$	4%	AB	1%	$\overrightarrow{AB}$	19%	ABC	17%	$\overset{\longmapsto}{ABC}$	19%

Task 6 Convex angle	∢ **81 %**	∡ 17 %	∠ 1 %

The set of symbols preferred by the teachers can be seen from the above table, it means using the arrow as accent in front of letters, the most distinctive difference is seen for a half line, in contrast to a straight line where it is almost the same. Having the choice of three options for the symbol of angle (∢ spherical angle, ∡ measured angle, ∠ angle) the respondents distinctly prefer common selection of symbols from textbooks written in Czech.

2.2.2 What symbol do you prefer?

Task 7: What symbol for **proper subset** do you prefer?
Task 8: What symbol for **subset** do you prefer?

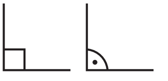
Task 7 Proper subset		Task 8 Subset	
$B \subset A$	**90 %**	$B \subseteq A$	**85 %**
$B \subsetneq A$	8 %	$B \subset A$	12 %

The strong majority of teachers prefer current symbols. Moreover, the confusion between the written symbols $\subsetneq$ and $\nsubseteq$ may occure while using them.

2.2.3 Are you willing to accept designation to the European standard?

Task 9: Are you willing to accept designation for **right angle** to the European standard ISO 800000-2?

accept	32 %
not accept	**68 %**



ISO ČSN

This task refers to the symbolic designation for right angle in geometric designs. Most of the teachers prefer the existing symbolism used in the Czech language textbooks, even though mathematical symbols according to ISO are commonly used in most software applications.

Numbers of respondents: Comparison of practice duration – respondents who do not accept the adoption of symbolic designation according to European standards:

practice duration [years]	
1-5	67 %
5-10	65 %
10-15	**86 %**
15-20	64 %
20-25	70 %
25-30	75 %
30-35	58 %
over 35	55 %

According to these data it appears that the rates of non-acceptance is similar for all teachers of duration of practice. Task 10: Are you willing to accept designation for **tangent function** to the European standard?

accept (preference of tan)	**71 %**
not accept (preference of tg)	29 %

For designation of the function tangent, willingness to accept another symbol is much higher than it is for right angle in task 9. Merely one third of the respondents prefer using of current, specifically Czech designation tg. Favorable situation for the changes is also given by the fact that symbol tan is used by most calculators, which teachers use in maths lessons.

Numbers of respondents: Comparison of practice duration – respondents who do not accept the adoption of symbolic designation according to European standards:

practice duration [years]	
1-5	33 %
5-10	18 %
10-15	21 %
15-20	32 %
20-25	41 %
25-30	25 %
30-35	31 %
over 35	18 %

The table of the percentage distribution of teachers, who don't support implementation of new symbols, clearly shows, that they are mostly represented by teachers with the teaching experience length between 15 and 25 years.

2.2.4 Definition of convex angle

Task 11: What definition of convex angle do you use in your practice?
A: *Part of plane, formed by two half lines diverging from a common point.*
B: *The intersection of two half planes with nonparallel edges.*
C: *Other definitions.*
D: *Not specified.*

A	**41 %**
B	**44 %**
C	6 %
D	8 %

Here, it can be merely stated, that the percentage of teachers using both definitions is at approximately equal rate.

2.2.5 Definition of deviation of the two lines

Task 12: What definition of deviation of two lines do you use in your practice?
A: *The deviation of the two lines is equal to the value of acute angle (or null, right angle) determined by the direction vectors of the lines.*
B: *The smaller of the pair of the supplementary angles.*
C: *Other definitions.*
D: *Not specified.*

A	**79 %**
B	9 %
C	1 %
D	11 %

Interestingly, that considerable number of teachers (9 %) didn't indicate in their formulation the substantial fact, that it regards to the size of angle, thus the definition is not formulated correctly.

2.2.6 Direction of the lines in a plane

Task 13: What is the direction of the lines in a plane?
A: *A set of each other parallel lines.*
B: *A set each other parallel half lines of the same orientation.*
C: *Not specified.*

A	**62 %**
B	32 %
C	6 %

Numbers of respondents: Comparison of second subject in combination with mathematics:

Second subject	A	B
physics	43 %	27 %
information technology	23 %	13 %
chemistry	12 %	22 %
descriptive geometry	43 %	27 %
geography	9 %	7 %
physical education	7 %	0 %
biology	5 %	15 %

It also shows the possible influence of another subject. There are higher numbers for physics and descriptive geometry. Also, in case of biology and chemistry the opposite proportion A/B, comparing to other subjects, can be clearly seen.

2.2.7 Definition of circle

Task 14: What definition of circle do you use?
A: *A set of all points in a plane that are at a given distance from a given point (the centre).*
B: *Symbolic definition:* $k(S;r) = \{X \in \rho; |SX| = r\}$

A	36 %
B	9 %
AB	**53 %**

Designation AB, AC, BC, ABC in task 14 to 17 means that the respondent in survey chose more options.

Number of respondents: Comparison of Grammar school and Specialized secondary school:

	Grammar school	Specialized school
A	19 %	17 %
B	8 %	1 %
AB	44 %	9 %

The results correspond to the type of school (general education / vocational training). Among the grammar school teachers there is a higher preference of using symbolic notations than texts.

Number of respondents: Comparison of practice duration:

practice duration [years]	A	B	AB
1-5	5 %	0 %	5 %
5-10	5 %	1 %	4 %
10-15	1 %	0 %	7 %
15-20	5 %	2 %	10 %
20-25	7 %	1 %	8 %
25-30	5 %	3 %	10 %
30-35	6 %	2 %	6 %
over 35	1 %	1 %	4 %

We assume, that the results could depend on the method of teachers' self-learning. For example, at a certain time the emphasis was put on symbolic notations in the context of implementation of sets in the school teaching.

2.2.8 Definition of axis of line segment

Task 15: What definition of axis of line segment do you use?

A: *A set of all points in a plane that have the same distance from both end points.*

B: *Symbolic definition:* $o = \{X \in \rho; |AX| = |BX|\}$, *where A, B are end points.*

A	40 %
B	11 %
AB	48 %

Number of respondents: Comparison of Grammar school and Specialized secondary school:

	Grammar school	Specialized school
A	21 %	19 %
B	9 %	2 %
AB	42 %	6 %

Number of respondents: Comparison of practice duration:

practice duration [years]	A	B	AB
1-5	5 %	0 %	5 %
5-10	5 %	1 %	4 %
10-15	2 %	0 %	6 %
15-20	5 %	2 %	9 %
20-25	8 %	1 %	7 %
25-30	6 %	2 %	10 %
30-35	6 %	4 %	5 %
over 35	1 %	1 %	4 %

As well as in task 14, we suggest, that the results could depend on the method of teachers self-learning. For example, at a certain time the emphasis was put on symbolic notations in the context of implementation of sets in the school teaching.

2.2.9 Definition of Euclidean altitude theorem

Task 16: What definition of Euclidean altitude theorem do you use?

A: *In a right-angled triangle the second power of the altitude on the hypotenuse is equal to the product of the lengths of segments formed by the foot of the altitude on the hypotenuse.*
B: *In a right-angled triangle the area of the square above the altitude equals to the area of the rectangle constructed from segments formed from the foot of the altitude on the hypotenuse.*
C: *In a right-angled triangle ABC with the right angle at the vertex C,* $v^2 = c_a.c_b$*, where* c_a *and* c_b *are the lengths of the line segments formed on the hypotenuse by the foot of the altitude v).*

A	5 %
B	7 %
C	54 %
AC	10 %
BC	18 %
ABC	5 %

2.2.10 Definition of Pythagorean theorem

Task 17: What definition of Pythagorean theorem do you use?

A: *In a right-angled triangle the second power of the hypotenuse is equal to the sum of the second powers of the legs.*
B: *The sum of the areas of the two squares on the legs (a and b) equals the area of the square on the hypotenuse (c).*
C: *If triangle ABC is right-angled with the right angle at C, then the legs a and b and the hypotenuse c satisfy the following relation:* $c^2 = a^2 + b^2$.

A	8 %
B	22 %
C	16 %
AB	5 %
AC	2 %
ABC	14 %

Interestingly, that while in Euclidean Altitude Theorem the most common choice is the definition using formula, in Pythagorean Theorem the text definition is prefered.

Number of respondents: Comparison of duration of practice for B and BC:

practice duration [years]	B	BC
1-5	3 %	11 %
5-10	21 %	4 %
10-15	3 %	14 %
15-20	13 %	25 %
20-25	18 %	18 %
25-30	18 %	18 %
30-35	24 %	9 %
over 35	0 %	1 %

3 Conclusion

The research has brought interesting results, which are going to be implemented by the Terminology Commission.

Acknowledgements

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Financial literacy - opinions of teachers from secondary school for pupils with mild mental disabilities

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Abstract. Paper outlines a brief pilot study of financial literacy and deals with the issue of financial literacy from the secondary school teacher point of view. In an increasingly risky and globalized marketplace, people must be able to make well-informed financial decisions. Many studies show the increasing indebtedness of Czech pupils (vulnerable group consists of people with mental disabilities) and the insufficiency of education in the area of financial literacy.

Keywords: financial literacy, pupils with mild mental disabilities.

Classification: M30.

1 Introduction

Poor financial choices/decisions can have a number of negative consequences. The money and the financial products are an integral part of today's world. People should be able to deal with them effectively.

Financial products such as current account, saving, private pension scheme, consumer or mortgage credit etc., form a natural part of our lives. We can say that each citizen (pupils, students, adults, seniors) should have the knowledge and skills that are important/necessary for her/his orientation on the issue of money and prices. [1]

"A person gets basic competences and literacy needed for life in society mainly at school. Within the compulsory education the elementary schools represent a united starting level for all pupils. Therefore our pupils should get basic competences from all areas of life there, which means a certain level of financial literacy too. Pupils of elementary schools (6 - 15 years old) get in touch with economic aspects of life (parents go to work - they earn money; they go shopping with parents - they learn they cannot afford everything, expenses are limited by family/ home budget; they are influenced by sales in shops or by TV advertisements at home; they are aware of private ownership; they try to manage their own finances etc.)." [3]

Helping students and young people understand financial issues is important, as younger generations are likely to face ever-increasingly complex financial products and services. They are also more likely to have to bear more financial risks in adulthood than their parents, especially in saving, planning for retirement and covering their healthcare needs.

Financial literacy is a core life skill for participating in modern society. Children are growing up in an increasingly complex world where they will eventually need to take charge of their own financial future.

International surveys show that students and young adults have amongst all groups the lowest levels of financial literacy. This is reflected by their general inability to choose the right financial products and often a lack of interest in undertaking sound financial planning. Even from an early age, children need to develop the skills to be able to choose between different career and education options and manage any discretionary funds they may have, whether from allowances or part time jobs. These funds may entail the use of savings accounts or bank cards. [4]

2 Defining the concept of financial literacy

The Government of the Czech Republic approved an updated version of the "National strategy of financial education" prepared by the Ministry of Finance of the Czech Republic in May 2010. This strategic document deals with the important principles of financial education, the roles of individual subjects in the financial education and presents an action plan for the financial education in the Czech Republic. This document is the central document for financial education in the Czech Republic.

"Financial literacy is a set of knowledge, skills, and attitudes of a citizen necessary for ensuring his/her own financial well-being and the financial well-being of his/her family within the present society, and for his/her active involvement in the market of financial products and services. A financially literate citizen is familiar with the issues of money and prices, and is able to manage his/her personal and/or family budget responsibly, including the management of financial assets and liabilities in consideration of changing life situations." [7]

The President's Advisory Council on Financial Literacy (PACFL), convened to "improve financial literacy among all Americans," defines financial literacy and financial education as follows:

- Financial literacy: the ability to use knowledge and skills to manage financial resources effectively for a lifetime of financial well-being.
- Financial education: the process by which people improve their understanding of financial products, services and concepts, so they are empowered to make informed choices, avoid pitfalls, know where to go for help and take other actions to improve their present and long-term financial well-being. [6]

Organization for Economic Co-operation and Development (OECD) defines financial education as *"the process by which financial consumer/investors improve their understanding of financial products and concepts and, through information, instruction and/or objective advice, develop the skills and confidence to become more aware of financial risks and opportunities, to make informed choices, to know where to go for help, and to take other effective actions to improve their financial well-being."* [5]

The financial literacy is a specialized part of a wider economic literacy. With the financial literacy are also associated:

- The numerical literacy,
- Information literacy,
- Legal literacy.

Financial literacy as a management of personal or family finances includes three components:

- Monetary literacy - the competencies necessary for management of cash and cashless money,
- Price literacy - the competencies necessary for understanding the price mechanism and inflation,

- Budget literacy - the competencies necessary for the management of the personal or family budget. [2]

3 Brief pilot research project

The brief pilot research was conducted at several secondary schools for pupils with mild mental disabilities in the Czech Republic. The target group consists of 24 teachers of mathematics from these schools (19 women and 5 men). Graduate training of teachers is not important because they teach at elementary schools for pupils with mild mental disabilities.

Teachers answered a few questions in the electronic questionnaire. The following tables show temporary results of the pilot research.

Age of respondents	Number of respondents	Percentage
to 30 years	2	8,33%
31-40 years	3	12,50%
41-50 years	10	41,67%
51-60 years	5	20,83%
61 years and more	4	16,67%
Total	24	100,00%

Table 1: Age of respondents

Graduate teaching qualification	Number of respondents	Percentage
Mathematics	11	45,84%
Czech language	3	12,50%
Nature science (history)	3	12,50%
English (language)	2	8,33%
Music, Art, Technical education	5	20,83%
Total	24	100,00%

Table 2: Graduate teaching qualification

Types of literacy	Number of respondents	Percentage
Financial literacy	24	100,00%
ICT	15	62,50%
Mathematical literacy	14	58,33%
Scientific literacy	11	45,83%
Reading literacy	10	41,67%
Economic literacy	5	20,83%

Table 3: Which types of literacy do you know?

Answer	Number of respondents	Percentage
Yes	8	33.33%
Rather yes	4	16,67%
Rather not	7	29,17%
Not	5	20,83%
Total	24	100,00%

Table 4: Can you define the financial literacy?

The next guestion: Define the term "Financial literacy".

It is interesting, that none of the respondents could define the term "Financial literacy". In the previous question 8 respondents answered that they could define this term.

Answer	Number of respondents	Percentage
Yes	15	62,50%
Rather yes	6	25,00%
Rather not	2	8,33%
Not	1	4,17%
Total	24	100,00%

Table 5: Is financial literacy important for pupils with mild mental disabilities?

Answer	Number of respondents	Percentage
Yes	16	66,67%
Rather yes	5	20,83%
Rather not	2	8,33%
Not	1	4,17%
Total	24	100,00%

Table 6: Do you develop financial literacy beyond the textbook?

Answers	Number of respondents	Percentage
Textbooks	24	100,00%
Internet	21	87,50%
Didactic games	15	62,50%
Their own materials	12	50,00%
Interactive whiteboard	8	33,33%
Competitions	6	25,00%
Mathematizaton of real situation	5	20,83%
Dramatization of the situation	3	12,50%

Table 7: How do you develop financial literacy in your class?

Answer	Number of respondents	percentage
At least 2-3x a week	7	29,17%
1x a week	9	37,50%
At least 1x every 14 days	6	25,00%
Less than 1x every 14 days	2	8,33%
Total	24	100,00%

Table 8: How often do you use other materials than textbook to develop financial literacy?

Answer	Number of respondents	Percentage
Management of households	20	83,33%
Money, currency, payment	17	70,83%
Financial products - risks and rewards	5	20,83%
Financial environment - consumer rights	0	0,00%

Table 9: Which area of financial literacy is for pupils the most important?

A surprising finding was that none of the respondents consider the area "financial environment - the rights of consumer" as most important.

4 Conclusion

The teaching and learning of mathematics makes important contributions to the development of financial literacy.

School mathematics is one of the teaching areas which have a significant role in supporting young people in becoming financially literate. This role is played out across the spectrum of the types of mathematical activity that happens in school - numerical, graphical and other information, constructing and using financial models to help make financial decisions (e.g. best buy), budgeting etc.

At elementary school for pupils with mild mental disabilities, it is appropriate to apply financial (and economic) education, so that after graduation at elementary school, pupils can orientate in the basic concepts of this area. Teaching of financial literacy should lead to the independence of the pupils and their better integration to intact population.

Pupils with mild mental disabilities often do not realize the true value of money, cannot manage money well, so they often become victims of various fraudsters.

Our study confirms that it is important to teach financial literacy at elementary school.

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Unconventional methods in teaching mathematics

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Abstract. In this contribution a concept of a new seminar intended for students of teaching study programmes will be presented. The aim of the seminar is that the students of teaching study programmes were able to find and create sources and tools for appropriate and effective motivation of pupils. For this work they can use less conventional forms and methods. We will present chosen proposals that have arisen in the seminar.

Keywords: Teaching methods, didactic tool, mathematics, motivation.

Classification: C70, F90.

1 Introduction

During the years 2011 and 2012 a thorough analysis of pre-graduate preparation of mathematics teachers was carried out at the Faculty of Natural Science at Ostrava University. A comparison with the conditions that were 10 years ago was also carried out. [2], [3]. One of the conclusions of the analysis was the identification of absent subjects complementing professional didactics and experience so that the teaching competences of students – future teachers of mathematics – were developed. Other surveys among teachers with professional experience have shown that one of the worst evaluated competences of the graduates of teaching study programmes is the ability to motivate students [4]. A reaction to the conclusions of the mentioned analyses and surveys was the formation of a new seminar focused on a new, modern, unconventional and also proved forgotten and neglected methods in mathematics teaching. The aim of the seminar is that the students of teaching study programmes were able to find and create tools themselves for appropriate and effective pupils' motivation and make use of less conventional forms and methods.

The skill to motivate is one of the crucial teacher's abilities. Positive motivation leads students to the fact that they do not have to learn but they want to. Motivation is divided into a short-term and a long-term motivation. Conciousness that the things I learn are useful for me and I will make use of them in my ordinary life or they are profitable for my profession that I want to do can be considered the long-term motivation. The short-term motivation predominates with pupils and students at secondary schools. The reasons are e.g. to gain the success, appreciation by a teacher, classmates, etc. (but also a negative motivation such as fear of a bad mark). Tools and methods that satisfy person's natural curiosity belong to the short-term motivation. They make use of playfullness, creativity and they present a natural motivation. Within the seminar we focused exactly on the creation of such didactic tools that make use of the last mentioned student's motivations.

2 Created didactic tools

Under the term of didactic tools we understand all material objects that ensure, determine and make the course of teaching process more efficient. It can be real objects (products, preparations), models, projections (paintings, symbols, static projection, dynamic projection), sound, touch or literary devices and programmes. [15]

Our aim was to create tools that:

1. attract student's attention; process of getting information starts through senses and enters the short-term memory. To start the process of learning we have to initiate the receipt of information by a pupil, we have to attract his/her attention. And this is not easy. Let me quote Skalková as she says "changes in the world and culture in which children grow up have been updated" [15]. Students are used to obtaining everyday information from media that are accompanied by strong visual and audiovisual elements. Therefore, it is very difficult to compete with information of such form within traditional school education.
2. build positive emotional bond between mathematics and its intermediary (teacher) and a student. A lot of students, and actually general public, consider mathematics to be an uninteresting discipline existing separately from ordinary life, and they consider mathematics teachers to be boring and uninteresting. Unconventional didactic tools and methods can change these opinions. Moreover, information gained on the basis of emotional experience are fixed in the student's memory in a better way.

2.1 Mathematical theatre and mathematical fairy tale

Mathematical theatre is a didactic tool that apart from performing cognitive goals in the sphere of mathematics develops other communicative skills, ability to perform, movement skills, etc. We can use theatre in mathematics in three basic forms; watching mathematical theatre, performing mathematical theatre and creation of mathematical theatre [6].

As for the first two we suppose that the script is known and we make use of the created theatre in a passive or active form. In case of the third form it is necessary to set a didactic goal and then decide whether the theatre is going to be only a dramatic interpretation of a real, confirmed mathematical reality or history or we can afford to create fictitious story based on a chosed mathematical content. The second option is more suitable for mathematically matured individuals, within the preparation of university students or teachers' further education as it puts high demands on understanding mathematical terms and relations, without which the transferred information would be confusing.[1]

Mathematical fairy tale has a similar character to a script for mathematical theatre. We can use an already created literary piece of work and bring the education alive by this unconventional tool. Or we can create the fairy tale ourselves which puts high demands on mathematical and didactic maturity of an author. Mathematical fairy tales are integrated into the education of lower grades when the first images of children are created. If a child creates a false or confusing image of something, it can lead to a false construction of a whole conceptual and relational structure built on the basis of given term.

I will mention an example based on a survey [7]. One of the chemical tasks of the questionnaire survey was to answer a very simple question: if limestone is soluble in water. This question was answered correctly that it is not soluble by 30 % of respondents; 48 % answered that it is soluble. We followed up these questions

[1] For more information on creating a mathematical theatre, including demonstrations, are available in [6].

during complementary dialogues with university students who were also asked to answer this question. Correct and incorrect ratio was approximately 1:1. When we checked up which reasons led the students to choose particular answer, we found out that those students who were convinced that limestone is not soluble in water had a visual representation of limestone rock connected to the term of limestone. Students who answered incorrectly had a visual image of limestone stalagmites connected to the term of limestone. And this was confusing because when stalagmites are formed, the limestone is disrupted not by water but by acid which results from the reaction of water and air carbon dioxide [16].

Similarly, we discussed a fairy tale with the students of teaching study programmes, the aim of which was to introduce children with number sets. Particular separated kingdoms presented number sets in the story. This interpretation represents a risk for the creation of original image because number sets are not disjunctive sets but natural numbers form a part of integral numbers, and those numbers form a part of rational numbers, etc.

When creating mathematical fairy tales and scripts to mathematical theatre it is thus necessary to eliminate risk and, on contrary, make use of the occasion to create the most effective image out of the mathematical entity and their attributes.

2.2 Mathematical comics and cartoons

Other didactic tools that can be used in teaching are connected to pictures. We can also make use of already created cartoons and comics to brighten the education up, to attract the attention but also to fix mathematical pieces of knowledge and repetition.

In the examples in Figures 1–3 it is obvious that students will not understand the pictures if they do not know the symbol of infinity, the term of absolute value or the meaning of cartesian coordinates for definite determination of the point location in the plane.

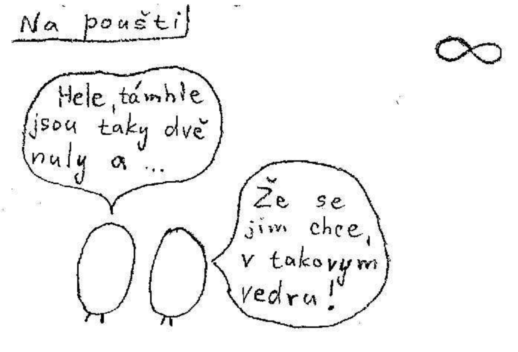


Figure 1: Cartoon of the author Lenka Ticháčková

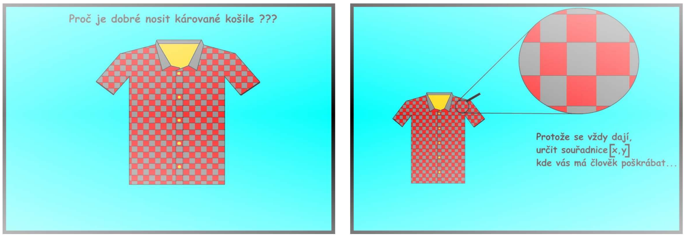


Figure 2: Cartoon of the authors Denisa Blachová & Matěj Rybář

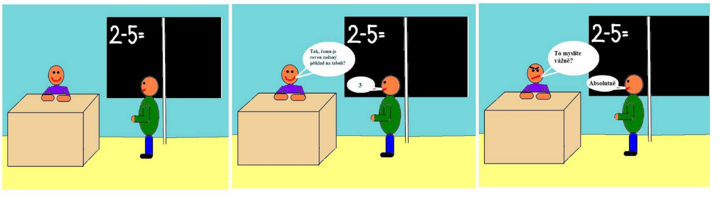
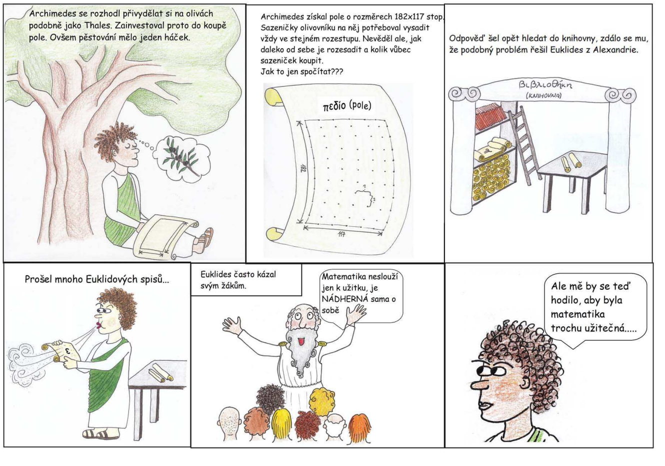


Figure 3: Cartoon of the authors Denisa Blachová & Matěj Rybář

We can make use of the comics in an exposure phase when we introduce the students with new problems or historical context. It is also possible to set a task that students try to find factual errors or find out what is put in the comics on the basis of documented historical facts and what is fictitious information complementing the story. This is the way how students will be stimulated to individual work, searching for information and working with other sources of information.



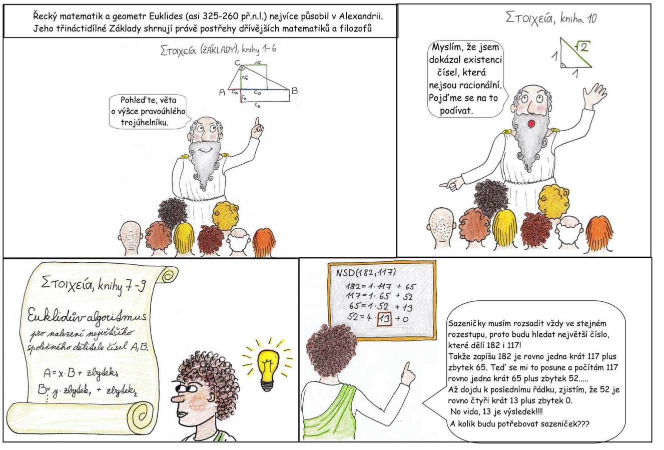


Figure 4: Part of mathematical comic; authors Magdalena Kaczurová, Jitka Pazderová & Martina Sikorová

Other possibility is to involve students straight into the creation of cartoons and comics, by which we will make use of inter-subject relations, develop students' fine arts abilities, imagination and creativity.

2.3 Mathematical games

A separate chapter describes didactic games. According to the Pedagogical dictionary Didactic game is an analogy of a spontaneous children activity that follows (for students not always in an obvious way) didactic goals. It can take place in a classroom, gym, playground, village or in the countryside. It has its own rules, it demands continuous control, final evaluation. It is intended for both individuals and groups of students where the role of a tutor has a wide range, from a head organizer to an observer. Its advantage is a stimulation charge because it awakens interest, increases pupils' involvement in performed activities, stimulates their creativity, spontaneity, cooperation and competitiveness, forces them to make use of various pieces of knowledge, skills and use life experience. [13]

Analogies of popular board and party games were created within the seminar. The games serve for practising basic mathematical knowledge and arithmetic skills (mathematical Bingo, board game start – finish) or for more demanding activities when a player has to imagine the mathematical object in a pantomimic way or by an appropriate drawing (mathematical Activities). Examples and detailed descriptions of games can be found in various students' final essays [1], [15] or in articles [8].

3 Conclusion

Created didactic tools usually connect several teaching methods in themselves, which in their mutual connection create unconventional way to student's activation and motivation to learn not only in mathematics lessons.

The creation of didactic tools itself has the importance for the preparation and further education of teachers themselves. In terms of the piece of knowledge type

Didactic tool	**Teaching methods – classification according to didactic aspect**	**Teaching methods – classification according to process aspect**
Theatre	Cooperative education, skills training	Motivation, exposition, fixation, application
Fairy tales	Definition, dialogue, dynamic projection, discussion	Motivation, exposition
Jokes	Projection	Motivation, fixation, application
Comics	Dynamic projections	Motivation, exposition , fixation
Games	Cooperative education, skills training	Motivation, fixation,application

Table 1: Possible use of created didactic tools from the point of view of teaching methods in terms of source of knowledge and piece of knowledge type (didactic aspect) and from the point of view of educational process phases (process aspect) [10]

we can mention the use of practical methods during teachers' preparation, from the point of view of their activity we often make use of a group work method or scholastic and problem methods. We involve fixation methods, application methods and last but not least motivation methods into the students' and teachers' educational process

Created didactic tools thus not only make learning process more effective from a pupil's point of view but also from the point of view of students of teaching study programmes and teachers within their lifelong learning.

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Two Important Heuristic Strategies: Reformulation and Generalization of Problems

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Abstract. One of the most important skills we need to develop in future teachers of mathematics is the skill of problem-solving. Strategies are tools for solving problems and since antiquity, mathematicians have devised strategies to help them solve problems. It is obvious from their work that even then, mathematicians knew the strategy of generalization and that of reformulation of the problem. We will show you examples of the use of these strategies dating from the time of ancient Greece.

Keywords: Problem, problem solving, problem-solving strategies, strategy of generalization, strategy of reformulation, Pythagorean theorem, regular pentagon, golden section, golden ratio.

Classification: D50, C30

Devoted to professor Petr Vopěnka

To be effective teachers of mathematics, one of the most important skills we need to develop is the skill of problem-solving. A problem can be viewed as a situation in which neither the solution nor the method of solution is readily apparent. One's level of mathematical experience will influence whether or not a particular situation is a problem – a challenging problem for a student may be a trivial exercise for a research mathematician.

In a problem-solving situation, a student has a goal, namely the solution to the problem. However, he may not have the means to reach it immediately. Solving the problem consists of constructing or discovering the means. This process sometimes involves asking questions (that may remain unanswered), making false starts, and encountering dead ends. However, all of these can provide ideas for solving the given problem and perhaps others. Strategies are tools for discovering or constructing means to reach a goal. Often, a problem can be solved in more than one way.

There are many strategies which mathematicians use when solving problems. We list several:

Investigative strategies:

- Trial and error
- Guess-check-revise
- Systematic experimentation

Others:

- Analogy
- Examination of a simpler case
- Generalization

- Concretization
- Working backward
- Examination of a related problem
- Drawing a sketch or diagram (geometric approach)
- Writing an equation (algebraic approach)
- Identifying a subgoal (nearer goal)
- Reformulation of the problem

Prof. Vopěnka contributed substantially to the creation of the following two examples. The first example will demonstrate the heuristic strategy of generalization and the second the strategy of reformulation of the problem.

For our example of the *strategy of generalization*, we choose the Pythagorean Theorem and one of its generalizations (there are many). This example will demonstrate that it is possible that a more general problem can be more easily solved than the original one. Let us start with the Pythagorean Theorem.

The Pythagorean Theorem: The area of the square constructed on the hypotenuse of a right triangle is equal to the sum of the areas of the squares constructed on the other two sides. This means that in any right triangle with hypotenuse c and legs a, b it is true that

$$c^2 = a^2 + b^2. \tag{1}$$

There are many of proofs of this theorem (one was found by a president of the United States when he was a student); we will show the proof which was constructed by Euclid in the third century BC.

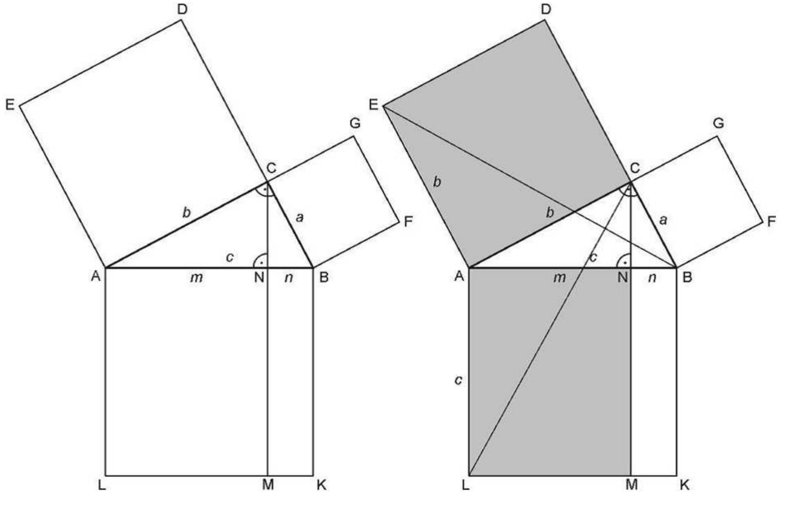

Figure 1:

Proof. Let ABC is a right triangle with a hypotenuse c (see Fig. 1 left). From the point C we construct a line perpendicular to the side AB and intersecting AB at N and LK at M. The point N divides the line segment AB into two segments having length m and n. The line CN divides the Figure 1 into two parts. We will deal first with the portion of the diagram to the left of CN. We will prove that the rectangle ALMN has the same area as the square constructed on side b of the triangle, namely the square ACDE. For this portion of the proof, examine triangles ABE and ALC in Fig. 1 right. These triangles are congruent since they each have sides of length b and c and $< \text{EAB} = < \text{LAC}$. Therefore the triangles have the same area. The side AE of triangle ABE has length b. The opposite vertex B is on the line CD, which is perpendicular to AC. Thus, the height of the triangle from side AE is the length of AC, namely b. Therefore, the area of triangle ABE is $b.b/2$. In the same way, the side AL of triangle ALC is of length c. The opposite vertex C is on line MN, which is perpendicular to AN. Thus, the height of the triangle from side AL is m. Therefore, the area of triangle ALC is $c.m/2$. Since the areas of the two triangles are the same,

$$b.b/2 = c.m/2$$

and so

$$b^2 = c.m.$$

That is, the area of rectangle ALMN is the same as that of square ACDE.

We have just proven (as a lemma to the Pythagorean Theorem) a theorem known as the Euclidean Theorem. It gives us a relationship between the area of the square on the leg b and the area of rectangle ALMN.

For the second part of the proof, in Fig. 1 left, draw line segments AF and CK. In a manner exactly analogous to the above, we show that triangles ABF and BKC are congruent, that triangle ABF has area $a.a/2$ and triangle BKC has area $c.n/2$ and so

$$a^2 = c.n.$$

That is, the area of rectangle NMKB is the same as that of square CBFG. But adding the areas of rectangles ALMN and NMKB, we obtain that of the square ALKB, which has area c^2. Thus,

$$a^2 + b^2 = cm + cn = c^2,$$

and our proof is complete. □

We now proceed to the Theorem of Pappus, a generalization of the Pythagorean Theorem. This proof followed Euclid's proof of the Pythagorean Theorem by 600 years.

The Theorem of Pappus: Let ABC be an arbitrary triangle and let ACDE and CBFG be parallelograms constructed on the sides AC and CB of the triangle (see Fig. 2a). Let H be the point of intersection of the lines ED and FG. If the parallelogram ABKL is constructed on side AB of the triangle such that the line segments AL and BK are parallel to and equal in length with the line segment HC, then

$$\text{area ACDE} + \text{area BFGC} = \text{area ALKB}.$$

Although one might expect the proof of this theorem to be even more complex than Euclid's proof of the Pythagorean Theorem, it will turn out to be simpler.

Pappus certainly knew of Euclid's proof and he used it in his elegant proof of this generalization.

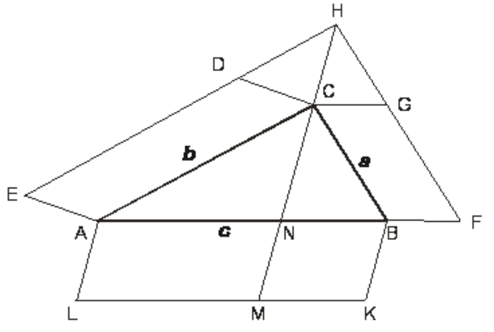

Figure 2a:

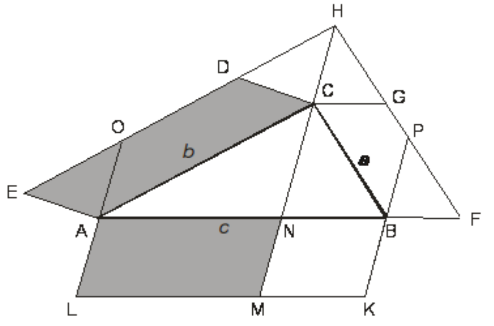

Figure 2b:

Proof. Extend line segment HC so that it intersects AB at N and LK at M (see Fig. 2b). This line divides the figure into two parts, and as in Euclid's proof above, we deal with each part separately. We begin with the part to the left of the line HM and will prove that the parallelogram ACDE has the same area as the parallelogram ALMN. Begin by extending the line segment AL so that it intersects EH at O (see Fig. 2b). Since parallelogram ACHO and parallelogram ACDE have the same base and the same height, their areas are equal. Furthermore, ACHO and ALMN have the same area; their bases, MN and CH are of equal length by construction and their altitudes are equal, namely the distance between the parallel lines HM and OL. Thus ACDE and ACHO have the same area. By writing the formula for the area of a parallelogram, we can use this fact to obtain a generalization of Euclid's Theorem.

In the same way that the point O was constructed, we can construct the point P on the line FH and following the pattern in the first part of this proof, show that the parallelogram BFGC has the same area as NMKB. Thus, we have

$$\text{area ACDE} + \text{area BFGC} = \text{area ALMN} + \text{area NMKB} = \text{area ALKB},$$

and our proof is complete. □

The Pythagorean Theorem is a special case of the Theorem of Pappus; if we begin with a right triangle and following the pattern of the proof of the Theorem of Pappus, first construct squares on the two sides of the triangle, then the further constructions in the proof of the theorem will yield a square constructed on the hypotenuse. Carrying out the details is a useful exercise. Similarly, if we begin by constructing a square on the hypotenuse of the right triangle and another on one of the sides, the further constructions will yield a rectangle on the other side of the triangle whose area is the sum of that of the two squares! Again, working through the construction is valuable.

It is important to see the strategies common to these proofs; in both, the figure was divided into two by the addition of a straight line. This is the *strategy of the auxiliary element.* Then, in each case, the areas of certain figures were computed by finding the areas of certain intermediate figures – again the strategy of the auxiliary element. Also, we found the techniques of the proof of the Pythagorean Theorem very useful in attacking the proof of the Theorem of Pappus. Here we used the *strategy of analogy* – what worked for one proof provided guidance in another. Thirdly, since the proof of the Theorem of Pappus was more transparent than that of its predecessor and since the Pythagorean Theorem is a special case, we see the value of the *strategy of generalization.* Lastly and perhaps most important is the *strategy of geometric representation.* Without the use of the figures, proofs of the theorems would be nearly impossible.

We now demonstrate the *strategy of the reformulation of the problem.* The example we use is again historical and from the work of the Pythagorean School who were masters of this strategy.

The symbol of the Pythagorean School was the regular pentagon (see Fig. 3). It was seen to represent secret wisdom, of which we will say more later.

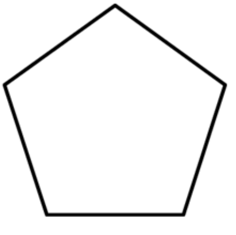

Figure 3:

The Pythagoreans set themselves the task of finding a geometric construction for the regular pentagon, which they found not to be a simple task. During their struggle, they found the relationship between the regular pentagon and the golden section ratio; it is the ratio of the length of a diagonal of the pentagon to the length of a side (see Fig. 4). The problem of the construction of the pentagon became that of constructing the golden section.

Problem 1: Construct a regular pentagon

has thus been reformulated into

Problem 2: Construct the golden section.

Once the second has been solved, the first becomes a triviality. If we specify the length of the diagonal, we can then divide the diagonal in the golden section ratio to obtain the length of a side of the pentagon. Thus, we can construct a pentagon whose diagonal is a given length. If we wish to construct a pentagon whose sides are a given length, we can simply make use of similarity.

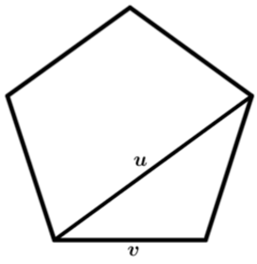

Figure 4:

Solving the problem of constructing the regular pentagon using the golden section ratio is an example of the strategy of identifying a subgoal. Often, the original problem may be solved when the solution to a somewhat easier or more familiar problem has been obtained. In that case, finding the solution to the easier problem has become a subgoal to the primary goal of solving the original problem.

A note on the golden section and the regular pentagon:

What is the golden section?

Given a line segment of length u, if the point Z divides this segment into two shorter segments of lengths x and y such that $u/x = x/y$ (see Fig 5), then we say that x and y are in the golden section ratio. We refer to division of the segment of length u in this way as the golden section.

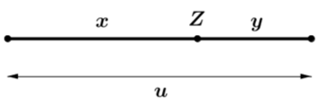

Figure 5:

How can the golden section be constructed?

Our task, given the segment of length u, is to find the point Z.

The Greek mathematicians constructed the golden section using a right triangle with sides of length u and $u/2$ as shown in Fig. 6. First, the point P is found by striking an arc of radius $u/2$ with center at B. The length of the segment PA is then the desired value of x. Finally, striking an arc of radius x and center A gives us the point Z, the golden section. The segment u is approximately 1.618 times the length of the segment x.

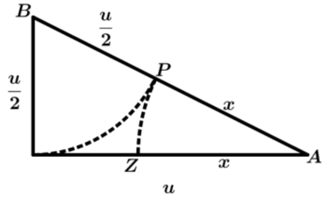

Figure 6:

How does the golden section occur in the regular pentagon?

We can see that quadrilateral ABCF in Fig. 7 left left is a rhombus with side of length v. In Fig 7 right we have constructed the smaller regular pentagon EFDGH with sides of length w. Since the large pentagon ABCDE and the small pentagon EFDGH are similar, corresponding lengths are in the same proportion so $u/v = v/w$. In Fig. 7 right, the point F is the golden section of the diagonal AD.

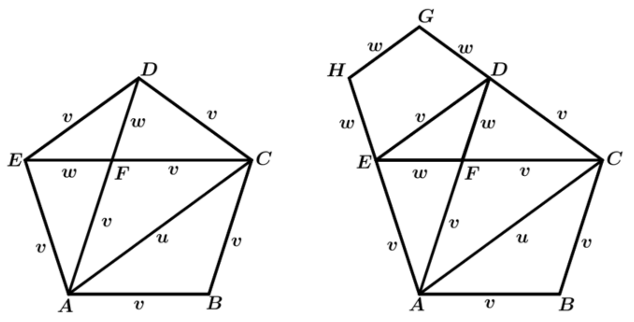

Figure 7:

The regular pentagon was to the Pythagoreans a symbol for secret wisdom. We may ask what wisdom or knowledge it hid. Perhaps it is two-fold: intersecting diagonals intersect at a point that divide them in the golden section. Also, the ratio of the length of a diagonal to the length of a side is the golden ratio, considered by some to be one of the most beautiful of mathematical concepts. But this is knowledge, not wisdom. Could it be that the wisdom lies in the fact that these mathematical truths are not obvious? Their beauty lies hidden, waiting to be discovered by anyone willing to study, explore, and overcome difficulties.

But yet another mystery lies in the pentagon and the golden section. We present it by asking and answering a question.

Question: If we let φ stand for the golden ratio (recall, φ is approximately 1,618), for what two natural numbers m and n is it true that $m/n = \varphi$?

Answer: There are no such numbers!

Proof. (by contradiction) Assume there are a pair of natural numbers whose ratio is φ. Let m be the smallest for which there is an n such that $m/n = \varphi$. Since we know that $\varphi > 1$, n is smaller than m. But since $m/n = \varphi$, then $m/n = n/(m-n)$ and we have found another pair of integers whose ratio is φ and where the larger of the two numbers is less than m. Thus we have a contradiction and our proof is complete. □

Consequence: The side and diagonal of a regular pentagon are incommensurate.

Another "mystery", one not pleasing to the Pythagoreans, was the fact that the lengths of the diagonal and side of the regular pentagon are incommensurable. This flew in the face of their belief that "all is number"; here were two lengths whose ratio could not be expressed in terms of whole numbers The man who discovered the incommensurate nature of the ratio of the diagonal and the side of the square to be incommensurable was said to be cursed by the Pythagorean school (and perhaps murdered) since the existence of incommensurate lengths was seen as a fatal flaw to their philosophy. Thus, the school concealed this “mystery” of the regular pentagon. The lengths of the diagonal and a side of a square are also incommensurate, but the proof is more complicated and lacks the beauty of that for the golden ratio. The incommensurability of the length of a side and the length of a diagonal of the regular pentagon became a closely-guarded secret of the Pythagoreans.

A more modern "mystery" is whether such incommensurable lengths can be found among the sides and diagonals of regular polygons. And that can lead to yet another investigation.

Conclusion: Heuristic strategies are commonly used by creative mathematicians when they solve problems. Indeed, many do so without even recognizing their use! Teachers of mathematics, should use these strategies when they solve problems with their students and clearly identify them when they are used. Then their students can begin to use these strategies themselves and enhance their own problem-solving skills.

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The symmetry of the parabola order three and four

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Abstract. The second order polynomial function $f_2 : \mathbb{R} \to \mathbb{R}$, $f_2(x) = ax^2+bx+c$, shortly the second order function has got its associated graph a parabola (order 2). The symmetry properties of this function are known as well by most of the secondary school students as well. This function has got as symmetry axis the line: $x = -\frac{b}{2a}$,in other words: $f(-\frac{b}{2a} - x) = f(-\frac{b}{2a} + x)$, is satisfied for all $x \in \mathbb{R}$. This symmetry is used to establish the relation between the coefficients and the position of the roots of the equation order 2, related to one or two fixed numbers, see for example [1], or [2]. The naturally arising question is if the polynomial function order 3 and 4 have got any symmetry or not? We will see in the sequel that the function $f_3 : \mathbb{R} \to \mathbb{R}, f_3(x) = ax^3 + bx^2 + cx + d$ has a symmetry point, we will denote it by $S(x_s, y_s)$, for which the relation: $f_3(x_s) - f(x_s - x) = f(x_s + x) - f(x_s)$ holds for all $x \in \mathbb{R}$. Similarly, we will show that the function $f_4 : \mathbb{R} \to \mathbb{R}, f_4(x) = ax^4 + bx^3 + cx^2 + dx + e$ has a kind of skew symmetry, which can be visualised due to the so called inflection line.

Keywords: Symmetry of polynomials, pointwise symmetry, axial symmetry

Classification: G10.

1 The symmetry of polynomial order three

Statement. The function $f_3 : R \to R, f_3(x) = ax^3+bx^2+cx+d$ has got the symmetry point $S(x_s, y_s)$, for which the relation: $f_3(x_s)-f_3(x_s-x) = f_3(x_s+x)-f_3(x_s)$, holds for all $x \in \mathbb{R}$.

Proof. The above mentionned point will be

$$S = \left(\frac{b}{3a}, f_3\left(-\frac{b}{3a}\right)\right),$$

in other words the inflexion point of the function. Its coordinates can be found as the root of the second order derivative:

$$f_3^{"}(x) = 6ax + 2b.$$

All we need to show is that:

$$f_3\left(-\frac{b}{3a}\right) - f_3\left(-\frac{b}{3a} - x\right) = f_3\left(-\frac{b}{3a} + x\right) - f_3\left(-\frac{b}{3a}\right),$$

for all real x.

We have to compute the following three values:

$$f_3\left(-\frac{b}{3a}\right), f_3\left(-\frac{b}{3a} + x\right) \text{ and } f_3\left(-\frac{b}{3a} - x\right)$$

Computing the details as follows:

$$f\left(-\frac{b}{3a}\right) = a\left(-\frac{b}{3a}\right)^3 + b\left(-\frac{b}{3a}\right)^2 + c\left(-\frac{b}{3a}\right) + d$$

$$f\left(-\frac{b}{3a}+x\right)=a\left(-\frac{b}{3a}+x\right)^3+b\left(-\frac{b}{3a}+x\right)^2+c\left(-\frac{b}{3a}+x\right)+d$$

$$f\left(-\frac{b}{3a}-x\right)=a\left(-\frac{b}{3a}-x\right)^3+b\left(-\frac{b}{3a}-x\right)^2+c\left(-\frac{b}{3a}-x\right)+d,$$

and

$$f\left(-\frac{b}{3a}\right)=-a\frac{b^3}{27a^3}+b\frac{b^2}{9a^2}-c\frac{b}{3a}+d$$

$$\begin{aligned} f\left(-\frac{b}{3a}+x\right) &= -a\frac{b^3}{27a^3}+3a\frac{b^2}{9a^2}x-3a\frac{b}{3a}x^2+ax^3+ \\ &\quad +b\frac{b^2}{9a^2}-2b\frac{b}{3a}x+bx^2-c\frac{b}{3a}+cx+d \end{aligned}$$

$$\begin{aligned} f\left(-\frac{b}{3a}-x\right) &= -a\frac{b^3}{27a^3}-3a\frac{b^2}{9a^2}x-3a\frac{b}{3a}x^2-ax^3+ \\ &\quad +b\frac{b^2}{9a^2}+2b\frac{b}{3a}x+bx^2-c\frac{b}{3a}-cx+d, \end{aligned}$$

thus

$$f\left(-\frac{b}{3a}+x\right)+f\left(-\frac{b}{3a}-x\right)=$$
$$=-2a\frac{b^3}{27a^3}+2\frac{b^3}{9a^2}-2\frac{bc}{3a}+2d=2f\left(-\frac{b}{3a}\right).$$

In consequence: $f(-\frac{b}{3a}+x)+f(-\frac{b}{3a}-x)=2f(-\frac{b}{3a})$ and this ends the proof.

The discussion of the nature of the roots of the third order equation becomes now possible. We will discuss the nature of the roots of the equation:

$$ax^3+bx^2+cx+d=0 \tag{1}$$

Let us take the real function $f_3:\mathbb{R}\to\mathbb{R}, f_3(x)=ax^3+bx^2+cx+d$ and the first order derivative of it $f_3'(x)=3ax^2+2bx+c$.

Denote by α and β the roots of the above first derivative.

Lemma. In the case $4b^2-12ac>0,\alpha$ and β are different real numbers, and we can distinguish 3 cases:

1. The roots of the equation (1) are all real and different if and only if:
$$f(\alpha)f(\beta)<0,$$
2. The roots are real but not different if and only if: $f(\alpha)f(\beta)=0$,
3. The equation (1) has got only one real root (and two conjugated complex roots) if and only if: $f(\alpha)f(\beta)>0$.

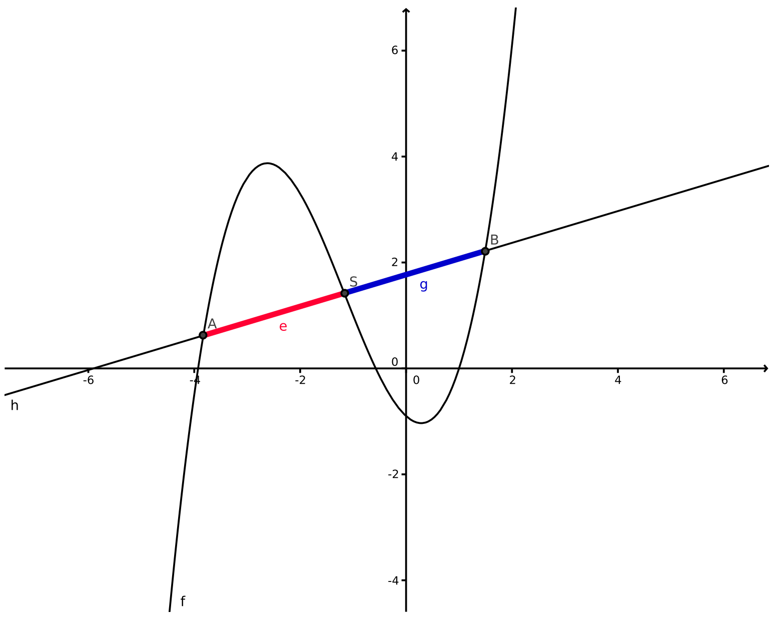

Figure 1: $a = 0.4, b = 1.4, c = -0.9, d = -0.9, p = 0.3$

Remark. The expression is $H = f(\alpha)f(\beta)$ symmetric in α and β.
Indeed:

$$\begin{aligned} H &= (a\alpha^3 + b\alpha^2 + c\alpha + d)(a\beta^3 + b\beta^2 + c\beta + d) = \\ &= a^2\alpha^3\beta^3 + ab\alpha^2\beta^2(\alpha+\beta) + ac\alpha\beta(\alpha^2+\beta^2) + ad(\alpha^3+\beta^3) + \\ &\quad + b^2(\alpha^2+\beta^2) + bc\alpha\beta(\alpha+\beta) + bd(\alpha^2+\beta^2) + c^2\alpha\beta + cd(\alpha+\beta) + d^2, \end{aligned}$$

consequently it can be expressed with the sum and the product of α and β.,

$$\alpha + \beta = -\frac{2b}{3a} \quad \text{and} \quad \alpha \cdot \beta = \frac{c}{3a}.$$

The computations lead to the expression.

$$H = \frac{4ac^3 - b^2c^2 + 4b^3d - 18abcd + 27a^2d^2}{27a^2}$$

which can be written

$$H = \frac{3(bc - 3ad)^2 + 4(ac^3 - b^2c^2 + b^3d)}{27a^2}.$$

Conclusion. As a consequence we can state that the polynomial

$$f_3 : \mathbb{R} \to \mathbb{R}, \ f_3(x) = ax^3 + bx^2 + cx + d$$

has got:

(a) three different real roots for $H < 0$,

(b) real, but not different roots for $H = 0$,

(c) One real, and two conjugated complex roots for $H > 0$,

and this result is not contradicting, but generalizing the similar result in the literature, see [3].

The Cardano's formula [3] starts with reducing the given equation to a form of $x^3 + px + q = 0$, and we look to find the solutions in the form of sum of u and v:

$$\begin{aligned} x_1 &= u + v \\ x_2 &= \varepsilon u + \varepsilon^2 v \\ x_3 &= \varepsilon^2 u + \varepsilon v \end{aligned}$$

where

$$\varepsilon = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

and

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}}, \qquad v = \sqrt[3]{-\frac{q}{2} - \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}}.$$

The expresion H will be as follows:

$$H = \frac{4p^3 + 27q^2}{27} \quad \text{or} \quad H = 4\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$$

and the same conditions are known: If $H < 0$ there are three different real roots, if $H = 0$, there are two roots coinciding and one more, real, while $H > 0$, it has two conjugated complex, and one real root.

2 The symmetry of polynomial order four

The polynomials degree four may or may not have inflection points. If a forth degree polynomial $f_4 : \mathbb{R} \to \mathbb{R}, f_4(x) = ax^4 + bx^3 + cx^2 + dx + e$ does have inflection points i and $j, i < j$, and consider a linear function $L(x)$, called *inflection line* is drawn through $(i, f_4(i))$ and $(j, f_4(j))$, the inflection line will meet the graph of the polynomial in two other points.

Let's denote their abscissas x$_L$ and x$_R$ assuming $x_L < i < j < x_R$.

Statement. The difference $p = f_4(x) - L(x)$ is a 4th degree polynomial as well, which has a vertical axis of symmetry [4].

Moreover

(1) the roots of p are $x_L < i < j < x_R$,

(2) $i + j = x_L + x_R$,

(3) $x_L = \frac{1+\sqrt{5}}{2}i + \frac{1-\sqrt{5}}{2}j, x_R = \frac{1-\sqrt{5}}{2}i + \frac{1+\sqrt{5}}{2}j$,

(4) $\frac{x_R - x_L}{j-i} = \sqrt{5}$,

(5) The areas of the three regions between the graphs of $f_4(x)$ and $L(x)$, respectively between the polynomial p and the x-axis are in proportion 1:2:1.

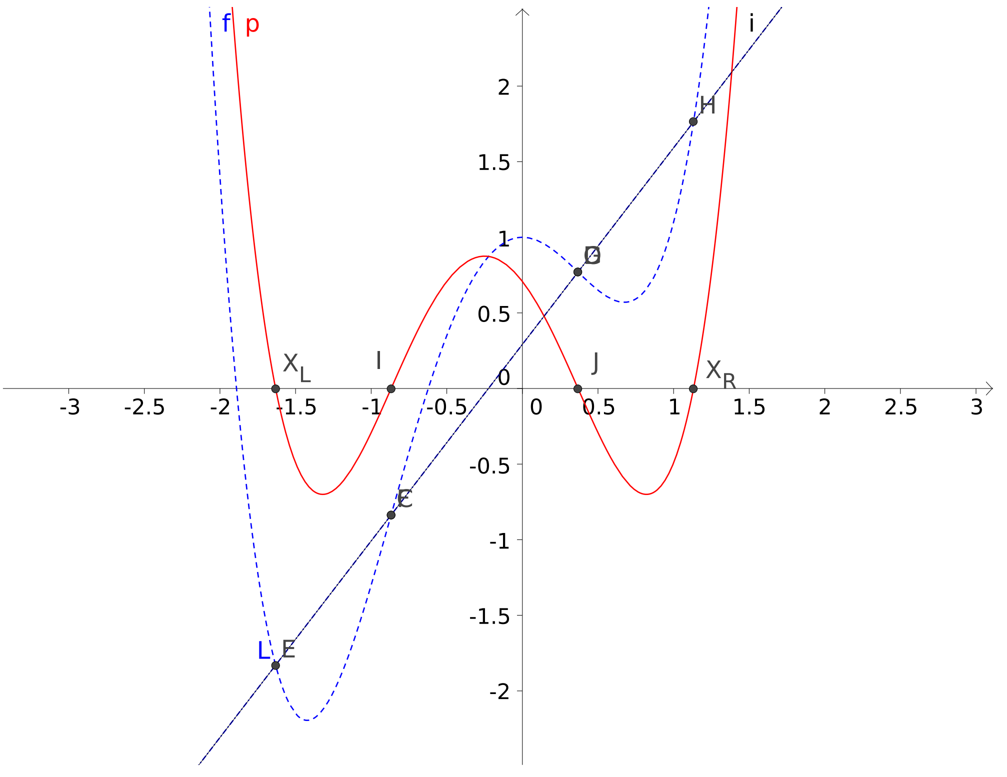

Figure 2: $a = 1.2, b = 1.2, c = -2.3, d = 0, e = 1$

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Stochastic graphs vs. in-school probability theory teaching process

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Abstract. In this paper we concern with a stochastic graph as a chart to a random game, his construction as mathematical task and his role in mathematical argumentation. It is the paper on didactics of mathematics. The organization of a phaze of mathematization, a phaze of calculation and a phaze of interpretation in mathematics for everyone are the subject of this article.

Keywords: stochastic graphs, probability, Penney-ante game

Classification: K50; K20, K30.

1 Probability theory vs. intuition

Mathematical research and discovery is not only a result of one's pure deduction, inductive thinking and analogy-based reasoning but it is also a result of intuitive thinking (see [24]). The formal approach towards mathematics is often opposed to the intuitive approach. Abstractions and schemas are contrasted to "seeing" and "perception" of general, important mathematical constructions and quantitative and space relations. The inspiration and beginning of all discoveries as well as the point that gives certainty in all kinds of reasoning and the author of new ideas, hypotheses or statements is "obviousness", "common sense", that is – intuition.

For a long time Freudenthal used to replace the word "intuition" with a phrase "shaping of mathematical objects" (see [6]). He was doing so because of a wide range of meanings that the word "intuition" has in different languages. Freudenthal also wrote (see [7]) that "intuitions without concepts are empty, and concepts without intuitions are blind".

Stochastic intuitions are the ability of drawing judgments and beliefs of probabilistic character without any conscious inference or even without perceiving the clues which justify that belief or judgement. It is an ability allowing us to estimate properly the probabilistic characteristics (the event's probability, the expected value, distribution or stochastic independence) of a given sample or population on the basis of incomplete data about the sample and without any (conscious) reasoning or analysis, when the estimation is based only on one's experience or knowledge.

The intuitive conclusions are the ones which we consider obvious, we draw them instantly, almost without thinking, without any reasoning, calculations or argumentations on the basis of images, schemes or situation models that we have in memory. Intuitive thinking is thinking about an abstract situation through its specific model (see [20]).

In [27], [28] and [29] we can find the research of psychologists A. Tversky and D. Kahneman which show that people do not have their probabilistic intuitions properly developed. Humans were not provided even with basic probabilistic intuitions through evolution.

Wrong probabilistic intuitions may be mathematically – based. They can be a result of lack of basic probabilistic, stochastic and combinatorial knowledge, but they can also rise from its poor acquisition (a formalized lecture does not eliminate mistakes in intuitive judgments). They can also have psychological background. A formal explanation of the probability theory and statistics rules is not enough to eliminate those "incorrect representations" in the process of probabilistic predicting, which is seen as an important pre-decisive process by psychologists. The psychological research show that in the process of predicting people do not use probabilistic arguments as much as they use some rules, principles and strategies.

Tversky and Kahneman analyzed the basis of incorrect representations (incorrect intuitions) in situations concerning probability estimations. They point out the divergence between a subjective probability (i.e. estimation of probability given by a person as his /her estimation of a chance of a given event to happen) and objective, normative probability resulting from a probabilistic model. They conducted the research as a part of a bigger project concerning problems of teaching mathematics. They studied the strategies used by people of different age and occupation while solving specific stochastic (combinatorial, as a matter of fact) problems.

J. M. Shaughenessy's research shows how vast is the role of personal contact between a person and empiricism (drawing lots, working with statistical data, using the pre-developped data, like the results of chance games, calculating frequencies, confronting the *a posteriori* judgments with the ones made *a priori*) in developing correct stochastic intuitions which appear in using heuristic strategies properly. The same research proves that teaching probabilistic theory in too formalized way, apart from statistics, omitting the empirical aspect of probabilistic issues and leaving out some classical paradoxes like problems – stochastic surprises does, not remove incorrect intuitions. Tversky and Kahneman emphasize the fact, that the same mistakes are made by "stochastically naive" students (the ones with no probabilistic experience) and adults – even ones who had graduated from advanced but formalized stochastic courses. They find mistakes of this kind made even by psychologists who have some knowledge of stochastics.

2 The functional teaching of mathematics

The idea of functional teaching is the basic strategy of didactically correct process of teaching-learning mathematics. It may also be seen as a basic strategy of discovering and creating mathematics by students (see [26]). It is a universal method, recommended in teaching different subjects, but in mathematics – because of an abstract and operative character of mathematical notions – it has got a particular meaning. In functional teaching we try to show mathematics from the notional side, not through the algorithms and rules, as it was in the mechanistic approach. The definitions, rules, reasoning or theorems are important, but they come later on, as a summary, a result of different activities, discovering and using algorithms. According to the integral approach, mathematics should grow from reality, everyday situations. In the functional method the objects and phenomena of the students' environment do not have to be the starting point of mathematical issues. Along with real situations we can use the ones artificially created, using special teaching aids as well as purely abstract problems. The care for precision and order, for clarity and understanding of mathematical issues, for the compatibility of school and scientific notions is vital in the functional teaching. The basis of the student's mathematical activity is his awareness of where in the "math construction" he actually is at the moment. The overriding aim of this teaching method is the student gaining operative knowledge not on the basis of chaotic trials of solving schematic problems or too "casual work", but through the student's activities carefully planned by the teacher. Only a well trained teacher, with a good knowledge of methodology can plan the student's work properly and lead the student to create sequent elements of mathematical knowledge, stressing "mathematical activity, working in mathematical world and its connection to reality, creative experience gathered by the student gradually through solving problems open for creativity at his level" (see [20]).

Through the functional teaching the constructive approach is accomplished. The student creates his own knowledge integrated with various materials and tasks, on the way of reach experience gathered in cooperation with the teacher and fellow students. However, it is not about the superficial shaping of mathematical issues leading to the answer to "what is it" question. It is about active study of techniques and methods that allow the student to solve "the how do we construct" problems. We can find the confirmation of this idea in Piaget's *Where does education* aim in an extended and supported by numerous research form. Piaget claims there that the basic condition of the whole mind shaping process, which is especially important in the matters that lead young learners to science, is using active methods of teaching. They allow the student to spontaneously search for solutions and demand each truth that is to be discovered to be rediscovered by the student and not only passed to him.

3 Probability versus stochastic games

Probability is present at every stage of teaching math teachers. But they often lack proper tools of introducing probability at school. This situation is eve a bigger challenge for primary and secondary school teachers. A real didactic suggestion is to introduce stochastic issues on the grounds of chance games that are often followed by lots of stochastic paradoxes. Solving different problems connected to those games leads to proper understanding of elementary characteristics and acquiring correct intuitions. Thanks to the paradoxes occurring in those games we can set didactic situations leading to didactic reflections both tor students and teachers. Although probability is present on the elementary and secondary stage of education of math teachers, mathematicians often lack specific tools for teaching probability. Even well trained math teachers, having broad knowledge of mathematics, usually need some additional professional training connected to teaching probability. General rules of teaching which are usually effective in other branches of mathematics are not necessarily as effective in teaching probability theory. This situation is even a greater challenge for primary school teachers. Although teachers do not need a very high level of mathematical knowledge, it is necessary for them to understand the basic notions of mathematics they teach at schools thoroughly, including deep understanding of relations and connections among different aspects of that knowledge (see [21]). The additional elements that are important in the professional teachers' knowledge are described in [1]:

a) epistemology: a reflection on meanings of different notions, like different meanings of probability (see [2]);

b) learning: foreseeing problems in the student's learning, mistakes, obstacles and strategies;

c) didactical means and methods: experience in good selection of examples and didactic situations; ability to analyze the textbooks, curricula and other documents critically; ability to adapt the statistics to different levels of education;

d) ability to engage the students in work and make them interested in what they do; taking their beliefs and attitudes into consideration;

e) interactions: ability to create effective communication in the classroom and using rating as a means of instructing students.

Classical paradoxes play a great role in teaching probability. Because of them we can organize some didactical activities for the math teachers. The aim of these activities is to provoke their reflection on the basic probabilistic notions. These activities also help the teachers understand the students' obstacles and difficulties in understanding probability and they allow them to expand their own methodological and didactical base.

Introduction of the stochastic graph into the probability teaching process is to create, develop and shape those correct stochastic intuitions in a proper way. Simultaneously, we build this process by introducing a specific kind of chance experiments and problems generated by them.

4 Penney's game and a stochastic graph

There are two possible results of a coin toss. We shall code them in such a way: o – *the result will be heads* and r – *the result will be tails*. We shall call the r result *a success* and the o result *a failure*. The result of k coin tosses, which is a k-arrangement of $\{o, r\}$ set we shall call *a series of successes and failures*, in short – *a series of k length*.

Let a and b be a defined series of successes and failures of k length. Repeating a coin toss as many times as needed to get k trial result make the a or b series is called *waiting for the a or b series* and marked as δ_{a-b}. Let us connect the events of:

A={waiting δ_{a-b} will finish with the a series},

B={waiting δ_{a-b} will finish with the b series}

with the δ_{a-b} chance experiment.

Let us mark the A event as $\{\dots a\}$ and its probability as $P(\dots a)$. The B event shall be marked as $\{\dots b\}$ and its probability as $P(\dots b)$.

In a short article [22] Walter Penney discusses repeating a coin toss as many times as needed to get three times heads or a heads-tails-heads series. Let $\delta_{ooo-oro}$ mean the described chance experiment. Penney suggests a lot game for two players. In the game the $\delta_{ooo-oro}$ experiment is conducted (it is not important who tosses the coin). One of the players wins if the experiment ends up with the *ooo* result, and the other player wins when the experiment ends with the *oro* result. The game described above we shall call $g_{ooo-oro}$. The fact that the *ooo* and *oro* series are equally possible to happen would suggest that the game is fair. But the probability that the waiting $\delta_{ooo-oro}$ will end up with the *oro* series is 0,6, while the probability that it will end up with the *ooo* series is 0,4. Penney finds the probabilities on a way of particular reasoning (see [23], p. 415) and he does not try to hide his being

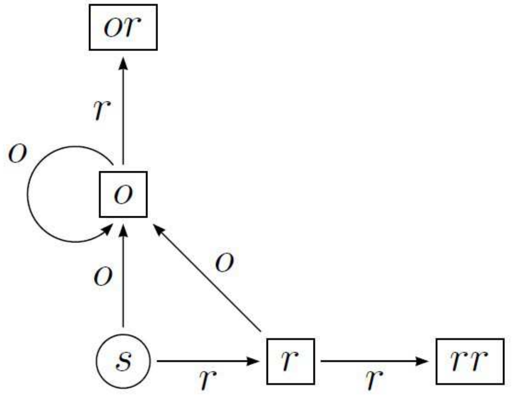

Figure 1: Stochastic graph - game g_{rr-or}

surprised by the fact that the game is not fair. The *oro* series gives the player a bigger chance to win that the *ooo* one. This is the interpretation of the results and the calculation on the real-life ground. So the *oro* series is called *better than the ooo one.*

The problem of the fairness of chance games in case of waiting for other pairs of series of heads and tails – those are called *Penney's games* – the issues connected to the paradox characteristics of the success-failure series in waiting for one of them to occur, as well as the problem of time needed for such waiting (meant as a period of time taken by the game, when time is measured with the number of coin tosses executed) are called *Penney's problems* in mathematical literature. Only in case of some pairs of heads and tails series the Penney's game is fair. Such series are called *equally good.*

Some of the results of research on the Penney's problems are gathered in [8] monograph and [25], [9], [10], [11], [12], [13], [17] and [18] articles.

A tool for examining the countable probabilistic spaces for waitings for success-failure series is a stochastic graph. Such a waiting for a series of successes and failures is a chance experiment of a random number of stages.

The research on the probabilistic space for the waiting for one of many success-failure series may be brought down to searching for the probabilities of reaching each of the absorbing levels. Waiting for a success-failure series is often interpreted as a homogeneous Markov's chain with the non-empty set of absorbing stages (see [15]) and it is suggested to use an iconic representation, along with the algebraical one, that is *a stochastic graph.* Traditionally, such calculations are based on sequences and differential equations. The essence of argumentations based on the stochastic graph is, among others, a reduction of cycles and loops on the graph (we call them reductions of the graph), or transition from a graph with unlimited number of passages to a limited-passage graph (see [14]). It is a development of methods and tools suggested long ago by Arthur Engel in [3], [4] and [5] (see also [16]).

The stages of a homogeneous Markov's chain can be interpreted as points on a plain and called *knots.* The knot that represents the beginning stage is called *starting knot*. Each knot representing an absorbing stage is called *edge knot*. If the probability of getting from a j stage to a k stage in one step is positive, then we connect those knots with the oriented section of a line or curve and we mark that section k. We call that section *an arc*. A graph constructed in such a way is an iconic representation of a Markov's chain.

At the beginning (before conducting the first stage of experiment) we place a pawn in the starting knot of the graph. If a stage ends with the j result we move the pawn along the j arc. The route of the pawn ends when it gets to an edge knot, that is at the rim of the graph (see [19]). Picture 1 shows a stochastic graph being a board of the g_{or-rr} game.

If the pawn gets to the o knot at any stage of the game, it is certain (the probability equals 1) that it will get to the knot (finish) *or* – that is the player waiting for this series wins. For the pawn getting to the *o* knot the heads must be the result of first or second toss, so the probability of this event is 0,5+0,25=0,75. The pawn gets to the *rr* knot only if the first and second toss result with tails, then the other player wins, and this happens with the probability of 0,25.

It is just one example of elementary, simple, but very elegant and making a great impression reasoning based on a stochastic graph. There are lots of such examples can be found in the quoted literature.

A natural generalization of discussed problems is replacing a coin toss with any chance experiment having two possible results of non-equal probability (that is a Bernoulli's trial) or a chance experiment having more than two results. Then we can discuss *the series of successes and failures* or *series of colors* (*flags*).

5 *The Waitings for flags* computer program

An interesting complementation of the discussed issues is a computer program called *Waitings for flags* (file cnf.exe), which allows us to gather statistical data in a quick and easy way and so formulate different assumptions on their basis.

The program has its limitations:

1) possible number of series: 1 to 4;
2) the number of results in a single experiment: 2 to 12;
3) the probability of each result in a single experiment: measurable, given with maximum accuracy to 12 decimal places.

After inserting the number of results in a single experiment (`the number of results for a` n`-trial`) their labels appear. They are: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B. The labels can be replaced with other available on the keyboard. We enter the probability of each result next to its label. To check if the probabilities sum up to 1 we shall click the `TOTAL` button. When we click `the classical distribution` button the program will automatically insert the same probability in every window. After setting the result labels in a single experiment we enter the number of color series (flags) and, in appearing windows, the color series coded with the result labels for a single experiment. When all the data is entered we click the `READY` button. The probabilities we look for and estimated time of the experiment (game) will show in a new window. In the right upper corner there is `a Simulation` window. After inserting the number of experiments we wish to simulate and clicking `START` a new window opens. It is a protocol of conducting a required number of experiments.

In the new window, in its upper part, there is a number of waitings resulting with specified series and their frequencies. We can simulate up to 1000 experiments.

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Difference of a function in the space of strictly monotonic functions

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Abstract. The space of strictly monotonic functions is defined as a set of all continuous real-valued functions of a real variable x from R which map one-to-one the interval R on the interval (a, b), where a and b are real or extended numbers. In this space the theory of Abel functional equations is studied. The Abel functional model reduces under specialization to a linear functional or difference equations. In the paper we introduce a new definition of a difference of a function which treats all classes of strictly monotonic functions.

Keywords: Space of strictly monotonic functions, difference of a function, linear difference equations, linear functional equations.

Classification: I20.

1 Introduction

We will introduce the set of strictly monotonic functions, where a method is presented for solving a new model equation. This method can be applied to solve, in particular, some classical linear functional and difference equations, for the classical theory see [1], [2], [2], [8] and [9].

Later in the paper we will study newly defined term of a difference of a function in the set of strictly monotonic functions.

Definition 1.1. *Given $a, b \in \bar{R}$, $a < b$. The set of all continuous functions on $(-\infty, \infty)$ which map one-to-one the interval $(-\infty, \infty)$ onto (a, b) will be denoted by the symbol S_a^b and called a* **space of strictly monotonic functions.**

Definition 1.2. *An arbitrarily chosen increasing function $X = X(x)$, $X \in S$, will be called a* **canonical function** *in S. The inverse to the canonical function X will be denoted by X^*.*

Definition 1.3. *Let $\alpha, \beta \in S$. The composite function $\gamma = \alpha(X^*(\beta(x)))$, shortly $\gamma = \alpha X^* \beta(x)$, will be called a* **product** *and denoted $\gamma = \alpha \circ \beta$.*

Remark 1.4. *The set S with the operation of multiplication $\circ$ forms a non-commutative group.*

Definition 1.5. *Let $\phi \in S$, ϕ increasing and $X \in S$ be a canonical function, $\phi > X$ on $(-\infty, \infty)$. The* **iterates of ϕ in S** *are given by*

$$\phi^0(x) = X(x)$$
$$\phi^{n+1}(x) = \phi \circ \phi^n(x), \quad n = 0, 1, 2, \dots$$
$$\phi^{n-1}(x) = \phi^{-1} \circ \phi^n(x), \quad n = 0, -1, -2, \dots,$$

$x \in (-\infty, \infty)$, where ϕ^{-1} is the inverse element according to $\circ$.

2 Difference of a function in S

We introduce a new definition of a difference of a function, which will treat all classes of strictly monotonic functions.

Let $X \in S$ be a canonical function. Let $\Phi \in S$, $\Phi(x) > X(x)$ for $x \in \mathcal{J}$. Let $f \in C_0(J)$.

Definition 2.1. *The number $f \circ \Phi^1(x_0) - f \circ \Phi^0(x_0)$, where $\circ$ denotes the multiplication in S, is called a* **difference of a function** $f = f(x)$ **relative to a function** Φ **at point** $x_0 \in \mathcal{J}$. *It is denoted by Δ_Φ. We write*

$$\Delta_\Phi f(x_0) = f \circ \Phi^1(x_0) - f \circ \Phi^0(x_0)$$

or briefly

$$\Delta_\Phi f(x_0) = f \circ \Phi(x_0) - f(x_0).$$

The point x_0 is the **initial point** *of the difference. The difference of a function f is defined in a natural way on a group of functions*

$$\mathcal{G} = \{\Phi^k(x)\}_{k=-\infty}^{+\infty}.$$

Definition 2.2. *The function*

$$\Delta_\Phi f(x) = f \circ \Phi^1(x) - f \circ \Phi^0(x). \tag{1}$$

will be called the **difference of a function** $f = f(x)$ **relative to the function** $\Phi \in S$**.**

Formula (1) implies $\Delta_\Phi f(x) \in C_0(\mathcal{J})$.

Lemma 2.3. *Let $f, g \in C_0(\mathcal{J})$, $a \in \mathbb{R}$. Then*

1. $\Delta_\Phi(f(x) \pm g(x)) = \Delta_\Phi f(x) \pm \Delta_\Phi g(x)$,
2. $\Delta_\Phi af(x) = a\Delta_\Phi f(x), \quad a \in \mathbb{R}$.

Proof.

$$\begin{aligned} \Delta_\Phi(f(x) \pm g(x)) &= (f \pm g) \circ \Phi^1(x) - (f \pm g) \circ \Phi^0(x) \\ &= (f \circ \Phi^1 - f \circ \Phi^0) \pm (g \circ \Phi^1 - g \circ \Phi^0) \\ &= \Delta_\Phi f \pm \Delta_\Phi g \end{aligned}$$

and

$$\begin{aligned} \Delta_\Phi af(x) &= af \circ \Phi^1 - af \circ \Phi^0 \\ &= a(f \circ \Phi^1 - f \circ \Phi^0) \\ &= a\Delta_\Phi f. \end{aligned}$$

Example 2.4. *Find $\Delta_\Phi f(3)$ if $f \in S$.*

Solution: $\Delta_\Phi f(3) = f \circ \Phi^1(3) - f \circ \Phi^0(3) = fX^*\Phi(3) - f(3)$.

Example 2.5. *Find the difference of a function* $f = aX^2(x) + bX(x)$ *relative to* $\Phi \in S$, *where* $\Phi > X$ *and* $X \in S$ *a canonical function, with* $a, b \in \mathbb{R}$.

Solution: $\Delta_\Phi f(x) = aX^2 \circ \Phi^1(x) + bX^1 \circ \Phi^1(x) - aX^2 - bX(x) = a\Phi^2(x) + b\Phi(x) - aX^2(x) - bX(x) = a[\Phi^2(x) - X^2(x)] + b[\Phi(x) - X(x)]$.

Especially in $S_{-\infty}^{\infty}$, *where* $X(x) = x$, $\Phi(x) = x + 1$, *then* $f = ax^2 + bx$, $\Delta f = a(x+1)^2 + b(x+1) - ax^2 - bx = a(2x+1) + b$.

2.1 Difference of a product

The difference of a function f relative to Φ is defined by the formula

$$\Delta_\Phi f = f \circ \Phi^1(x) - f \circ \Phi^0(x).$$

Lemma 2.6. *The difference of a product of two functions* u, v *relative to* Φ *is defined by the formula*

$$\Delta_\Phi uv = v \circ \Phi(x) \Delta_\Phi u(x) + u(x) \Delta_\Phi v(x). \tag{2}$$

Proof.

$$\begin{aligned} \Delta_\Phi uv &= (uv) \circ \Phi - uv \\ &= (u \circ \Phi)(v \circ \Phi) - uv \\ &= (u \circ \Phi)(v \circ \Phi) - uv \circ \Phi + uv \circ \Phi - uv \\ &= [u \circ \Phi - u] v \circ \Phi + u[v \circ \Phi - v] \\ &= v \circ \Phi \Delta_\Phi u + u \Delta_\Phi v. \end{aligned}$$

Lemma 2.7. *The difference of a product of three functions* u, v, w *relative to* Φ *is defined by the formula*

$$\Delta_\Phi uvw = v \circ \Phi w \circ \Phi \Delta_\Phi u + uw \circ \Phi \Delta_\Phi v + uv \Delta_\Phi w.$$

Proof. If we set $uv = U$ then according to (2) we get

$$\Delta_\Phi Uw = w \circ \Phi \Delta_\Phi U + U \Delta_\Phi w$$

and because

$$\Delta_\Phi U = \Delta_\Phi uv = v \circ \Phi \Delta_\Phi u + u \Delta_\Phi v$$

then

$$\begin{aligned} \Delta_\Phi uvw &= w \circ \Phi [v \circ \Phi \Delta_\Phi u + u \Delta_\Phi v] + uv \Delta_\Phi w \\ &= v \circ \Phi w \circ \Phi \Delta_\Phi u + uw \circ \Phi \Delta_\Phi v + uv \Delta_\Phi w. \end{aligned}$$

Lemma 2.8. *The difference of a product of* n *functions* $u_1, u_2, \ldots, u_n$ *is given by*

$$\begin{aligned} \Delta_\Phi u_1 u_2 \ldots u_n \quad = \quad & u_2 \circ \Phi u_3 \circ \Phi \cdots u_n \circ \Phi \Delta_\Phi u_1 \\ & + u_1 u_3 \circ \Phi u_4 \circ \Phi \cdots u_n \circ \Phi \Delta_\Phi u_2 \\ & \vdots \\ & + u_1 u_2 \cdots u_{n-2} u_n \circ \Phi \Delta_\Phi u_{n-1} \\ & + u_1 u_2 \cdots u_{n-1} \Delta_\Phi u_n. \end{aligned}$$

Proof. The formula can be proved by the mathematical induction. It is valid for $k = 2$. Suppose it is valid for k and we will prove it is valid for $k+1$. Define $U = u_1 u_2 \dots u_k$, $V = u_{k+1}$. Then $\Delta_\Phi UV = V \circ \Phi \Delta_\Phi U + U \Delta_\Phi V$. Thus

$$\begin{aligned}
\Delta_\Phi u_1 u_2 \cdots u_k u_{k+1} &= u_{k+1} \circ \Phi \Delta_\Phi u_1 \cdots u_k + u_1 \cdots u_k \Delta_\Phi u_{k+1} \\
&= \{u_2 \circ \Phi u_3 \circ \Phi \cdots u_k \circ \Phi \Delta_\Phi u_1 \\
&\quad + u_1 u_3 \circ \Phi u_4 \circ \Phi \cdots u_k \circ \Phi \Delta_\Phi u_2 \\
&\quad \vdots \\
&\quad + u_1 u_2 \cdots u_{k-1} \Delta_\Phi u_k\} u_{k+1} \circ \Phi \\
&\quad + u_1 u_2 \dots u_k \Delta_\Phi u_{k+1} \\
&= u_2 \circ \Phi u_3 \circ \Phi \dots u_k \circ \Phi u_{k+1} \circ \Phi \Delta_\Phi u_1 \\
&\quad + u_1 u_3 \circ \Phi u_4 \circ \Phi \dots u_k \circ \Phi u_{k+1} \circ \Phi \Delta_\Phi u_2 \\
&\quad \vdots \\
&\quad + u_1 u_2 \dots u_{k-1} u_{k+1} \circ \Phi \Delta_\Phi u_k \\
&\quad + u_1 u_2 \dots u_k \Delta_\Phi u_{k+1}
\end{aligned}$$

2.2 Difference of a determinant

Consider a 2×2 determinant

$$D^2 = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = f_{11} f_{22} - f_{12} f_{21}.$$

If we apply the rule for a difference of two functions we get

$$\begin{aligned}
\Delta_\Phi D^2 &= f_{22} \circ \Phi \Delta_\Phi f_{11} + f_{11} \Delta_\Phi f_{22} - f_{21} \circ \Phi \Delta_\Phi f_{12} - f_{12} \Delta_\Phi f_{21} \\
&= \begin{vmatrix} \Delta_\Phi f_{11} & \Delta_\Phi f_{12} \\ f_{21} \circ \Phi & f_{22} \circ \Phi \end{vmatrix} + \begin{vmatrix} f_{11} & f_{12} \\ \Delta_\Phi f_{21} & \Delta_\Phi f_{22} \end{vmatrix}.
\end{aligned}$$

Theorem 2.9. *The difference of the determinant D^k, where*

$$D^k = \begin{vmatrix} f_{11} & \cdots & f_{1k} \\ \vdots & & \vdots \\ f_{k1} & \cdots & f_{kk} \end{vmatrix},$$

can be expressed as a sum of k determinants

$$
\Delta D^k = \begin{vmatrix} \Delta f_{11} & \cdots & \Delta f_{1k} \\ f_{21}\circ\Phi & \cdots & f_{2k}\circ\Phi \\ \vdots & & \vdots \\ f_{k1}\circ\Phi & \cdots & f_{kk}\circ\Phi \end{vmatrix} + \begin{vmatrix} f_{11} & \cdots & f_{1k} \\ \Delta f_{21} & \cdots & \Delta f_{2k} \\ f_{31}\circ\Phi & \cdots & f_{3k}\circ\Phi \\ \vdots & & \vdots \\ f_{k1}\circ\Phi & \cdots & f_{kk}\circ\Phi \end{vmatrix}
$$

$$
+ \begin{vmatrix} f_{11} & \cdots & f_{1k} \\ f_{21} & \cdots & f_{2k} \\ \Delta f_{31} & \cdots & \Delta f_{3k} \\ f_{41}\circ\Phi & \cdots & f_{4k}\circ\Phi \\ \vdots & & \vdots \\ f_{k1}\circ\Phi & \cdots & f_{kk}\circ\Phi \end{vmatrix} + \cdots + \begin{vmatrix} f_{11} & \cdots & f_{1k} \\ \vdots & & \vdots \\ f_{(k-1)1} & \cdots & f_{(k-1)k} \\ \Delta f_{k1} & \cdots & \Delta f_{kk} \end{vmatrix}. \tag{3}
$$

The determinants are denoted as D_1^k, D_2^k, ..., D_k^k. Then we can write

$$
\Delta D^k = \sum_{i=1}^{k} D_i^k.
$$

Proof. The formula is valid for $n = 2$. We will use mathematical induction. Let us consider the determinant of the $(k+1)$-th order

$$
D^{k+1} = \begin{vmatrix} f_{11} & \cdots & f_{1(k+1)} \\ \vdots & & \vdots \\ f_{(k+1)1} & \cdots & f_{(k+1)(k+1)} \end{vmatrix}.
$$

Expand the determinant by cofactors using elements of the first row

$$
D^{n+1} = f_{11}M_{11} + \ldots + f_{1(k+1)}M_{1(k+1)},
$$

where M_{1i} denotes the minor of element f_{1i}, $i = 1, 2, \ldots, k+1$. Let us calculate the difference

$$
\begin{aligned}
\Delta D^{k+1} &= \Delta\{f_{11}M_{11} + \cdots + f_{1(k+1)}M_{1(k+1)}\} \\
&= \sum_{i=1}^{k+1} \{M_{1i}\circ\Phi\Delta f_{1i} + f_{1i}\Delta_\Phi M_{1i}\}.
\end{aligned}
$$

Minors M_{1i} are determinants of k-th order and according to the induction hypothesis, formula (3) holds.

First we calculate $\sum_{i=1}^{k+1} M_{1i} \circ \Phi \Delta f_{1i}$. We get a determinant D_1^{k+1} of $(k+1)$-th order

$$D_1^{k+1} = \begin{vmatrix} \Delta f_{11} & \cdots & \Delta f_{1(k+1)} \\ f_{21} \circ \Phi & \cdots & f_{2(k+1)} \circ \Phi \\ \vdots & & \vdots \\ f_{(k+1)1} \circ \Phi & \cdots & f_{(k+1)(k+1)} \circ \Phi \end{vmatrix}.$$

Now we will calculate

$$\sum_{i=1}^{k+1} f_{1i} \Delta_\Phi M_{1i}.$$

Each factor $\Delta_\Phi M_{1i}$ of term i in the preceding summation equals the following sum of determinants:

$$\begin{vmatrix} \Delta f_{21} & \cdots & \Delta f_{2(i-1)} & \Delta f_{2(i+1)} & \cdots & \Delta f_{2(k+1)} \\ f_{31} \circ \Phi & \cdots & f_{3(i-1)} \circ \Phi & f_{3(i+1)} \circ \Phi & \cdots & f_{3(k+1)} \circ \Phi \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{(k+1)1} \circ \Phi & \cdots & f_{(k+1)(i-1)} \circ \Phi & f_{(k+1)(i+1)} \circ \Phi & \cdots & f_{(k+1)(k+1)} \circ \Phi \end{vmatrix}$$
$$+ \begin{vmatrix} f_{21} & \cdots & f_{2(i-1)} & f_{2(i+1)} & \cdots & f_{2(k+1)} \\ \Delta f_{31} & \cdots & \Delta f_{3(i-1)} & \Delta f_{3(i+1)} & \cdots & \Delta f_{3(k+1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{(k+1)1} \circ \Phi & \cdots & f_{(k+1)(i-1)} \circ \Phi & f_{(k+1)(i+1)} \circ \Phi & \cdots & f_{(k+1)(k+1)} \circ \Phi \end{vmatrix}$$
$$\vdots$$
$$+ \begin{vmatrix} f_{21} & \cdots & f_{2(i-1)} & f_{2(i+1)} & \cdots & f_{2(k+1)} \\ f_{31} & \cdots & f_{3(i-1)} & f_{3(i+1)} & \cdots & f_{3(k+1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta f_{(k+1)1} & \cdots & \Delta f_{(k+1)(i-1)} & \Delta f_{(k+1)(i+1)} & \cdots & \Delta f_{(k+1)(k+1)} \end{vmatrix}.$$

We multiply the previous display by f_{1i}, then add all the terms for $i = 1, 2, \ldots, k+1$ to obtain a cofactor expansion of a determinant of order $k+1$.

Thus

$$\sum_{i=1}^{k+1}\{M_{1i}\circ\Phi\Delta f_{1i}+f_{1i}\Delta_\Phi M_{1i}\}$$

$$= D_1^{k+1} + \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1(k+1)} \\ \Delta f_{11} & \Delta f_{12} & \cdots & \Delta f_{1(k+1)} \\ f_{31}\circ\Phi & f_{32}\circ\Phi & \cdots & f_{3(k+1)}\circ\Phi \\ \vdots & \vdots & \vdots & \vdots \\ f_{(k+1)1}\circ\Phi & f_{(k+1)2}\circ\Phi & \cdots & f_{(k+1)(k+1)}\circ\Phi \end{vmatrix}$$

$$+ \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1(k+1)} \\ f_{21} & f_{22} & \cdots & f_{2(k+1)} \\ \Delta f_{31} & \Delta f_{32} & \cdots & \Delta f_{3(k+1)} \\ \vdots & \vdots & \vdots & \vdots \\ f_{k1}\circ\Phi & f_{k2}\circ\Phi & \cdots & f_{k(k+1)}\circ\Phi \\ f_{(k+1)1}\circ\Phi & f_{(k+1)2}\circ\Phi & \cdots & f_{(k+1)(k+1)}\circ\Phi \end{vmatrix}$$

$$\vdots$$

$$+ \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1(k+1)} \\ \vdots & \vdots & \vdots & \vdots \\ f_{k1} & f_{k2} & \cdots & f_{k(k+1)} \\ \Delta f_{(k+1)1} & \Delta f_{(k+1)2} & \cdots & \Delta f_{(k+1)(k+1)} \end{vmatrix}$$

$$= \Delta D^{k+1}.$$

3 Conclusion

In this paper we deal with a difference operator in the space of strictly monotonic functions S.

In this space a method is presented for solving a new model equation. This method can be applied to solve, in particular, some classical linear functional and difference equations.

Because difference equation theory uses a difference operator to express certain results, the notion of the difference operator is defined for Abel functional equations on the space of strictly monotonic functions S. Results are obtained for the difference operator, which reduce, when the model equation is a difference equation, to classical known results.

Algebraic models directly related to $\alpha(f(x)) = \alpha(x) + 1$, which is the classical Abel functional equation, are studied in [5]. The Abel functional equation model in S is able to simultaneously model both k-th order linear difference equation with constant coefficients [4], and first order linear difference equations with constant and nonconstant coefficients [7]. For the solution structure for functional equations in S see [6]. Some applications appear, which show how to do the uniform modeling of classical equations.

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Use of Mathematics in Art Education at Elementary School

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Abstract. The author mentions art expression of an artist Zdeněk Sýkora whose work she follows up in art education at elementary school. She points out terms creativity and creative school and defines basic features of activity learning. She describes methodology of mathematics education of professor Hejný and shows twelve principles of this method. Beside that the author presents her own work–art education with use of mathematics, especially theory of probability.

Keywords: Creativity, creative school, activity learning, art education with use of probability.

Classification: M10.

Despite the fact it is year 2015, current elementary education is based on National Programme of Education in Czech Republic (so called *White Book*) published in year 2001 by Czech Ministry of Education, Youth and Sport. This National Programme puts emphasis on development of inter-subject relationship in elementary education, active education and various forms of inter-subject integration like inter-subject themes. My work with students is focused on integration of mathematics and art including applied art. My motto is: *Students should produce more than reproduce.*

I have chosen mathematical areas–combinatory and probability. In this sense I have followed art expression of Zdeněk Sýkora (1920–2011) and his geometric structures. Mr. Sýkora strictly used three basic shapes–triangle, square and semicircle as well as I do. His pictures are based on searching maximum variations of individual elements position and their combination. By means of combinatory Mr. Sýkora explores possible relations inside picture.

As a teacher at elementary school I work on development of pupils' creativity and activity learning. The term creativity occurs very often in literature. However it is difficult to define creativity in brief. According to VladimuAr Smékal [1] creativity is complex characteristic of personality. It becomes apparent from a person's work and acts. They are characterised by novelty and stimulating effect and have potential. The author proclaims that the output of creative work is characterised by novelty, originality, utility and value of the final product. We cannot expect that student creates something unique in praxis. The main points are student's effort and that he/she has to solve specific problem and via this process he/she learns something new and finishes the process successfully. We should bear in mind that original/unique product is not the main target of student's creative activity. The main idea is to create and arrange conditions for optimal personal development in order to apply personal abilities.

Creativity is closely connected with activity learning because the project Creative Schools defines ten basic points of activity learning [2]:

1. Students are unique and original personalities.
2. By means of activities students are led to think independently, to find new things and to be responsible.

3. Positive motivation.
4. Proceed from easy to difficult and use student's knowledge.
5. Education is connected with real life. Use of all senses.
6. Use of inter-subject relations.
7. Students are led to discuss and communicate.
8. Self-control, feedback, work with mistake.
9. Eliminate overstretch, use differentiation.
10. Positive evaluation and self-evaluation.

In my professional praxis I continue in method of professor Milan Hejný. The method is based on 12 basic principles that he puts together in order to form comprehensive concept which enables pupils to discover mathematics themselves [3]:

1. Forming of scheme. Child knows things we have not taught him/her.
2. Work in various environment. We learn on the basis of repetitive visits.
3. Blend of themes. No isolation of mathematical rules.
4. Personal development. Support of independent thinking.
5. Real motivation. *I do not know but I want to know.*
6. Real experience. Child's own experience.
7. Enjoy mathematics.
8. One's own knowledge has greater value than transferred knowledge.
9. Teacher's role. Guide and presenter.
10. Work with error. Prevent fear.
11. Adequate challenges based on child's ability.
12. Support of cooperation. Knowledge goes out from discussion.

Creativity and method of professor Hejný have many mutual features and I think it is correct. Approaches towards education should not vary because we work with identical object–student.

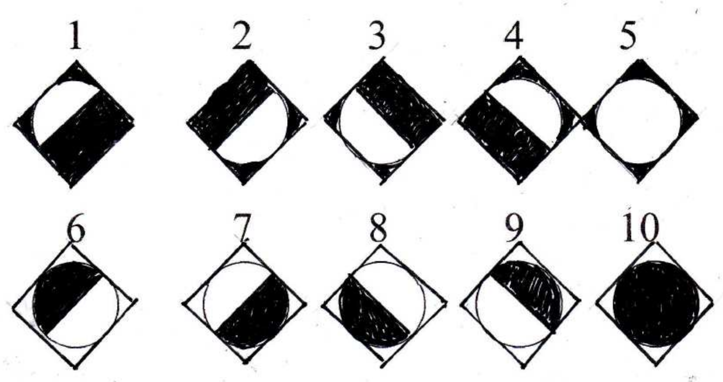

Figure 1: Ten geometric shapes marked by numbers from 1 to 10

I would like to present my educational methods at elementary school. We prepare 10 geometric shapes that are marked by numbers from 1 to 10, see Fig. 1.

We put all shapes to drawing drum and students are drawing one shape and they note down its number. They put the shape back to drawing drum. Than students paint one shape after another into a square net, see Fig. 2.

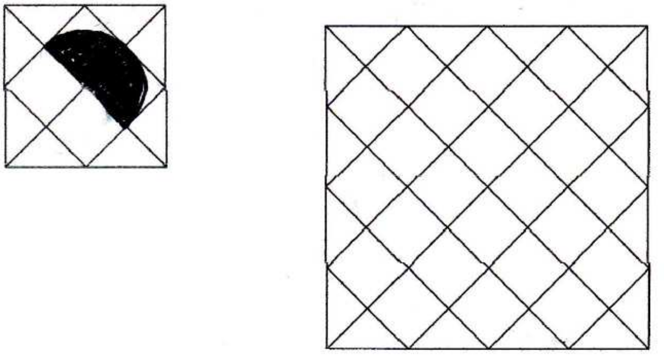

Figure 2: Shapes painted into a square net

Students are creative and they are trying to find other way of getting 10 shapes e.g. they have only shapes 1, 2, 3, 4, 5 and they can flip a coin. Front side of the coin – F represents black shape and back side of the coin – B represents white shape. Students are drawing shapes from 1 to 5 one by one. After that they flip the coin in order to know whether the shape will be white or black.

Figure 3: Example to motivate student's creativity

There is other option. Shapes are marked only by numbers 1, 5, 6 and 10. When student draws shape number 1 or 6 he draws tetrahedral prism. Each side symbolizes different position of a given shape.

Example of author's work that can motivate student's creativity is given in Figure 3.

In cooperation with students we can find other way of shape's drawing e.g. by means of decagon. We teach students to draw a decagon by means of known construction. Students draw the decagon on carton. Each angle is marked by numbers from 1 to 10 and each number represents one given shape. Students scissor the decagon and perforate it in middle with stick. They made a drawing drum that looks like a spinning top. Students spin this *toy* and got numbers that represent given shapes. At the end students verify that number of given shapes marked by numbers from 1 to 10 is approximately equal to 1/10 of all drawn attempts.

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A cosine function

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Abstract. The paper deals with the fact that in the certain phase of high school students' mathematical development their teachers are forced to make the formalization of trigonometric functions, for the geometrical definition, which students usually meet, is intuitive – the existence of the bijection of the interval $[0, 2\pi]$ on the set of all points of the unitary circle is assumed. There is a cosine function defined by means of d'Alembert's functional equation in this article.

Keywords: cosine, cosine of a difference identity, cosine of a sum identity, functional equation, d'Alembert's equation.

Classification: I20.

1 Introduction

This textbook is intended as an attempt to make mathematics teachers and high school students acquainted with the construction of some new mathematical terms within mathematical analysis. There are functional equations used for this purpose here.

2 Solution of cosine equation

The main objective of this paper is to generalize a "common" cosine function so its continuous version appeared to be a "common" function cos. To generalize cosine function $\cos : \mathbf{R} \to \mathbf{R}$ cannot be discused without understanding the notion of a cosine equation and its solution. We need to find therefore all solutions of the cosine equation (often called also d'Alembert's equation)

$$f(x+y) + f(x-y) = 2f(x)f(y), \tag{1}$$

where $f : \mathbf{R} \to \mathbf{R}$, for all $x, y \in \mathbf{R}$.

Let's mention already now that this is not only a question of mere generalization of the "common" cosine function. From the wider point of view we could understand this problem as follows:
The area of a reflexion of knowledge in mathematics can be divided to:

- the area of an idealized reality
- the area of a formal elaboration.

In mathematics, in the area of an idealized reality, which originated historically primarily, real objects are built intuitively. On the other hand in the area of a formal elaboration are built axiomatically. The transformation of a mathematical term from the first to the second area is evidently step to exact mathematical expression. It is possible to approach a formation and a development of terms in teaching either that we'll consequentially pursue historical evolution of the term or we'll proceed from the contemporary understanding of the term in the branch of science. The first approach is mostly unbearable, because it needs two much time, the second one is tributary to the maturity of students. That is why we often have to choose a compromise. But we may never suffer so that the effort in the simplification would contribute to the falsification of the scientific interpretation of the term. Concretely in our case this is the question of the term of the cosine function. The geometrical definition, which students usually meet, is intuitive. The existence of the bijection of the interval $[0, 2\pi]$ on the set of all points of the unitary circle is assumed. That is why in the certain phase of their mathematical development we are forced to make the formalization of this term. First, let us start with well-known cosine (and sine) of a difference and sum identities.

2.1 Cosine identities

We start with developing the well-known identity for the cosine of the difference of two numbers x,y. Let a real number x be in the interval $[\pi/2, \pi]$ and a real number y be in the interval $[0, \pi/2]$. Let us consider the unit circle with the center in the origin. Then points X and Y with coordinates $(\cos x, \sin x)$ and $(\cos y, \sin y)$ are on this circle and for arc length $a = x - y$ holds $a \in [0, \pi]$.

By means of the distance formula the expression for the distance XY is as follows

$$XY = \sqrt{(\cos x - \cos y)^2 + (\sin x - \sin y)^2}.$$

We can simplify:

$$XY = \sqrt{2 - 2(\cos x \cos y + \sin x \sin y)}.$$

Further, let's imagine rotating our circle above so that point Y is at $(1, 0)$. In spite of the fact the coordinates of point X are now $(\cos a, \sin a)$, the distance XY has not changed! (Make a picture if necessary).

Now we can use the distance formula for the second time:

$$XY = \sqrt{(\cos a - 1)^2 + (\sin a - 0)^2}.$$

After simplification we get

$$XY = \sqrt{2 - 2\cos a}.$$

Equating our two expressions for the distance XY, we have

$$\sqrt{2 - 2(\cos x \cos y + \sin x \sin y)} = \sqrt{2 - 2\cos a}.$$

Solving this equation for $\cos a$ gives

$$\cos a = \cos x \cos y + \sin x \sin y.$$

Setting up $a = x - y$ we have the equation

$$\cos(x - y) = \cos x \cos y + \sin x \sin y.$$

The above formula is valid when a is the length of the shortest arc from X to Y. Given any real numbers x and y, the length of the shortest arc from X to Y is not always $x - y$; it could be $y - x$. However, because the cosine function is even, it has to be $\cos(y - x) = \cos(x - y)$. Thus, $\cos a$ is always equal to $\cos(x - y)$ and the last formula holds for all real numbers x and y; this formula is the identity we sought:

$$\cos(x - y) = \cos x \cos y + \sin x \sin y.$$

The cosine sum formula could be derived easily from the one we have just made. Indeed, since $\cos(x+y) = \cos[x-(-y)]$ we have from our derived identity

$$\cos(x+y) = \cos[x-(-y)] = \cos x \cos(-y) + \sin x \sin(-y).$$

Since $\cos(-y) = \cos y$ and $\sin(-y) = -\sin y$, we get the following identity

$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

Now, for any real t consider $\cos(\frac{\pi}{2} - t)$ we can use the derived identity for the cosine of a difference to simplify in the way as follows:

$$\cos(\frac{\pi}{2} - t) = \cos\frac{\pi}{2}\cos t + \sin\frac{\pi}{2}\sin t = 0.\cos\frac{\pi}{2} + 1.\sin t = \sin t.$$

Thus we have proved the identity

$$\sin t = \cos(\frac{\pi}{2} - t).$$

Now put $t = \frac{\pi}{2} - \alpha$. Then we have

$$\sin(\frac{\pi}{2} - \alpha) = \cos(\frac{\pi}{2} - (\frac{\pi}{2} - \alpha)) = \cos\alpha,$$

which yields the identity

$$\cos\alpha = \sin(\frac{\pi}{2} - \alpha).$$

Using the derived identities we can write $\sin t = \cos(\frac{\pi}{2} - t)$ and therefore

$$\sin(x+y) = \cos(\frac{\pi}{2} - (x+y)) = \cos((\frac{\pi}{2} - x) - y))$$

$$= \cos(\frac{\pi}{2} - x)\cos y + \sin(\frac{\pi}{2} - x)\sin y = \sin x \cos y + \cos y \sin y.$$

Thus we have found the identity

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

It is not difficult to derive the formula for the sine of a difference - we can use the above identity for the sine of a sum. Namely, substituting $-y$ for y we have:

$$\sin(x + (-y)) = \sin x \cos(-y) + \cos x \sin(-y).$$

Simplifying gives us finally

$$\sin(x-y) = \sin x \cos y - \cos x \sin y.$$

2.2 Cosine equation

Let's sum the cosine of a difference identity and the cosine of a sum identity:

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

plus

$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

The results is as follows

$$\cos(x+y) + \cos(x-y) = 2\cos x \cos y.$$

If we write here instead of the function cosine any function $f : \mathbf{R} \to \mathbf{R}$, for all $x, y \in \mathbf{R}$, this equation looks like this:

$$f(x+y) + f(x-y) = 2f(x)f(y).$$

The last equation is a special case of so called functional equation. Functional equations are (claimless accuracy) equations, both sides of which are terms formed by a finite number of unknown functions and by a finite number of independent variables. The peculiarity of functional equations, compared with other equations (algebraic, differential, integral etc.), is that one functional equation can contain more unknown functions in the sense that all unknown functions can be determined from it. Just this fact plays the most important role especially in the construction of new elementary functions – so called generalized elementary functions.

We have seen that the cosine function $\cos : \mathbf{R} \to \mathbf{R}$ is one of solutions of the cosine equation (1). There are other solutions, for example $\cosh x$, $f(x) = 0$, $f(x) = 1$, however. Our first problem is therefore to solve equation (1). It can be done in a quite natural way. Assume first that the equation has a solution f and from that assumption we shall deduce (if possible) what form the solution must have. Then we must verify that the function obtained does solve the equation.

Thus, suppose that f is a non-constant solution of (1). Since the domain of f is the whole set $\mathbf{R}$, (1) holds for any two real numbers x and y. In particular, for $y = 0$ and an arbitrary x. Hence

$$2f(x) = 2f(0)f(x)$$

for every x, which easily yields that $f(0) = 1$. Putting $x = 0$ we get

$$f(y) + f(-y) = 2f(y)$$

and

$$f(y) = f(-y).$$

We have shown that f is an even function. Putting $x = y$ we get

$$f(2x) + f(0) = 2f^2(x)$$

and

$$f^2(x) = \frac{1 + f(2x)}{2}. \tag{2}$$

Unfortunately these are the last reasonable necessary conditions we can find for the time being. However, if a function f which solves the functional equation (1) is assumed to have some appropriate additional property, we can use a standard approach to deriving results (their proofs are not the subject of this paper):

1) Let f is a non-constant continuous solution of (1) satisfying

$$|f(x)| \leq 1$$

for each $x \in \mathbf{R}$. Then f is of the form

$$f(x) = \cos Ax,$$

where A is an arbitrary real number.

2) Let f is a non-constant solution of (1) satisfying

$$|f(x)| \geq 1$$

for each $x \in \mathbf{R}$. Then f is of the form

$$f(x) = \frac{1}{2}(a^x + a^{-x}),$$

where $a \neq 1$ is an arbitrary positive real number.

We can summarize:
Solutions (continuous) of d'Alembert's equation (1) are:

- $\cos Ax \quad (A \in \mathbf{R})$
- $\cosh_a x \quad (a \in \mathbf{R}^+,\ a \neq 1)$
- 0 (i.e. zero function).

There is another, much more difficult problem, if we are looking for all solutions (not only continuous) of d'Alembert's equation (1); it is not again a purpose of this paper devoted to high school students to explore it, however.

3 Conclusion

We can summarize that we have prepared by means of this paper to (partly) understand the notion of GENERAL cosine function:

The general cosine function is a complex-valued function on an Abelian group G of the form

$$\frac{h(x) + (h(x))^{-1}}{2}$$

where h is a homomorphism of G into the multiplicative group of non-zero complex numbers...

Acknowledgements

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Using a computer in the course of studying a convergence of series

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Abstract. The Harmonic Series has an irreplaceable meaning for students while studying a convergence of series. It is a really convenient way to help students create their conception of convergence or divergence of series as well as their notions about the speed of divergence. The paper presents a practical example leading to the harmonic series and points out the suitability of using a computer for solving specific kinds of tasks and problems.

Keywords: Harmonic series, Euler's formula.

Classification: I20, R20.

1 Introduction

The harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

is one of the most celebrated infinite series of mathematics. As a counterexample illustrates, the convergence of terms to zero is not sufficient to the convergence of a series.

Such an elementary idea to prove the divergence of the series could be this one:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \dots =$$
$$= 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \dots\right) \geq$$
$$\geq 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \dots = +\infty$$

The original author of the proof is probably Nicolas d'Oresme, who dealt with it in his treatise called *Questiones super Geometrian Euclidis* already in the 14^{th} century [2].

There are many ways to prove that the sum of the harmonic series is infinite. For further studies, we can recommend Kifowit and Stamps [1], who demonstrate altogether twenty such proofs. The harmonic series could be described as a "border" series, since every other series of the type given below is convergent:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} < +\infty \quad \text{for arbitrary} \quad \varepsilon > 0.$$

To determine the N−th partial sum of the harmonic series, we usually use the *Euler's formula*

$$\sum_{n=1}^{N} \frac{1}{n} \doteq \ln N + \gamma,$$

where γ is the *Euler's* or the *Euler-Mascheroni's constant*, which value is aproximately 0.577216. It is generally a great surprise for the students to find out how inconceivably slow the speed of the divergence is. Even a really efficient computer does not lead us to expect the sum of $+\infty$. Just to illustrate it, if we used a computer that sums up 10^{10} elements in a second and the summing had started 13.5 billion years ago (straight after the big bang), we would receive the sum less than 100 today.

2 The harmonic series in a daily life

The following example demonstrates that we meet the harmonic series even in those situations, where we would not expect it at all.

Example:
We have a pack of n identical cards, which we lay down towards the edge of a table. Let H_n be an overhang over the edge of the table. What is the maximum possible value of H_n so that the cards do not fall down? What is the maximum possible value of H_∞, i.e. if we had a limitless amount of cards?

Solution:
Let us suppose that the cards are of the lengths of 2. As we can see in Figure 1, the maximum value of H_1 is reached in that case, when the centre of gravity of the card is lying straight above the edge of the table. The overhang is equal to one half of the card's length, so $H_1 = 1$. How does the situation change if we add another card?

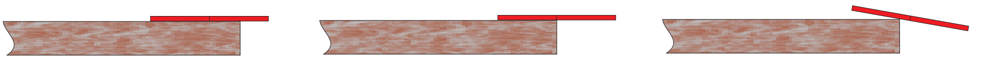

Figure 1: Position of a card considering the edge of the table

When we add a card and we do not want the cards to fall down, the centre of gravity of these two cards has to lie on axis O, which is going through the edge of the table (see Figure 2). Then $H_2 = 1 + \frac{1}{2}$.

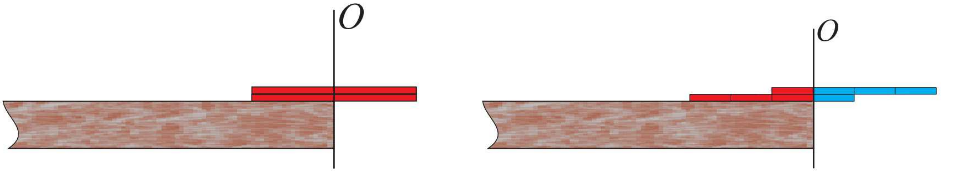

Figure 2: System of two cards

It is similar for three cards – the centre of gravity of such a set of cards has to lie on axis O as well. This condition is fulfilled as long as the "mass" to the left of axis O is equal to the "mass" to the right of axis O (Figure 3). Then, evidently, $H_3 = 1 + \frac{1}{2} + \frac{1}{3}$.

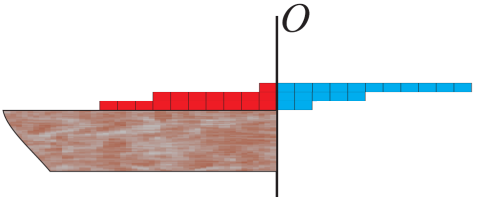

Figure 3: System of three cards

Generally speaking, let us have a balanced system of $n-1$ cards and let us add the n–th card. We can see that the particular moves of the cards generate the harmonic sequence and the searched sequence H_n is the sequence of partial sums of the harmonic series. That is

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

and

$$H_\infty = \infty.$$

That means that it is possible, in theory, to build infinitely many cards on each other (where every card is shifted considering the previous card) and the system of the cards would still be balanced.

3 Conclusion

The example above is just one of much more real situations, where we recognize a mathematical principle. Even though a computer is usually a great tool to help us calculate and figure out many kinds of problems, there are situations where using a computer is not as efficient as we might expect. The use of computer in this particular example could simply lead to the wrong conclusion and we would label the series as convergent.

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Using of Tangram in the mathematics education at primary school

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Abstract. Geometry is difficult for pupils at every stage of education. Because of this reason is important to develop space imagination from the early age and to use proper didactic tools for this. One of them is puzzle Tangram that is ideal connection of game and didactic tool.

Keywords: Geometry, space imagination, early age development, Tangram.

Classification: G10; U60.

1 Introduction

After decades of educational activities we confirmed with the colleagues the fact, that geometry makes a problem to pupils and also students at all degrees of education. It is a serious problem, because it is related for example with readiness of students to choose within the orientation of university study faculties of technical orientation. We have seen that it is not sufficient to deal with this problem only at the 2nd degree of primary school, for which as well as for the secondary schools we prepare at our faculty future teachers. Achievement of good results at geometry depends also on good spatial imagination, including geometrical. In the publication (Půlpán, Kuřina, & Kebza, 1992) we learn a lot of interesting facts; the following definitions of vision and imagination will be from this publication:

Normally we understand imagination as "ability to develop and create visions". A vision is then an image, created in our mind, based on a previous perception, by mental activity, or based on experience.

Z. Půlpán, F. Kuřina and V. Kebza characterize the geometric imagination as a summary of capabilities, concerning our images about shapes and mutual relations between geometric objects in space (Půlpán, Kuřina, Kebza, 1992).

The results of pedagogical research indicate that for development of a spatial imagination is very suitable already a pre-school age. According to J. Piaget is such first genetically suitable period 5–7 years. Children can devote to developing of spatial imagination during various games and educational activities. We gained many good results in this area. As J. Molnár has stated in his publication:

Generally it is possible to say, that at the entrance to the school the geometric imagination of children is at much higher level, than mathematics curriculum for elementary schools presuppose (Molnár, 2009).

We consider as problematic an attitude to education of geometry just at the 1st stage of a primary school. It is a pity that thus we miss a genetically suitable period, as it was stated. Just here we can see the reasons of insufficient results in geometry.

Each degree of education is important regarding mathematical orientation, development psychology, and also the level of a spatial imagination defined by Van Hiele for given age category as visual, analytic and spatial.

During the pre-school age and in the 1st year of a primary school we can talk about the visual level of children. Children recognize geometrical shapes regarding concrete visual images, such as the square like a frame of the picture, rectangular like the door, circle like a round, etc.

Pre-school age and the 1st year of a primary school is important in term of propedeutics of mathematical terms, such as equality, similarity, axial symmetry, and so on.

In the 2nd-4th year of a primary school we already observe the analytic level of geometrical thinking. Here we create the terms, such as straight line, ray, line segment, plane, perimeter and surface of simple plane objects.

In the 4th year the children progress to the abstract level of geometrical thinking. They deduce here the contents of terms and they start to use the elementary logic rules for example at classification of quadrilaterals.

2 Activities for development of geometrical imagination using Tangram

A puzzle Tangram enables us to realize and also to develop a spatial imagination in many tasks. It represents a very good connection of game and a didactic material. It is a square, divided to 7 pieces: five triangles, (two biggest, one middle and two smallest), the square and the parallelogram.

It is possible to work with Tangram in two ways: To compose individual parts into predetermined outlines and to create figures of the people, animals, known objects, things, geometrical shapes, according to their own imagination and note down the results.

We can explain to pupils the relationships between individual pieces of the Tangram at the age-appropriate level. For example, we can point out, that we play only with isosceles triangles having equal angles, which are similar, the individual pieces are identical by some parties, etc. Thus, Tangram can be used on many issues directly in teaching geometry. However, we can realize through it for example also competitions in composing and various didactic games as well. We have dealt with these topics in the following publications:

For a pre-school age: (Uherčíková, 2014).

For 1st and 2nd degree of a primary school: (Brincková, 1996).

For the 2nd degree of a primary school and for the secondary school: (Brincková, Uherčíková, & Vankúš, 2013).

Samples of tasks from the publication (Brincková, Uherčíková, & Vankúš, 2013):

Task 1: Create from all parts of Tangram house according your fantasy.

Task 2: Create some object by putting the parts of Tangram in the way, that each part touches other with the side of the same length. Name this object!

Task 3: Create from two puzzles Tangram object with axial symmetry. Name this object!

Task 4: Create two rectangles with different perimeter by putting together two biggest, one middle and two the smallest triangles from the Tangram puzzle. Do they have the same surface?

Task 5: Create all possible objects using one middle and two smallest triangles from Tangram.

Task 6: Proof using Tangram Pythagorean theorem!

Solutions can be found in the publication (Brincková, Uherčíková, & Vankúš, 2013) available online at the address **www.comae.sk/netradicnemetody.pdf**.

Acknowledgements

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Creative science – the way how to improve the knowledge and skills of children

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Abstract. In the last years, the natural sciences are not very popular, although the institutions of non-formal education have been founded in Europe (with the support of the EU), offering insight into these subjects in a popular way. The success is based on an excellent concept of a programme, which must meet a number of seemingly unimportant parameters. In the paper there will be discussed not only the creation of the overall concept of activities but there will be also introduced the particular activities which were created specifically for the newly opened Science and technology center in Ostrava - creative science, science boxes, lessons and more.

Keywords: Science, non-formal education, creative science, mathematics.

Classification: D40.

1 Introduction

Development of an opinion on education (both general and natural-science education) has undergone many stages throughout the history in terms of its content, methods as well as the attitude of a student and teacher in the educational process (e.g. [1], [2]). It is irrelevant which approach we tend to; however, it is important to understand that the education has always been and will always be the synthesis of several different entities – school, family, institutions of non-formal education and society as well. Rapid growth of information, new technologies and, above all, incredible acceleration of time has been bringing us into the situation when, unfortunately, each of these entities does only its job and there is a lack of time for mutual cooperation and interconnection. Schools are rushed by Framework Educational Programmes; families spend minimum time with their children (and out of that minimum only a negligibly little time is dedicated to education); in general, society makes everything faster at the expense of quality.

Children's handwriting has been getting worse; many of them are not able to tie their shoelaces or perform some of the activities which have been ordinary until recently. As we are always in a hurry, we do not provide enough space to train and practise basic skills and we replace them with another, quicker solution. We copy materials for children at schools so that we are not loosing the time with writing; we substitute shoelaces with velcro fastening; we replace a lengthy experiment in the classroom with a video. We are looking for a simple, problem-free, safe and above all fast way everywhere. Unfortunately, at the expense of experience.

We often call that absolutely incorrectly the "interactivity" and we boast about how modern we are. In fact we are just replacing reality with something imaginary which, however, leaves no experience, no emotions in ourselves. It is precisely the experience which we as well as our children are desperately short of at the moment – we have no time to gain experience. This approach needs to be changed because they are exactly the deep emotions and experience that represent the key elements for remembering the events, monitoring the processes around us, deducing subjective conclusions. Whenever we experience something awakening strong emotions of us, we tend to think about it. And thinking about something brings questions and questions require answers. Trying to find answers is then nothing else than a natural process of education.

2 Institution of non-formal education

At present the primary subjects being able to provide experiences are institutions of non-formal education – science centres, discovery centres and others. Their aim is to bring science closer in a playful way, to stimulate interest in these areas, offer the opportunity to discover the laws of nature and processes by means of interactive exhibitions which form the heart of the centres. Nevertheless, exhibitions usually stay for several years and therefore an important aspect of the success of such an institution is a well-prepared supporting programme. Supporting programme includes many activities, programmes and modules (short and long term) broadly aiming at different target groups (children, students and general public) and focussing on different topics. What I want to focus on in this paper is one of the activities – the so called creative science.

2.1 Creative science

Creative Science is a comprehensive system of modules designed for students which has been successfully running for already several years in the Science and Technology Centre in Ostrava's Lower Vítkovice district. The idea of how to make gaining the experience stronger by means of producing a variety of physical and mathematical toys or by performing experiments has become the motivation to develop the creative science as a separate complex within the supporting programme conception.

Each module of the creative science includes production of one item or one experiment which can be implemented anywhere – at home, at school, with friends or in a science centre. If the project is implemented in an educational centre or at school, it is important for a particular student to be able to take it home with him/her. Commonly available materials are being used all the time; in most cases we are trying to recycle. In terms of the time, a module is designed to take 3x45 minutes and it involves motivation, device production itself and feedback (comparison of products, discussion about modification options, etc.). Project documentation includes worksheets for students, production instructions as well as the step-by-step methodology for teachers in case they would like to include the module into their teaching.

Project always aims at introducing one primary phenomenon. Other laws may be introduced secondarily but the number and level of them must always be chosen with respect to the age of children. The project offers just a basic idea; many results remain hidden and it is up to the children/students to reveal them and learn through observation. An important part in this journey to knowledge are errors. Projects are purposely designed so that students make errors and if their product does not work, they have the opportunity to reveal the problem and correct it. If a child starts building, manufacturing or implementing something on his/her own, and if he/she is interested in the result, he/she will want to know more. Either via school, in an educational centre, or from other available sources.

Basic criterion to propose a new project is obviously:

- safety;

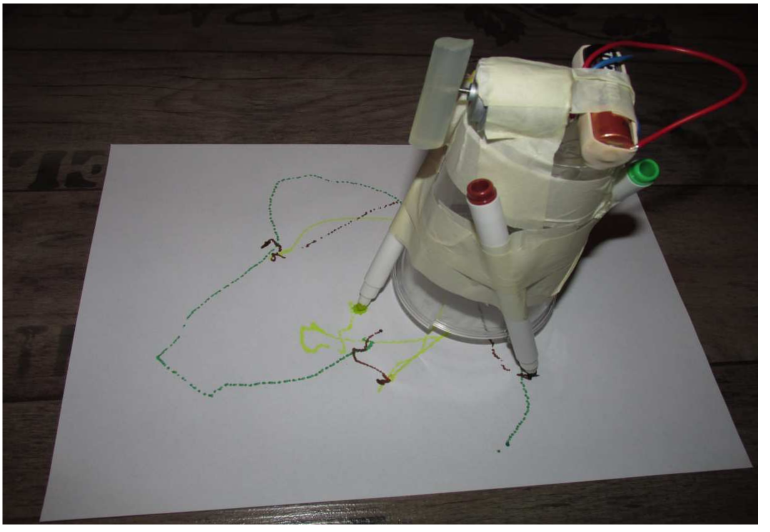

Figure 1: Creative science – the painting machine.

- low material demandingness (we often try to recycle common household materials);
- a wide range of the target group (activities should be adjusted according to age, skills and professional orientation);
- we never use prefabricated products; every step must be designed so that every student is able to perform it on his/her own;
- projects must both respect the individuality of a student and offer scope for cooperation.

Projects are always determined for a particular age group and focus on specific themes which appear in educational plans. Some can be modified with respect to special requirements (for example being used for general public in thematically oriented weekends). Individual projects that originated in the Science and Technology Centre aim at pupils and students of primary and secondary schools and include all natural sciences. We are trying to design projects with the possibility of various designs and modifications in order to create space for each personality, for each individual. This variability often brings new ideas – students want to distinguish from their friends and suggest their own design or even to make their own product from provided components.

For example a project aiming primarily at functioning of a small electrical engine (secondarily at the circuit) has brought several different and often very creative ideas. The first idea was to construct something that is moving due to a small engine – we chose a cup of yogurt. To make the product amusing, we provided it with colored markers which then drew patterns on paper while moving (see Figure 1).

Children liked the product, however, they found it useless (although it is funny, it is just for fun). While discussing about where an engine is used, children mentioned fans most often. So let's make your own fan! We used an empty deodorant container,

Figure 2: Creative science – the fan.

Figure 3: Creative science – the bees.

small engine and cardboard for the fan propeller. (It was very nice to develop discussion on aerodynamics of paper wings – how can the shape of wings effect a fan efficiency?) The result can be seen in Figure 2.

While arranging projects for younger students, there was an idea of a moving beetle (in those days similar toys were a big hit). Size of the engine was just perfect for using an empty kinder egg container and as it was yellow, an idea of making a bee emerged spontaneously. Result can be seen in the Figure 3.

During the assembly some children hit the engine center of gravity and so the bee was not moving even thought the engine was running. It was a perfect opportunity to discuss why actually a bee is moving in an uncoordinated manner. What will happen if one of the aircraft engines begins failing? Why can it be dangerous?

Many similar projects focused on mathematics (particularly on symmetry, golden section, encryption, solid figure surfaces, perspective and others) as well as on other natural sciences.

3 New experience

3.1 Positive aspects

The most significant positive aspect of the creative science is the fact that the activity has completely met the purpose – children like creating, trying out different alternatives, thinking the ideas over, looking for answers. They are pushing the limits of their options and are not afraid of starting more complex projects. Creative science has become one of the most popular parts of the supporting programme and the projects are used:

- in the morning times for schools (registration via the booking system);
- at weekends for general public and particularly families. Simpler and less expensive projects are offered then because of many visitors (which is difficult to be estimated in advance) attending the location. More complex projects are carried out using science boxes;
- some projects are designed thematically with respect to special events: Easter, Halloween, etc.

Projects for schools are designed in combination with a short commentary on the field which it is focused on. At the end there is enough time for the discussion with students – they sometimes propose various modifications of their own and they have the opportunity to perform their ideas using web applications even at home and send pictures to the science centre then. The best of them will get a special reward. Schools have shown such a great interest that our capacity has been exceeded. School groups often keep coming back for other modules for both the same and different science fields.

3.2 Critical moments of creative science

Besides the positive aspects the creative science has many pitfalls. One of them is the difficulty in creating projects – it is necessary to generate new ideas all the time which requires the existence of a well-functioning team of experts. The team must be composed of people educated in a specialized field but it is also necessary to bear in mind that the product is intended for a wide range of talented children (or public) both intellectually and manually and the level must be adapted to that. Unfortunately not every scientist is able to select a general, really interesting phenomenon out of his/her field and is not always able to present it in so that it is understandable even for a small pupil. Sometimes the project fails to be smoothly included in the offer (badly estimated level of time consumption); nevertheless, minor drawbacks can be eliminated with instructors on the basis of test runs.

Second critical factor is the fact that the modules are never led by project authors but by instructors. Instructors are usually students (seasonal staff) who often do not have sufficient experience (not even any natural science education is required). Although the staff undergo some training before instructing in projects, it still may not reveal all secluded places of the project. It is necessary to put emphasis on choosing the instructors/animators for particular areas as well as specific age groups of students or participants. Instructors need to be communicative, have an interest in the entrusted project which he/she will study as much as possible in their own interest. These drawbacks can be partly eliminated by the fact that project proposals will include all possible risks (why a student's attempt for an experiment can turn out to be unsuccessful and how to proceed in such situation). Uniformity of proposals is ensured by a prescribed template provided; this template includes much information intended only for the operator.

Last negative aspect is also its high financial performance. Even though most material is recycled, or less costly, with regard to the number of students who daily attend individual modules, this amount accumulates and grows rather quickly. Yet the requirement for taking the product home is essential and may not be compromised (the emotional side strongly affects that the students get back to the product again at home and will make similar items on their own). Non-formal education institutions are often operated with the financial support of the region and so the question of money is not that acute; nevertheless, implementation of these projects into the teaching process (or becoming a part of afterschool activities) could be more complicated for that reason.

4 Conclusion

Current hectic time brings us new tools and we do not mean only in the area of education. Nevertheless, many of them can be a good servant but evil master. Time is becoming our enemy; and it is not always practical to build quantity over quality. Some activities have always required and will still require its time for training and absorption. In case we hurry up too much with them, the apparent time saving will return in a negative way later on in the future. It is also important to be dissatisfied with the good feeling that children enjoy the activity, but to have a desire to look deeper – does the activity really pursue the objectives? For example if children enjoy working on the computer in mathematics, do they really think about geometry in that moment? Or do they actually think about which icon they should click on?

Present time has also provided us with many institutions which can be a successful partner for school or family in the educational process. Do not be afraid to use them! Any activity that makes children explore the world around them in their free time, for whatever reason it is, is a positive step ahead. Let's give the students real, genuine experiences and emotions and then everything else comes naturally.

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Development of pupils' attitudes towards mathematics

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Abstract. Our paper is dealing with the results of research on pupils' attitudes towards mathematics. We study changes in the attitudes between pupils in different grades of lower secondary education. Analysis of these changes is the source of information beneficial for potential improvement of these attitudes.

Keywords: Mathematics education, attitudes towards mathematics, beliefs.

Classification: C20; C70.

1 Introduction

The attitudes toward mathematics are important factors those are influencing the results of mathematics education (Zimmerman & Bandura, 1994; Pajares, 1996; Marsh & Yeung, 1997; Skaalvik & Skaalvik, 2004; Skaalvik & Skaalvik, 2006). The research in the area of these attitudes is therefore relevant and needed. One of the interesting topics in this area of research is development of the pupils' attitudes towards mathematics during the school education. The goal of the research presented in this paper was to study this development in the lower secondary education (pupils' age 10–15). The results of this research are beneficial for mathematics teachers and also researchers in the theory of mathematics education.

2 Attitudes towards mathematics

Before we continue with the description of our research we will specify the definition of the attitudes that we use in our paper. We will use so called multidimensional definition. By this definition individual's attitudes towards mathematics are defined in a complex way by the emotions that he/she associates with mathematics, by individual's beliefs towards mathematics, and by how he/she behaves (Hart, 1989; Zan & Di Martino, 2007). This definition shows that attitudes are complex and rich notion. In our paper we deal mostly with the more stable part of pupils' attitudes, with individual's mathematics-related beliefs. They form the system of beliefs, that can be define as the implicitly or explicitly held subjective conceptions students hold to be true about mathematics education, about themselves as mathematics learners, and about the mathematics class context (Op 't Eynde & De Corte, 2003). Students' beliefs about mathematics education contain beliefs about mathematics, mathematics learning and teaching. Beliefs about themselves as mathematics learners consist of goal orientation beliefs and self-efficacy and self-concept beliefs. Beliefs related to the mathematics class context are beliefs about the role of the teacher, the role of the students and about socio-mathematical norms and practices in the mathematics class. (De Corte & Op 't Eynde, 2002)

3 Research description

In this part of the paper we will describe the research. As was mentioned before, the goal of our research was to study development of pupils' attitudes towards mathematics during lower secondary education. The research tool to study the attitudes was the questionnaire. We developed it on the basis of the questionnaires used in previous researches on the attitudes, in some of those we participated too (De Corte & Op 't Eynde, 2002; Andrews at all, 2007, 2008, 2011; Vankúš & Kubicová, 2010). The questionnaire used in the research had 16 items. They were 6 scales Likert type. The scales were: Strong agree; Agree; Partially agree; Partially disagree; Strong disagree. Items were examining 4 areas of mathematics-related beliefs; each area consisted of 4 items. The areas were: The liking of the mathematics; Beliefs on the usefulness of mathematics; Pupils' mathematics self-beliefs; and Self-evaluation of pupils' effort in the mathematics. Considering definition of the mathematics-related beliefs mentioned before, the first two areas of our questionnaire are part of students' beliefs about mathematics education and last two areas are part of beliefs about themselves as mathematics learners. Complete text of the questionnaire can be found in our publication (Vankúš, 2014, pp. 134–135), available for free download on the web address: www.comae.sk/efektivnost.pdf. The items for each area are listed in the fig. 1.

Area	Items
The liking of the mathematics	I am fond of mathematics. I am glad about mathematics. Studying mathematics is pleasure for me. I like mathematics.
Beliefs on the usefulness of mathematics	I need mathematics in various situations of my life. Mathematics will help me to find the job. Mathematics is useful for me in the real life. Mathematics increases my possibilities to get the job.
Pupils' mathematics self-beliefs	Mathematics is very easy for me. I am very good in mathematics. I am sure that I can understand everything we study on mathematics. I am sure I can grasp everything we learn on mathematics classes.
Self-evaluation of pupils' effort in the mathematics	I put lot of effort to mathematics. I work very hard on mathematics. I strive really hard in mathematics. I try to work the best I know in mathematics.

Figure 1: Items of the questionnaire used in the research

The research sample was 154 pupils from the state lower secondary school in Bratislava. Pupils were attending 5th–9th grade of school education; their age was 10–15 years. Questionnaire was administrated on the mathematics lesson. Afterwards we evaluated the answers, marking them with natural number of points between 6 points and 1 point. 6 points was the value for the most positive response (strong agree) and 1 point was the most negative response (strong disagree). Positive responses indicated positive beliefs towards mathematics. For further statistical analysis we then computed for each pupil final score from the each area of questionnaire, adding the scores from the 4 items, belonging to that area. Then we did test for normality of our data using Shapiro-Wilk test. All the results for each grade of pupils are in the fig. 2.

Grade / Area	5th $n = 43$	6th $n = 25$	7th $n = 26$	8th $n = 37$	9th $n = 23$
The liking of the mathematics					
Mean	17.35	15.32	12.04	13.00	11.87
Standard deviation	4.64	3.50	5.81	5.02	4.18
Normality test, W; p less than 0.10	0.95 not normal	0.94 normal	0.90 not normal	0.97 normal	0.97 normal
Beliefs on the usefulness of mathematics					
Mean	20.86	19.36	14.54	17.51	15.43
Standard deviation	2.66	3.84	5.09	4.05	3.85
Normality test, W; p less than 0.10	0.89 not normal	0.86 not normal	0.96 normal	0.96 normal	0.92 normal
Pupils' mathematics self-beliefs					
Mean	17.65	15.00	13.04	14.54	12.56
Standard deviation	3.80	4.36	4.23	4.60	4.69
Normality test, W; p less than 0.10	0.97 normal	0.96 normal	0.97 normal	0.96 normal	0.93 normal
Self-evaluation of pupils' effort in the mathematics					
Mean	19.77	18.04	15.58	15.03	13.61
Standard deviation	3.25	2.98	4.71	4.53	4.42
Normality test, W; p less than 0.10	0.92 not normal	0.95 normal	0.96 normal	0.95 normal	0.96 normal

Figure 2: Results of the used questionnaire

To study more closely the changes in our results between the grades of lower secondary education we computed differences between the means of each two subsequent grades. To find out if the changes are statistically significant we used nonparametric Mann-Whitney U-test, because our data have mostly not normal distribution, as we could see in fig. 2. The computed changes and results of the U-test are in the fig. 3.

Changes between grades	5th-6th	6th-7th	7th-8th	8th-9th
The liking of the mathematics				
Difference of means	**-2.03**	**-3.28**	0.96	-1.13
p (Mann-Whitney U-test)	0.06	0.09	0.58	0.47
Beliefs on the usefulness of mathematics				
Difference of means	-1.50	**-4.82**	**2.98**	**-2.08**
p (Mann-Whitney U-test)	0.20	0.00	0.02	0.04
Pupils' mathematics self-beliefs				
Difference of means	**-2.65**	**-1.96**	1.50	**-1.98**
p (Mann-Whitney U-test)	0.02	0.09	0.21	0.08
Self-evaluation of pupils' effort in the mathematics				
Difference of means	**-1.73**	**-2.46**	-0.55	-1.42
p (Mann-Whitney U-test)	0.04	0.05	0.61	0.25

Figure 3: Changes between grades; numbers in bold are statistically significant with p less than 0.10

4 Discussion

From the results in the fig. 3 we can see, that there are statistically significant (with p less than 0.10) changes in all our areas of attitudes towards mathematics during the lower secondary education. Namely, there are decreases: in the area The liking of the mathematics between 5th and 6th grade, and between 6th and 7th grade; in the area Beliefs on the usefulness of mathematics between 6th and 7th grade, and 8th and 9th grade; in the area Pupils' mathematics self-beliefs between 5th and 6th grade, and 8th and 9th grade; and in the area Self-evaluation of pupils' effort in the mathematics between 5th and 6th grade, and 6th and 7th grade. The increase was just in the area Beliefs on the usefulness of mathematics between 7th and 8th grade. The above mentioned results are in accordance with the results of some other researchers on the attitudes towards mathematics. They have found out that there is decrease in the pupils' positive attitudes towards mathematics during the process of the school education (Ma & Kishor, 1997). Possible explanation of this could be difficulty of mathematics problems, too high pace of the education, language not adequate for pupils, negative attitudes of teachers etc. (Philippou & Christou, 1998). Our research showed that the decreases are in all areas of our questionnaire. The reasons for the decreases can be for all areas similar or there can be more factors, those can act separately or in coherence. To find this is the task for future researches that we want to realize using both quantitative and qualitative ways to find out more about this interesting and important matter.

5 Conclusion

In our paper we presented the research on pupils' attitudes towards mathematics. We studied changes in the pupils' attitudes in different grades of lower secondary education. We have found out, in accordance with previous researches, the decline in the positive attitudes towards mathematics. In the future research we plan to study more closely the changes in the attitudes towards mathematics during early years of primary education, to find out in what age the decline in positive attitudes begins to be significant. Then we plan to do some qualitative analyses to find the possible reasons for this and also we plan to implement some active methods of mathematics teaching to find out if we can overcome these negative tendencies. We hope that this will be helpful in the effort to improve pupils' attitudes towards mathematics and so improve in some ways the quality of mathematics education as such.

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Geometrical notions for primary school according Franz Močnik

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Abstract. The historical texbook „Geometrical notions for primary schools" offers very interesting instructions for every teacher that explains gemoetry. The author Franz Močnik (1814–1892) introduces us teaching, where students discovers new knowledge by themselves. Everything is very demonstrative and students can better understand context in lesson.

Keywords: František Močnik, geometry, cube, tetrahedron.

Classification: G10, G20.

1 Introduction

In this book [3] the author presents a clear and comprehensive method of teaching geometric shapes and general geometry with the best possible results. According to the author, the main goal of teaching geometric shapes is not only to introduce pupils to the most important spatial objects and their basic properties, but also to use this knowledge in practice. After all, it is not our aim to prepare pupils for the academic study of geometry in the future but to teach them skills they will need in everyday life. This is why the presentation of new subject matter should be followed with its practical use. Subject matter should be presented as a well-arranged, rounded off whole. The methods described in this book enable pupils to observe objects and to explore their properties and provides them with good preparation for problem solving in real life.

2 Geometrical notions according Franz Močnik

To make the presentation of geometric shapes as effective as possible, every teacher should follow these three principles:

- Establishing a clear idea of spatial objects and defining their basic properties,
- Drawing simple geometric figures,
- Comparing spatial objects with regard to their size; i.e. measurements of such figures and calculations

The author approaches the presentation of this subject matter in two different ways. The first method divides geometry into planimetric geometry (flat-plane geometry) and solid geometry (geometry of solid bodies and figures). After these two essential terms are explained, plane geometry is introduced first to be gradually followed with the geometry of spatial objects. The other type of subject matter presentation uses the principle of demonstrating individual solids and of gradually deducing basic terms and properties. This book deals with the latter method and teaches pupils to explore interrelations and properties in order to be able to solve practical tasks. This is why teachers have to remember to include in their lesson those terms that pupils fail to discover themselves, to cover gaps in the subject matter being explained in order to present a comprehensive body of knowledge. Drawing practice and geometric shapes are inseparable. The author recommends hand drawings, mainly of the grids of solids, while he finds the use of a ruler and a drawing-compass less appropriate. The presentation is closed with area and volume calculations. The book is divided into two parts. The first part introduces a graphic analysis of solids and figures within them. The role of the teaching during their presentation is that of a guide, as pupils discover most knew knowledge on their own. The teacher asks targeted questions and tries to keep pupils' thoughts in the „right direction". The second part is linked to the first one and deals with the calculation of the area and volume of solids. As far as the first part of the book is concerned, I would like to point out the first chapter which presents a graphic analysis of the cube. Another interesting chapter was chapter three which introduces a graphic analysis of a regular tetrahedron. The notions introduced in this textbooks is possible to find in [2]. Let us discuss both chapters in detail later on.

2.1 Cube

A graphical analysis of the cube is the first step on a way towards the continual acquisition of the knowledge of geometric shapes. The author finds it most appropriate to begin presentation by showing pupils a cube model and to put it in a visible place, e.g. on the teachers' desk. Using this illustrative aid, the teacher and pupils interact to deduce the following geometric terms. Pupils understand, at first sight, that a cube is a space enclosed from all sides; hence they perceive it as a solid object. Like every other solid object, a cube has three dimensions (length, width, height). Since the cube is bounded by 6 faces, it is also called a hexahedron. Every one of these faces is flat and so is called a planar surface. Students easily infer that planar surfaces have two dimensions (length and width). The teacher informs pupils about the differences between a base surface and a lateral surface and also helps students deduce the fact that a cube contains three pairs of parallel planes. The teacher further states that every face confining a cube creates its surface and so it is a polygonal surface.

Furthermore, pupils should focus on the individual faces of the cube. Every face is enclosed by four edges. An edge forms where two faces meet. Pupils easily work out, by using a model, that a cube has 12 edges. The teacher should point out that a cube is an angular (square) structure. Every edge of the cube is a straight one dimensional (length) line. The cube has four subsidiary and eight vertical edges. The teacher allows pupils to realize their parallelism, parallelism with their respective planes and the lengths of the cube's edges. We also examine the relation between perpendicular planes and then between edges and faces. The teacher should inform pupils about the concept of quadrilaterals, regular and linear quadrilaterals and should also mention that polygonal lines form the perimeter of every face of the cube. The teacher should closely examine the end points of the edges of the cube and call them vertices. A vertex is where three edges meet and so the cube has eight vertices and every individual face of the cube has four vertices. This point (vertex) has no dimension. Here, purposefully, the teacher informs pupils about the location where two edges meet and introduces the new term angle. Pupils figure out that every face has four angles and that the whole cube has twenty-four angles. This is where the teacher fills in about the vertex of an angle, the arms of an angle and the right-angle. Pupils should discover that every quadrilateral has four angles and so it can be called a quadrangle. A regular and right-angled quadrangle is called a square. The faces of the cube are only made up of squares. Pupils get to know different types of squares and realize that they are similar in shape but not in size. Later on, they are introduced to the term identical squares. Another relevant element introduced during the presentation of new geometric shapes is the grid of the cube. However, the teacher must remember to draw it beforehand. The teacher uses the method of continually assembling neighboring in a plane, thus creating a continuous planar surface. The author suggests drawing the grid on paper, cutting it out, and covering the cube model with it. To give pupils a clearer idea, they should be given a chance to try to assemble their own cube using a grid they have drawn. To enable pupils to deepen the newly acquired knowledge, it is advised to work with the special positions of the cube towards the horizontal plane on which the cube has been placed before. Pupils should find out whether all of the previously taught terms and relations between them also apply to a cube that touches the horizontal plane with one edge or vertex only. It is at this point that the author advises teachers to dedicate more time to the presentation of the cube because most terms used in explanation are introduced for the very first time. Before introducing more terms, the teacher must have pupils revise the previously terms to check whether they sufficiently understood and internalized the subject matter taught before. The subsequent part of the chapter explains several other relevant terms and further elaborates on the terms pupils are already familiar with. The role of the teacher is dominant during this part of the teaching process as he spends most time introducing the precise names of already known subject matter, or, as the case may be, corrects wrong pre-concepts.

2.2 Straight lines

Pupils should notice that a straight line is not clearly bounded by one point only because three edges originate in the cube vertex. Therefore, the straight line is unambiguously bounded by two points. If pupils imagine that a straight line is like a trace created by the movement of a point, they find out that the line is indefinite while the edge of the cube is bounded by two points called endpoints. This straight line is part of the indefinite straight line. This is a moment when the teacher steps in and calls the indefinite line a line segment and the previously mentioned points its end points. Then, using illustrative examples, the teacher guides pupils to the conclusion that a straight line is the shortest line connecting two points. The teacher also shows straight lines that area called vertical, horizontal and oblique. To do so the teacher uses the cube model to clearly demonstrate parallel, concurrent and skew lines. In the end, pupils are introduced to the measurement of line segments by means of various aids and also to the approximate division of a line segment into two identical parts. It is advised to include a number of interesting examples in presentation. Such examples are presented in this chapter. They help to practice acquired knowledge and can become a source of motivation for further work.

2.3 Planes

Pupils, still using the cube model, should notice that three different planes intersect one vertex and that two different planes intersect one edge of the cube. Therefore, the plane is unambiguously bounded by three points that do not lie on one line.

The author considers the use of a sheet of paper even more illustrative. Pupils can use the cube model when determining vertical and horizontal planes. Later on the teacher introduces oblique planes too. Observing the cube, pupils clearly see the straight lines that are parallel or concurrent (perpendicular), or parallel or intersecting planes. At this point the teacher steps in and calls the point where two planes meet an intersection. In the end, all the subject matter taught must be practiced thoroughly using a sufficient number of examples. Only then can new chapters be studied.

2.4 Regular tetrahedron

The illustrative analysis of a regular tetrahedron helps pupils to further expand their knowledge and skills. Like with the cube, a tetrahedron model is used. The teacher positions the model so that one of its faces is turned towards pupils. Once again, the role of the teacher is that of a guide who helps pupils discover new facts and acquire new knowledge. Pupils should now be able to easily determine that this solid figure is called a tetrahedron because it consists of four (tetra) faces (hedron), whereas all of its faces are planes. The horizontal face is called a base and the oblique faces are called lateral faces. The surface area of the tetrahedron is, like with the cube, a polygonal surface. The teacher lets pupils calculate that the tetrahedron has six edges and four vertices. Pupils are encouraged to notice that every face is bounded by three edges; i.e. straight lines of the same length, and that there are no two parallel edges. Each face has three vertices. Realizing the number of angles of every face, pupils are able to call this face a triangle and to establish that the tetrahedron has twelve angles that are equal and acute. This is why all faces are regular equilateral triangles and the tetrahedron itself is called regular tetrahedron. Pupils try to draw the grid of a regular tetrahedron and in order to better remember the terms taught they compare this tetrahedron and a cube. They place both models next to each other and begin to compare both solids, focusing on their faces, edges, vertices and angles (i.e. comparing their number, direction and size). The subsequent part explains a few more relevant terms and further elaborates on the terms that are already known to pupils. The role of the teacher is dominant at this stage as he spends most time giving the exact names of the objects pupils are already familiar with.

2.5 Triangle

Every triangle has three sides and three angles. The teacher shows pupils that every face has two contiguous angles and one opposite angle. It is important to draw pupils' attention to the fact that the sum of the lengths of any two sides must be greater than the length of the third side. The teacher shows pupils the sum of all three angles (using a straight angle). This presentation is followed with a brief discussion about the size of the sides of various triangles. Then, triangles are categorized according to side length (equilateral, isosceles and scalene triangles). The same method is applied to categorize triangles according to the measures of their interior angles (acute, rectangular and obtuse). All the subject matter taught must be practiced thoroughly using a sufficient number of examples.

2.6 Calculation of area and solids

Geometry mostly deals with the determination of areas and volumes. However, the teacher should not merely present formulas to pupils as he should rather try to help pupils to derive them from pupils' previous knowledge. Furthermore, the subject matter taught must be practiced thoroughly because it will help pupils better remember what they have discovered and internalize it through repetition. This is why calculations are to be solved at school but also at home. The teacher should assign tasks and homework with an increasing level of difficulty.

Task 1.

Calculate the area of irregular polygon $ABCDEF$ (see the figure).

Solution: The author provides two types of solution. The area of an irregular polygon can be found by first breaking the polygon up into triangles. Then the area of each triangle is calculated and the totals are added. This means that diagonals are used to break polygon $ABCDEF$ up into 4 triangles. The size of their bases and heights are determined by measurements.

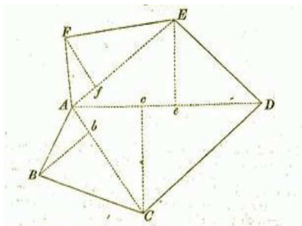

Figure1.

The other solution is as follows. The two farthest vertices are connected with a straight line. Then a perpendicular is dropped from all other vertices to this straight line. This breaks up the polygon into right triangles and trapezes. Their areas are calculated and the totals are added.

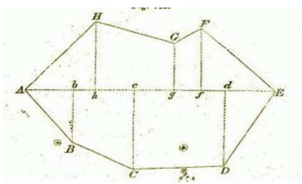

Figure2.

Task 2.

Calculation the volume of cube $ABCDEFGH$ with an edge length of $2m$.

Solution: The bottom can be broken up into 2 x 2 = $4m^2$. Every square meter of the base is covered by one cubic meter. In this way we create a $4m^3$ layer on the base. The height of the layer is 1m. Because the whole cube is 2m high, it contains two $4m^3$ layers; i.e. 2 x 2 x 2 = $8m^3$.

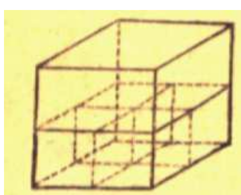

Figure3.

3 Conclusion

We highly appreciate the author's approach to teaching and to handling subject matter to be taught. His method of teaching can be used to make geometry, which is perceived as rather difficult by many pupils, more accessible and understandable to pupils (see also [1],[4]). We believe that if pupils discover and establish most relations between facts themselves, the subject matter taught will become more entertaining for them. This book provides a host of examples and thoughts to be utilized during lessons in contemporary schools. Another creative approaches is possible to find in [5] and [6].

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CLIL in mathematics lessons

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Abstract. The approach CLIL is a relatively new trend in education. It is a combination of content and language learning. The article is focused on using the CLIL in the Czech Republic and it presents research results of the influence on students´ motivation in mathematics lessons with CLIL.

Keywords: CLIL, mathematics, education, ICT.

Classification: D40, M10.

1 Introduction

Education should always reflect the current social trends, the requirements of the labour market and the development of the society and technology. Today, we live in an era of migration in order to find better employment, information technology and information explosion. Accordingly, there are increasing requirements for some basic skills in the present-day society. People need to be able to critically evaluate information, speak foreign languages and, at least to some extent, work with the computer, which is an inherent part of many jobs. The CLIL approach may contribute to the improvement of students' language skills.

2 CLIL approach

The CLIL abbreviation was used for the first time by David Marsh in 1994. It is the abbreviation of the concept of Content and Language Integrated Learning. The author of the concept, David Marsh, defines CLIL in one of his recent publications as follows:

"Content and Language Integrated Learning (CLIL) is a dual-focused educational approach in which an additional language is used for the learning and teaching of content and language with the objective of promoting both content and language mastery to pre-defined levels." (Marsh, 2012)

As can be seen above, the author defines the concept as a dual-focused educational approach in which a foreign language is used for the learning and teaching of a non-language subject. Skills are improved both in the foreign language and in the non-language subject in question.

CLIL can bring many advantages. One of the advantages is for example the expansion of field-specific vocabulary. In language courses as such, there is little time for specific topics relating to scientific subjects such as Mathematics, Physics and Chemistry. Students then lack such vocabulary, which may prevent them from studying or working abroad in many cases. This may also be related to their insufficient confidence in being able to discuss specialist topics in a foreign language. If the CLIL approach is integrated into teaching, it will help students to train and become more confident in terms of foreign-language discussions or studies. Another advantage lies in the possibility of making lessons more interesting, as some activities conducted in a foreign language may vary the stereotype.

Naturally, CLIL constitutes certain disadvantages as well. One of them is the risk that students will be demotivated, especially in two cases. The first case is a situation where students are already experiencing problems with a non-language subject and the integration of a foreign language might mean that they will give up outright – "How am I supposed to solve the task in English when I don't even know how to do it in Czech?" The second case involves students who are talented in non-language subjects but who have problems with learning foreign languages – "I would probably know the solution in Czech but I can't do it in English." Both these cases can be addressed by positively motivating the students to try and solve tasks and participate in activities. Another disadvantage, which is a major negative according to the following research, lies in the relatively high demands placed on the teachers in terms of preparation and even the language skills of the teachers themselves. Insufficient language skills can only be improved by further education and enhancement of the communication skills of the teacher in the foreign language. As for higher demands on preparation, those can be very easily eliminated today. I will discuss these options in more detail in the last chapter.

3 CLIL in surveys

Even though the issue of CLIL is relatively new in the Czech environment, there are relatively many different surveys with a focus on the integration of a foreign language into the learning and teaching of non-language subjects. I will mention the conclusions of some of them in the Czech Republic and abroad. Marylin Hunt of the Educational Institute at the University of Warwick, Coventry, UK examined the popularity of lessons conducted using the CLIL approach in 2011. 67 % of the students liked the lessons, 66 % liked the activities, and 63 % were looking forward to the following lessons conducted using CLIL. (Hunt, 2011) Similarly, in 2012, Pavlína Hořáková inquired into the issue as part of her dissertation thesis and found out that students who had experienced lessons taught using the CLIL approach viewed foreign languages more positively than students who had never experienced such lessons. (Hořáková, 2012) However, the survey results are frequently accompanied by the fear that it will be difficult and time-consuming to prepare such lessons. One of such examples is the research of Josephine Marie Moate of the University of Jyväskylä, who conducted a survey in 2011 and stated that the first two years of teaching using the CLIL approach had been difficult and demanding for most of the respondents (teachers). Nevertheless, the majority of teachers had then taken a fancy to the method and stopped seeing it as a burden. (Moate, 2011) Similar results were obtained by Světlana Hanušová under the project entitled "CLIL in Czech Educational Practice". The respondents (teachers) admitted that the integration of a foreign language into the teaching of non-language subjects was beneficial and advantageous. At the same time, however, they were unwilling to participate in the project and start using the approach actively. (Hanušová, 2012)

In my preliminary research, in which I was concerned with the current situation in terms of the use of CLIL at elementary schools and lower grades of grammar schools in the Olomouc Region and the South Moravia Region, I found out that

CLIL was used by only 10 % of schools in the Olomouc Region and 13 % of schools in the South Moravia Region. These data were obtained from the managements of the individual schools. In both regions, English was predominant and was generally used in a wide range of subjects. The schools in the South Moravia Region mostly used CLIL in the teaching of Mathematics, Music and Art. The questionnaire-based survey also found that Mathematics taught using CLIL was slightly more popular than Mathematics without the CLIL approach. The assumption that students might fear tasks set in English was not confirmed. Moreover, the students felt that they were more active in lessons based on the CLIL approach. Most of them were also aware of the benefits of the integration of English into Mathematics in terms of their future studies and employment. (Wossala, 2014)

It follows from the selected surveys that students and teachers see the integration of a foreign language into non-language subjects as beneficial, but the fear and unwillingness of teachers to integrate CLIL into their lessons are still prevalent.

4 CLIL together with ICT

As was mentioned in the chapter about CLIL, one of the reasons why teachers are reluctant to integrate CLIL into their lessons is the rather demanding nature of the preparation of such lessons, both in terms of time and content. When preparing an activity which integrates a foreign language, it is necessary to focus on the content in more detail, create the relevant vocabulary which will be new to the students, and prepare the activity so that it corresponds to the cognitive skills of the individual students, if possible. This may discourage many teachers, as indicated by the surveys mentioned in the second chapter.

However, currently there are many projects that not only aim at the examination of the influence of the integration of a foreign language into the learning and teaching of non-language subjects but also help create various methodological materials, which make it easier to prepare such lessons. In some cases, prospective teachers are already trained in such teaching methods during their studies at the Faculties of Education. An example may be the Department of Mathematics of the Faculty of Education of Palacký University in Olomouc, which offers a course entitled English Mathematical Terminology. It is based on CLIL and the university students of teaching fields can thus master English mathematical terminology and also the ways of integrating English into the teaching of Mathematics.

Teachers have yet another very effective tool at their disposal that can assist them in preparing and conducting their lessons. The tool consists in modern technologies. Modern technologies have become not only everyday part of the lives of most people, i.e. also the present-day youth, but also part of the educational process. Nowadays, most classrooms are equipped with data projectors and interactive boards, and most schools have computer classrooms. Recently, teachers have begun using tablets, which should further increase the efficiency of the use of ICT in teaching and make it more up-to-date. The actual advantages and disadvantages of using tablets in teaching will only be revealed over time; nevertheless, such technologies already constitute benefits for teachers who want to integrate a foreign language into their lessons. Many teachers who are considering the use of tablets criticise the lack of available applications in the Czech language. However, this does not necessarily present a problem in most cases. For instance, there are numerous high-quality Biology programs with pictures, animations, 3D models etc. These include Corinth Micro, Know Your Body and My Incredible Body. The teacher can use the graphical aspects of the applications, at any rate. And of course, they can also include the English biological terminology in their lessons. And what about Mathematics? There is also a wide range of applications, frequently in English. One of them is Mathematics designed for the Android operating system.

There is a number of English words which the students can learn when using the application – e.g. functions, equations, linear function with point and slope, etc.

Figure 1: User environment of the Mathematics (Google Play)

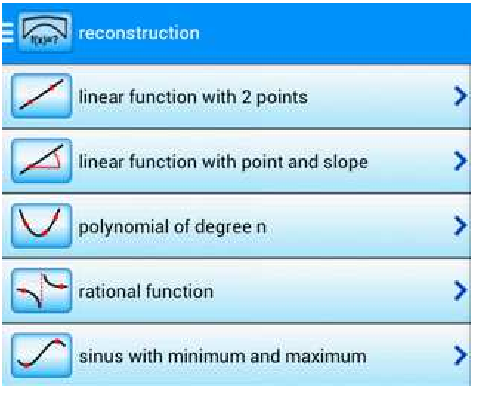

Figure 2: User environment of the Mathematics (Google Play)

Another application is Formulas, again for Android. As can be seen, this application offers not only formulas for calculations but also a wide database of English mathematical terminology – e.g. triangle, area, perimeter, square, etc. Students can learn the vocabulary and also different labelling of perimeters, volumes, areas etc.

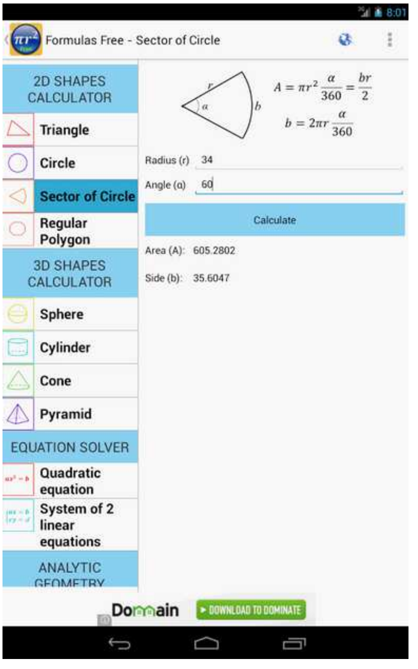

Figure 3: User environment of the Formulas (Google Play)

As a matter of course, there are many applications for younger children as well, to help them understand basic mathematical operations; again, such applications assist in the development of foreign-language vocabulary. One of such applications is iMath designed for the Windows 8 operating system.

The above programs constitute only a small portion of what teachers can use in teaching their non-language subjects. There are similar applications for subjects such as Physics, Chemistry, and Geography, among other things. The effective use of modern technologies, selected high-quality programs and well-chosen teaching methods together form a useful tool which makes lessons more attractive and also facilitates the teacher's work.

5 Conclusion

As the above research and surveys indicate, CLIL is a relatively popular approach in the educational process. Unfortunately is little-used in particular due to the relatively high demands on preparation. Modern technology and suitably chosen programs may eliminate such demands and apply multiple current trends in education – CLIL and ICT – at the same time. There are many high-quality teaching applications in foreign languages that can be used by teachers of a wide range of non-language subjects to improve the quality of their lessons.

Figure 4: User environment of the iMath (Windows Store)

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Mathematical talent

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Abstract. The article dealing with talent, specifically mathematical talent, is intended for general overview of the topic. The authors describe the identification of gifted individuals, the pedagogical attitude and they specifically focus on mathematical gifts which is the main point of this paper. The conclusion of the article, the authors are introduced with their own research probe in mathematical gift which was conducted for children in kindergarten, principal research will be investigate on pupils in the first to third grade of elementary school.

Keywords: talent, mathematics, shape, keyword.

Classification: D40.

1 Introduction

The paper presents the mathematical talents and accelerated progress of children from preschool up to the third year of first grade elementary school. I would like to address this issue because talented or just more attentive pupils in Czech schools are oversighted and for most teachers this problem creates barriers in the traditional approach to teaching.

It is a very complicated issue because symptoms of talent are very individual. It may discover peculiarities that are common to more individuals, but there are also individual characteristics. In the article there is mentioned several times a permanent work with talented people because they need to develop their skills. There is an initial development for the talent's development itself, we continue the development of talented individuals to fulfilment, improve and for higher goals.

2 Talent

Talent is a term with a wide range of characteristics is not always easy to identify. According to V. Fořtík and J. Fořtíková (2007) a talent is defined as follows:

It could be:

1. qualitative summary of individual abilities which determine the successful execution of activities;
2. general abilities or general elements of capabilities which undermine the possibilities of person, the level and characteristics of its activities;
3. intellectual potential or intelligence; holistic individual characteristics of cognitive capabilities and ability to learn;
4. summary of aptitudes, inborn givens, speech levels and properties inborn ingredient;
5. talent, an existence of internal conditions for achieving excellence in pursuits.

There is a need to expose the talents and accelerated progress as soon as possible to develop the work of the pupil's abilities, knowledge and skills in the topic. We must approach individually to the gifted. It is a pupil with special needs in the class.

Talent could be divided according to a subject: language, mathematics, science (physics, chemistry, biology), etc. The relations between the subjects are very narrow, rarely happens that interested individual exceeds these limits.

2.1 Mathematical talent

Mathematical talent is unique and a teacher often recognizes it. The first symptoms are discovered in early childhood and when individual development is accelerating. When an elementary school is beginning the talents are displayed with a greater ability in mathematical operations and logical thinking. An individual pupil's specific approach to the teaching goes with it.

As an example there could be given a mathematical sentence. The result of this sentence must be discovered by the pupil himself/herself (with no formula beforehand) who helps to the development of his/her divergent thinking. There is an emphasis on the development of logical thinking, individual creativity, inbounding form mechanically learned ways of solution and development of an individual progress to the issue.

The characteristics of these pupils are the same as intellectually gifted but also they have specific characteristics related to mathematics. The general symptoms include independence, creativity, rich vocabulary, longer attention, curiosity, excellent memory and what the pupils are interested in.

Mathematically gifted children are very quickly able to compare anything (by size, name or other criteria), they tend to constantly look for logical reasons for things, discuss and analyse (for example why we have five fingers), they love to think new variety of quizzes and games up. They usually have developed the ability to solve problems, like thinking about abstract ideas, leading scientific experiments, they are the best in solving complex problems and constantly asking for something. (Havigerová, 2011, p. 93)

The specific symptoms summarize the huge interest in mathematics, awareness of mathematical relationships and develop divergent thinking based on a creative approach to the tasks. Thy symptoms use relationships between knowledge, which are used by the older children.

2.2 Own research probe

Since April 2015 I have started a research probe about determination of the accelerated development and mathematical talent of children in preschool age. My research probe was participated by twelve respondents: two of them have a deferral of school, one girl is seven years old and nine of them are going to the first grade of elementary school in September this year.

I have created some worksheets about five jobs.

The first task was focused on a detection shape and finding two identical shapes.

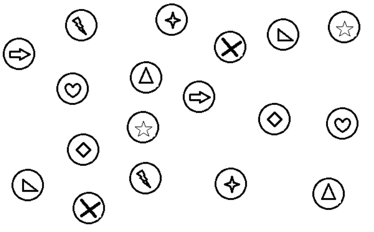

Figure 1: Find two identical beads and paint them with the same color.

The second task was focused on counting to ten and matching quantity up to the numbers.

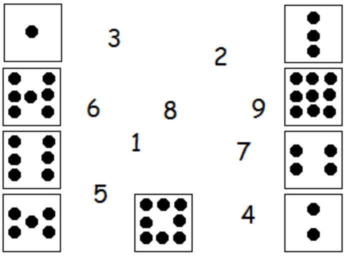

Figure 2: Find the correct number and connect it with quantity of dots on a dice.

The third task concerned the counting of geometric shapes in the picture.

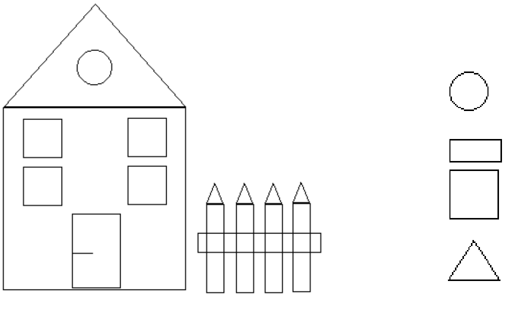

Figure 3: Calculate rectangles, squares, circles and triangles which are in the picture. Number of geometric shapes write as number on the marked site.

Pupils counted triangles in the fourth task.

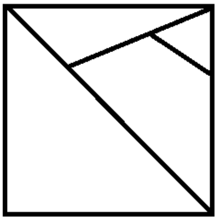

Figure 4: How many triangles are in the picture?

They drew their own picture from predefined geometric shapes in the fifth task.

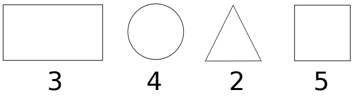

Figure 5: Draw your picture, use all these geometric shapes.

A challenge of each job is different. The first exercise was the easiest for children because it was done faultlessly and during a short time. I saw the first differences between the children at intervals – two preschool children completed the task in two minutes and the others were finishing it gradually. During the second task some problems occurred because the general education program for kindergarten excludes knowledge of numbers from 1 to 10. The error did not occur in joining numbers with dices but in the actual counting dots on a dice. For seven children the greatest difficulty was counting the number of dots on dices and the connection was without problem.

The hardest was the third task which I had to explain individually. Nobody understood the task and everybody failed. They did not realize that for example a triangle does not have to be only a roof but also a fence. I set fourth task to the children according to the time because some tasks were more challenging for them and because every task took the children another time interval. The last task was not successful at all. Instead of the children draw a picture they only copied the defined shapes. I had thirty minutes because the teacher had another program in the kindergarten. So I could not divide this worksheet into two parts and it was one of the reasons why the preschool children were tired.

After the completion and assessment the worksheets I found that there are no gifted individuals in mathematics in our test class in the kindergarten. However there are three children who have accelerated development more than other participants. These preschool children could not complete the third and the last tasks. They did the first and second task without a fault but they did not complete the third and fifth task correctly and some preschool children did the fourth job because they worked faster than other participants.

I would like to direct another research in selected elementary school where these participants will start their elementary education. It is an elementary school in Děčín which has 542 students in this school year. Pupils from this school take part in various competitions (about sports or knowledge) in which the school results are at the top. It is one of the best schools in this town.

The future research will be realized by the test methods as the first research probe. Further I would like to use direct observation which is an important part during working out worksheets. I want to use the interview for talented pupils in mathematics to find out more information about them, about their development and their interests. It could be very interesting to do a questionnaire for parents of the gifted children. The questionnaire applies to the process of education and development of talent at home. Selection of other methods is based on the discovery of mathematical talent among the pupils.

3 Conclusion

Talent is a complex attribute which needs to be worked with and developed with the children. Symptoms are individual and often hardly recognizable. It is very important to develop even a small sign of accelerated evolution or talent.

Mathematical talent compared to other kinds, language and scientific, is easily recognizable as artistic or sports talent. We find the same characteristics in mathematical talent as the intellectually gifted individuals. They are independence, creativity, rich vocabulary, curiosity and an excellent memory. We add specific attributes to these general characteristics of the mathematically gifted which are excessive interest in mathematics and developing divergent thinking based on creative approach to problem solving.

We should develop mainly ourselves because we are the ones who give the pupils an opportunity for their self-fulfilment and the development of all these characteristics, general and specific. The talent is a great specificity which does not occur frequently in the population and therefore we have these pupils leading to constant self-improvement which is our mission.

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