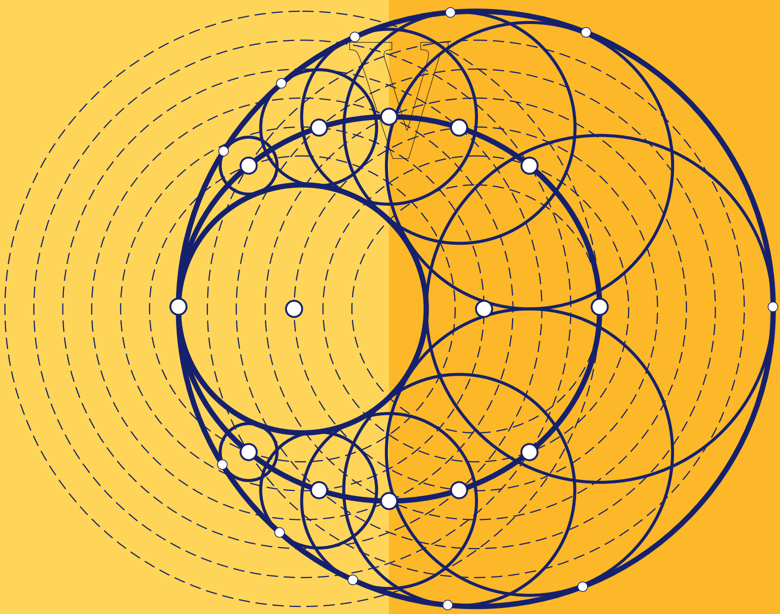
MATHEMATICA IV

EDITOR

Martin Billich



Scientific Issues

MATHEMATICA IV

The publication was published with the support of
the Catholic University in Ružomberok
Faculty of Education, Department of Mathematics

This volume is published thanks to the support of the grant
KEGA 001UJS-4/2011

Scientific Issues
Catholic University in Ružomberok

MATHEMATICA IV

Martin Billich

VERBUM
Ružomberok 2012

MATHEMATICA IV, Scientific Issues

Typesetting by LaTeX
Martin BILLICH

Graphical Project of a Cover
Martin PAPČO

Place Andrej Hlinka 60, 034 01 Ružomberok
http://ku.sk, verbum@ku.sk, Phone: +421 44 4304693

ISBN 978-80-8084-954-2

CONTENTS

PREFACE

This volume of 28 texts presents the authors, who took part in the Polish-Czech-Slovak Mathematical Conference, which was held at the Faculty of Theology of the Catholic University in Ružomberok, Spišská Kapitula, June 5 – 8, 2012. The main goal of the conference was international exchange between researchers in mathematics and mathematics education. The primary support for the conference was the participation of the PhD students from the PhD School in mathematics education of the Palacký University, Czech Republic, Constantine the Philosopher University in Nitra, Slovakia and other researchers and teachers from universities in Visegrad countries. The participants presented their research results, and possibilities of future cooperative research.

The primary objective of *MATHEMATICA IV* is to present results some of these PhD students and experienced researchers and teachers. It is nowadays important exchange between these groups of researchers, because PhD students bring new methods (for example in ICT aided education) and experienced researchers bring experience and good practices for the teaching. New findings in pure mathematics can also support mathematics education. We hope that potential readers (teachers and researchers in mathematics and mathematics education) of this publication can find many inspirations for their educational and research work.

Martin Billich

The Tests of Triangles for Age Group between 15–17

Ludmila Benešová, Jakub Tláskal, Josef Molnár

Department of Algebra and Geometry, Palacký University in Olomouc
17. listopadu 1192/12, 771 46 Olomouc, Czech Republic
lidabene@seznam.cz
jakub.tlaskal@seznam.cz
josef.molnar@upol.cz

Abstract. A pilot study spatial imagination is submitted to students 1st and 2nd year-class of secondary school and students of the same age at grammar school. Problems about spatial imagination allow teacher to specify level of student's mathematical abilities. The aim of this pre-research is to compare knowledge of students of secondary school and students at grammar school.

Keywords: Spatial imagination, Test of Triangles.

Classification: D60.

1 Introduction

Geometrical imagination lies in skill to imagine geometrical figures and their attributes. We can move or fold geometrical figures in our head with the assistance geometrical imagination.

2 Test of Triangles

We tested 440 students from these three schools:

- grammar school in Zlín,
- grammar school in Lanškroun,
- technical school in Olomouc.

The test took 20 minutes and included 40 figures. The task was *to divide each figure with one straight line. The parts moved in their head to equilateral triangle.*

2.1 Grey & White test

This test was component of Jana Slezáková's dissertation (2011). We did little change of this test and we compare our results with results of Jana Slezáková. Each figure was colored in the original test. We used two types of test grey and white. We must find out if black and white tests have different results then coloured tests. Also we compared grey test and white test and the grey test had better results then white test.

Confrontation grey & white

	Grammar school Lanškroun		Technic school Olomouc	
Type of Test	White	Grey	White	Grey
Mistakes	6,22	4,85	9,06	7,33

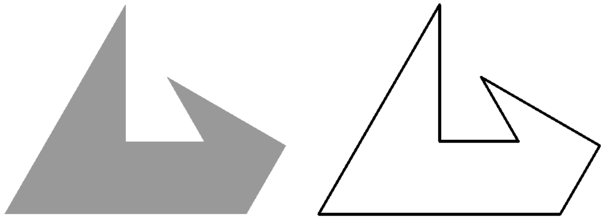


Figure 1: Example of figure from grey and white test

2.2 Grammar & Technical school

We compare each school with the others schools. Also we compare classes of school. We can see results in next table. This table shows percentage success for each task. We tested organization figures. Figures were organization from easy to difficult. This organization was good. There were some exceptions as figure 3, 5, 13. Different figures are difficult for individual students.

Confrontation classes

Class	1		2			3	
School	lanskroun	olomouc	lanskroun	zlin	olomouc	zlin	olomouc
Task No.							
1	100,0 %	98,7 %	100,0 %	93,3 %	95,5 %	97,2 %	92,5 %
2	100,0 %	100,0 %	97,8 %	97,3 %	97,0 %	100,0 %	95,0 %
3	100,0 %	97,3 %	97,8 %	89,3 %	84,8 %	95,8 %	92,5 %
4	97,8 %	96,0 %	100,0 %	94,7 %	93,9 %	97,2 %	90,0 %
5	91,1 %	89,3 %	93,5 %	84,0 %	87,9 %	90,3 %	87,5 %
6	100,0 %	93,3 %	100,0 %	93,3 %	95,5 %	98,6 %	97,5 %
7	97,8 %	97,3 %	97,8 %	93,3 %	92,4 %	98,6 %	92,5 %
8	91,1 %	84,0 %	95,7 %	90,7 %	90,9 %	94,4 %	92,5 %
9	91,1 %	93,3 %	97,8 %	90,7 %	92,4 %	94,4 %	97,5 %
10	95,6 %	89,3 %	100,0 %	92,0 %	92,4 %	98,6 %	67,5 %
11	97,8 %	89,3 %	93,5 %	88,0 %	89,4 %	97,2 %	90,0 %
12	93,3 %	85,3 %	95,7 %	89,3 %	92,4 %	97,2 %	90,0 %
13	84,4 %	89,3 %	93,5 %	82,7 %	93,9 %	95,8 %	92,5 %
14	93,3 %	92,0 %	100,0 %	89,3 %	92,4 %	95,8 %	97,5 %
15	84,4 %	88,0 %	91,3 %	93,3 %	90,9 %	97,2 %	92,5 %
16	97,8 %	89,3 %	100,0 %	90,7 %	90,9 %	97,2 %	87,5 %
17	97,8 %	89,3 %	91,3 %	92,0 %	93,9 %	97,2 %	95,0 %
18	97,8 %	86,7 %	89,1 %	88,0 %	92,4 %	94,4 %	90,0 %
19	100,0 %	90,7 %	93,5 %	92,0 %	90,9 %	95,8 %	95,0 %
20	84,4 %	78,7 %	82,6 %	86,7 %	86,4 %	94,4 %	95,0 %
21	91,1 %	85,3 %	82,6 %	78,7 %	86,4 %	93,1 %	87,5 %
22	84,4 %	78,7 %	91,3 %	86,7 %	87,9 %	93,1 %	80,0 %
23	93,3 %	81,3 %	91,3 %	92,0 %	90,9 %	95,8 %	87,5 %
24	84,4 %	81,3 %	87,0 %	80,0 %	83,3 %	86,1 %	85,0 %
25	84,4 %	81,3 %	89,1 %	82,7 %	87,9 %	80,6 %	82,5 %

Class	1		2			3	
School	lanskroun	olomouc	lanskroun	zlin	olomouc	zlin	olomouc
Task No.							
26	84,4 %	82,7 %	87,0 %	78,7 %	83,3 %	91,7 %	77,5 %
27	82,2 %	73,3 %	87,0 %	85,3 %	81,8 %	91,7 %	75,0 %
28	93,3 %	77,3 %	82,6 %	77,3 %	78,8 %	87,5 %	70,0 %
29	68,9 %	69,3 %	71,7 %	65,3 %	80,3 %	79,2 %	55,0 %
30	73,3 %	60,0 %	73,9 %	61,3 %	66,7 %	79,2 %	55,0 %
31	68,9 %	76,0 %	78,3 %	60,0 %	72,7 %	79,2 %	65,0 %
32	68,9 %	62,7 %	73,9 %	65,3 %	65,2 %	72,2 %	57,5 %
33	86,7 %	69,3 %	71,7 %	65,3 %	60,6 %	84,7 %	52,5 %
34	55,6 %	65,3 %	60,9 %	66,7 %	59,1 %	77,8 %	60,0 %
35	66,7 %	68,0 %	60,9 %	57,3 %	47,0 %	76,4 %	57,5 %
36	86,7 %	77,3 %	82,6 %	80,0 %	66,7 %	93,1 %	75,0 %
37	51,1 %	61,3 %	73,9 %	66,7 %	51,5 %	75,0 %	40,0 %
38	66,7 %	41,3 %	73,9 %	62,7 %	42,4 %	76,4 %	42,5 %
39	82,2 %	50,7 %	73,9 %	58,7 %	36,4 %	72,2 %	32,5 %
40	51,1 %	46,7 %	69,6 %	48,0 %	36,4 %	62,5 %	40,0 %
Total	85,5 %	80,2 %	86,8 %	80,7 %	80,0 %	89,4 %	77,7 %

2.3 Unexpected solutions

Students resolved some figures specially. They found some solutions with which we didn't suppose. Some solutions were bad because they changed straight line and line segment.

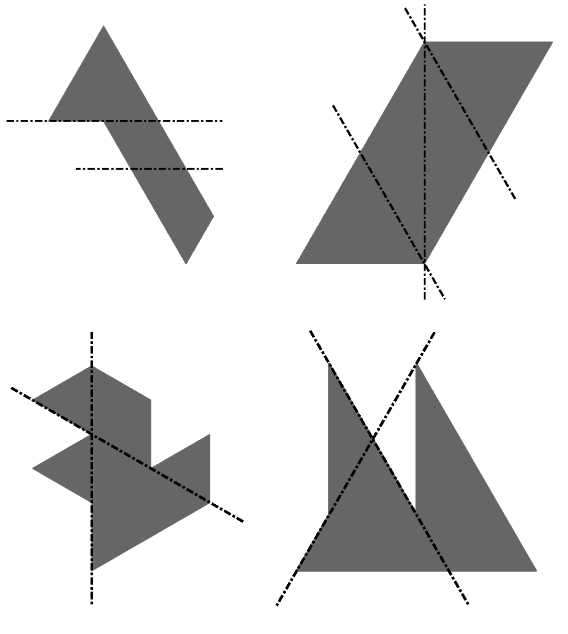


Figure 2: New good solutions

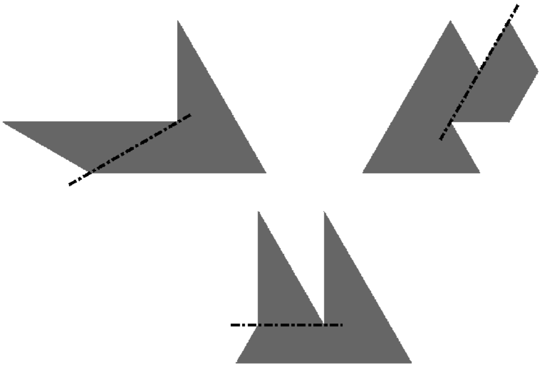


Figure 3: Not correct solutions

3 Conclusion

There weren't many differences between these schools. There were many students with 40 points. Interesting results were between grey and white tests. White test had better results by all of schools.

References

[1] J. Molnár: *Rozvíjení prostorové představivosti (nejen) ve stereometrii,* VUP, Olomouc, 2004.

[2] J. Slezáková: *Geometrická představivost v rovině (dissertation),* PřF UP, Olomouc, 2011.

[3] J. Molnár: *Testing of goemetrical imagination,* Mathematics XVI, Scientific Issues, Jan Dlugosz university of Czestochowa, 2011.

Some logical problems connected with theorems

Grzegorz Bryll, Grażyna Rygał

Pedagogical Department, Jan Długosz University in Częstochowa
Waszyngtona 4/8, 42-200 Częstochowa, Poland
g.rygal@op.pl

Abstract. Pupils have problems with a logical understanding of proofs of theorems. The paper presents an attempt to explain such notions as a direct theorem, contrapositive of a theorem, a contrary theorem, and converse of a theorem. The logical laws allowing us to prove theorems correctly are discussed.

Keywords: Implication, contrary implication, converse implication, sufficient condition, necessary condition.

Classification: 00A35.

Parallel to mathematical difficulties pupils at school also meet logical difficulties. Often just they prevent a true understanding of mathematical content. In particular, this concerns problems connected with theorems.

1 Negation of implication and contrary implication

Often pupils identify the notion of "a contrary implication" and "negation of implication" referring to the notion of the opposite number and even are inclined to accept equivalence

$$\neg(A \Rightarrow B) \Leftrightarrow (\neg A \Rightarrow \neg B),$$

whereas the equality

$$\neg(A \Rightarrow B) \Leftrightarrow (A \wedge \neg B).$$

is true.

Hence, the negation of an implication $A \Rightarrow B$ is the statement $\neg(A \Rightarrow B)$ which is equivalent to the statement $A \wedge \neg B$, whereas the implication contrary to the implication $A \Rightarrow B$ is the implication $\neg A \Rightarrow \neg B$ which is equivalent to the statement $A \vee \neg B$.

For example, the contrary implication to the implication "If a quadrangle is a rhombus, then its diagonals are orthogonal" is the implication: "If a quadrangle is not a rhombus, then its diagonals are not orthogonal." The first implication is true, but the second implication is false.

In this connection, it is worth while to discuss with pupils all implications connected with a direct implication $A \Rightarrow B$ (a direct theorem). They are: a contrary implication $\neg A \Rightarrow \neg B$ (a contrary theorem), a converse implication $B \Rightarrow A$ (converse of a theorem), and contrapositive of an implication $\neg B \Rightarrow \neg A$ (contrapositive of a theorem).

Connection between these implications can be stated using the general contraposition law:

$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P).$$

On the basis of this law and the law of double negation we obtain four contraposition laws:

$$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A) \tag{1}$$

$$(A \Rightarrow \neg B) \Leftrightarrow (B \Rightarrow \neg A) \tag{2}$$

$$(\neg A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow A) \tag{3}$$

$$(\neg A \Rightarrow \neg B) \Leftrightarrow (B \Rightarrow A) \tag{4}$$

In the scheme below the implications having the same numbers are equivalent:

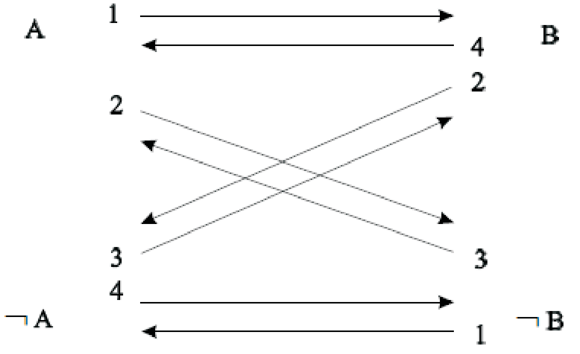


Figure 1

Some pupils think mistakenly that if a direct implication is true, then a contrary implication is also true. We should convince a pupil that if a direct theorem has been proved, then a contrapositive theorem should not be proved as it is also true (the law (1)).

Similarly, if converse of a theorem has been proved, then a contrary theorem should not be proved as it is also true (the law (4)).

In the textbook [2], when proving the direct theorem on a comparative criterion of number series, contrapositive of the theorem is unnecessarily proved.

However, it sometimes occurs that all the theorems connected with the direct theorem are true. We have such a situation in a case of the following theorem: "*If a given number is divisible by 3, then the sum of all the individual digits is divisible by 3.*"

In the case of theorem being equivalence $A \Leftrightarrow B$, two theorems should be proved: $A \Rightarrow B$ and $B \Rightarrow A$. Sometimes, the equivalence proof can be carried out. For example, equality $(X \cup Y)' = X' \cap Y'$ from the set theory can be proved as follows:

$$a \in (X \cup Y)' \Leftrightarrow a \notin X \cup Y \Leftrightarrow \neg(a \in X \cup Y) \Leftrightarrow$$

$$\neg(a \in X) \wedge \neg(a \in Y) \Leftrightarrow a \notin X \wedge a \notin Y \Leftrightarrow$$

$$a \in X' \wedge a \in Y' \Leftrightarrow a \in X' \cap Y'.$$

Here the corresponding implications have the form

$$a \in (X \cup Y)' \Rightarrow a \in X' \cap Y',$$

$$a \in X' \cap Y' \Rightarrow a \in (X' \cup Y)'.$$

In the case of closed system of theorems the proof simplifies significantly. To prove the equivalence $A \Leftrightarrow B,\ B \Leftrightarrow C,\ C \Leftrightarrow A$ it is sufficient to prove implications $A \Rightarrow B,\ B \Rightarrow C$ and $C \Rightarrow A$, as the following statement is a tautology

$$[(A \Rightarrow B) \wedge (B \Rightarrow C) \wedge (C \Rightarrow A)] \Rightarrow$$

$$[(A \Leftrightarrow B) \wedge (B \Leftrightarrow C) \wedge (C \Leftrightarrow A)].$$

This law can be substantiated by the transitive property for an implication:

If $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \gamma$, then $\alpha \Rightarrow \gamma$.

2 Necessary conditions and sufficient conditions

Pupils are often encountered with formulations: "*the necessary and sufficient condition*", "*if and only if*", "*it is necessary and sufficient*", "*always and only if*", "*is equivalent*", "*in this and only in this case if*". These formulations have the same meaning and are used alternately by mathematicians.

As the necessary condition for a statement A we mean every sentence which follows from the statement A, while as the sufficient condition for the statement A we mean every sentence from which the statement A follows.

The necessary conditions and the sufficient conditions can be illustrated as follows (an arrow means the implication relation):

$$\begin{array}{c} A \\ \swarrow \quad \downarrow \quad \ldots \searrow \\ \underbrace{B_1 \quad B_2 \quad \ldots \quad B_m}_{\substack{\text{necessary conditions} \\ \text{for a statement } A}} \end{array} \qquad\qquad \begin{array}{c} \substack{\text{sufficient conditions} \\ \text{for a statement } A} \\ C_1 \quad C_2 \ \ldots \ C_n \\ \searrow \quad \downarrow \quad \swarrow \\ A \end{array}$$

An example of necessary conditions:

$$\begin{array}{ccc} & \text{A number } l \text{ is divisible by 4} & \\ \swarrow & \downarrow & \searrow \\ l \text{ is an even number} & \begin{array}{c}\text{A number } l \\ \text{is divisible by 2}\end{array} & \begin{array}{c}\text{A number built from} \\ \text{two last digits of } l \\ \text{is divisible by 4}\end{array} \end{array}$$

An example of sufficient conditions

$$\begin{array}{ccc} \begin{array}{c}\text{A number } l \text{ is divisible} \\ \text{by 9}\end{array} & & \begin{array}{c}\text{The sum od digits of a number } l \\ \text{is divisible by 3}\end{array} \\ \searrow & & \swarrow \\ & \text{A number } l \text{ is divisible by 3} & \end{array}$$

Some of conditions can be necessary and sufficient conditions at the same time. Often mathematical theorems are formulated as necessary and sufficient conditions. For example "A number l is divisible by 9 if and only if the sum of its digits is divisible by 9."

"A circle can be inscribed into a convex quadrangle always and only if the sums of opposite sides of this quadrangle are equal."

If a statement B is the necessary and sufficient condition for a statement A, then the equivalence $A \Leftrightarrow B$ is true, i.e. two implications are true: the direct implication $A \Rightarrow B$ and the converse implication $B \Rightarrow A$.

In the case of a theorem formulated as the necessary and sufficient condition, pupils often restrict themselves to a proof of necessary condition or to a proof of sufficient condition only. To assert that a given set Z is a locus with a property A the following two theorems should be proved:

"*If a point belongs to the set Z, then it has the property A*" (the direct theorem; the necessary condition), and

"*If a point has the property A, then it belongs to the set Z*" (the converse theorem; the sufficient condition).

Instead of proofs of the direct and converse theorems the proofs of the direct theorem and the contrary theorem (the rule (4)) can be given.

To show that locus of points equidistant from arms of an angle is its bisector the following two theorems should be proved:

"*If an arbitrary point is equidistant from arms of an angle, then it lies on its bisector.*"

"*If an arbitrary point lies on the bisector of an angle, then it is equidistant from arms of this angle.*"

3 Proofs by contradiction

Formally the proof by contraction is easier than the direct proof.

In the proof of a theorem $A \Rightarrow B$ by contraction the statement A is taken as an assumption, and the statement $\neg B$ is taken as an assumption of the proof by contradiction. Obtaining two contradictory statements C and $\neg C$, we infer that the theorem is true.

Some pupils cannot understand why after obtaining statements C and $\neg C$ we infer that the implication $A \Rightarrow B$ is true. Hence, we propose the following proof of the implication $A \Rightarrow B$ by contradiction. As a starting step of the proof we assume the statement $\neg(A \Rightarrow B)$. Suppose that the given implication is false. From the law of the negation of an implication

$$\neg(A \Rightarrow B) \Leftrightarrow (A \wedge \neg B) \tag{5}$$

we obtain the statements A and $\neg B$ which in the traditional proof are initial statements. After obtaining contradiction, i.e. two contradictory statements C and $\neg C$, we infer that the statement $\neg(A \Rightarrow B)$ is false as only true stements follow from true statements, while the statement $C \wedge \neg C$ is false. Therefore, the statement $A \Rightarrow B$ is true.

Example. The proof by contradiction of the theorem

$$(a^2 \text{ is an even number}) \Rightarrow (a \text{ is an even number})$$

can be carried out as follows:

1. $\neg[(a^2 \text{ is an even number}) \Rightarrow (a \text{ is an even number})]$
   $\{$the assumption of the proof by contradiction$\}$

$$\left.\begin{array}{l} 2.\ \ a^2 \text{ is an even number} \\ 3.\ \neg(a \text{ is an even number} \end{array}\right\} \qquad \{1 \text{ the law of negation of an implication } (5)\}$$

4. a is an odd number $\{3\}$

5. $a = 2k_1 + 1, \quad k_1$ is an integer $\{4\}$

6. $a^2 = (2k_1 + 1)^2 = 4k_1^2 + 4k_1 + 1 = 4k_1(k_1 + 1) + 1$ $\{5\}$

7. a^2 is an odd number $\{6\}$

8. $\neg(a^2 \text{ is an even number})$ $\{7\}$
   contradiction $\{2, 8\}$

References

[1] I.C. Gradstejn: *Priamaja i obratnaja teoriemy* (izd. 3). Moskwa, 1959
[2] F. Leja: *Rachunek różniczkowy i cakowy*. PWN, Warszawa, 2008
[3] W. A. Pogorzelski: *Klasyczny rachunek zdań* (wyd. 2). PWN, Warszawa, 1973
[4] Z. Ziembiński: *Logika pragmatyczna* (wyd. 2), PWN, Poznań, 1960

Arithmetical and Geometrical Sequences of Higher Degrees in any Arithmetic mod m

Grzegorz Bryll, Robert Sochacki

Institute of Philosophy, University of Opole
Katowicka 89, 45-061 Opole, Poland
rsochacki@uni.opole.pl

Abstract. In this paper we consider arithmetical and geometrical sequences of higher degrees in any arithmetic mod m. Moreover formulas for any term of such sequences and formulas for sums of its n initial terms are presented.

Keywords: arithmetic mod m, arithmetical and geometrical sequences of higher degrees.

Classification: 11B50; 11Y70.

More important properties of arithmetical and geometrical sequences of higher degrees are also valid in any arithmetic mod m. Before we give properties of these sequences, we recall some basic properties of arithmetic mod m. Let m be an arbitrary natural number greater than 1. Any modulo operation finds the remainder of the division of one number by another. Given two positive numbers, n (the dividend) and m (the divisor), n modulo m (abbreviated as n mod m) can be thought of as the remainder, on division of n by m. For instance, the expression "5 mod 4" is equal to 1, because 5 divided by 4 leaves the remainder of 1. Generally, any arithmetic modulo m (in short mod m) is an arithmetic of remainders of the division of any positive number by m. All possible remainders of such a division are the numbers $0, 1, \ldots, m-1$. Let $[n]_m$ denote a remainder of a division of n by m. Let also $Z_m = \{0, 1, \ldots, m-1\}$. On the set Z_m we define arithmetic operations as follows:

$$a +_{(m)} b = [a+b]_m,$$

$$a \cdot_{(m)} b = [a \cdot b]_m,$$

$$a -_{(m)} b = c \iff b +_{(m)} c = a,$$

$$a :_{(m)} b = c \iff b \cdot_m c = a, \quad (b \neq 0).$$

The additive inverse of a number a and the multiplicative inverse of a number a are defined as follows:

$$b = -_{(m)} a \iff a +_{(m)} b = 0,$$

$$b = (\frac{1}{a})_{(m)} \iff a \cdot_{(m)} b = 1, \quad (a \neq 0).$$

Of course, we have:

$$-_{(m)} a = 0 -_{(m)} a,$$

$$(\frac{1}{a})_{(m)} = 1 :_{(m)} a, \quad (a \neq 0).$$

The operations mentioned above in the arithmetic mod 5 have the following tables:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

-	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

:	0	1	2	3	4
0		0	0	0	0
1		1	3	2	4
2		2	1	4	3
3		3	4	1	2
4		4	2	3	1

a	0	1	2	3	4
$-a$	0	4	3	2	1

a	0	1	2	3	4
$\frac{1}{a}$	-	1	3	2	4

Let us notice that in an arithmetic mod p, where p is a prime number, a division of any number by p always gives a unique result. In another case, where p is not a prime number, the result of such a division may not exist or may not be unique. For instance, in the arithmetic mod 10 the result of the division $5:4$ does not exist since there is no such $x \in Z_{10}$ that satisfies the condition $4 \cdot x = 5$. Moreover, in this system we have $2:4 = 8$ and $2:4 = 3$, so indeed the division mod 10 is not unique.

Now we will give some definitions. Let N be the set of all natural numbers. Any function $f : N \to Z_m$ is called a sequence. Let $f(n) = a_n, n \in N$. In the set of all sequences we consider arithmetical and geometrical sequences in the ordinary sense and arithmetical and geometrical sequences of higher degrees.

Definition 1. *Any sequence $\{a_n\}$ is called an arithmetical sequence in an arithmetic mod m iff for any term of this sequence (except the first) the condition holds:*

$$a_n = a_{n-1} +_{(m)} r, \quad n > 0. \tag{1}$$

As in the ordinary arithmetic, an arbitrary term may be expressed as [1]

$$a_n = a_0 + [n]_m \cdot r, \quad n \geq 0, \tag{2}$$

where the operations $+$ and $\cdot$ are treated as mod m. Any arithmetical sequence in an arithmetic mod m is a periodic sequence. The maximum length of the period of such a sequence can be m. Let us consider the arithmetical sequences in which we have: $a_0 = 3,\ r = 2$ and $a'_0 = 2,\ r' = 3$. In the arithmetic mod 4 the successive elements of these sequences are: $3, 1, (3, 1)$; and $2, 1, 0, 3, (2, 1, 0, 3)$, respectively. Their lengths of periods are 2 and 4, respectively.

For any sequence $\{a_n\}$, where $a_n \in Z_m$, we can calculate a sequence of the first finite differences $\{\Delta^1 a_n\}$ in the following way: $\Delta^1 a_0 = a_1 -_{(m)} a_0$, $\Delta^1 a_1 = a_2 -_{(m)} a_1$, $\Delta^1 a_2 = a_3 -_{(m)} a_2, \ldots$. Similarly, we define sequences of successive differences: $\Delta^{k+1} a_i = \Delta^k a_{i+1} -_{(m)} \Delta^k a_i$, $i \in N$, $k \in N \setminus \{0\}$.

Definition 2. *Any sequence $\{a_n\}$ is called an arithmetical sequence of the k-th degree ($k \geq 1$) if and only if the sequence $\{\Delta^k a_n\}$ is constant and differs from the zero sequence.*

[1] If an aritmetical sequence $\{a_n\}$ is a function $f : N \setminus \{0\} \to R$, then an arbitrary term of this sequence has the form $a_n = a_1 + (n-1) \cdot r$.

To determine any sequence of successive differences one can draw up the following table:

$$
\begin{array}{c|cccccc}
a_0 & & & & & \\
 & \Delta^1 a_0 & & & & \\
a_1 & & \Delta^2 a_0 & & & \\
 & \Delta^1 a_1 & & \Delta^3 a_0 & & \\
a_2 & & \Delta^2 a_1 & & \Delta^4 a_0 & \\
 & \Delta^1 a_2 & & \Delta^3 a_1 & & \ddots \\
a_3 & & \Delta^2 a_2 & & \vdots & \\
 & \Delta^1 a_3 & & \vdots & & \\
a_4 & & \vdots & & & \\
 & \vdots & & & & \\
\vdots & & & & &
\end{array}
\tag{3}
$$

For example, the sequence $2, 0, 2, (2, 0, 2)$ is the sequence of the 2-nd degree in the arithmetic mod 3. The table of successive differences has the form:

$$
\begin{array}{c|cccc}
2 & & & & \\
 & 1 & & & \\
0 & & 1 & & \\
 & 2 & & 0 & \\
2 & & 1 & & \ddots \\
 & 0 & & 0 & \\
2 & & 1 & & \\
 & 1 & & 0 & \\
0 & & 1 & \vdots & \\
 & 2 & \vdots & & \\
2 & \vdots & & & \\
\vdots & & & &
\end{array}
\tag{4}
$$

Just as in the ordinary arithmetic (see [4])[2] any term of an arithmetic sequence of the k-th degree in an arithmetic mod m has the form:

$$
a_n = \left[\binom{n}{0}\right]_m a_0 + \left[\binom{n}{1}\right]_m \Delta^1 a_0 + \cdots + \left[\binom{n}{k}\right]_m \Delta^k a_0. \tag{5}
$$

For this formula the differences $\Delta^1 a_0, \ldots, \Delta^k a_0$ can be read from Table (3). By Table (4), for the sequence $2, 0, 2, (2, 0, 2)$ we have: $a_0 = 2$, $\Delta^1 a_0 = 1$, $\Delta^2 a_0 = 1$, $\Delta^k a_0 = 0$ for $k > 2$. Using formula (5) we obtain:
$a_n = \left[\binom{n}{0}\right]_3 \cdot 2 + \left[\binom{n}{1}\right]_3 \cdot 1 + \left[\binom{n}{2}\right]_3 \cdot 1$, that is $a_n = 2 + [n] + 2[n(n-1)] = 2[n]^2 + 2[n] + 2$, since $1 : 2 = 2 \pmod 3$.

[2] The formula (5) looks a bit different if we consider sequences as functions $f : N \setminus \{0\} \to Z_m$.

The following sequence:[3] $a_n = f([n]_m) = c_k[n]_m^k + c_{k-1}[n]_m^{k-1} + \cdots + c_1[n]_m + c_0$ corresponds to a polynomial $f(x) = c_k x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$, where $c_k \neq 0$, $c_i \in Z_m$, $i \in N$, in an arithmetic mod m. For example, in the arithmetic mod 3, the sequence: $a_n = 2[n]_3^2 + 2[n]_3 + 2$ corresponds to the polynomial $f(x) = 2x^2 + 2x + 2$. In the arithmetic mod 5, the following sequences:

$a_n = 3[n] + 2$, i.e. $2, 0, 3, 1, 4, (2, 0, 3, 1, 4)$;
$a_n = [n]^2 + 1$, i.e. $1, 2, 0, 0, 2, (1, 2, 0, 0, 2)$;
$a_n = 2[n]^3 + [n]^2 + 1$, i.e. $1, 4, 4, 3, 3, (1, 4, 4, 3, 3)$;
$a_n = [n]^4 + [n]^2 + 4$, i.e. $4, 2, 1, 1, 4, (4, 2, 1, 1, 4)$;

correspond to the polynomials: $f_1(x) = 3x+2$, $f_2(x) = x^2+1$, $f_3(x) = 2x^3+x^2+1$, $f_4(x) = x^4+x^2+4$, respectively. The above sequences are the arithmetical sequences of the degree: $1, 2, 3, 4$, respectively.

For any arithmetical sequence of the k-th degree in an arithmetic mod m the sum of its n initial terms (see [2]) can be calculated. For this purpose one can use the formula:

$$S_n = \left[\binom{n+1}{1}\right]_m a_0 + \left[\binom{n+1}{2}\right]_m \Delta^1 a_0 + \cdots + \left[\binom{n+1}{k+1}\right]_m \Delta^k a_0. \qquad (6)$$

If a sequence $\{a_n\}$ is an arithmetical sequence of the first degree, then by (6) we obtain the formula $S_n = \left[\binom{n+1}{1}\right]_m a_0 + \left[\binom{n+1}{2}\right]_m \Delta^1 a_0$, i.e.

$$S_n = \left[\frac{n+1}{2}\right]_m [2a_0 + n \cdot r]_m \qquad (7)$$

where $r = \Delta^1 a_0$. Let us consider the sequence $a_n = n^2$ in the arithmetic mod 7. The period of this sequence has the length of 7. We will calculate the sum of the elements of the sequence belonging to the first period. Since the sequence $\{a_n\}$ is the arithmetical sequence of the second degree, so, by (6) the sum of its n initial terms is:

$$S_n = \left[\binom{n+1}{1}\right]_7 a_0 + \left[\binom{n+1}{2}\right]_7 \Delta^1 a_0 + \left[\binom{n+1}{3}\right]_7 \Delta^2 a_0. \qquad (8)$$

To determine the terms $a_0, \Delta^1 a_0, \Delta^2 a_0$, a table of succesive differences may be drawn up. From this table we find that: $a_0 = 0, \Delta^1 a_0 = 1$, $\Delta^2 a_0 = 2$, $\Delta^k a_0 = 0$ for $k > 2$. Thus, formula (8) has the form:

$S_n = \left[\binom{n+1}{1}\right]_7 \cdot 0 + \left[\binom{n+1}{2}\right]_7 \cdot 1 + \left[\binom{n+1}{3}\right]_7 \cdot 2 = \left[\frac{n(n+1)}{2}\right]_7 \cdot 1 + \left[\frac{(n-1)n(n+2)}{6}\right]_7 \cdot 2.$

For $n = 7$ we obtain $S_7 = 0$.

As in the ordinary arithmetic one can prove that:

Theorem 1. *Any sequence is an arithmetical sequence of the k-th degree in an arithmetic mod m if and only if this sequence is characterized by a polynomial of the k-th degree.*

[3] We understand any power a^n in an arithmetic mod m as: $a^0 = 1, a^1 = 1, a^n = \underbrace{a \cdot \cdots \cdot a}_{n}$, if $n > 1$. It should be noted that in general $a^n \neq a^{[n]_m}$. For example, $2^{[4]_3} = 2^1 = 2$, $2^4 = (2 \cdot 2) \cdot (2 \cdot 2) = 1 \cdot 1 = 1$.

In an arithmetic mod p (p is a prime number) we can limit ourselves to the study of polynomials, at the most - of the $(p-1)$-th degree. This conclusion follows from the fact that:
$x^{(p-1)k+1} \equiv x, \quad x^{(p-1)k+2} \equiv x^2, \quad \cdots, \quad x^{(p-1)k+(p-1)} \equiv x^{p-1}, \quad k \in N.$
So, for the initial prime numbers we have:

- in the arithmetic mod 2: $x^{k+1} \equiv x,$
- in the arithmetic mod 3: $x^{2k+1} \equiv x, \quad x^{2k+2} \equiv x^2$
- in the arithmetic mod 5: $x^{4k+1} \equiv x, \quad x^{4k+2} \equiv x^2, x^{4k+3} \equiv x^3, \quad x^{4k+4} \equiv x^4.$

Therefore, the following quantities of polynomials can be considered:

- in the arithmetic mod 2: 2 linear polynomials;
- in the arithmetic mod 3: 6 linear polynomials, 18 polynomials of the 2-nd degree;
- in the arythmetic mod 5: 20 linear polynomials; 100 polynomals of the 2-nd degree; 500 polynomials of the 3-rd degree and 2500 polynomials of the 4-th degree.

To these polynomials there are arithmetical sequences of corresponding degrees.

Definition 3. *Any sequence $\{a_n\}$ in an arithmetic mod m is called geometrical if and only if each term of this sequence (except the first) is given by a multiple constant q of the previous one.*

So we have:

$$a_n = a_{n-1} \cdot_{(m)} q, \quad n > 0. \tag{9}$$

The n-th term of any geometrical sequence is given by the formula:

$$a_n = a_0 \cdot_{(m)} q^n, \quad n \geq 0. \tag{10}$$

In the arithmetic mod 7 the geometrical sequence with $a_0 = 5, q = 3$ has the form: $5, 1, 3, 2, 6, (5, 1, 3, 2, 6)$. The period of this sequence has the length of 5. Also in the case of the sequence of the form $a_n = 1 \cdot_{(11)} 4^n$ we have the following sequence: $1, 4, 5, 9, 3, (1, 4, 5, 9, 3)$ with the same period's length.

In our further considerations we will consider geometrical sequences in which $a_0 \neq 0$ and $q \neq 0$. It is easy to see that if $a_0 = 0$, then we obtain the zero sequence (for an arbitrary q), and if $a_0 \neq 0$ and $q = 0$, then the sequence has the form: $a_0, 0, 0, \ldots$. We will also assume that we are dealing with any arithmetic mod p (p is a prime number). For any sequence $\{a_n\}$ ($a_n \neq 0$) we form a sequence $\{q_{nk}\}$ ($k \in N \setminus \{0\}$) given by the terms:

$$q_{n1} = \frac{a_{n+1}}{a_n}, \quad q_{n2} = \frac{q_{n+1,1}}{q_{n1}}, \quad \ldots, \quad q_{nk} = \frac{q_{n+1,k-1}}{q_{n,k-1}}, \quad (n \in N).$$

The terms of each particular sequence are arranged in the succesive columns of the following table of successive quotients:

$$
\begin{array}{c|ccccc}
a_0 & & & & & \\
 & q_{01} & & & & \\
a_1 & & q_{02} & & & \\
 & q_{11} & & q_{03} & & \\
a_2 & & q_{12} & & q_{04} & \\
 & q_{21} & & q_{13} & & \ddots \\
a_3 & & q_{22} & & \vdots & \\
 & q_{31} & & \vdots & & \\
a_4 & & \vdots & & & \\
 & \vdots & & & & \\
\vdots & & & & &
\end{array}
\tag{11}
$$

Definition 4. *Any sequence $\{a_n\}$ in an arithmetic mod p is called a geometrical sequence of the k-th degree ($k \geq 1$) if and only if the sequence $\{q_{nk}\}$ is a constant sequence of elements different from 1.*

Any constant sequence is called a geometrical sequence of the 0-th degree. Thus, any ordinary geometrical sequence with a quotient $q \neq 1$ is a geometrical sequence of the first degree. One can prove that (see [3]):

Theorem 2. *If $\{a_n\}$ is a geometrical sequence of the k-th degree (in an arithmetic mod p), then the n-th term of this sequence may be expressed by the succesive quotients as follows:*

$$a_n = a_0^{\binom{n}{0}} \cdot q_{01}^{\binom{n}{1}} \cdot q_{02}^{\binom{n}{2}} \cdot \ldots \cdot q_{0k}^{\binom{n}{k}}, \tag{12}$$

i.e.

$$a_n = a_0 \prod_{i=1}^{k} q_{0i}^{\binom{n}{i}}. \tag{13}$$

For the sequence $1, 4, 4, 1, (1, 4, 4, 1)$ in the arithmetic mod 5 the table of the succesive quotients looks as follows:

$$
\begin{array}{c|cccc}
1 & & & & \\
 & 4 & & & \\
4 & & 4 & & \\
 & 1 & & 1 & \\
4 & & 4 & & \ddots \\
 & 4 & & 1 & \\
1 & & 4 & \vdots & \\
\vdots & 1 & & & \\
 & \vdots & 4 & & \\
 & & \vdots & &
\end{array}
\tag{14}
$$

From this table it follows that: $a_0 = 1, q_{01} = 4, q_{02} = 4, q_{0r} = 1$ for $r > 2$. Thus, this is the sequence of the 2-nd degree. Using formula (12) we can give the formula for the n-th term: $a_n = 1^{\binom{n}{0}} \cdot 4^{\binom{n}{1}} \cdot 4^{\binom{n}{2}}$, i.e. $a_n = 4^n \cdot 4^{\frac{n(n-1)}{2}} = 2^{2n} \cdot 2^{n(n-1)} = 2^{n(n+1)}$. As we stated earlier, in order to investigate polynomials in an arithmetic mod p we can limit our considerations to polynomials, at the most - of the $(p-1)$-th degree. Let us notice that any sequence $\{a_n\}$ of the form:

$a_n = r^{c_s n^s + c_{s-1} n^{s-1} + \ \ldots \ + c_2 n^2 + c_1 n + c_0}$

where $0 \neq s \leq p-1, r \neq 1$, is a geometrical sequence of the s-th degree, since the sequence $\{q_{ns}\}$ of successive quotients ($q_{ns} = r^{c_s \cdot s!}$) is a constant sequence.

Let $d^\circ(W_i)$ denote the degree of a polynomial $W_i(n)$ of a variable n. For polynomials $W_1(n), W_2(n), \ldots, W_l(n)$ we adopt the following symbol:

$v = \max(d^\circ(W_1), d^\circ(W_2), \ \ldots \ , d^\circ(W_l))$.

It is easy to verify that if a sequence $\{a_n\}$ has the form:

$a_n = r_0 \cdot r_1^{W_1(n)} \cdot r_2^{W_2(n)} \cdot \ldots \cdot r_l^{W_l(n)}$, where $r_i \neq 1, i \in N$,

then it is the geometrical sequence of the v-th degree. So, for example, if $p > 2$ (p is a prime number), then the sequence $a_n = 2^{n^2}$ is a geometrical sequence of the 2-nd degree, and in the case of $p > 3$, the sequences $a_n = 3^{\binom{n}{2}}$, $a_n = 2^n \cdot 3^{\binom{n}{2}}$ are also sequences of the 2-nd degree.

References

[1] G. Bryll: *Some remarks on arithmetical sequences of higher degrees* (in Polish). Zeszyty Naukowe WSP w Opolu, Matematyka 7 (1969), 23–37.

[2] G. Bryll: *Application of the method of succesive differences in making inductive hypotheses* (in Polish). Prace Naukowe WSP Częstochowie, Matematyka VII (1999), 79–83.

[3] G. Bryll, R. Sochacki: *Remarks on geometrical sequences oh higher degrees.* Prace Naukowe WSP w Częstochowie, Matematyka 8 (2000-2001), 115–118.

[4] G. Bryll, R. Sochacki: *A uniform treatment of arithmetical and geometrical sequences of higher degrees.* Acta Universitatis Purkynianae, (Czech-Polish Mathematical School 2001), 72 (2001), 65–72.

[5] J. Cihlář: *Aritmetické a geometrické posloupnosti vyššich řádů.* Acta Universitatis Purkynianae 12, Czech–Polish Mathematical School, Ústi nad Labem, 30–31 January 1995, (1996), 107–112.

[6] E. B. Dynkin, V. A. Uspienskij: *Matiematičeskije biesiedy.* Gos. Izd. Tech-Teoret. Lit., Moskva-Leningrad, 1952.

Projects using Multimedia Computers for an Integration of a Disadvantaged Pupils

Lukáš Círus

Faculty of Education, Jan Evangelista Purkyně University
Hoření 13, 400 96 Ústí nad Labem, Czech Republic
cirus@pf.ujep.cz

Abstract. This article is focused on the possibilities of communication reinforcement among children from different socio-cultural environment while using ICT tools in class. The article uses the *"Informační centrum pro turisty"* project, deepens the key competencies such as communicative, social and personal competencies.

Keywords: ICT, project tuition, RVP ZV, multicultural education.

Classification: D40, R10.

1 Introduction

By the school year of 2007/2008 the *"Rámcové vzdělávací programy pro základní vzdělávání"* (Framework Education Programme for Elementary Education) came into practice. These RVP are going to substitute current educational programs (*Základní škola* – Primary School, *Národní škola* – lower grades of Primary School, *Obecná škola*, etc.) A main idea of RVP is to provide schools and its teachers with better autonomy in classes and with possibility to react due to real school conditions and pupils' needs.

"Not only advantages a computer brings into pupils' life but also many disadvantages in case of unreasonable, frequent and long-term use. Parents and teachers should be the proper guides and help and motivate young pupils to use computers in a good and appropriate way."[1]

One of the most problematic areas is a communication among children from different socio-cultural environment and their peers. Many children are ashamed of such a "handicap", they either communicate with their friends from the same environment or they do not communicate at all.

Our contemporary politics is aimed at integration of disadvantaged children, either disabled or socially handicapped, into common schools; such as declared in the White Book of Primary education ("*Bílá kniha základního vzdělávání*").

We all are surrounded by modern technical tools and the majority of us use them every day but that is not the case of socio-culturally disadvantaged families. These families may not have the chance to use all up – to date technologies which can have great impact on the children integration. Schools have immense chance to remove the communicative barrier among children and help the disadvantaged children by the use of modern technologies.

All mentioned above is not valid in two cases of different socio – economic and ethnic classes. For instance, Vietnamese families have no big problems concerning money; they fight with language barrier and different cultural background. I

[1] ČERNÁK, Igor. Informatická výchova na 1. stupni ZŠ (současný stav). In: Zborník z 3.ročníka konferencie s medzinárodnou účastou Informatika v škole a praxi, 10.9.-12.9.2007, Ružomberok, Katolícka univerzita v Ružomberku, Pedagogická fakulta, s. 160-164.

have personal experience with Vietnamese children who had troubles even with this subject.

2 A Compensation of disadvantage proposal

Such trouble needs to be solved from the real beginning to avoid its deepening. Also, future teachers must be ready for these situations and so we are getting ready to solve them during our courses of *Didactics aspects of information technologies.* This course is part of new university study for future teachers at 1^{st} grade of primary school at Faculty of Education of Jan Evangelista Purkyne University in Usti nad Labem.

There are plenty of possibilities to help these pupils; one of them could be organization of a group that would motivate pupils in gaining the basis in computing.

Another possibility could be organizing the projects that require cooperation among children; every child has stable position and role in the group. Future success of a project depends on their cooperation because everyone is important for the group.

These projects are nowadays very popular at many schools, concerning the White Book theses and the RVP ZV (*Framework Education Programme for Elementary Education).*

3 ICT in the Framework Education Programme for Elementary Education

Owing to rising need to acquire the basic computing skills, the field of Information and Communication Technologies studies has been included into curriculum of 1^{st} and 2^{nd} grade of Primary Schools.

The educational field of ICT is focused on handling the computer technology, mainly searching and processing the information founded on the Internet and other digital media. It propagates a method of easily accessible information that is not limited by time or place. It serves as a tool for reducing the load of information, knowledge up- dating and also, it is a valuable part in many educational areas of primary school.

What are the main goals of this educational field? First, the information recognition and understanding and its sources; second, use of the computer technology and application software. The computer technology as a device for natural phenomena shaping; a respect for intellectual property.

A curriculum is divided into 1^{st} and 2^{nd} grade. The 1^{st} grade is set as follows:

- Basic usage of computer
- The information research, communication
- The information processing and use

These skills are improved at the 2^{nd} grade:

- The information research, communication
- The information processing and use

There are outputs and curriculum formulated in each unit.[2]

4 The Key Competencies

Because of the RVP ZV (*Framework Education Programme for Elementary Education*), a new terminology appears as well. The Key Competencies are one of them. They can be defined as more complex capabilities that can be used not only in one's life but learning, too. The Key Competencies overlap individual educational fields and change the approach towards education. The RVP ZV module defines six Key Competencies:

1. Learning Competency
2. Problem-Solving Competency
3. Communication Competency
4. Social and Personal Competency
5. Civic Competency
6. Professional Competency

The pupils reach these competencies in fields of knowledge, skills and attitudes.

It is impossible to achieve the competencies without the cooperation between individual school subjects and cross-curricular projects because the Key Competencies overlap every educational field.

5 A Project "*Tourist Information Centre*"

It is not achievable for the school subject Information and Communication Technologies studies to develop the Key Competencies on its own. We need to take advantage of it, link it with other subjects and with practice.

One of the possibilities could be a project "*Informační centrum pro turisty*" (*Tourist Information Centre*).

Main task for the target group, the 5^{th} class pupils, is to set up the Tourist Information Centre.

Brief steps of the project:

1. Introducing the project, e.g. the class creates and finds out the information about our region that could be attractive for the 1^{st} grade classes; presents the places of interest and answers their potential questions.
2. The pupils need to know the target group they are finding the information for. Then, they have several possibilities how to do it. They either go to the local Tourist Information Centre with their teacher or they choose pupils who would visit the Tourist Information Centre on behalf of the whole class (which can be taken as an alternative task). The other option is to use the Internet and find several IC to compare. They make notes about interesting and inspiring ideas to apply them later to their own project.
3. Due to the information found the pupils should identify the areas of their basic knowledge about the region so they are able to provide with them their clients.
   (a) For example:
      i. The information about companies, organizations and services in the city
      ii. Bus and train timetables, transport information

   iii. Accommodation, catering, cultural surveys
   iv. Selling the maps, postcards, promotional material
   v. Sightseeing information, places of interest, etc.

4. It is visible that the class needs to be divided into several groups with its own assignment.
5. Classroom division into groups:
   (a) Randomness – every pupil draws a number from a non-transparent bag (there are numbers that correlates with the real amount of groups and the total amount of leaflets correlates with the real number of pupils in the class).
   (b) Intentional mingle – the teachers create diverse groups consisting of pupils with significant differences among each other (e.g. gender, different personality, experience, knowledge). These groups help the pupils to be able to cooperate and learn and gain some new skills.[3]
6. Each group should appoint their leader and sort out individual roles. Some pupils will look for the information; some of them will process it or there is an opportunity for parallel searching and processing the information, which depends on each group.
7. Each group handling one particular field should plan an offer and know the information sources to find out the answers for their client's questions.
8. The offer would be typed and printed. It is apt to post an electronic version on a school web page.
9. As soon as the information and its presentation are ready to show, the pupils set up their Information Centre Stand at their school that would be available for the peers during the breaks.
10. Each group has its turn in the Information Centre Stand due to its particular services.
11. The other teachers assign their pupils with tasks they should accomplish at the Information Centre Stand. The outputs required are posters in classrooms with tourist cultural and historical places of interests that were created by pupils.

Realization of a first phase of project – founding the Information Centre by the competent pupils - depends on the fact that it is either separated two days or one week project or if we use the classes of National History and Geography (*Vlastivěda*) and Computer Technology Studies. It is important when we need to know the duration of the project. Personally, I would support the idea of long term preparation and the realization and processing the information in groups to one Project Day.

The second part when the peers go to the Information Centre Stands to get the information should be two or three weeks long. It can become part of a whole-school competition when in months chosen the pupils should focus on tourist and historical information of an individual region.

There would be a final evaluation of the poster presentations of individual classrooms.

6 The Tourist Information Centre as a support for the Key Competencies Development

All six Key Competencies have been improved in our project:

1. Learning Competency – The pupils are able to find and gather information, classify them and use the outputs gained. They can use the computer technology and the Interned as a valuable tool in problem–solving tasks.
2. Problem-Solving Competency – The pupils have the task assigned (a problem to be solved), not the solution; they had to solve it by cooperating among each other. The solution was the outcome of their mutual cooperation and procedures.
3. Communication Competency – The pupils were guided by the teacher, they agreed on a particular structure of a group, they appointed their leader and sorted out the individual tasks. They improve their communicative skills in working teams and among the other peers in the Information Centre.
4. Civic Competency – The pupils became involved in the social life of their town and region, they perceive its positive and negative sides, and it makes them think about the possible improvement. There is also the opportunity to advise the local government with the most valuable ideas.
5. Professional Competency – The pupils can work with the Computer Technology and the Internet, they can find the information.

7 Conclusion

I believe that this kind of a project has a chance to develop and strengthen the communication among our pupils across the social groups; it can improve the Key Competencies and show the pupils a practical use of ICT in a helpful context of regional information. It also develops the communicative, interpersonal and civic qualification and skills in the problem-solving.

Teacher does not often realize that the ordinary projects can serve as valuable and supportive tools in integration of children from different socio-cultural environment.

References

[1] Spilková, V.: *Proměny primárního vzdělávání v ČR,* 1. vyd. Praha: Portál, 2005. 311 s. ISBN 80-7178-942-9.
[2] Kol.: *Rámcový vzdělávací program pro základní vzdělávání,* Praha: VUP, 2005.
[3] Geoffrey, P.: *Moderní vyučování* Praha: Portál, 1996. 380 s. ISBN 80-7178-070-7.

The Strategy of Reformulation of a Problem

Petr Eisenmann, Jan Kopka, Jiřina Ondrušová, Jiří Přibyl

Faculty of Science, Jan Evangelista Purkyně University
České mládeže 8, 400 96 Ústí nad Labem, Czech Republic
petr.eisenmann@ujep.cz, jan.kopka@ujep.cz
jirina.ondrusova@ujep.cz, jiri.pribyl@ujep.cz

Abstract. This contribution describes one of the general heuristic problem solving strategies - strategy of reformulation of a problem. This strategy is illustrated by one historical excursion (Pythagorean construction of regular pentagon) and four examples of current educational practice.

Keywords: Strategy of Reformulation, Solving Strategies, Problem Solving.

Classification: D50.

1 Introduction

Since the Ancient times, Mathematicians have invented various strategies to help solve their problems. Such strategies serve as devices enabling us to reach our target most easily when moving from the initial situation. There are a large number of such strategies nowadays. We are to deal with just one of them; this is the strategy of reformulation of a problem, for more details and other strategies, see [6].

We may state the strategy is used when reformulating a given problem into a new one, sometimes an utterly different problem, of which the solving is easier for us and where the solution is a direct solution to the initial problem or it forwards us to the solution. Such category covers a problem translation from one mathematical discipline language into a language of a different mathematical discipline. This transition is to be in particular used in cases where the new translation offers a more achievable solution.

2 Strategy of reformulation

2.1 Brief history

The strategy of reformulating a problem is not a new one at all; to convince the reader, let us show an example dating to the times of Ancient Greece – the very beginning of the period – the Pythagoras' times, see [6] for more details. Mathematicians of the period were masters of using this strategy; it is not a novelty that their symbol was a regular pentagram, with two upward vertices (Fig.).

Figure 1: Pentagram from Agrippa's book ([3])

The Pythagoreans were keen to construct their symbol – the regular pentagram. They were unable to do so until they found its hidden secret – the incommensurability of line segment, which is set in the proportion of the golden ratio, meaning the ratio between the diagonal length and the side.

Thus the problem of constructing a regular pentagram[1] was reformulated into a new problem (see the below diagram):

Problem 1: Construct a regular pentagram.

$\big\downarrow$ Reformulation

Problem 2: Divide the line segment in the golden ratio proportion.

As seen in Fig. , there is a certain connection between problems 1 and 2. The golden ratio is a ratio between line segment lengths ($a : b = b : c = c : d$), where $a = |GJ|$, $b = |CG|$, $c = |DJ|$ and $d = |CD|$. The regular pentagram became a symbol to the Pythagoreans due to its incommensurability of line segment and the multiple golden ratio; see [5] for more detail.

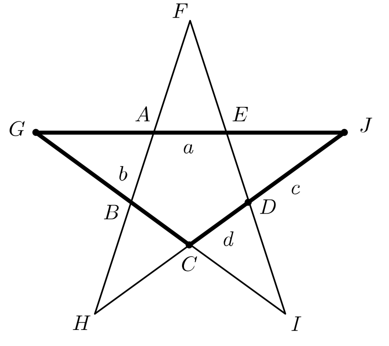


Figure 2

As the Pythagorean School solved problem 2, the construction of a regular pentagon was rather a simple matter; the given example showed a way of reformulating an initial problem into a different one, of which the solution enabled solving the original problem.

The solution to problem 2 belongs to secondary school mathematics and is to be found in its related literature, too. E.g. [1] offers a structured solution.

The solution to problem 1 is based on the problem 2 solution.

Let us set an arbitrary line segment a. If the line segment is divided in the golden ratio proportion, we come to a new line segment b and thus construct an isosceles triangle (with the other leg $c + d$). Should the leg be divided in the proportion of the golden ratio, a line segment is set, of which the layout to the leg b as well as the line segment a gives us another two point. Once the points A, B, C, D, E are

[1] An alternative being a task of constructing a regular pentagon.

interconnected, a pentagon comes to existence and with a line drawn through the points, we reach a pentagram $FGHIJ$.

Let us stay in the past for another while. It is a fact that classic problems of geometry (trisection of an angle, doubling of cube, quadrature of circle, constructing a regular n-gon, rectification of the circle) found in Ancient Greece in the times between Pythagoras and Euclid, had been a core issue remaining unsolved for a long period of time. It took two millennia to prove there is no way to solve these problems using a Euclidean construction; this meaning it is impossible to deal with the problem using a straightedge (or ruler without marks) and a compass. The solution was to be found only when mathematicians reformulated the problem using an analytic method, or speaking more accurately, translated their problem into algebra.

To sum up: The first problem was transferring the initial problem into a new one, a very differently phrased one. Solving the new issue helped solve the initial problem. The second problem is of a different type: The original geometry problems were translated into algebra. The algebraic devices offered a solution to the translated problem and consequently returned it to geometry.

Having finished our historical introduction, we are about to see an illustration of this method while solving some tasks in school maths.

2.2 Task 1

A swimmer went against the river flow and at an A point he lost his swimming cap. He went on swimming for another 5 minutes when he eventually changed his direction and went down with the stream. He reached the cap under a bridge 500 metres away from A. At what speed does the river flow? For original see [6].

Let us proceed in the standard way first:

Solution 1 (direct method): u meaning the swimmers speed (assuming his speed was constant) and v being the flowing speed. If t is the time the swimmer needed to go down with the stream. Thus t may be expressed in two ways:

$$t = \frac{500 - 5v}{v} \qquad \text{time of cap}$$

$$t = \frac{5(u - v) + 500}{u + v} \qquad \text{time of swimmer}$$

Thus:

$$\frac{500 - 5v}{v} = \frac{5(u - v) + 500}{u + v}$$

Therefore:

$$\begin{aligned} 5uv - 5v^2 + 500v &= 500v + 500u - 5v^2 - 5vu \\ 10vu &= 500u \\ vu &= 50u \\ v &= 50 \end{aligned}$$

Answer: The river flows at 50 m/min.

Solution 2 (reformulated problem): If we conceive the water is stagnant and the cap does not move from A, in theory, the bridge moves to A at the speed of flow. It takes the swimmer 5 minutes to go in one direction and then another 5 minutes in the opposite direction (though it is the same speed and stretch), and he "catches up" with his cap in point A under the moving bridge. The bridge has travelled 500 m in 10 minutes, it means it went at $500/10 = 50$; with 50 m being the speed of the river flow.

Answer: The river flows at 50 m/min.

2.3 Task 2

Decide which fraction is greater than the other one: $\frac{125}{126}$ or $\frac{124}{125}$.

Solution of reformulated problem: We should have two cakes of pizza of the same size (identical circles), and one is to be divided into 125 identical pieces; the other being segmented into 126 pieces. According to the fact we divided identical objects, the parts are smaller with the second pizza (the same size is divided into more identical particles). If we take a single piece of pizza away, there will be 124 pieces in the first pizza and 125 pieces in the other one. Because we have taken a smaller pizza piece, the remaining cake of pizza is greater.

Answer: $\frac{125}{126} > \frac{124}{125}$.

The following task belongs to recreational mathematics and is to be found in [7].

2.4 Task 3 – Dudeney's Puzzle

There is a table of 5×5 squares, with numbered playing stones 1 to 25, following the diagram as depicted (Fig.):

7	24	10	19	3
12	20	8	22	23
2	15	25	18	13
21	11	5	9	16
17	4	14	1	6

Figure 3

The point of the puzzle is to draw the stones up, starting in the bottom left corner and ending in the upper right corner, with the least possible number of passing shots. One move is to shoot two stones only (for example 9 – 7).

Solution: Let re-formulate the task into a different language. The initial situation may well be described as follows:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & \dots \\ 17 & 4 & 14 & 1 & 6 & 21 & 11 & 5 & 9 & 16 & 2 & 15 & 25 & 18 & 13 & 12 & 20 & \dots \end{pmatrix}$$

This is a single permutation description. Our target is to find the lowest number of transpositions (we exchange two stones only at a time) which we need to be able to create identical permutation. Our situation may be divided into segments as reads below:

- Cycles – $(1, 17, 20, 23, 10, 16, 12, 15, 13, 25, 3, 14, 18, 8, 5, 6, 21, 7, 11, 2, 4)$, $(19, 22, 24)$
- Fixed points – 9

A 3-cycle consisting of two transpositions $(19, 24)$ and $(22, 19)$ is to be transferred to fixed points, which takes us to a resulting situation of having placed four elements $(9, 19, 22, 24)$ out of 25.

Now it is time to decompose the remaining cycle. If we proceed systematically – that is proceeding individual transpositions and placing one element at a time, we learn the final number of transpositions needed to create identical permutation is 22. The division into 3-cycles also leads us to 22. Zapletal in [7] says, that best solution is 19.

2.5 Task 4

There are n (n is positive integer) designated points on the circumference of the circle and every couple of points is interconnected with a chord. We assume there are no three mutually intersecting chords in a point within the circle; how many intersections are there within the circle?

Solution: We start with experimenting. There are designated cases for $n = 3, 4, 5$ in Fig. . There is no intersection point for $n = 3$, one crossing point with $n = 4$ and 5 cross points with $n = 5$. (If we depict a diagram for $n = 6$, we find 15 intersects.) When looking at case $n = 4$ (in Fig. , we have no trouble realizing that every inner intersection is defined by four designated points on the circle; we may therefore reformulate our problem into combinatorics (here it means a way of expressing the problem in a different language):

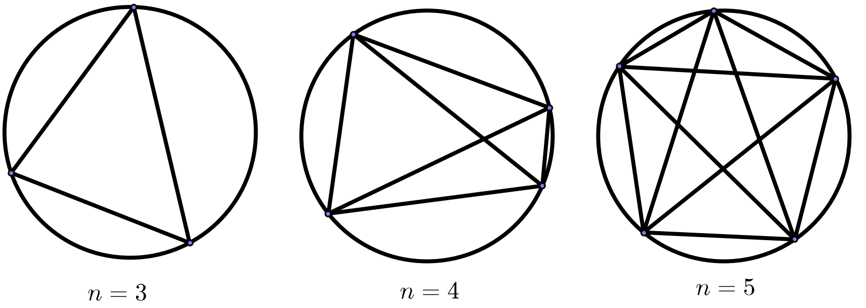


Figure 4

Reformulated problem: How many 4-element subsets has a set of n elements?
Solution: From combinatorics we know that there are $\binom{n}{4}$ subsets.

Answer: There are $\binom{n}{4}$ points of intersection within the circle. That is $\frac{n!}{4!(n-4)!}$ intersects.

E.g. within $n = 7$, there are $\frac{7!}{4!\cdot 3!}$ of intersects, which leads us to 35.

3 Conclusion

It has been stated in the opening section earlier on that mathematicians have created a number of strategies helping them to solve problems. The above described strategy has been taken advantage of in the past as well as at present day. If we wish our pupils as well as students to become better problem shooters under the influence of school maths, there is a strong need to practice various strategies; to be efficient at using the strategy of problem reformulation, we need to have reasonable knowledge together with experience in the branch, and/or in the branch in which the new problem is expressed.

Acknowledgements

The authors has been supported by Czech Science Foundations (grants 407/12/1939).

References

[1] Golden ratio. In: *Wikipedia: the free encyclopedia* [online]. San Francisco (CA): Wikimedia Foundation, 2001 – [cit. 2012-05-14]. Available from: `http://en.wikipedia.org/wiki/Golden_ratio`.

[2] KOPKA, J. *Ako riešit' matematické problémy.* Verbum: Ružomberok, 2010. ISBN 978-8080845636.

[3] Pentagram. In: *Wikipedia: the free encyclopedia* [online]. San Francisco (CA): Wikimedia Foundation, 2001 – [cit. 2012-05-14]. Available from: `http://en.wikipedia.org/wiki/Pentagram`.

[4] POLYA, G. *How to Solve It: A New Aspect of Mathematical Method.* Princeton University Press: Princeton, 1973. ISBN 978-0691023564.

[5] VOPĚNKA, P. *Rozpravy s geometrií.* Panorama: Praha, 1989.

[6] VRÁBEL, P. *Heuristika a metodológia matematiky.* FPV UKF: Nitra, 2005. ISBN 978-8080508401.

[7] ZAPLETAL, M. *Kniha hlavolamů.* Albatros: Praha, 1983.

The innovation of a course Introduction to Mathematics

Věra Ferdiánová, Zuzana Václavíková

Faculty of Science, University of Ostrava
30. dubna 22, 701 03 Ostrava, Czech Republic
vera.ferdianova@osu.cz
zuzana.vaclavikova@osu.cz

Abstract. In the talk we will survey the innovation of a basic mathematical course, Introduction to Mathematics. This subject is intended for students of the Faculty of Science of the University of Ostrava in bachelor programmes Mathematics, Applied Mathematics, Computer Science and Applied Computer Science in all forms of study. We particularly describe the innovation using modern ICT components that help students better understanding of the subject.

Keywords: Innovation, analyses.

Classification: B45.

1 Introduction

A course Introduction to Mathematics (UTELO) is included as a compulsory subject in the 1st year's winter semester of study bachelor of Applied Mathematics and Mathematics Faculty of Science at the University of Ostrava. Students must complete a compulsory courses: Basic linear algebra and geometry, Mathematics analysis, Basic of mathematics, Introduction to Mathematics in the winter semester. The aim of the course is to obtain the logical theory of set construction and mathematics - sets, relations, maps, basic algebraic structures, basic properties of number fields.

2 The development of a course Introduction to Mathematics in years 2002-2012

A course Introduction to Mathematics didn't exist before year 2002. Basic of mathematics was included in courses Mathematic Logic and Theory of set. Study program Applied mathematics was changed on Bolones model and it was required to create a preparatory subject. It means, the 5-years master's degree study programs was changed to 3-years bachelor study and 2-years master study. It was created a basic course of mathematics, which is mandatory for students of bachelor study program. The name of new course was Introduction to theory of sets and logics (UTELO) and it contained following core parts: First-order Language, Intuitive theory of sets, Relations, Zermel-Fraenkel axiomatic system and Ordinal and Cardinal numbers. Course had two hours of plenary lecture and two hours of practice lecture per week, it was concluded by an examination. The course was changed in year 2005, because we asked for re-accreditation of bachelor study program Application mathematics and we changed the study subject Teaching for secondary school. This subject was divided into double discipline bachelor study (Mathematics in combination with another subject) and follow-up master's degree study Teaching for secondary school. In the new education curricula, it was requirement to include basic math course to 1st year's bachelor's programs. Students obtain a knowledge of formal structure of a mathematic's construction and it makes them easier the construction in other subjects, e.g. Mathematics analysis or Algebra. It was found out, that the preparatory course would be useful for students of Informatics. The course Introduction to theory of sets and logics must be changed according to the requirements of following teaching. The pilot version of course had too high knowledge level, so it was changed. The student control wider field of knowledge, without details. The pilot version contained basics of predicate logic and Zermel-Frankel axiomatic system (set, operations with sets, cartesian product, relations, cardinality of the set) and was expanded by functions. Weekly hours subsidy was changed to two hours plenary lecture, the conclusion remained in form of oral examination.

Approximately, 5-8% of students had the course as mandatory optional (B). The Fig. 1 shows a success of the course. Though the students rated the course as beneficial, the statistics show, that mean grade was 1.76 and even 28% of students didn't try to complete the course. For the rest of students it wasn't problem to complete the course, because some areas they knew from secondary school.

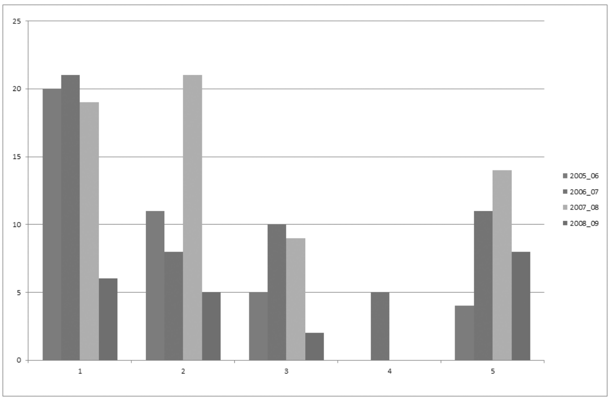


Figure 1: Success of students in 2005-2008: Degrees 1-3 successfully completed the course, 4 unsuccessfully completed the course, 5 didn't try to comlete the course

In 2008, they was accredited bachelor study programs Investment advice and Applied informatics. The original course UTELO was divided into two new courses - Introduction to study of mathematics with shortcuts UTELO and UTML. The difference of this two courses was in their difficulty. The course UTELO remained mandatory (A) for study fields Applied mathematics (AM), Mathematics in economy (AME), Investment advice (IP) and double discipline study of Mathematics in combination with another subject. The field Informatics (IN) has this course as mandatory optional (B). The course UTML was created as mandatory optional (B) for the field Applied Informatics (AI). Syllabus of course contains this parts:

1. Propositional logic, symbolic mathematical notation (why do we write symbolically), a formula, the truth value, logical connectives, work with formula.
2. Intuitive introduction to set theory, what is axiomatic system and why it was created, the system construction of mathematics - what is the definition, theorem, lemma.
3. Proofs, types of proofs - direct, indirect, dispute, mathematical induction.
4. Sets, intersection, union, difference, De Morgan's rules. Disjoint sets.

5. The intuitive term of cardinality - the count of elements of sets, finite sets, countable and uncountable sets.
6. Ordered pairs, unordered pairs, Cartesian product of sets, relations, properties of relations.
7. Ordered relation: partial and complete ordered, Hasse diagram, the largest and the maximum element, supremum, the smallest and minimal element, infimum.
8. Divisibility on the set of integers, modulo n residual class.
9. Equivalence relation, decomposition into equivalence classes.
10. Maps, properties of maps, function, the domain of function, range of values and functions, inverse functions.
11. Algebraic structures, the difference set - structure, a binary operation on the set and their properties (commutativity, associativity, distributivity), examples of basic algebraic structures.
12. Basic properties of number fields - integers, rational numbers, real numbers (irrational, transcendental), complex numbers.

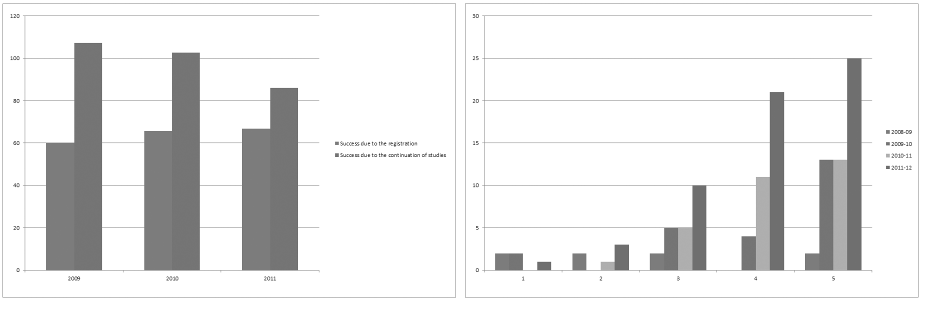


Figure 2: Success of students in course UTELO (left) and in course UTML (right)

One can see from the syllabus of course, that there was big changes. The content must be modified, because secondary schools use Educational framework RVP. According to the Educational framework, the secondary schools have more freedom in distribution of curriculum from the content and hours subsidy point of view. The result is that students from different schools have different mathematical literacy. Unfortunately, the situation is that even school leavers from high schools have different level of mathematical literacy. Another change came in conclusion of the course, now the course is concluded by a credit. Achieved knowledge is tested in written test.

Students evaluate course very positively and they consider it as useful for further study. One can see from the graphs that throughput is optimal. In last years, it was reduced a little to 86%, which is standard decrease in 1st year of bachelor study. We wanted to find out the composition of students from the study fields point of view in this academic year. We found out that 40% of students is from double discipline study and even 12 % is from the field Informatics. The count of students, which does not have mathematics as their main field, was increased.

The course UTML has one hour of plenary lecture and one hour of practice lecture per week, the concept is identical with course UTELO, but it is concluded by an exam.

The throughput of course is following: up to 41% of student didn't try to conclude the course, 22% of students is classified as fail. The main reason is in hours subsidy. If the student misses one hour, it shows in other hours. Students don't use study materials, even if they have it. The paradox is, that the course is quite popular and the students sign it up. Therefore, we decide to innovate this course according to practical things, which motivate students. We have very good feedback from students from previous years, if the material is processed by interactive or multimedia form. The innovation will be in following key areas:

The innovation of lectures with application to the technical

- Sets: data structures, cardinality of set=allocation of array
- Propositional logic: simplification of circuits
- Residual classes modulo 2: binary system, logical circuit
- Display: data analysis
- Numeric fields: datatypes, complex numbers in circuits

Use of technical instruments in practice lectures

- Free available simulators of logical circuits

In last years, when was performed this innovation in other courses [1], the success and attendance to the course was increased. Motivational examples can seem trivial, but don't forget, that students graduate at high schools, commercial schools and industrial schools. So the composition of students is really diverse in 1st year of study.

3 Conclusion

The article approximated the evolution of course during last 10 years. It is worth mentioning, that the preparatory courses were not in count before year 2002. All changes were conditioned by some fact, e.g. accreditation or re-accreditation of fields, the change of conception in teaching at high schools. This course demonstrates, that it developed and modified to requirements of times. We must state that some processes in education are very lengthy and it is good, that a reaction to some situations was flexible.

Acknowledgements

The first author has been supported by CEEPUS and SGS07/PrF/2012: Geometric mechanic a optimalization.

References

[1] FERDIÁNOVÁ, V.: *Popularizace lineární algebry pro informatické obory,* Sborník Užití počítačů ve výuce, České Budějovice, 2011.

Magic squares on mobile phone

Leonard Frobisher, Jan Kopka, Marián Trenkler

58 Buckstone Avenue, Leeds
LS 17 5HP, England
ljfrobisher@btinternet.com

Faculty of Science, Jan Evangelista Purkyně University
České mládeže 8, 400 96 Ústí nad Labem, Czech Republic
Jan.Kopka@ujep.cz

Faculty of Education, Catholic University in Ružomberok
Hrabovská cesta 1, 034 01 Ružomberok, Slovakia
trenkler@ku.sk

Abstract. Investigation is not only a method for a teacher to use in teaching mathematics, but also for a student to use in learning the subject. We show how to construct a magic square using the numbers on a mobile phone with the help of investigation. We then consider how the same problem can be approached in the early years of elementary school giving rise to many related problems.

Keywords: Magic squares, Investigation in School Mathematics.

Classification: D53; C40.

The book *Investigation in School Mathematics* [1], published in 2006, was written for teachers, didacticians and students – future teachers of mathematics. This book describes:

• several important problem solving strategies – especially investigative ones and explains what investigation in school mathematics is;

• examples of problems which can be solved using the Investigations approach;

• mathematical situations which can be approach through investigations.

Investigations, as described in [1], is a method of teaching and learning mathematics which permits students to enter and penetrate more deeply into the world of mathematics that most other teaching approaches fail to do. The book has a strong practical complexion. It also illustrates for teachers and students what an investigation is before placing them in the position of learners when they are forced to take a fresh and invigorating look at school mathematics from a new perspective. School mathematics abounds with problems which can be turned into investigations. But when and how to use them in the classroom as a teaching approach must be left in the hands of individual teachers.

One of the attractive features of investigating a problem is the potential for children of all ages and abilities to have access to the problem. This is achieved by modifying both the problem and the investigative approach in order that **every** child achieves a level of success and satisfaction.

We now show an interesting and well known problem from [1] which can be solved with the help of an investigative approach. This is suitable for older children.

Since ancient times magic squares have fascinated mathematicians. The most well known magic square is that constructed by Albrecht Dürer on copperplate in 1514. In this article we discuss one simple example of a magic square which assumes only knowledge of the decomposition of a number into 3 natural numbers. This has application when teaching children about addition and the decomposition of natural

1	2	3
4	5	6
7	8	9

Figure 1: Mobile phone

numbers, and when they practise and apply these two aspects of mathematics in more challenging situations.

The number keys on mobiles have this square arrangement (see Figure). We introduce the problem by asking:

What is the sum of the three numbers in

- *the second column?*
- *the second row?*
- *the main diagonal?*
- *the other diagonal?*

We find that the four sums are all 15 and that the sums of the other rows and columns are not equal to 15.

Problem 1: Try now to move one or more of the number keys, 1 to 9, so that the sum of each row, each column and each diagonal is 15.

A 3×3 square arrangement of nine numbers where the sum of each row, each column and each diagonal is the same, is called a magic square. The sum is called the magic number of the square. In the case of our square the magic number is 15.

Investigating the problem

• We can start by positioning the numbers 1 to 9 randomly in the 3×3 square. This experimental strategy often results in no solution. We call this trial and error.

• A more reliable and effective strategy involves correcting errors and using insight so improving the possibility of finding a solution. We call this trial and improvement.

Both of these strategies may or may not lead to a solution, but always to a better understanding of the structure of the problem, which in itself is likely to lead to other strategies.

One such strategy begins by asking if it is possible to make a magic square using the nine numbers 1 to 9, and if so what its magic number will be. Some children may even ask if there are different magic numbers for the magic square using the same set of numbers 1 to 9. (We must always take great care not to assume that what we consider obvious is also obvious to children.)

The following questions guide children in the direction of answers to the two questions:

• *What is the total of the nine numbers, 1 to 9 ?* $(1 + 2 + \cdots + 9 = 45)$

• *As the nine numbers are shared equally between three rows (also three columns), what is the sum of the three numbers in each row?* $(45 : 3 = 15)$

This establishes that if a magic square exists, then its magic number must be 15.

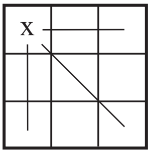


Figure 2: The contribution of a number

We are now in a position to investigate where each of the numbers 1 to 9 should be positioned in the 3×3 square. Children sometimes need guidance in the form of questions to assist them with an investigation. Only one question should be asked at a time to allow children the opportunity to think about investigative strategies for answering a question.

- *Which four numbers must be in the four corner boxes?*
- *Which four numbers must be in the middle boxes along each side?*
- *Which number must be in the centre box?*

We consider the first question.
A number in any corner contributes to the sum of three decompositions one row, one column and one diagonal, as shown in Figure .

The magic number of our previously described square is 15. So we can search for all possible decompositions of 15 into three numbers which are chosen from $1, 2, 3, \ldots, 9$.

When 1 is one of the three numbers then we have only two decompositions.

$$15 = 1 + 5 + 9$$
$$15 = 1 + 6 + 8.$$

We have shown that for 1 to be a corner number it must be a member of at least three decompositions. Hence, 1 cannot be a corner number. Because a number in the middle of a side belongs to only one row and one column (two decompositions), then 1 must be in a middle square of a side.

When 2 is one of the three numbers then we have three decompositions.

$$15 = 2 + 4 + 9$$
$$15 = 2 + 5 + 8$$
$$15 = 2 + 6 + 7.$$

Because a number in a corner square belongs to one row, one column and one diagonal, then 2 must be in a corner square.

When 5 is one of the three numbers then we have four decompositions.

$$15 = 5 + 1 + 9$$
$$15 = 5 + 2 + 8$$
$$15 = 5 + 3 + 7$$
$$15 = 5 + 4 + 6.$$

Because a number in the centre square belongs to one row, one column and two diagonals, then 5 must be in the centre square. Repeating the decompositions to include $3, 4, 5, \ldots, 9$ we can conclude that

2	9	4
7	5	3
6	1	8

Figure 3: Magic square

$2, 4, 6, 8$ must be in corner squares;
$1, 3, 7, 9$ must be in middle of side squares;
5 must be in the centre square.

We can now put the numbers 1 to 9 into appropriate boxes in the 3×3 square and check the sums in rows, columns and diagonals to see if we have made a magic square. After several experiments using this approach a regular strategy can be seen for completing 3×3 magic squares. Figure shows one possible magic square using the numbers 1 to 9.

Please check to see if this is true.

Question: Can you find any more solutions?

Answer: As 3×3 square has together 8 reflections and rotations (including the identity) we can construct, using Figure , seven more magic squares. We leave these for you to complete. Do you consider any of the eight solutions to be equivalent? How would you define equivalent in this situation?

Exercise 1: How many keys on the mobile (Figure) need to be moved to get anyone of the magic squares?

Answer: Only the number 5 remains in its original position so 8 of the keys need to be moved.

Exercise 2: Given an empty 3×3 square with 5 in the centre box, how many numbers, including their positions, do you need to know to uniquely determine a magic square.

Answer: After experimenting you may discover that you need a minimum of 2 other numbers with the three following conditions:

1. they must not be in the same row, column or diagonal;
2. either the even numbers $2, 4, 6, 8$ should be in corner boxes and the odd numbers $1, 3, 7, 9$ should be in middle of side boxes;
3. the sum of the numbers is not equal to 10. (Why?)

Remark: Although a magic square arrangement on your mobile would be interesting to have it would not be very practical!

Our original problem can now be changed by using, for example:

- the numbers 2 to 10,
- any 9 even numbers,
- any 9 prime numbers.

For which of these sets of numbers do you think it is possible to make a 3×3 magic square?

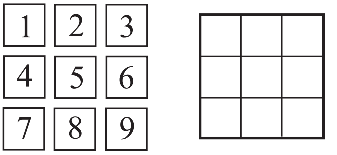


Figure 4: Number cards

What necessary and sufficient conditions does a set of 9 numbers have to satisfy in order to make a magic square?
We now look at other possible investigative approaches that you may wish to use with younger children with the original problem unchanged.

Children in the early years of elementary school find it helpful to have number cards 1 to 9 which they can move around from square to square using either a trial and error or, preferably, a trial and improvement strategy. Each child should have a set of the nine number cards and a large 3×3 grid as shown in Figure .

The material overcomes the difficulties created by the fixed positioning of a written number which children find inhibits their thinking and the mental operation of intuition. They respond positively to the freedom and openness of having such materials. However, one consequence of this approach is that children need to be taught the importance of recording both successful and unsuccessful trials. In a perfect world one would wish children to devise their own method of recording: this is possible with older children, but in the early learning stages of investigative work teachers should provide recording sheets. One such is described later.

We now look at changing the problem by removing some of the requirements of problem 1, in particular, not asking for rows, columns and diagonals to sum to 15. Instead, we allow children to place the nine numbers randomly in any of the squares. However, our ultimate target is that the children achieve the magic square with the magic number of 15. Let us explain the process by which this can be attained.

Problem 2: Put the nine numbers in the grid squares in any way you wish.

With every child having completed this we ask these questions:
• Who has a row / column / diagonal of three numbers with the greatest sum?
• Has anyone got two rows / columns / diagonals which have the same sum?

This becomes the next problem followed by successively more difficult problems each being an investigation in their own right.

Problem 3: Put your numbers in the squares so that two of your rows / columns / diagonals have the same sum.

Problem 4: Put your numbers in the squares so that all of your rows / columns / diagonals have the same sum.

Problem 5: Put your numbers in the squares so that every row, every column and every diagonal has the same sum.
The solution to problem 5 is in fact our magic square, but has been achieved without providing the information that the eight equal sums are 15, the magic number.

When working on Problems 2 to 5 children need a structured recording sheet. The one shown in Figure works well with the circles used to record the sums. (Only

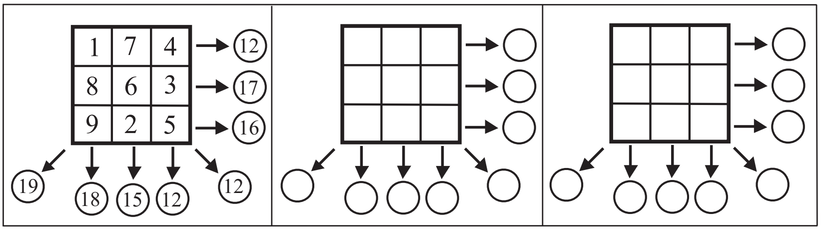


Figure 5: Recording sheet

the first row of the sheet is shown.)

Throughout children's schooling they are asked to solve or investigate problems that have been devised by teachers or appear in textbooks. We should look to change this with children posing their own problems, often arising out of ones they have investigated. Here are some you will find children suggest. You may like to investigate each one. We cannot guarantee that each has a solution. Perhaps you can prove that they do or do not.

Problem 6: Put your numbers in the squares so that every row, every column and every diagonal has an even sum.

Problem 7: Put your numbers in the squares so that the eight sums in the rows, columns and diagonals are consecutive numbers.

Problem 8: Put your numbers in the squares so that the eight sums in the rows, columns and diagonals form an additive sequence.

Most mathematical theories have both an experimental and inductive character. Their beginnings arise out of tentative searching and speculative trial and error; they gain a deductive character only after they a period of investigation. If we wish our students to have experiences of how mathematics evolves, then we should respect how mathematical theories come into existence, how they develop, and how they finally gain their formal nature. Too frequently students are only exposed to mathematics in its final and approved form. Using Investigations is one method of teaching that can substantially contribute to students being involved in the full range of the development of a mathematical theory. Investigations also provide students with insights into what it is like to be a mathematician and to experience mathematical thinking at work.

References

[1] Orton, A., Frobisher, L.: *Insights into Teaching Mathematics,* London, Cassell, 1996

[2] Kopka, J.: *Zkoumání ve školské matematice,* Ružomberok, KU Ružomberok, 2006

[3] Trenkler, M.: *Magické štvorce a kocky – zdroj námetov na vyučovanie matematiky,* Disputationes Scientificae Universitatis Catholicae 2 (2002), č.1, 88-94

Australian mathematics textbooks - the adaptation of the topic *Three-dimensional Objects*

Daniela Guffová

Matej Bel University
Tajovského 40, 974 01 Banská Bystrica
daniela.guffova@umb.sk

Abstract. New Australian mathematics curriculum was developed in the years 2008 – 2010. Due to this, ICE-EM Mathematics series of textbooks for Year 5 – Year 10 were developed. This paper is focused on the topic *Three-dimensional Objects*. A short overview of its inclusion in Australian and Slovak mathematics curricula in Year 5 – Year 7 is given and the ways of its different adaptation in Australian and Slovak mathematics textbooks for Year 5 are demonstrated.

Keywords: curriculum, textbook, three-dimensional object.

Classification: 97D99; 97G40.

1 Introduction

Australia was amongst top 10 OECD countries in the assessment of mathematical competence in Pupils International Standard Assessment (PISA) in 2006 [3]. However, they did not find these results sufficient and the ambition to be one of the best countries in PISA led to a creation of the Melbourne Declaration on Educational Goals for Young Australians in 2008. It claims that new world-class curriculum should be created in Australia and it also mentions that not every learning area is of equal importance at all levels of education, but English and mathematics play key role in every year of schooling [2]. To meet the educational goals of Melbourne Declaration new Australian Curriculum has been developed [7].

2 Australian mathematics curriculum

The Australian Curriculum, Assessment and Reporting Authority [10], that collaborates with teachers, principals, governments, education authorities, professional education associations, community groups and with the general public too, is responsible for the new Australian curriculum. The development of the Australian Curriculum has consisted of three phases [10]:

1. development of Foundation to Year 10 Australian Curriculum for mathematics, science, English and history (2008 - 2010);
2. development of Foundation to Year 10 Australian Curriculum for geography, languages and the arts (2010 - 2012);
3. development of Foundation to Year 10 Australian Curriculum for health and physical education, information and communication technology, design and technology, economics, business, civics and citizenship (2011 - 2013).

The *Australian mathematics curriculum* [9] consists of four parts according to the age of pupils:

- early childhood (Foundation - Year 2);
- late childhood (Year 3 - Year 5);
- early adolescence (Year 6 - Year 8);

- late adolescence (Year 9 - Year 10).

To organise the learning areas, the Australian mathematics curriculum is divided into three strands:

- *Numbers and Algebra* where students explore the number representation, computation, patterns and relationship.
- *Measurement and Geometry* aimed at developing an understanding of size, shape, relative position and movement of two- and three-dimensional objects.
- *Statistics and Probability* focused on recognising and analysing data and drawing inferences.

Australian mathematics curriculum is a basic document for mathematics teachers that helps them to plan and organise mathematics lessons and to assess their pupils.

One relevant difference between Australian and Slovak curricula is the time allocation on mathematics and numeracy. In Australia the minimum teaching times for mathematics were mandated by DECD (Department for Education and Child Development) to assist in the lead up to implementing the Australian Curriculum. From the start of 2011, Years 5 to 7 students should spend a minimum of 300 minutes per week on mathematics in Australia [11]. Unfortunately, Slovak pupils in Years 5 to 7 spend only 180 minutes (four 45-minute lessons) per week on mathematics on average according to the Slovak curriculum framework.

3 Three-dimensional objects

With such a small amount of time allocated to mathematics at lower secondary level of education in Slovakia, there is a lack of time that can be devoted to some topics. Among these neglected topics is the important topic *Three-dimensional objects*. In the table Slovak curriculum and Australian curriculum for this topic are compared.

	Slovak curriculum	Australian curriculum
Year 5	Cube, rectangular prism (as propedeutics). Construction of 3D objects made of building blocks. Construction of 3D objects under specified conditions.	Drawing conventions for 2D representations of 3D objects. Sketch representation of objects from different viewpoints, knowing that the same two-dimensional shapes can be drawn in different orientations.
Year 6	*The topic* Three dimensional objects is forgotten.	Properties involving line, length, angle and surface of common 3D objects (prism, pyramid, sphere, cylinder) and part and composite shapes. Construct accurate representations of 2D shapes according to specification (e.g. using drawing instruments and software) and 3D objects from plans, nets and isometric diagrams.

Year 7	Some ways of construction of 3D objects representations. Image of cube and rectangular prism, visibility of edges. Solids composed of cubes and rectangular prisms and their visualisations. Net of cube and rectangular prism.	Properties involving line, length, angle and surface of common 3D objects (prism, pyramid, sphere, cylinder) and part and composite shapes. Solve problems requiring knowledge of geometric properties and transformations of common 2D shapes and 3D objects. Draw different views of prisms and solids formed from combinations of prisms.

One can observe that pupils in Australia can gain much more knowledge on three-dimensional objects than in Slovakia. Moreover, many facts have been excluded from Slovak curriculum since 2008 (cf. [4], [5]), although Slovak pupils achieved just average results in PISA 2003 and 2006 [3].

4 Mathematics textbooks

Based on changes in the Australian mathematics curriculum, new mathematics textbooks for Year 5 up to Year 10 were rewritten [1]. Every book was written by a team consisting of both mathematicians and teachers of mathematics. It covers all of the required content, as well as additional topics that are relevant and essential for a robust understanding of the subject. For every year a set of two books was written. The focus is primarily to ensure a thorough understanding of concepts and ideas presented. In addition to basic content based on the Australian curriculum, the textbooks include other information, enriching the traditional teaching of mathematics and encouraging students to a deeper study of mathematics.

According to new Slovak curriculum new textbooks for mathematics were created in Slovakia, too. However, they were not released in time and therefore were not available for pupils in schools for quite a long time. Besides that some of the topics are just briefly presented, which means that it is very hard to establish robust understanding mostly for the pupils who need more time to master new mathematical concepts. We demonstrate the difference between Slovak and Australian learning materials on mathematics textbooks for Year 5.

There are just five pages devoted to the topic *Three-dimensional objects* in Year 5, although Slovakia obtained very bad results in *Space and shape* area in PISA [3]. In Slovakia, pupils are just building blocks during several lessons and they should recognize just some three-dimensional objects in the Year 5. Moreover, the only real three-dimensional object they can see in those five pages is a football ball.

The situation is very different in Australia, as in their mathematics textbooks for Year 5 there is a short preparation followed by the motivation showing some examples of three-dimensional objects that pupils know from their own experience (e.g. car, box of chocolates, can of soup) (see Figure). After the motivation subchapters on polyhedra and prisms, cylinders and pyramids presenting the basic knowledge are included. Every chapter contains a lot of exercises for the whole class and after mastering these tasks pupils should work individually following the exercises for individuals in their textbooks (see Figure). By doing so they can acquire a thorough

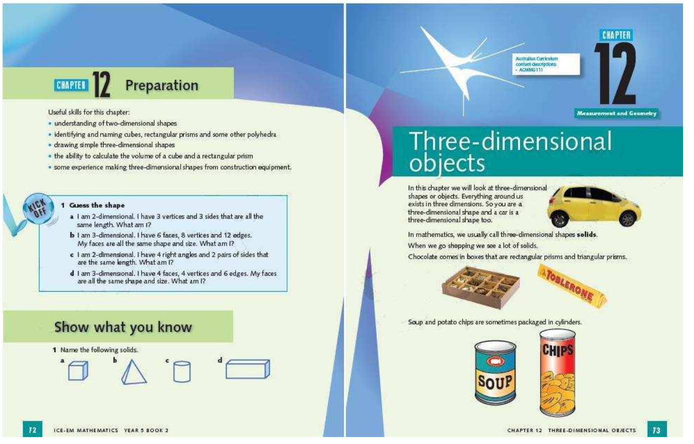


Figure 1: Motivation to the topic *Three-dimensional objects* in Australian textbook

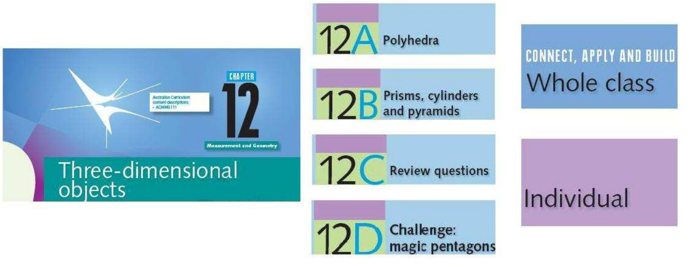


Figure 2: Structure of a chapter *Three-dimensional objects* in Australian textbook

understanding of three-dimensional objects which they can verify in review part of the chapter.

5 Conclusion

As we have illustrated, the situation in mathematical education is not very optimistic in Slovakia. There is not enough time allocated to mathematics at schools, the textbooks are not delivered to schools in time and those available are not very sufficient for the profound understanding. Therefore the question is why not to follow the Australian example and their reform of mathematics education. Following the Australian mathematics curriculum and adapting their textbooks in Slovakia would give an opportunity to Slovak teachers to teach better and to Slovak pupils to learn and understand more.

References

[1] ICE-EM Mathematics Australian Curriculum Edition. Cambridge University Press, 2012

[2] Melbourne Declaration on Educational Goals for Young Australians. Ministerial Council on Education. Employment, Training and Youth Affairs, 2008
[3] PISA 2006: Science Competencies for Tomorrow's World. Executive summary. Paris, OECD, 2007
[4] Učebné osnovy MATEMATIKA pre 5. – 9. ročník základnej školy. MŠ SR, 1997. Available online at `http://www2.statpedu.sk/`
[5] Štátny vzdelávací program MATEMATIKA (Príloha ISCED 2) ŠPU, 2010. Available online at `http://www.statpedu.sk/files/documents/svp/2stzs/isced2`
[6] Australian Government - school education. Available online at `http://australia.gov.au/topics/education-and-training/school-education`
[7] The Shape of the Australian Curriculum (Version 2). ACARA, 2010
[8] The Australian Curriculum. Available online at `http://www.australiancurriculum.edu.au`
[9] Australian mathematics curriculum. Available online at `http://www.australiancurriculum.edu.au/Mathematics`
[10] Development of Australian curriculum. Available online at `http://www.acara.edu.au`
[11] Australian curriculum implementation timelines and time allocations for learning areas. DECD, 2011
[12] Australian Mathematical Science Institute. Available online at `http://www.amsi.org.au`

GeoGebra as a motivational tool for teaching and learning mathematics, informatics and physics

Ján Gunčaga, Janka Majherová, Martin Janček

Faculty of Education, Catholic University in Ružomberok
Hrabovská cesta 1, 034 01 Ružomberok
jan.guncaga@ku.sk, majherova@ku.sk, jancekma@gmail.com

Abstract. We present new goals defined in new Slovak curriculum ISCED for Mathematics, partially in Physics at primary and secondary schools. Several students' works that use the GeoGebra software (by the students at Catholic University in Ružomberok - future teachers) will become a concrete example for motivation with ICT during the educational process. At the same time, we suggest some project possibilities for research and application of the GeoGebra in the education and in the work with students - future teachers or students at secondary schools. Finally, the engagement of teachers at primary and secondary schools in the GeoGebra teaching as a part of their lifelong learning will be discussed.

Keywords: Computer Based Math and Physics Education, GeoGebra Institute, ISCED, work with students - future teachers, motivation.

Classification: R20, U50, M50

1 Introduction

According to Oldknow and Taylor (see [11]) we can identify at least three reasons to promote an integration of Information and Communication Technologies (ICT) in the teaching process of mathematics at schools:

- Desirability: The use of ICT may stimulate pupils' motivation and curiosity and encourage them to develop their problem-solving strategies. Regarding teachers, the use of ICT can improve their efficiency, provide more time to address students individually, or stimulate re-thinking of their approach to teaching and understanding.
- Inevitability: Many fields of publishing have moved from a printed to an electronic form. This fact applies to conference proceedings, reference works such as encyclopedias, small-circulation textbooks, special journals, etc.
- Public policy: Slovak National Curriculum ISCED 1, 2 and 3 classifies Mathematics as a school subject, which is a part of the group called Mathematics and Working with Information.

2 The process of gaining knowledge and GeoGebra

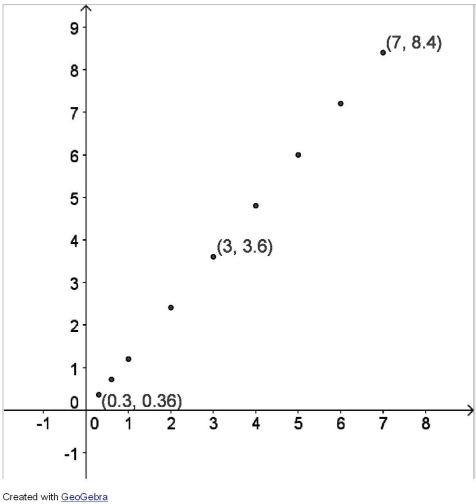
The process of gaining knowledge in mathematics education (see [3]) is based on stages. It starts with motivation and its cores are two mental lifts: the first leads from concrete knowledge to generic knowledge and the second from generic to abstract knowledge. The permanent part of the gaining of knowledge process is crystallisation, i.e. inserting new knowledge into the already existing mathematical structure. The whole process has following stages:

1. motivation
2. isolated models
3. generic model(s)
4. abstract knowledge

5. crystallization
6. automation

Motivation is the tension which occurs in a person's mind as a result of the discrepancy between the existing and desired states of knowledge. The discrepancy comes from the difference between "I do not know" and "I need to know", or "I can't do that" and "I want to be able to do that", sometimes from other needs and discrepancies, too. GeoGebra can be used an effective motivational tool because this software can represent a lot of mathematical notions and their relationship more visible in a dynamical way.

The pupils experiences about some mathematical notion is possible to use as an isolated model. For example the praise of apples according the wage of apples we can represent be following table and graph prepared in GeoGebra (1kg apple 1,20 Euro). First, in this case we can represent a function as a set of isolated points.



	A	B
1	The wage of apples	Praise in Euro
2	0.3	0.36
3	0.6	0.72
4	1	1.2
5	2	2.4
6	3	3.6
7	4	4.8
8	5	6
9	6	7.2
10	7	8.4
11		
12		
13		
14		

Figure 1

It is similar than by old Babylonian mathematicians, who studied the position of planets and represent their movement with the set of isolated points (picture of the positions of planets). In this stage (isolated models) the pupils in the school can measure the temperature, the high of the river in the time and so on. Their measures write in the table, which GeoGebra can very easy represent graphically.

In the third stage (generic models) we can try to find the curve which obtain the points and can lead to the graph of the function, which represent the concrete relationship. Abstract knowledge can be in this case the definition of the function, which we received in spontaneous and natural way from the previous isolated and generic models.

Crystallization and automation is work with concrete functions and also with functions which are not continuous and have more complicated shape.

3 The method of generating problems

According to Kopka in [6] many educators say that the main goals of teaching mathematics are: – the development of logical thinking – the development of creative thinking – the development of an autonomous person – the development of the ability to solve problems. GeoGebra allows to implement these goals in mathematics education.

The method of generating problems (see [13]) seems to be suitable for this purpose (due to its systematically creating sets of internally connected problems). Student activities and instructions have to be regarded as complementary factors in the learning process. These factors both are necessary and must be systematically related to one another so that optimal progress may occur.

The aim of our method is to create areas in which the students may-using the result of guided teaching-move as independently as possible, and in which he/she may develop their own initiatives. The student is considering his own problem and he could ask to assist for help as far as necessary. By this way he can obtain basis for further work. After a problem has been completely solved and clarified the teacher together with students are thinking about further questions and generate problems which are related to the problem just solved. Thus the original problem acts as a generating problem; we will call it generator problem (GP). Related problems are obtained by analogy, variation, generalization, specialization etc. The group of all new problems together with their GP will be called the set of generated problems of the GP or the problem domain of GP.

This method is possible to demonstrate with GeoGebra for example by the generator problem – Pythagorean theorem in grid paper (see Figure 2).

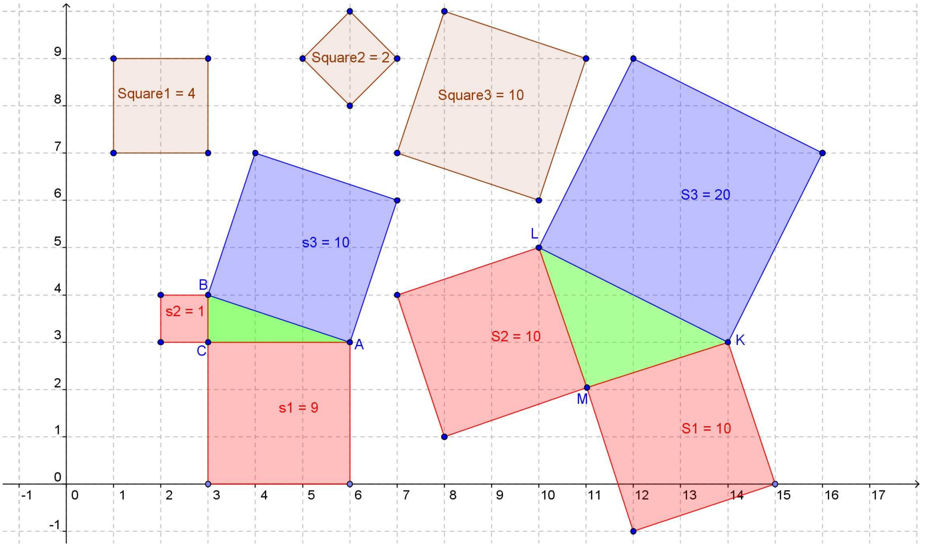


Figure 2: Pythagorean theorem in grid paper

4 Solving some geometrical tasks in GeoGebra

The new Slovak curriculum ISCED 2 for lower secondary level includes multiple educational competencies for teaching of geometry. The pupils should be able e.g. to construct and describe the basic geometrical figures or to specify properties of their particular elements (relationship of sides, diagonal, triangle inequality and so on).

They should know basic geometrical transformations, axial and central symmetry, the relationship between figure and its picture in transformation and how to analyse and solve application geometrical tasks with a use of mathematical know-how. Pupils obtain a geometric imagination as an ability to explore geometric figures and their properties, to abstract geometric properties from the particular objects, to have a perception of geometric shapes, and to be able to imagine geometric figures and their relationships.

Since the new curriculum in mathematics has been introduced in Slovakia only recently, textbooks satisfying these new rules are still missing. The materials from Wiki can help teachers to educate pupils in these new conditions. During this process of development, our aim is to cooperate with the teachers at schools, pupils and university students - future teachers. Some examples can be also found in works [1], [12] or on the website http://geogebra.ssgg.sk.

Now we can present some works, which are applicable in teaching of geometry. These examples have been prepared by students from the training programs for mathematics at secondary school. The materials are focused to explain new notions with support of interactive GeoGebra applets. The first example belongs to the framework of teaching of the plane geometrical figures (parallelograms). On the Figures 3 and Figure 4, we can see an applet for the construction tasks for a parallelogram and rhomboid, in the case that we have given some sides and angles. (available at http://www.geogebra.org/en/upload/files/Slovak/Ranostaj/Prezentacia.html)

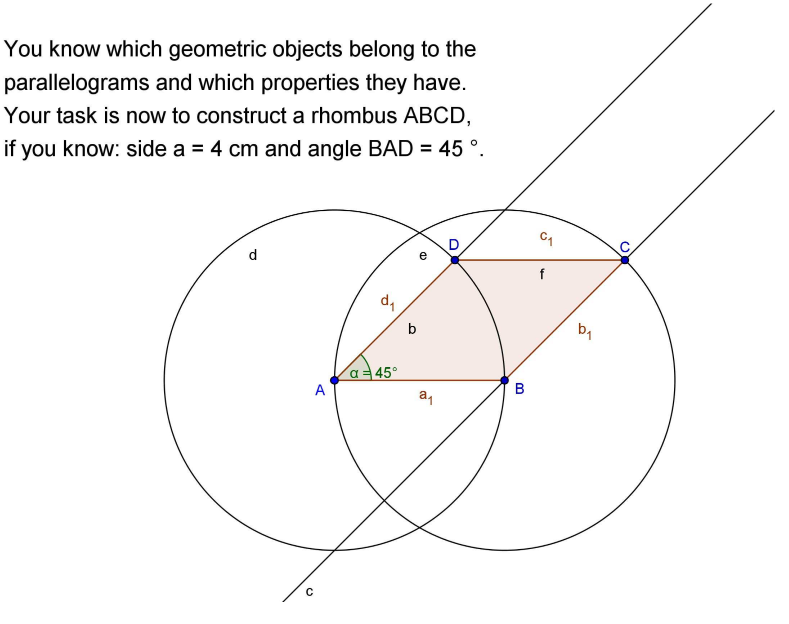


Figure 3: Parallelogram

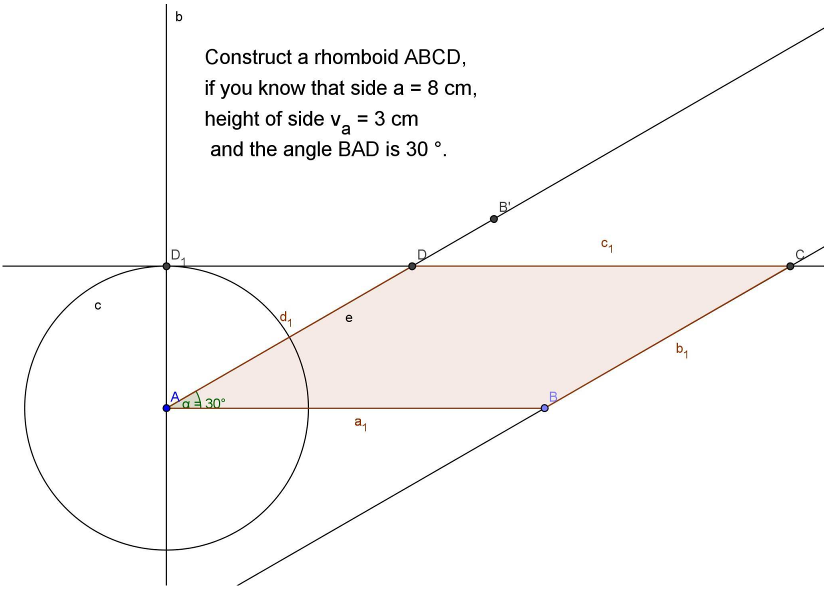


Figure 4: Rhomboid

The second example describes how to draw a cube in the plain using the rule that the side edges have the half length compared to the front edges (see Figure 5). Playing the construction is an important advantage in this teaching unit (see in [5]).

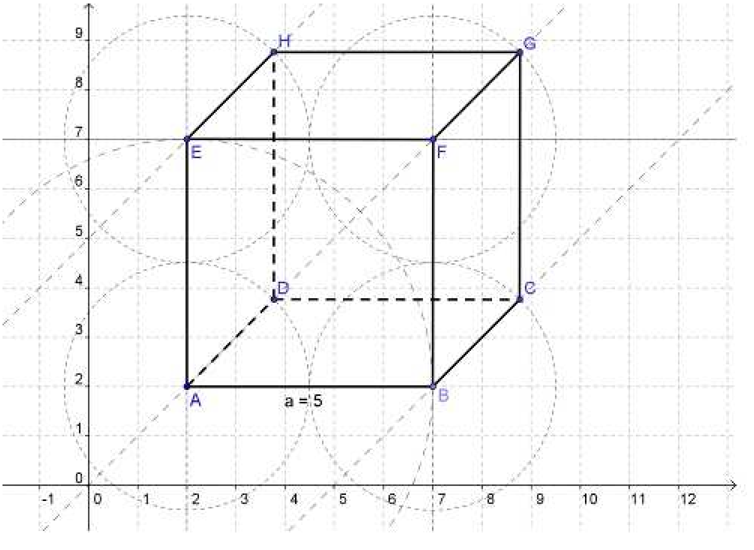


Figure 5: Cube

GeoGebra can be used as an effective tool to motivate pupils as it can visually represent a variety of mathematical notions and their relationships in a dynamic way. Within the interdisciplinary relationships, we can use GeoGebra also in the subjects of computer science or physics in the secondary education.

5 Modeling of plants and GeoGebra in informatics teaching

We suggest topic "Modeling on the computer" with further educational targets:

- Student understands a concept of a model and principles of model creation with the help of informatics tools
- Student specifies individual components of a model
- Student designs order of steps during a creation of model
- Student masters a implementation of a model on the simple examples in graphical or programming environment

In our proposal, an objective of education is to help students to create a simple model of certain part of a real world, in our case world of plants, with the help of informatics tools. On the example of the plant's model, the pupils understand the concept of the model and the principle of its creation using computer visualization. By the creation of the model, pupils also apply knowledge from geometry and biology (bilateral relationships of geometric shapes, morphology of plants, etc.). In teaching of mathematics, especially geometry, there are possibilities for visualization and simulation processes using a computer. Graphical possibilities of didactic software packages allow students to work with mathematical models of objects, e.g. graphs of functions or surfaces and geometric elements (see [2]) or to graphically visualize different algorithmic procedures.

In our methodology we go by the following principles of visualization. According Jirotková in [4] pupils obtain geometrical imagination as ability to:

- explore geometric figures and their properties,
- abstract geometric properties from particular objects,
- have an idea of geometric shapes,
- imagine geometric figures and their relationships.

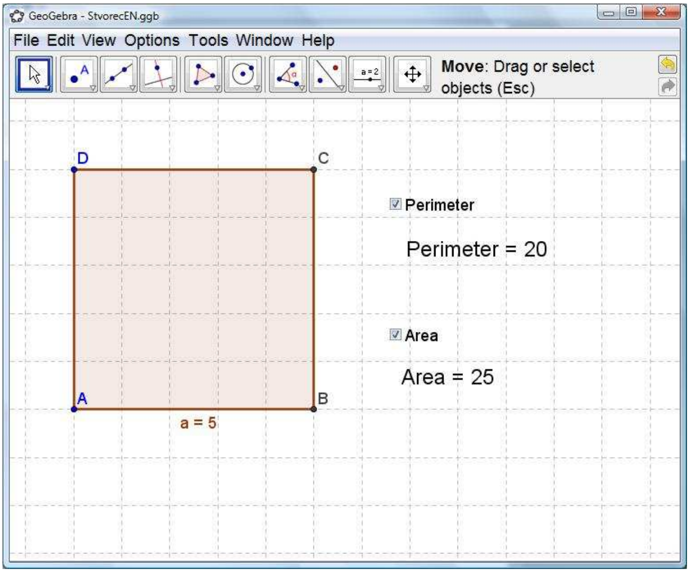


Figure 6: Picture of square in GeoGebra

Free software GeoGebra can be used as an effective motivational tool also in informatics education and can effectively assist teachers in enhancement of pupils' cognitive process. The figure 6 represent an example of environment of Geogebra with output for an option to change the length of square's side as well as to display its perimeter and area.

In order to model the flowers in Geogebra, we use the following commands in environment: New point, Segment between two points, Segment with given length from point, Reflect object about line, Reflect object about point, Angle with given size and Rotate object around point by angle. Pupils can create pictures of different plants and they use the knowledge about plain figures, angle and orientation in the plain (Figure 7 and 8).

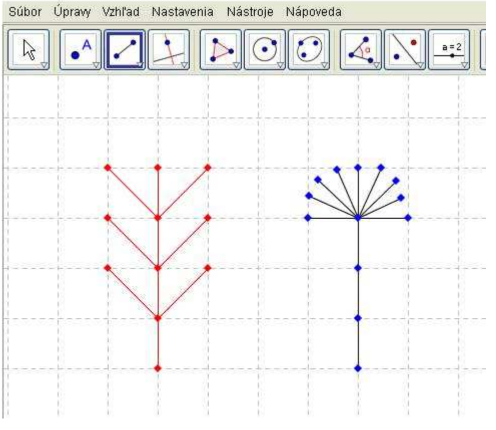


Figure 7: Sharpes of flowers in Geogebra

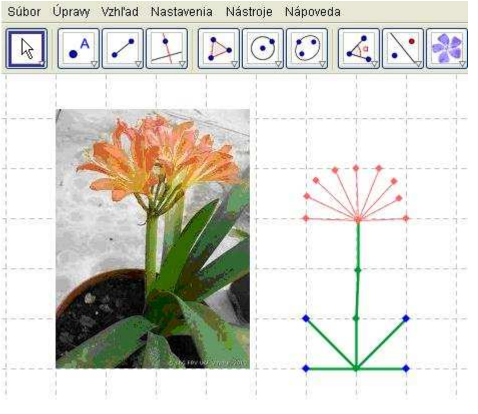


Figure 8: Picture of square in GeoGebra

6 Geogebra in teaching of physics

In physics, several educational competencies of the pupils are also defined in the Slovak curriculum ISCED 2 for lower secondary level. The pupils should be able to: "explain the force as a form of mutual interaction of two objects" advance in the analysis of different notions, effects (physical parameters, laws), "find a relationships between the physical parameters".

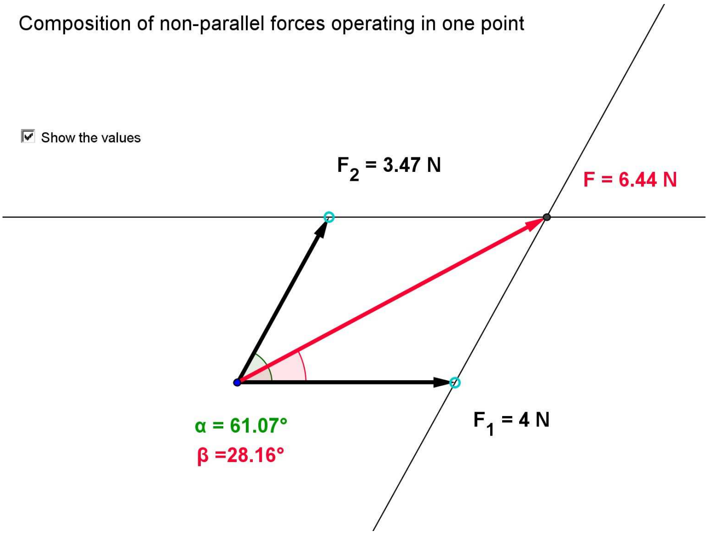


Figure 9: Composition of forces

In Figure 9 we present the representation of a composition of non-parallel forces (notice here a connection to pupils' knowledge of mathematics). Consequently this composition is used on Figure 10 to illustrate a motion on the inclined plane (available at http://edu.gsa.sk/jancek/?id=40200).

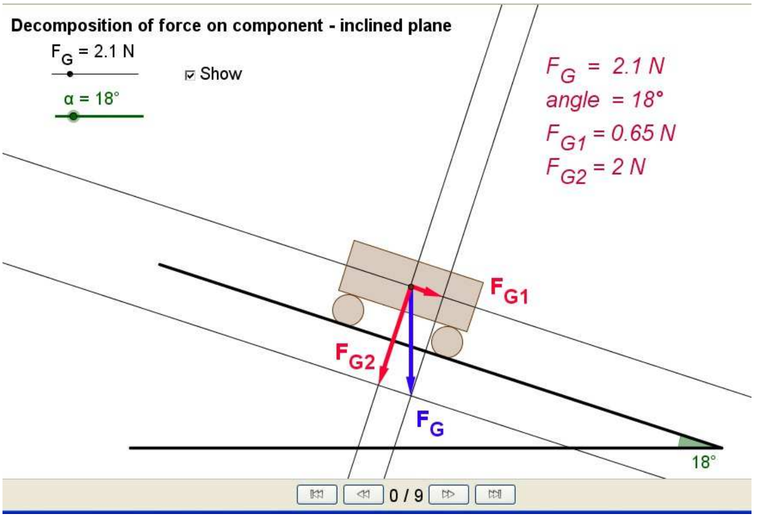


Figure 10: Motion on the inclined plane

7 Conclusion

There are numerous possibilities for a visualization and simulation processes using a computer while teaching of mathematics, computer science and physics. Graphical possibilities of didactic software allow students to work with models of different objects. Students can apply knowledge gained in learning stage while looking for solutions of different problems. Moreover, they can visually observe the result and thus understand and adopt basic concepts and notions more easily.

Creation of the model and its visualization by computer allows pupils to gain specific experiences regarding the use of mathematics, physics and informatics in a practical life. The relationships between mathematics, informatics and other subjects, which are supported by GeoGebra, are a very important part of integration of ICT in education.

In our paper we describe different examples of using GeoGebra in mathematics, informatics and physics education in lower secondary level. Materials in GeoGebra can effectively assist teachers in supporting the pupils' cognitive process. Pupils can develop its formal and logical reasoning, cooperation and communication. They will gain skills that are necessary for the research work, e.g. an ability to implement a simple research project, to formulate a problem, to look for the solution and cause context, and to learn how to use various methods of problem solving.

Acknowledgements

This paper has been supported by grant KEGA 001UJS-4/2011

References

[1] M. Billich: *Úlohy o dotykoch v programe GeoGebra,* In: Proceedings of Symposium on Computer Geometry SCG2010, Vol. 19, STU Bratislava, s. 11–16.

[2] J. Fulier et all.: *Niektoré aspekty reformy matematického vzdelávania,* UKF, Nitra, 2011

[3] M. Hejný, G. Littler: *Transmissive and constructivist approaches to teaching,* In: Creative Teaching in Mathematics: Charles University, Prague, 2006, pp. 11–34

[4] D. Jirotková , G. Littler: *Insight into pupils' structures of mathematical thinking through oral communication,* In: Marriotti, A. (ed.): Proceeding of CERME3, CD-ROM, Charles University, Prague, 2003.

[5] J. Kopáčová: *Investigative activities - student skills development tool.* In: Paths of democracy in education, Bratislava: Comenius University, 2003, pp. 147–150

[6] J. Kopka: *Problem Posing and learning of Mathematics,* In: Mathematical Investigations in School Mathematics 1, Oslo, Norway, University of Oslo, 1997, pp. 1–7.

[7] L. Koreňová: *Modelling of well-known historic mathematical problems solved using GeoGebra in secondary mathematics ICT in mathematics education* In: Proceedings of the Conference on the History of Mathematics and Teaching of Mathematics, Comenius Faculty College of the University of Miskolc, 2012

[8] P. Körtesi: *GeoGebra Institute of Miskolc Didactical Research Group of the Department of Analysis of the University of Miskolc in the framework of the European Virtual Laboratory of Mathematics,* In: IMEM 2009, Ružomberok: Catholic University, 2009, pp. 582–585

[9] G. Krech, I. Krech: *On some nontransitive relation,* In: Proceedings of ICPM'09. Liberec, 2009

[10] A. Kútna, H. Palásthy: *Application of information technology in teaching mathematics,* In: Billich, M. (ed.): Mathematica III, Ružomberok: Catholic University, 2009, pp. 47–52

[11] A. Oldknow, R. Taylor: *Teaching Mathematics using Information and Communications Technology.* Continuum, London - New York, 2003

[12] Š. Tkačik: *Why logarithms?* In: Matematyka XII prace naukowe, Czestochowa: AJD, 2007, pp. 429–434

[13] Ch. E. Wittman: *The Alpha and Omega of Teacher Education: Organizing Mathematical Activities,* In: The Teaching and Learning of Mathematics at University Level: An ICMI Study, Kluwer Academic Publishers, Netherlands, 2001, pp. 539–552

Using invariants and semiinvariants in the solving of mathematical problems

Radek Horenský, Jaroslav Švrček

Faculty of Science, Palacký University in Olomouc
U Vrby 1709/9, 785 01 Šternberk, Czech Republic
horensky@gymst.cz, svrcek.jaroslav@upol.cz

Abstract. The thesis focuses on the possible use of non-traditional methods for solving of mathematical problems. The main methods published in the work are those, which uses constancy of some quantities values and techniques which are based on keeping of certain but not always readily seen features of the observed objects. The first part of the work provides key definitions such as invariants, semiinvariants, Fermat's Method of Infinite Descent. The second part of the work contains some of didactic tests focusing on abilities relating to invariants and semiinvariants in the problem–solving.

Keywords: Invariant, semiinvariant, methods of problem–solving.

Classification: D50, C30.

1 Introduction

The thesis focuses on the possible use of non-traditional methods for solving of mathematical problems. The main methods published in the work are those, which uses constancy values of some quantitative properties and procedures which are based on keeping of certain but not always readily seen features of the observed objects.

2 Definitions

The first part of the work appropriately defines some of the basic methods used when solving mathematical problems, such as method of invariants, semiinvariants and one of the varieties of this method Fermat's Method of Infinite Descent. The methods of invariants and semiinvariants belong to methods which often do not require extensive mathematical apparatus. The difficulty of these problems is given by new, non-traditional thoughtful procedures, which the optimal solution can be discovered with, find the relations among the transformed quantitative properties and use the relevant obtained information to solving the problem.

Definition 1

Invariant is a quantity or a quality which doesn't change when we repeat some procedure, transformation.

Definition 2

Semiinvariant is a quantity which is doesn't increase or decrease when we repeat some procedure, transformation.

Specific method of semiinvariant is Fermats Method of Infinite Descent (FMID):
Let k be a nonnegative integer. Suppose that:

- whenever $P(m)$ is true for an integer $m > k$, then there must be some smaller integer j, $m > j > k$ for which $P(j)$ is true.

Then $P(n)$ is false for all $n > k$.

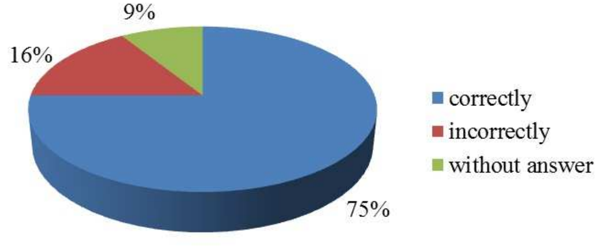
3 Didactic tests

The second part of the work evaluates some author's notes and experience of work with talented pupils and summarises some results following from the questionnaires. The series of the problems focused on the certain topics were given one by one to the students of the second grade of gymnasiums and to choosen pupils of mathematical voluntary lessons and seminars. It is obvious from the results of the research that the systematic work with the pupils supplemented by the suitably chosen problems has a significant influence on the development of the students' thinking.

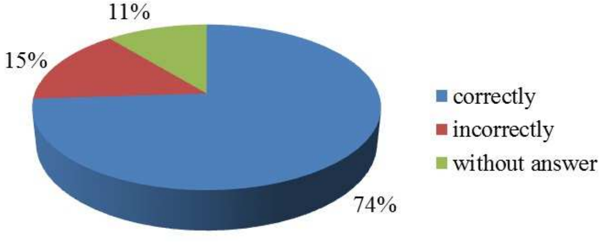
3.1 Didactic test 1

Determine which of the following statements are *always* true.

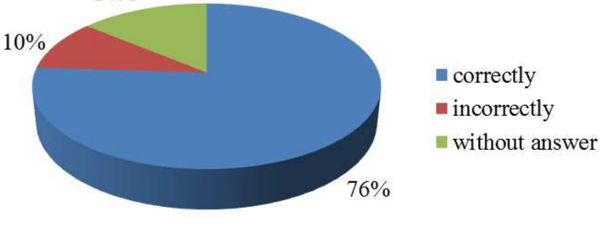
- Rest of division each one of nonnegative integer by 2 is the same as rest of division sum of its digits by 2.



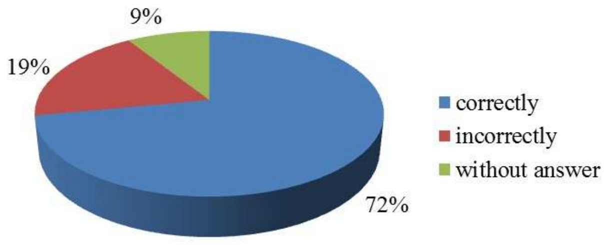
- Rest of division each one of nonnegative integer by 3 is the same as rest of division sum of its digits by 3.



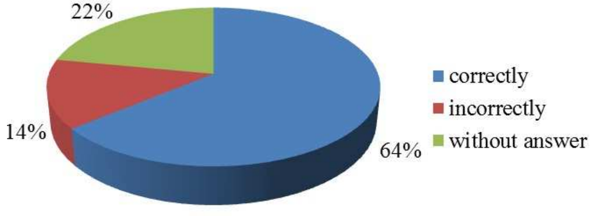
- Rest of division each one of nonnegative integer by 6 is the same as rest of division sum of its digits by 6.



- Rest of division each one of nonnegative integer by 9 is the same as rest of division sum of its digits by 9.



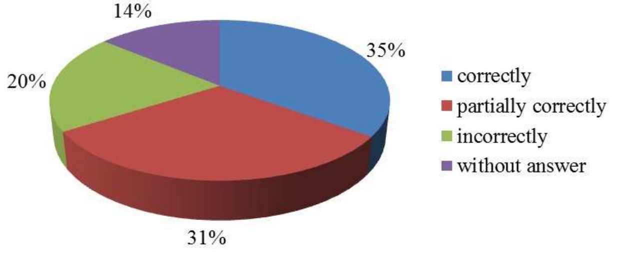
- Rest of division each one of nonnegative integer by 27 is the same as rest of division sum of its digits by 27.



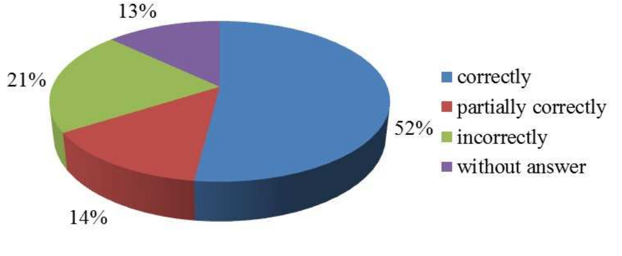
3.2 Didactic test 2

Determine which of the following statements *may be* true. Determine conditions when the statement are true.

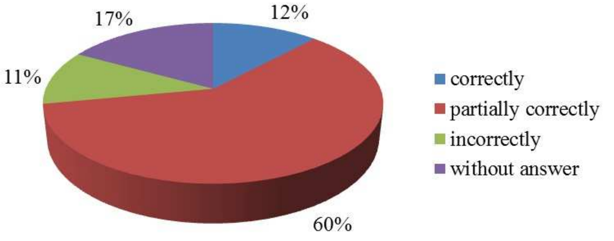
- Rest of division each one of nonnegative integer by 2 is the same as rest of division sum of its digits by 2.



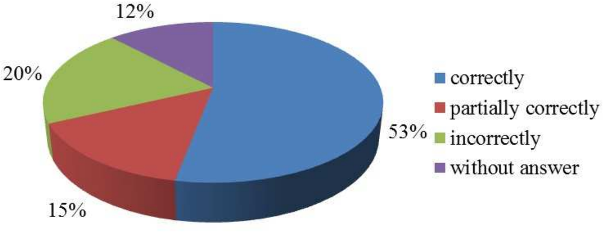
- Rest of division each one of nonnegative integer by 3 is the same as rest of division sum of its digits by 3.



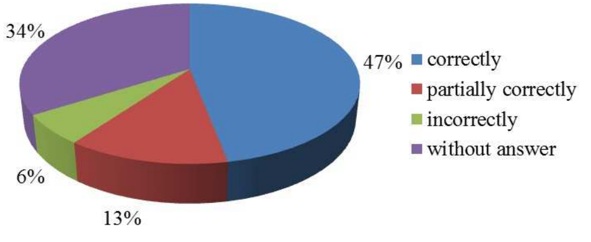
- Rest of division each one of nonnegative integer by 6 is the same as rest of division sum of its digits by 6.



- Rest of division each one of nonnegative integer by 9 is the same as rest of division sum of its digits by 9.

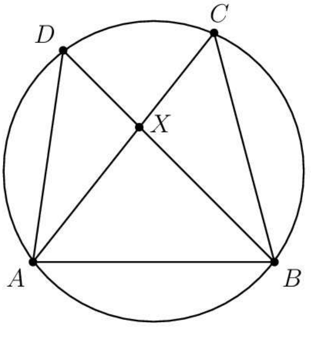


- Rest of division each one of nonnegative integer by 27 is the same as rest of division sum of its digits by 27.



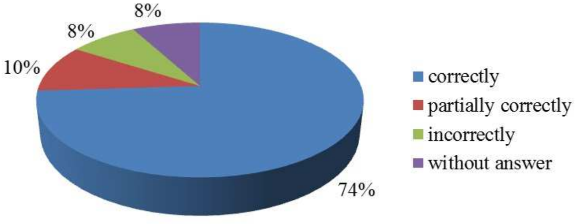
3.3 Didactic test 3

There are a chord AB and points C and D in the circle (see the picture) and X as a point of intersection line segments AC and BD.

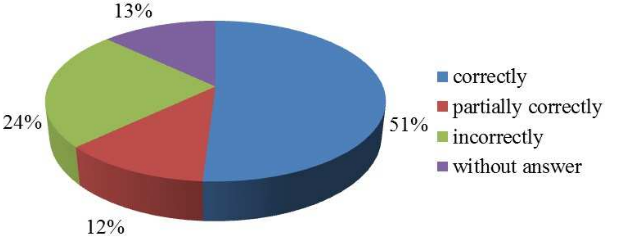


Determine which of the following statements are *always* true. If the statement *may be* true determine conditions when it is true.

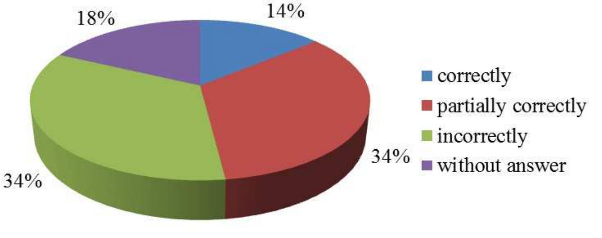
- $|\angle ABC| + |\angle BAC| = |\angle ABD| + |\angle BAD|$



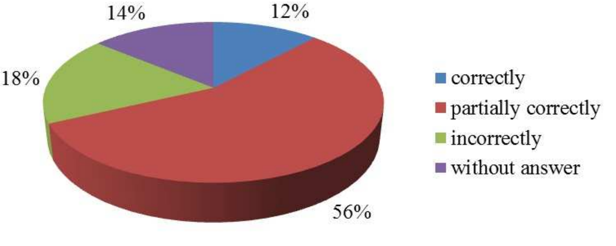
- $|AX| \cdot |CX| = |BX| \cdot |DX|$



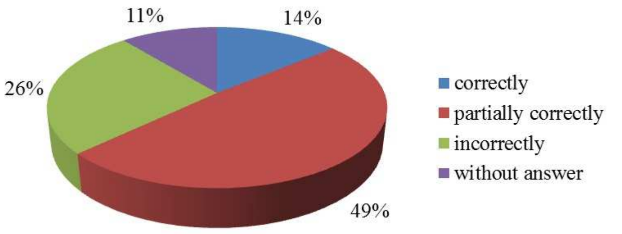
- $|AX| \cdot |BX| = |CX| \cdot |DX|$



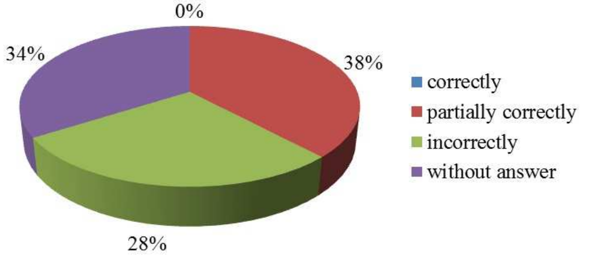
- $S_{AXB} = S_{CXD}$



- $|AC| \cdot |BC| = |AD| \cdot |BD|$



- $|AD|^2 + |BC|^2 = |AB|^2 + |CD|^2$



4 Conclusion

Thus, this work and tests can serve to teachers of the secondary schools as a collection of the mono–thematically aimed problems when working with talented students. It can be also used by students for their self–study, for better understanding of a certain topic, it gives them possibility to approach the issues from the different frame of references.

References

[1] A. Engel: *Problem-Solving Strategies,* Springer–Verlag, New York, 1998

[2] J. Herman, R. Kučera, J. Šimša: *Equations and Inequalities: Elementary Problems and Theorems in Algebra and Number Theory,* Springer–Verlag, New York, 2000.

[3] R. Horenský: *Invarianty a poloinvarianty ve školské matematice,* MFI, vol. 17, 2007/08, no. 2, s. 73-80, Olomouc.

[4] R. Horenský: *Invarianty, poloinvarianty a gradované řetězce matematických úloh,* MFI, vol. 18, 2008/09, no. 10, s. 577-587, Olomouc.

[5] J. Švrček: *Tvorba a využití gradovaných řetězců matematických úloh,* Vydavatelství UP, Olomouc 2008.

Games in the Maths Education

Vlastimil Chytrý

Pedagogická fakulta UJEP v Ústí nad Labem
Hoření 13, 40096 Ústí nad Labem, Czech republic
vlastimilchytry@centrum.cz

Abstract. This contribution describes the inclusion of selected mathematical games into the lessons as well as factors influencing these games. Last but not least, this contribution describes research cracking how you can affect an individual level of logical thinking through these games.

Keywords: Mathematical games, Logical thinking.

Classification: C40; D40.

1 Introduction

Wrong judgements of students, that are not caused by poor knowledge of mathematics, but of imperfection in their logical[1] thinking, appear in mathematical lessons very often. It would be desirable that pupils and students were able to formulate reversed or modified sentences. But they are able to do that without mistakes occasionally. Unclear understanding of the meaning of the quantifier "there exists exactly one object" make them troubles with evidence of unicity. They make mistakes especially in negative of formulas with quantifiers. In practise we can see the same instructional practices for decades, especially with transmissive way of leading the lesson. I believe that is necessary to make extensive changes in the management of learning activities of students towards greater possibilities of applying their individual characteristics and attitudes. I think that the ideal tool for this change is the game[2].

Teaching mathematics is not focused enough on specific benefits of gaming activities to the development of mathematical thinking[3]. In this article I will examine the influence of playing mathematical[4] games such as Mastermind, Sudoku and NIM on the development of logical thinking of pupils from elementary and high schools, and several other factors influencing the level of logical thinking.

2 Methodology and Research Objectives

In my research I primarily focused on two types of research investigations. The first is a compilation of criteria according to which will be available games evaluated to choose those that are suitable for integrate into teaching.

The second area is research using multiple questionnaire method in a relatively large sample of students, which brings data, which are then statistically processed.

[1] "Higher form of thinking than thinking depended on the specific activity; correct reasoning under the laws of formal logic, where the base is derivation from general to specific, so called deduction, and from specific to general, in other words induction." ([3])

[2] Dutch historian and cultural anthropologist Johan Huizinga, characterizes game in his major work Homo Ludens as follows: "The game is free negotiation or employment within a clearly defined time and space, which is held by freely adopted, but absolutely binding rules, has a goal in itself and entails the excitement and joy as well as consciousness of differences from everyday life." ([4], p. 37)

[3] "Is a cognitive approach to a problem that is both logical and mathematically sound "

[4] Martin Gardner defined a mathematical game in Scientific American as follows: "A mathematical game is a multiplayer game whose rules, strategies, and outcomes can be studied and explained by mathematics."

Research was attended by 443 respondents. Considering that the testing of logical thinking and intelligence of individuals took place in two days (two school lessons), not everyone of tested students attended both lessons. For this reason, logical test of logical thinking was developed for the 428 respondents.

Respondents from experimental and control groups, who were selected based on a multi-stage random selection, first wrote the entrance test of logical thinking. Then each experimental group played mathematical games for ten hours (one hour per week) and each work of pupils and students were recorded in detail worksheets. After ten weeks of playing mathematical games all respondents attended output test of logical thinking. The difference between the level of logical thinking as measured in input and output test was examined in control and experimental groups. IQ of individuals was measured by Raven's Progressive Matrices Test.

Each of the tests focused on logical thinking level included fourteen questions, some of them were divided into sub-questions. Overall respondents answered 24 questions that can be divided according to their focus into the following three areas:

- Finding number patterns (hereinafter reffered to as NP)
- Finding geometric patterns (hereinafter reffered to as GP)
- Working with elements of formal logic (hereinafter reffered to as FL)

Each of these areas corresponded to several questions which were evaluated by 1 – right, 0 – wrong. Based on average ratings of each of the questions from all areas I got for each respondent an evaluating vector of three coordinates, where each of them corresponds to an individual's likelihood of correct answers to the question in this area. Obtained evaluation vector is in the form (NP, GP, FL). Determining vector for these groups was obtained by calculating the average of all evaluation vectors of all pupils from control and experimental groups.

The research objective was to verify the following two hypotheses:

H_{01}: Positive changes in the logical thinking of pupils can be achieved by medium-term influence of mathematical games.

H_{02}: Significant factors affecting the level of logical thinking of individual are particularly intelligence, type of attended school, school evaluation of mathematics and level of playing logical games.

3 Development of Criteria for Game Selection

I had to solve a problem of selection of games suitable for development of logical thinking, at the beginning of my research. It was necessary to establish certain criteria that enable the selection of suitable games and differ them from such games, which are only aimed at motivating or at specific area of mathematics. Following areas happened to be important: motivational potential of games, the existence of an optimal strategy, variability in difficulty of the game, the presence of chance and the number of players. Games, according to these criteria, were a priori assigned on the expected value of simple scales, the actual range and criteria were specified during the research. Based on these criteria were chosen game Mastermind, NIM and Sudoku.

4 Collective research

The following table describes the further division of respondents, who were ought to attend the input and output test of logical thinking.

Evaluation group	Type of school	The frequency
The control group	Elementary school	56
	High school	38
	Total	**94**
The experimental group	Elementary school	41
	High school	31
	Total	**72**
	The total sum	**166**

Table 1: Frequency of respondents for the control and experimental group

The following table illustrates what values of logical thinking were reached by elementary and high school students in the control and experimental group.

Type of school	Evaluation group	Evaluation vector		
		NP	GP	FL
Elementary school	Control	0,55	0,70	0,46
	Experimental	0,34	0,57	0,41
High school	Control	0,86	0,90	0,85
	Experimental	0,44	0,69	0,42

Table 2: Evaluation vector of input test of logical thinking for the control and experimental group of elementary and high school students

The big difference between the control and experimental groups of high school students is primarily due to the fact that students from control group attend grammar schools.

5 Research Results

Table shows the difference between the level of logical thinking of elementary and high school students at the input and output test.

It is surprising that experimental group from both types of school exhibits a relatively high value of the improvement, even if the games were played with the children for only ten school lessons. For high school students is a positive shift in the level of logical thinking more significant.

For verification of the second hypothesis it was necessary to apply different levels of logical thinking to each component of the evaluation vector. At each game showed up a different factor as the important one. According to this factor the game was subsequently evaluated. Sudoku was evaluated based on the correct completion of all fields. In case of Mastermind the evaluation was based on using previous moves. For NIM was important finding a winning strategy.

	Evaluation vector		
Evaluation group	NP	GP	FL
Elementary school			
Control group	$-19\,\%$	$-1,45\,\%$	$5,6\,\%$
Experimental group	**11 %**	**31 %**	**19 %**
High school			
Control group	$-7,69\ \%$	$2,92\ \%$	$-3,10\,\%$
Experimental group	**52 %**	**24 %**	**34 %**

Table 3: The difference between the input and output vector for elementary and high schools at the control and experimental groups

Based on these requirements, it was possible to assign two values to each of games: 0 – mishandled the request or 1 – handled the request.

The following illustration demonstrates what dependencies were investigated during the research. In this article I will focus only on those that affect the level of logical thinking.

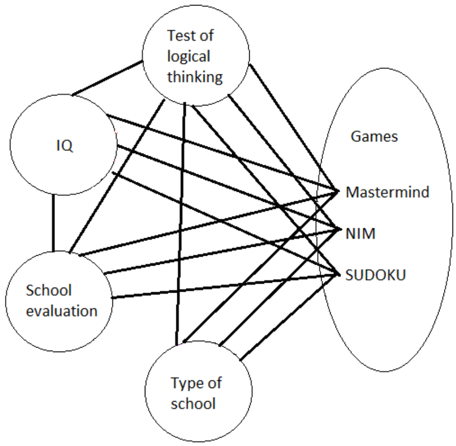


Figure 1: Factors affecting the level of logical thinking

To find the dependence between the level of playing games, which is evaluated with symbols 0 and 1, and the level of logical thinking, which corresponds to the evaluation in metric variables, the Mann-Whitney Test was used for paired values at a significance level of 5 %. Table demonstrates how were different components of the evaluation vector affected by different types of games.

There was no evidence that there is an effect of the Mastermind game on the level of logical thinking, like in the case of the NIM game, where the dependence showed in only one case. In contrast the Sudoku game has an appreciable effect on the level of logical thinking. The fact that the effect is more pronounced in elementary school is in my opinion due to the fact that during the experiment was easier to motivate

	Component of the evaluation vector	Mastermind	Sudoku	NIM
The level of logical thinking in elementary school	NP	NO $p = 0,14336$	**YES $\boldsymbol{p = 0,001174}$**	**YES $\boldsymbol{p = 0,031603}$**
	GP	NO $p = 0,14336$	**YES $\boldsymbol{p = 0,000113}$**	NO $p = 0,199247$
	FL	NO $p = 0,15755$	**YES $\boldsymbol{p = 0,004340}$**	NO $p = 0,940505$
The level of logical thinking in high school	NP	NO $p = 0,81510$	NO $p = 0,637355$	NO $p = 0,050807$
	GP	NO $p = 0,94672$	NO $p = 0,238602$	NO $p = 0,533452$
	FL	NO $p = 0,97335$	**YES $\boldsymbol{p = 0,029246}$**	NO $p = 0,428020$

Table 4: The effect of playing games on the components of the evaluation vector

them and they played these games with more passion.

To find the dependence between the level of logical thinking and IQ of individuals, where there is the dependence of two metric variables, the Spearman's Correlation Coefficient was used. To test the dependence between the level of logical thinking on the school evaluation the Pearson's Chi-Square Test at a significance level of 5% was used.

	The component of the evaluation vector	IQ of individual	School evaluation in mathematics
The level of logical thinking in elementary school	NP	**YES $\boldsymbol{p = 0,000002}$**	NO $p = 0,0991$
	GP	**YES $\boldsymbol{p = 0,000001}$**	NO $p = 0,2439$
	FL	**YES $\boldsymbol{p = 0,000749}$**	NO $p = 0,3927$
The level of logical thinking in high school	NP	**YES $\boldsymbol{p = 0,000001}$**	**YES $\boldsymbol{p = 0,0387}$**
	GP	**YES $\boldsymbol{p = 0,000001}$**	**YES $\boldsymbol{p = 0,0115}$**
	FL	**YES $\boldsymbol{p = 0,000001}$**	**YES $\boldsymbol{p = 0,0883}$**

Table 5: The effect of IQ and school-level evaluation on level of logical thinking

The finding that IQ of individual affects the individual level of logical thinking is not surprising. Research points to the interesting fact that the school evaluation on mathematics has a different character on the elementary and high school. While in high school it is influenced by the level of logical thinking of students, at elementary school does not reflect the level of their logical thinking at all.

6 Conclusion

The research found that in general the level of logical thinking is low, but can be affected in the positive way. Based on the ten school lessons, the significant improvement in logical thinking in the areas of finding numerical patterns, geometric patterns and work with elements of formal logic was shown. The different effect of selected games on the level of logical thinking was also demonstrated. I think that the level of logical thinking of individual can be positively influenced in long run by the properly selected games.

Acknowledgements

The author has been supported by project IGA UJEP.

References

[1] DUNLAP, J.: *Mathematical Thinking.* [online], 2011, poslední revize 31.5.2011 [cit. 2011-06-03]. Dostupné z `http://www.mste.uiuc.edu/courses/ci431sp02/students/jdunlap/WhitePaperII.doc`

[2] GARDNER, M.: Mathematical Games. *Scientific American.* 1979, roč. 241, č. 3, s. 22–32.

[3] HARTL, P., HARTLOVÁ, H.: *Velký psychologický slovník.* 4. vyd. Praha: Portál, 2010. 800 s. ISBN 978-80-7367-686-5.

[4] HUIZINGA, J.: *Homo ludens.* 2000. 297 s. ISBN 80-7272-020-1.

Didactic tools and their use in teaching

Magdalena Janků

Faculty of Education, Palacký University in Olomouc
Žižkovo náměstí 5, 771 40 Olomouc, Czech Republic
makda@seznam.cz

Abstract. The paper presents the basic goals and strategies of the anticipated dissertation named "Use of material didactic tools in teaching". It also presents the results of a preliminary research.

Keywords: Didactic tools, teaching aids.

Classification: C70.

1 Introduction

The education and training process is truly a complex one, based on mutual interactions between the provider and the receiver of the education content (Grecmanová, 1998). This process is influenced by many factors, the three main being the teacher, the student and the education content. However, other categories intervene as well, such as the educational goal, organizational conditions, material conditions and teaching methods and practices (so called methodological conditions).

In modern education, it is almost impossible to limit the education process to a mere verbal communication. Moreover, the principle of clarity has been one of the main didactic principles since the times of J. A. Comenius, who put emphasis on the involvement of as many senses as possible. *"If possible, the educators should allow the receivers of the education content to perceive the latter by all the senses (not just hearing or vision) and thus facilitate a rational cognition"* (Grecmanov, 1998, p. 154). The didactic tools are therefore of high importance as they contribute to the effectiveness of the education and training process. Every teacher should have a general knowledge of various didactic tools; more specifically of the teaching aids they can make use of in their lessons. However, the application of these aids shall not become the very aim of teaching, but only the means of it.

The aim of this paper is to specify the material didactic tools, introduce the possible applications of the latter in mathematics lessons, and to familiarize the readers with the survey preceding the anticipated dissertation.

2 Didactic tools and their function

Didactic tools are all resources, both material and immaterial, that teachers have at their disposal in order to achieve the set learning objectives. *"The immaterial didactic tools are represented by teaching methods and forms of teaching and learning. The material didactic tools, on the other hand, include teaching aids, teaching equipment, teaching techniques, school supplies, etc."* (Malach, 2004, p. 7). Didactic tools facilitate the class management for the teachers; they help them to make lessons more interesting and better motivate the pupils, whereas pupils themselves, once provided with these tools, get to understand and master the subject matter in a much better and quicker way.

There exist various classifications of material didactic tools. As an example, the classification of Malach (cited Kalhous, Obst, 2002) is submitted. This author divides the didactic tools into five categories as they follow:

1. Teaching aids,
2. Technical teaching tools,
3. Organizational and reprographic technologies,
4. Teaching facilities and equipment,
5. Teacher's and pupil's equipment.

As the issue of material didactic tools is very extensive, it would not be possible for a dissertation to cover all aspects of it. Therefore, we will focus only on one sub-category that is to say on teaching aids.

These are considered the most important category of material didactic tools, and are characterized in the Pedagogical dictionary as "objects mediating or imitating reality, thus making for a greater clarity or facilitating the teaching process" (Průcha, 2009, p. 332). They differ from other didactic tools mainly due to their close interconnection with the content. Their impact on the learners via their didactic function is direct, unlike the predominantly indirect effect of technical means.

Didactic tools are made use of at all stages of the education process, starting from the mental preparation for learning, through acquiring new knowledge and creating skills, consolidating and deepening of the latter, and the usage of the knowledge acquired, up to the verification and evaluation of pupils' performances. Nevertheless, the inclusion of material didactic tools into teaching can never be pointless, as they should carry out the following basic functions:

- *Informative function* - didactic tools support or implement the acquisition of knowledge, that is to say knowledge creation; they present, specify and represent the subject matter and thus play an important role in the development and conceptualization of ideas;
- *Formative function* - didactic tools induce practical and mental activities of pupils, develop pupil‘s activity, independence, and creativity;
- *Instrumental function* - didactic tools are used as tools for acquiring the learning data (via demonstrations, experiments, manipulations), as instruments for facilitating communication in lessons, and as means for pupils to perform various activities without the direct intervention of teachers (Rambousek et al. 1989, p. 19).

3 Teaching aids in mathematics lessons

Teaching mathematics involves the usage of mainly abstract concepts, which are often difficult to understand for pupils and make the process of conceptualization rather demanding for them. Yet mathematics has a basis in real life, as each abstract concept results from a real situation. Within the framework of the education process, teaching aids thus act as materialized concepts of abstract notions. Moreover, the handling of a teaching aid by the pupil not only leads to the development of logical thinking, but also to the improvement of practical skills and fine motor skills. Finally, by properly choosing a teaching aid, pupil‘s combinatorial thinking, spatial imagination, creativity, fantasy, and ability to cooperate and work in the group are evolved.

As the subject matter of mathematics is truly a wide-scope one, a wide range of teaching aids may be used, among the classical being textbooks, methodological guides for teachers, and workbooks for students, less traditional being in the form of tangible objects (polystyrene applications and geometric shapes attachable on a magnetic board), small objects (buttons, beads, etc.), demonstration notice boards, models of geometric bodies, models of money, drawing tools (rulers, compasses, French curves) and measurement tools, square grids with a set of coordinates, number axes, various counters, geometric puzzles (tangram, Columbus egg), building blocks (Lego system POLYDRON, GEOMAG), cubic blocks, card files (dominoes, pairs), dice, calculators, games, games based on puzzles or other types of board and card games, brain-teasers, computer learning programs (Novak, 2003).

The current market offers a wide range of didactic tools and it is necessary for the teachers to be knowledgeable about them and to use them. The use of multimedia tools, such as interactive whiteboards, personal computers and computer programs in teaching has recently become a trend, which brings about a certain decline in use of so called “classical” equipment, such as various construction sets and board games, the involvement of which can nonetheless increase the effectiveness of the education process as well. Most of these are applied in teaching at lower primary schools, where clarity is particularly important when shaping abstract notions. However, many of these utilities and games can also be used in teaching mathematics at higher primary schools.

4 Investigation survey

The aim of the empirical part of the thesis is to determine the use of which teaching aids teachers prefer in their classes, whether rather the “classic” or the “modern” ones, and what problems related hereinto they face. Furthermore, the influence of the length of teaching experience, the gender and the type of school on the use of teaching aids shall be examined. The data shall be collected by means of a non-standardized questionnaire comprised of open as well as closed items, and subsequently made subject to a statistical treatment in order to verify the initial hypotheses.

As a part of the pre-dissertation research, a questionnaire survey was conducted in order to characterize the use of teaching aids and technical devices in classes at primary schools. The target group were teachers of primary schools; the method applied was a non-standardized questionnaire.

The proposed questionnaire was anonymous and contained a total of 16 items, four of them being contact, searching for the teachers’ gender, age, length of teaching experience and the subjects taught, 5 being open and 7 closed, of which four polytomic (enumerative or listing), one scale, one dichotomic, and one semi-closed (Chráska, 2007). The questionnaire was distributed among teachers at primary schools from Hodonin district who attended the Project day in Rohatec in May 2012. Cited below are some of the results brought about by the data analysis. For clarity’s sake, a graphical representation of the data by means of the frequency histograms is provided in the annex, too. It is however necessary to say that the respondents did not always follow the instructions and circled more options than allowed, which may have influenced the interpretation of the results.

In total, 23 teachers got involved in the survey, of which 5 men and 18 women. The age structure of the sample more or less corresponded to the common distribution and is clearly demonstrated in Figure 1. Seven respondents were between 31 and 40 years old, seven between 41 to 50 years old and seven were over 50. One respondent had already reached the retirement age and one was between 20 and

30 years old. The length of the respondents' teaching practice is demonstrated in Figure 2. Two teachers had been teaching for less than 5 years, three stated a 6 to 10 years' experience, six had been involved in the field from 11 to 20 years, and seven from 21 to 30 years, and 5 respondents had more than 31 years' practice.

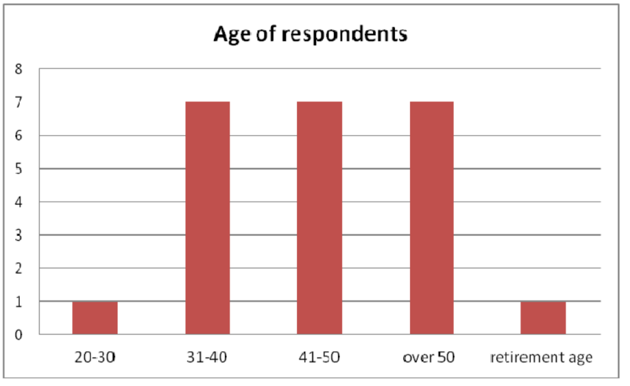


Figure 1: Age of respondents

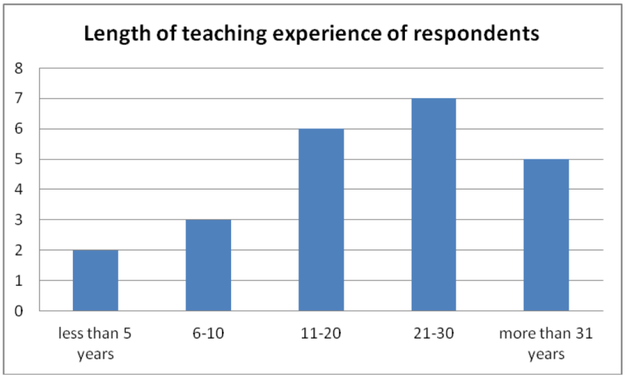


Figure 2: Length of teaching experience of respondents

Out of the total of the 23 respondents, 8 were teachers at lower primary school, the other 15 taught various subjects at higher primary schools, most of them teaching mathematics (8 respondents), Physical Education (5), Physics (3), English (3), Crafts (3), Arts (3), Czech (2), Civics (2), Geography (2), Music (2), Information Education (2), as well as History, German, Chemistry, Biology, National History and Geography or Health Education.

The aim of the next two questions was to determine which teaching aids are regarded by the teachers as "traditional" or "classic" and which are rather perceived as "modern". These questions were open and brought about a wide range of responses, especially regarding the "traditional" aids. As shown in Figure 3, teachers most often mentioned wall paintings, books and models, across all subjects. As "modern" tools, the interactive whiteboard and the computer were mostly cited (Figure 4). Surprisingly, some respondents labelled as "traditional" CD and DVD players. However recent this technology may be, it has clearly become perfectly naturalized over the last few years, which may lead to a hypothesis that even such a novelty as the interactive whiteboard surely is now, might be, in a few years, described as a traditional utility by the teachers.

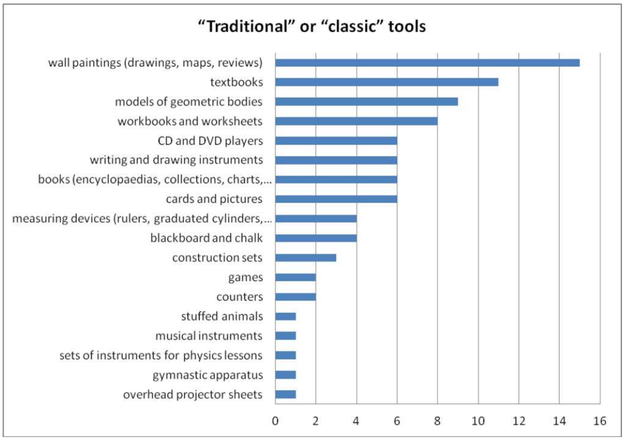


Figure 3: Tools regarded by the respondents as "traditional" or "classic"

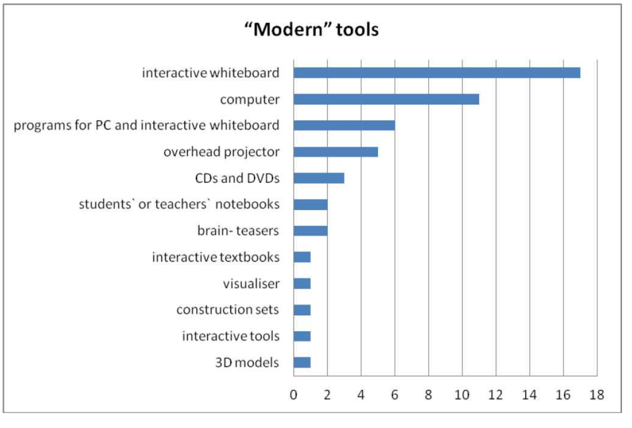


Figure 4: Tools regarded by the respondents as "modern"

Other questions searched for the responses regarding the groups of teaching aids that teachers use in the classroom directly. The respondents were provided with a selection of 7 categories, that is to say audio teaching aids (CD), tangible teaching aids (cuts, technical equipment), imitations (functional models), visual teaching aids (wall paintings), written teaching aids (textbooks, professional literature, charts), audiovisual teaching aids (films, demonstrations, educational programs), and cybernetic teaching aids (animations, simulations, presentations). The teachers involved were allowed to choose more than one option. As most frequently used aids were labelled the written, visual and audiovisual ones (see Figure 5). As regards the technical equipment, the classic blackboard was stated (see Figure 6), as the most commonly applied multimedia technology, the interactive whiteboard was elected, closely followed by the computer (see Figure 7).

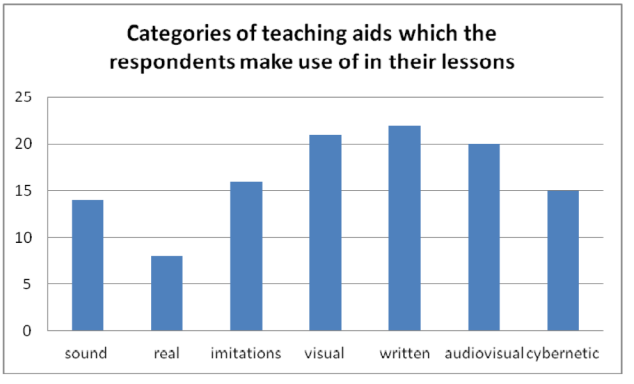


Figure 5: Categories of teaching aids which the respondents make use of in their lessons.

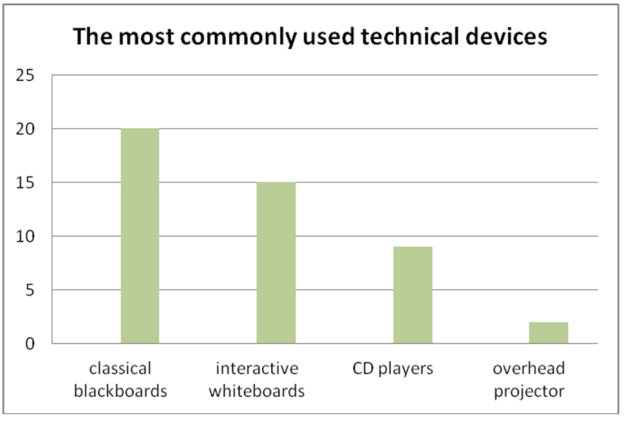


Figure 6: The most commonly used technical devices.

17 respondents expressed their satisfaction with the equipment of their schools with teaching aids, while 6 respondents voiced dissatisfaction in this respect. However, not all these answers necessarily reflect the real state of things regarding the equipment of schools with educational materials, as they may only be the teachers' subjective opinions. The aim of yet another item in the questionnaire was to determine how the teachers manage their lesson should they not have a suitable tool for teaching a particular topic. 21 respondents claimed they would create their own

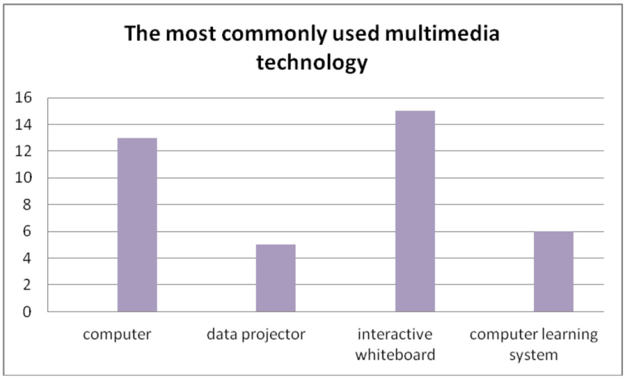


Figure 7: The most commonly used multimedia technology.

tool, which speaks for a high level of inventiveness and creativity of the latter. The next question concerned the sources of inspiration needed for creating and using the teaching aids. Most respondents (20) quoted the internet as a place they refer to most often when searching for appropriate tools, among other sources were literature (12), courses (10), the experience of their colleagues (7) or their own life experiences, training, and/or conferences.

As the above stated answers of the respondents show, many teachers use as many teaching aids as they can in a particular situation. As regards the frequency of using particular teaching aids, the classic blackboard still remains at the forefront, as most respondents admitted using it in almost every lesson of theirs. Very often textbooks and worksheets are made use of, regardless of the subject taught. Almost as often as these, the interactive whiteboard or computers are applied. In compliance with the present results of the pre-research, it is therefore possible to presume that there will be no statistically significant difference between the frequency of use of the "classic" and "modern" teaching aids, them being used equally in lessons.

5 Conclusion

The material didactical tools are an integral component of the educational process. They enable teachers to improve the quality and the effectiveness of teaching. Especially in mathematics, where abstract concepts are often dealt with, it is appropriate to make use of teaching aids for the sake of clarity. As there is a wide range of them, each teacher can choose the very type of tools that match his or her teaching style. However, the question still stands whether the teachers in their classes really do use the teaching aids in the way they claim to and how the pupils themselves see the problem. The answers to these questions may thus be subject to a further investigation.

Acknowledgements

The paper was prepared with the financial support from Palacky University, within the framework of a student grant competition titled "The level of functional thinking of student teachers of mathematics at higher primary school", under the registration number PdF_2012_020.

References

[1] J. Dostál: *Učební pomůcky a zásada názornosti.* 1.vyd. Votobia, Olomouc, 2008
[2] H. Grecmanová, D. Holoušová, E. Urbanovská: *Obecná pedagogika 1.* 1. vyd. Hanex, Olomouc, 1998
[3] M. Chráska: *Metody pedagogického výzkumu.* Grada, Praha, 2007
[4] Z. Kalhous, O. Obst: *Školní didaktika.* 1. vyd. Portál, Praha, 2002
[5] J. Malach, M. Mikošek: *Tvorba a užití didaktických médií* [online]. Ostrava, 2004. `http://www.osu.cz/fpd/kik/dokumenty/itvv/qtudm/qtudm.pdf`
[6] B. Novák: *Vybrané kapitoly z didaktiky matematiky 1: pro učitelství 1. stupně ZŠ.* 1. vyd. UP, Olomouc, 2003
[7] J. Průcha, E. Walterová, J. Mareš: *Pedagogický slovník.* 6. aktualiz. a rozš. vyd. Portál, Praha, 2009

Training mathematics teacher profession ten years ago and now

Petra Konečná

Department of Mathematics, Faculty of Science, University of Ostrava
30. dubna 22, 70103 Ostrava
petra.konecna@osu.cz

Abstract. In the last ten years significant changes have occurred in academic disciplines for teaching profession, mostly thanks to changes in structures of the study programmes. This paper aims at providing a thorough comparison of pregraduate policies in the secondary-level mathematics teacher training at the University of Ostrava ten years ago and today.

Keywords: Structure of the study programme, academic majors in teaching, mathematics, pregraduate policies in teacher training, field of study.

Classification: Primary 97A30; Secondary 97B40.

1 Introduction

In accordance with the Czech Republic's accession to the European Higher Education Area process and particularly since the joining of the Sorbonne and Bologna Declarations of 1998 - 1999, there has been a massive change in academic programmes and fields of study. Establishing a three-grade university education system was one of the main objectives. It has been ten years now since the initiation of the process and there has been an intensive discussion as to whether it is a change for good or bad [3]. With the five to seven years' hindsight, the above-mentioned process can be assessed from the viewpoint of the academic fields in teaching. The question of sensibility of restructured teaching programmes comes to forefront again, and with that the return to five-year master's degrees. The Accreditation Commission of the Czech Ministry of Education (MEYS) is dealing with a growing debate about the subject and assesses the overall policy and quality of pregraduate teacher training in general education courses at primary and secondary schools. "It's not just about whether structured studying system should be kept, or accreditation of five-year study programmes allowed, but it's about creating a comprehensive policy concept of university education for teachers (including all relevant standards) and corresponding funding." [2]

2 The Process of Restructuring the Academic Disciplines for Teaching Profession in the Czech Republic

The Accreditation Commission of the Czech Republic responded to the Bologna Declaration requirements by a discussion about undergraduate study programmes during its session on 24th and 25th October 2000. It postulated that, in cases where it is sensible, all prepared applications for accreditation renewals should respect the stipulations of the Bologna Declaration, while at the same time the undergraduate study must lead either to acquiring skills necessary for pursuing the occupation (profession-related study programmes) or a general bachelor's degree education that enables continuation on a master's degree programme. Concurrently it drew attention to problematic placements of bachelor's degree teachers from the labour market, legal and factual viewpoints. [1] In the Czech Republic the discussion was halted by a new amendment of the university law that put structured studies firmly in place and only exceptionally allowed unstructured study (as in the case of medical doctors and lawyers). [7] Despite initial doubts about the sense of the whole process within the scope of the teaching study programmes and further unresolved matters, the period of 2002 - 2006 witnessed a gradual reorganisation of master's degree study programmes specialising in the secondary education and primary education from the Year 5 up. This situation is the same today.

The restructuring process continued unsynchronised not only among universities but even among different departments of a single university. Therefore, consensus was required for changes to pregraduate educational policy for teaching profession. In spring 2004 a committee was appointed that finalised and proposed a framework policy for pre-gradual training of teachers at primary and secondary schools [6]. The policy defined general requirements for teacher training, and four options for enhancing it by pedagogical and psychological education were approved and lined up according to the committee's preference. The basic difference among the options was in what form the pedagogical and psychological part would be applied. The first two options were used primarily. Option no. 1 integrates the pedagogically-psychological component and professional training in both the bachelor's and the subsequent master's level in its compulsory phase despite the fact that the graduates of the bachelor's study programme do not obtain the teacher's qualifications. The subsequent master's study builds on the professional knowledge and that is also where the core of the teacher's training lies. Option no. 2 has in essence a very similar design, though on the bachelor's level it focuses more on the knowledge foundations of one or two specialisation courses of a particular field. Thus the compulsory part consisting of courses from a chosen field has strictly a non-teaching character. Those students who parallelly want to prepare themselves for the subsequent master's teaching level can take up an elective block of Pedagogy-Psychology component analogous to the Option no. 1.

As it is apparent from the summary of the accredited study programmes [4], most universities in the Czech Republic embraced the first two options. Option no. 1 was most popular with pedagogical faculties and faculties of arts, while science faculties preferred Option no. 2. There were some faculties that went for accreditation of both types of bachelor programmes. Reorganisation of the teaching study programmes at the University of Ostrava (UO) was carried out between 2004 and 2006. Just like most other universities, the Faculty of Science and the Faculty of Arts at UO chose the Option no. 2, while the Pedagogical Faculty leaned towards the Option no. 1 with the compulsory focus on teaching on bachelor's level.

3 Comparing Pregraduate Teacher Training at University of Ostrava ten years ago and Now

The area of educating teacher's profession recognises five basic training components:

- Field and Course-Related; Teaching at primary schools from Year 5 up and secondary schools has usually 2 specialisation courses,
- Course and Didactic; For Teaching at primary schools from Year 5 up and secondary schools it is usually didactics of specialisation courses;
- Pedagogy-Psychology; This part covers especially general pedagogy and didactics, general psychology, developmental psychology, social psychology, pedago-

gical psychology, etc.;

- Pedagogical Placements at Schools;
- General Education; That is general knowledge foundations, for instance philosophy, computer skills, communication skills, language proficiency, etc. [7]

In former years the above-mentioned components were not individually analysed for the purposes of the teacher education programmes nor were any optimum proportions set among them. Strictly speaking the educational programmes for teachers were split into three parts:

- Parts based on the academic field that consisted of the Field and Course-Related component, Course and Didactic component, Pedagogical Placement at Schools component and sometimes also the General Education component. There were usually 2 parts like that and they depended on the combination of specialisation courses.
- The so called Common Basis part consisting of Pedagogy-Psychology component often accompanied by the General Education component or part of it.

Number of credits or an hourly funding can serve as examples for a basic comparison. As the launching of the credit system came later, the courses' credit values at the beginning of the tracked period don't always correspond with the ECTS principles, which means that they don't take into account the total average time load per student but merely duplicate the hourly funding for a course. The price of credits per a course usually equaled 1 - 1.5 times hourly funding depending on demands level of a course. The principle was however not applied to all courses. That is why the comparison by the number of credit points is only applicable as an overall comparison among different parts of the studying programmes or alternatively to describe the scope of elective blocks in different specialisations. For a more detailed comparison within the programme parts, the hourly funding for different courses was used as a benchmark thanks to its better information value within the whole tracking period.

When comparing different parts of the programmes, the attention is focused especially on the compulsory ones that are crucial for the student's teaching qualification. After that, the results will separately be supplemented by information about the structure of elective blocks on offer. It is provided to give a better idea of the whole content and extent of the programmes.

3.1 Common Basis

The comparison of the pregraduate teacher training, common to all academic main fields, is shown in detail in [5]. It is a part of the Common Basis, which is the Pedagogical-Psychological component of the teacher training, combined with the General Education component.

It is apparent from the comparison results of the credit-based funding that overall there has only been a small shift in credits from the programme parts in favour of the Common Basis and further general qualifications of students in the elective block. This kind of student's development through elective courses is mainly present on bachelor's degree level.

These results correspond to the fact that within the Pedagogy-Psychology component students who were preparing themselves for their teaching jobs in the year

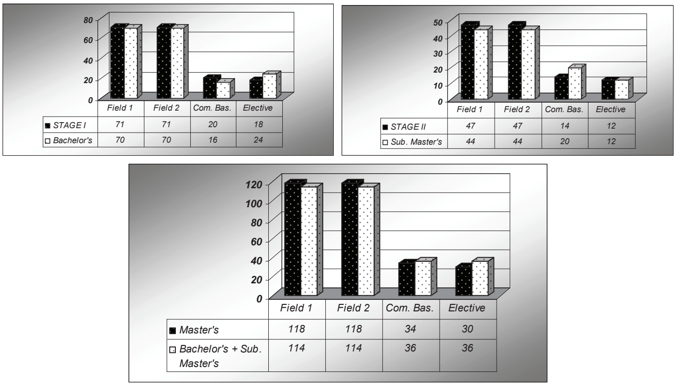


Figure 1: The credit numbers comparison in the first stage (first three years of study) and the second stage (the fourth and the fifth year) of master's teaching programmes and in bachelor's and subsequent master's degree programmes provided for teachers

2000/2001 were required to complete 10 hours of education and moreover they could choose compulsory electives in the total amount of 22 credits.

In contrast to that, in 2009/2010 it was up to 19 hours of compulsory education and choosing courses with the total of 25 credits.

After a more detailed analysis of the Common Basis part structure it can be observed that the restructuring of the long master's teaching programmes brought about an increase in the number of hours students must complete. That in particular goes for the Pedagogy-psychology part that is more focused on the subsequent master's degree, and General Education part, mathematics and information processing literacy parts as well as gaining communication and studying skills - all conducted more on the bachelor's level. [5]

3.2 Mathematics

The following paragraphs are going to deal with the so-called academic main field part and specifically Mathematics as the specialisation course. As it was already mentioned above in the comparison by the number of credit points, in the last 10 years there has only been a slight decrease in total credits from the profession or Field-Related part at the expense of the general education part. From the global view, no significant differences can be observed. A more detailed analysis of the whole academic field part structure however shows much more fundamental changes.

We are going to split the Field and Course-Related part into groups. The first three groups will comprise the most relevant branches of mathematics that are incorporated in the curricula. They are Mathematical Analysis, Algebra and Geometry. Other courses, the so-called preparational courses, make another independent group. They form a transition between the secondary school level and the university level mathematics. The remaining courses don't have an allocation and are simply called "Other Mathematical Courses". Courses are selected and tracked with regard to the latest trends, especially in the area of improving language proficiency and computer

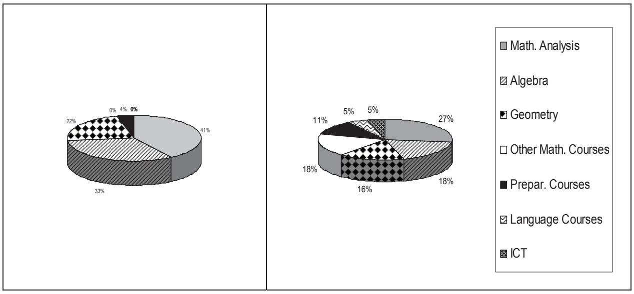


Figure 2: Percentages of grouped courses in the first stage of the compulsory academic field part in 2000/2001 and bachelor's level in 2009/2010

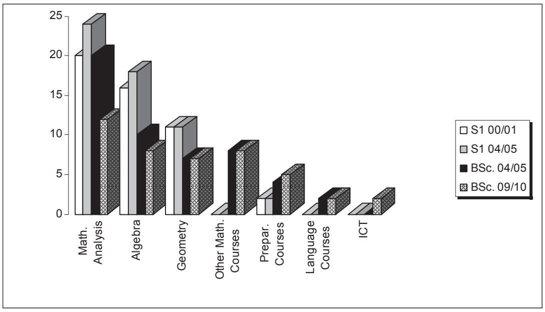


Figure 3: Development in percentages of grouped courses in the first stage of the compulsory academic field part in the period of 2001 – 2010

skills. The second stage, or rather the subsequent master's level concentrates on didactic perspectives on mathematics and pedagogical placement.

If we look at the structure of the compulsory academic main field part in the first stage of the master's level in 2000/2001 from the perspective of the above-mentioned groups and if we compare it to the same data in the year 2009/2010, we can see significant changes. The former concentrated on courses from the algebra group (33%), geometry (22%), and the dominant group consisting of mathematical analysis (41%). This arrangement was further only completed by the group of preparational courses with 4%. In the course of ten years the groups were gradually balanced (algebra 18%, geometry 16%, mathematical analysis 27%) and more mathematical and general knowledge courses were added (other mathematical courses 18%, preparational courses 11%, the academic field-related English 5%, computer work 5%).

Efforts to enhance the academic field part with more courses, so that individual areas of mathematics are represented equally, are reflected in the structure of the compulsory electives on offer.

Analogously, if we make the comparison for the second stage of the master's pro-

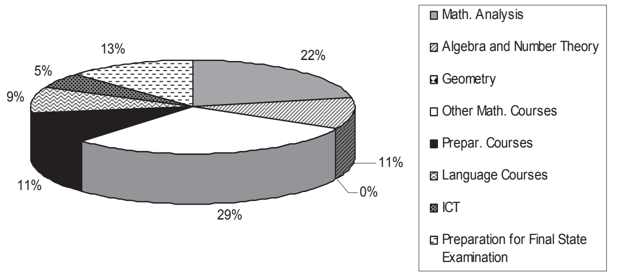


Figure 4: Present structure of the offered electives in the academic field part of the bachelor's programme shown by different course groups

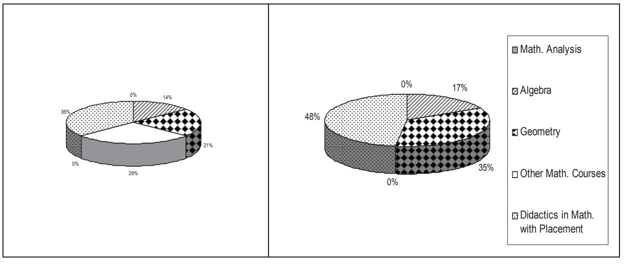


Figure 5: Percentage representations of grouped courses in the second stage of the compulsory academic field part in 2000/2001 and corresponding master's programme in 2009/2010

gramme in 2000/2001 and the corresponding master's level in teaching mathematics at secondary schools in 2009/2010, the trends observed are largely reversed. From the original narrowly constructed structure of Algebra (14%), Geometry (21%), and Other Mathematical Courses (29%) supplemented of course by components of didactic and pedagogic placement (36%), all other mathematical courses disappeared completely in the last ten years and the current structure is Algebra (17%), Geometry (35%), and Didactics of Mathematics with Placement (46%).

The predominance of the latter group is quite in order if you realise that this is the studying level where the core of the teacher preparation should be. However, if you take into account the total number of credits of this compulsory Field-Related part (only 47 credits equalling 22 hours of education and 2 weeks of continuous placement) and complete it with the set of courses in the compulsory elective block, you will find a totally unsuitable arrangement of the academic Field-Related part for the ensuing master's programme.

4 Conclusion

The reorganisation of the five-year master's programmes for teachers at secondary level education produced a slight increase of the Common Basis part at the expense

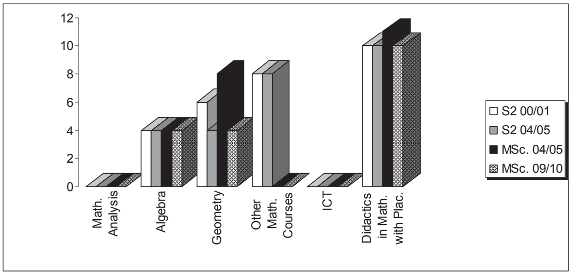


Figure 6: Development in percentages of grouped courses in the compulsory academic field of the second stage and the corresponding master's programme in years 2001 – 2010

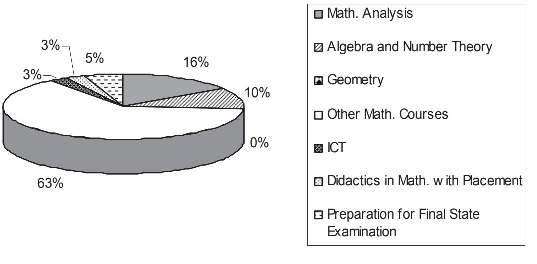


Figure 7: Present structure of the offered electives in the academic field part of the ensuing master's programme shown by different course groups

of the Field-Related parts. At the same time the bachelor's programme enables students to achieve wider qualifications within the elective block of courses. The offer of courses in the Common Basis part was enlarged especially by the Pedagogy-Psychology component and General Education component. The minimum number of Pedagogy-Psychology courses after completion of both degrees of study has almost doubled. The Academic Field-Related part of the bachelor's degree level has enhanced its mathematics base. Individual areas of mathematics are represented in a balanced way both in the form of compulsory and elective courses. This allows for a much more appropriate qualification profile of the bachelor's degree graduates. Looking back at the structure of the first stage in 2001, the resulting changes are positive. Unlike the bachelor's degree programme, the subsequent master's programme is unbalanced. Apart from the academic field-related didactics and the placement requirement for compulsory courses, students have no additional choice of other courses suitably amplifying their training to work as the math teachers. Despite offering geometry as one of the compulsory courses, planimetry, stereometry and software tools for secondary school level geometry are hopelessly missing. The compulsory elective course range is inundated with highly specialised mathematical courses that have little relation to the Field and Course-Related component and the Course and Didactic component of the teacher's professional training. The structure of this programme level is inappropriate and it must be significantly modified.

References

[1] Accreditation Commission Minutes, October 2000, p. 3 to 4 [online]. Available at http://www.akreditacnikomise.cz/cs/zapisy-ze-zasedani.html

[2] Accreditation Commission Minutes, April 2012, p. 75 [online]. Available at http://www.akreditacnikomise.cz/cs/zapisy-ze-zasedani.html

[3] Aula, the Journal for the University and Science Policy, volume 19, 01/2011, CSV, Praha. ISSN 1210-6658

[4] Complete Summary of the Accredited Study Programmes [online]. Available at http://www.msmt.cz/vzdelavani/akreditovane-studijni-programy-vysokych-skol-s-uvedenim-kodu-studijnich-programu-a-oboru

[5] P. Konečná: *Preparation on the teacher's profession before 10 years and today,* Proceedings of the International Conference KONOD 2011, p. 72 to 79. ISBN 978-80-8084-791-3

[6] Pregraduate Teacher Training Policy, Primary and Secondary Schools [online]. Available at http://www.msmt.cz/Files/AK/koncepce1.htm

[7] National qualification framework for tertiary education teaching area [online]. Available at http://qram.reformy-msmt.cz/kvalifikacni-ramec/oblasti-vzdelavani

Number Triangles
A Topic for School Investigation

Jan Kopka, George Feissner

Přírodovědecká fakulta UJEP v Ústí nad Labem
České mládeže 8, 40096 Ústí nad Labem, Czech republic
Jan.Kopka@ujep.cz

State University of NY, College at Cortland
Cortland, New York, 13045, USA
gffeisner@hotmail.com

Abstract. Solve following problem: Consider a square grid on which a path can be tracen along the horizontal and vertical lines. How many different routes are there from a given point S to a point F? Solution of this problem can lead to the famous Pascal's triangle. In this article we want to show how to get other interesting number triangles.

Keywords: Investigation, square grid, triangular grid, cubic grid, Fibonacci numbers, tribonacci numbers.

Classification: D50; Z30.

1 Introduction

In this article we show how to construct several interesting number triangles in a very natural way. We also investigate some of their properties. These constructions can serve as a vehicle for further mathematical investigations, since they have very simple beginnings but lead to interesting relationships.

2 Number Triangles

We begin by investigating shortest routes on a square grid[1].

Problem 1. Consider the square grid shown in Figure 1. We wish to trace paths from point S to each of the other points in the grid, moving only along the horizontal and vertical line segments. We wish to make our paths as short as possible. How many shortest paths are there from the point S to another point F?

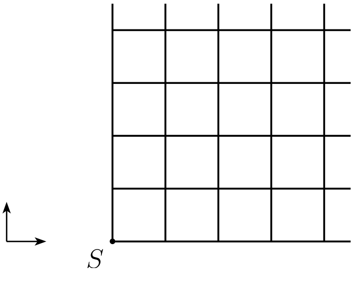


Figure 1

[1] See [2].

Solution. It is easy to see that we can move only horizontally to the right and vertically upwards. Any other movement would result in a needless lengthening of our path. Thus, for the points along the bottom edge of the grid, there is only one shortest path to each, namely the path along the bottom edge. Thus, we label each of these points with a 1. A similar argument allows us to label the points on the left edge of the grid in the same way. For a reason that will soon be clear, we will label the point S with a 1 also (We could also argue that there is only one shortest path from S to S; to follow it, we do nothing at all!).

For any point P in the grid not on an edge, the last step in a path must come from either the point immediately below it or the point immediately to its left. If there are k shortest paths to the first of these and l to the second, then any shortest path to P must consist of one of these, plus the addition of one horizontal or one vertical segment. Thus, the number of shortest paths to P is $k + l$. Remember, we are interested in the number of shortest paths, not their length!

Using this principle, we can systematically label each point in the grid with the number of shortest paths. We show this in Figure 2.

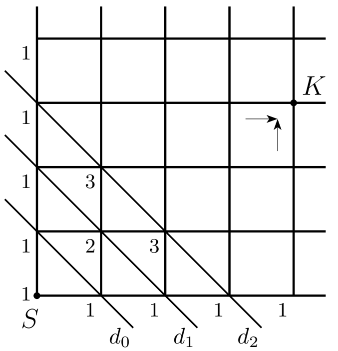


Figure 2

If we rotate Figure 1 so that the point S is at the top of the figure and the diagonals shown in Figure 2 become horizontal, we see that the labels form the famous Pascal's triangle!

Remark. Once we see this wonderful fact, we see that the label at each point can be calculated without knowing the labels of the points to the left and below; the number at the point with coordinates (m, n) is $(m + n)!/(m! \cdot n!)$.

The numbers $\binom{t}{s}$ are known as binomial coefficients and their properties have been extensively studied. Here, we will mention only an interesting relationship between them and the Fibonacci numbers.

First, we rewrite Pascal's triangle as shown in Figure 3, then add up the numbers along the diagonals shown. The sums, shown to the left, are the Fibonacci numbers 1, 1, 2, 3, 5, 8, ... Although we do not show it here, this fact can be proven using mathematical induction.

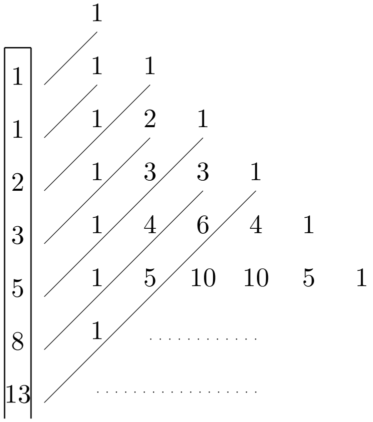


Figure 3

For our next problem, we add a dimension to the problem of counting shortest routes[2]. We now consider a cubic grid (grid goes on forever to the right, upward, and to the back), shown in Figure 4, and limit motion along the grid to moving to the right, upwards, and towards the back of the grid.

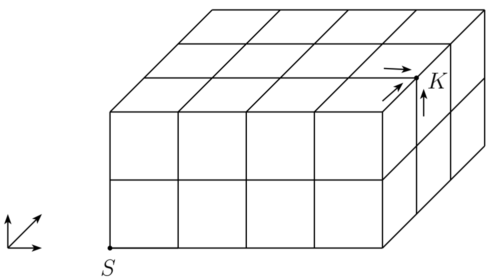


Figure 4

Problem 2. In the three-dimensional cubic grid, how many shortest paths are there from the point S to another point K?

Solution. We can reason as we did for the two-dimensional case. All the points along the edges of the grid, including S will have the label 1. The points in the bottom layer, the left-facing side, and the front will just be labeled as before with values from Pascal's triangle. For a point "inside" the grid, the last segment of any shortest path to it must come either from the point to its left, the point below it, or the point "in front" of it. Thus, we need only add the values at these three points.

[2]See [3].

If we follow this procedure, the triangles for the second and third layers are (see Figure 5):

1	1
2 2	3 3
3 6 3	6 12 6
4 12 12 4	10 30 30 10
5 20 30 20 5	15 60 90 60 15
....	....

Figure 5

What relationships are there between Pascal's triangle and these? Comparing Pascal's triangle with the triangle for the second layer, we see that the latter was obtained by multiplying the first row of Pascal's triangle by 1, the second row by 2, the third row by 3, and so on. The second triangle is obtained in the same manner, but now the multipliers are 1, 3, 6, 10, 15, ... These are the triangular numbers, which are found in the third diagonal of Pascal's triangle. If Pascal's triangle is rewritten as in Figure 3, then these numbers are in the third column.

Exercise. Using the process described in the solution to Problem 2, find the triangle for the fourth layer of the grid and show that fits the pattern shown above. This allows us to make a conjecture:

Conjecture. The triangle obtained from the nth layer of the three-dimensional grid is obtained by multiplying each row of Pascal's triangle by the numbers in its nth diagonal.

We'll now return to two dimensions and consider a variation of Problem 1. We replace the rectangular grid with a triangular one (it exteds infinitely in all direction away from S), shown in Figure 6. We'll consider two different rules for moving in the grid. In the first, labeled a), we allow only movement horizontally to the right or upward and to the right. In the second, labeled b), we also allow movement upward and to the left.

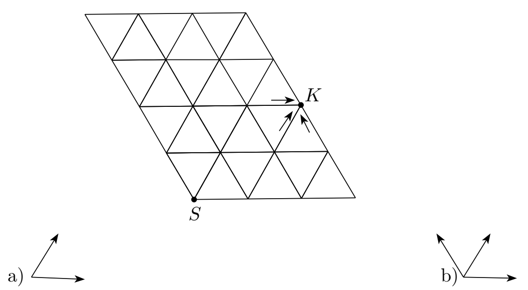


Figure 6

Problem 3. For each of the two rules of movement, how many shortest paths are there from S to each of the points in the grid?

Solution. We leave it as an exercise for the reader to show that if movement rule (a) is used, then again, we produce Pascal's triangle. For rule (b), again points on the boundaries and also S will get the label 1. As shown in Figure 6, every other point in the grid can be reached from three other points, so it is necessary to add the values at each of these three points to obtain the value at the given point. Following the procedure for the two previous problems, we obtain the triangle (see Figure 7):

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 3 & & 1 & & \\
 & 1 & & 5 & & 5 & & 1 & \\
1 & & 7 & & 13 & & 7 & & 1 \\
 & & & & \cdots\cdots & & & &
\end{array}
$$

Figure 7

As before, we rewrite the triangle and add the numbers along the indicated diagonals. The sums we obtain, 1, 1, 2, 4, 7, 13 ... are known as the tribonacci numbers. Beginning with the fourth, each is the sum of the three preceding it. This is shown in Figure 8.

	1					
1	1	1				
1	1	3	1			
2	1	5	5	1		
4	1	7	13	7	1	
7	1	9	25	25	9	1
13	1	....				
24	....					

Figure 8

We can restate Problem 1 and Problem 3b using “infinite chessboards”. Unlike the usual chessboard with eight rows and eight columns of squares, our infinite chessboards contain infinitely many rows and infinitely many columns, each with infinitely many squares. We picture them as beginning at the lower left and extending infinitely far to the right and upwards. Figure 9 shows such a chessboard. The “beginning” square is labeled A.

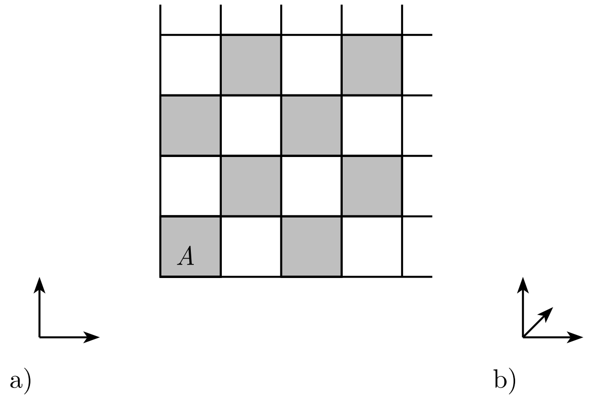


Figure 9

Problem 1’. A rook may move any number of squares in a straight line, either horizontally or vertically. It may not move diagonally. We place a rook in square A and ask the length of the shortest path the rook can take from that square to each of the others. Each square can then be labeled with the length of the shortest path leading to it.

Problem 3b’. We replace the rook on square A with a king, but we use new rules (not the traditional chess rules) for the king’s legal moves. He may move any number of squares in each move, but horizontally only from left to right, vertically only upwards, and diagonally only upwards and to the right. Again, for each square in Figure 9, we ask for the length of the shortest path from square A to that square and again label the square with the length of the shortest path.

For our last problem, we use a different infinite chessboard. As before, the board extends infinitely far upwards. But now, it extends infinitely both to the left and to the right. We illustrate this in Figure 10, labeling a starting square in the first row as square B.

Problem 4. We place a king on square B and again ask for the lengths of shortest paths to each of the other squares, but we change the king’s set of legal moves. Now, he may not move horizontally at all, but may move vertically upwards, diagonally up and to the right, and also diagonally up and to the left.

Solution. Our previous method of attack continues to work well. As was true for the triangular grid, some of the squares are not reachable; for instance, none of the

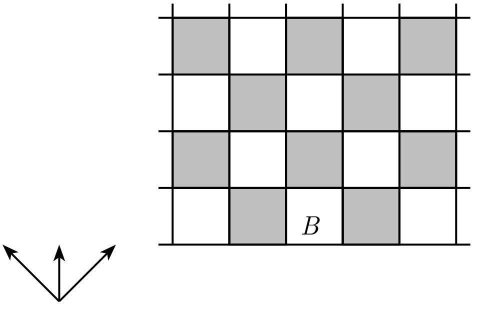


Figure 10

squares in the same horizontal row as square B can be reached! Again, for some of the reachable squares, three numbers must be added to obtain the length of the shortest path. For some, we must add two and for others, only one! Yet another exercise for the reader is to show that the resulting number triangle is the one we show in Figure 11.

				1				
			1	1	1			
		1	2	3	2	1		
	1	3	6	7	6	3	1	
1	4	10	16	19	16	10	4	1

. . . .

Figure 11

Interestingly enough, when we rewrite the triangle and add along the diagonals as before, we once more meet the tribonacci numbers! This is shown in Figure 12.

As the beginning of an independent investigation, we suggest the reader allow the queen in Problem 4 to move horizontally as well as vertically and diagonally.

3 Conclusion

All the problems shown here can be investigated by students at many levels of study. For younger students, the investigation may go no further than creating the number triangles. For those more advanced, conjectures can be made and tested

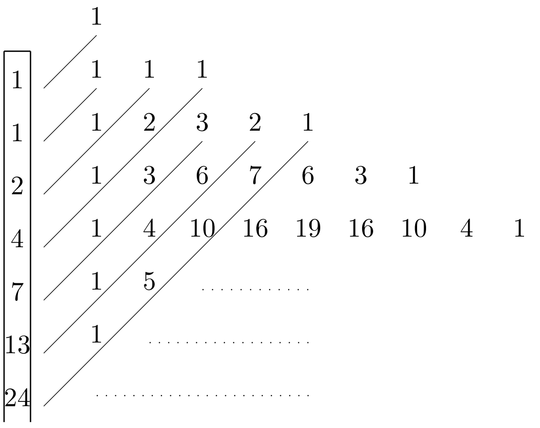


Figure 12

and for those familiar with mathematical proofs, attempts can be made to prove the conjectures. Here, a student has the opportunity not only to "do" mathematics, but to discover new mathematics.

Acknowledgements

The first author has been supported by project GAČR P407/12/1939.

References

[1] Cofman, J.: *What to solve?* New York, Oxford University Press, 1990.
[2] Kopka, J.: *Ako řešit matematické problémy.* Ružomberok, KU, 2010.
[3] Kopka, J.: *Výzkumný přístup při výuce matematiky.* Ústí n. L., UJEP, 2007.

Use of Linux and Open Source Software in education

Pavol Lajčiak

Faculty of Education, Catholic University in Ružomberok
Hrabovská cesta 1, 034 01 Ružomberok
Pavol.Lajciak@ku.sk

Abstract. This article points at possibilities of using Open Source Software in education. It focuses mainly on possibilities of creation and spreading media and multimedia contents, such as pictures, animations, sounds and movies using open source software. It points on possibilities of using software in conversion of media files. Next, it shows how created contents can be spreaded by portal LMS Moodle and RSS channels.

Keywords: aggregation, icecast, video, audio, mp3, ogg vorbis, video, ogg theora.

Classification: R69, R79; P20.

In the beginning, when God created the heavens and the earth, the earth was a formless wasteland, and darkness covered the abyss, while a mighty wind swept over the waters. (Genesis 1:1-2)

In the beginning was the Word, and the Word was with God, and the Word was God. He was in the beginning with God. (John 1:1-2)

Introduction

In the beginning of the computer era computers were huge, expensive and very slow. They were programmed in machine code. Programing and debugging of programs in machine code were tedious and difficult.

This situation led to searching possibilities, which could solve the state. It led to developing of operating systems.

Releasing of operating systems and evolution of high level programing languages such as C, C++, Java, Pascal and others, stimulated and simplified developing of computer applications and shorten their development time.

By establishment of FSF, GNU Project and releasing of GNU/Linux Operating System, new era began in computer industry and software development. There is enormous amount of code all over the internet, which can be reused. This is one of the advantages of FOSS (Free and Open Source Software). Nowadays, it is an era of FOSS. A lot of computer applications are covered by one of the Open Source Licenses, which allow software to be freely used, modified and spread.

There are many FOSS applications, which are designated for using in education. These applications are different in quality and price.

This article focuses on FOSS applications which can be used in education with focus on creation of education contents.

1 Short introduction to GNU/Linux and FOSS

In 1984 the GNU Project was launched to develop a complete Unix-like operating system which is free software: the GNU system. GNU's kernel isn't finished, so GNU is used with the Linux kernel.

Linux stands as name for the kernel of operating system developed by Linus Torvalds used in GNU/Linux operating system.

GNU/Linux is is used for naming of Unix-like operating system. GNU/Linux consists of the Linux kernel, open source tools - usually set of libraries, utilities from the GNU project and others tools, which encapsulate kernel of the operating system. [1]

There are many developer groups, which build their own GNU/Linux distribution. There are many of Open Source Licenses. On the web page of Open Source Initiative can be counted 72 licenses of open source software. Some of the most commonly used open source licenses are GNU GPL, GNU LGPL and BSD license. [2]

Advantages are scalability, security, stability, open source, constant updating and improvement, thousands of applications in distribution itself, no or reduced payments for software licenses. Disadvantages are incompatibility with some newer hardware devices, no drivers for some hardware/devices, not the same default graphics user interface in all distributions. GNU/Linux distributions can be categorized to many groups, for example by their purpose, packaging system type, way of installation, window manager (GNOME, KDE), etc. There are distributions for general use, Live CD distributions, distributions for scientists and engineers, distributions for education (Skolelinux, Freeduc, Edubuntu), distributions for sound, audio, music and movie creation, and distributions for another purpose. Big list of GNU/Linux distributions can be found on Distrowatch [3] or LWN [4] website.

2 Way of installing and using of GNU/Linux distribution

There are several methods how to install or try GNU/Linux.

First way of using GNU/Linux is running GNU/Linux Live distribution. GNU/ Linux Live distribution can be run directly from CD, DVD, USB stick or memory card. Some of these Live GNU/Linux Live distributions can be also installed on computer hard drive. GNU/Linux Live distribution can be downloaded from Internet and burned to CD or DVD media or ordered directly from the website for free or for some payment. When Live CD/DVD is used, content of you hard disk is not touched, but on the other hand system runs slower in comparison to system installed on hard disk. The best known GNU/Linux Live distributions are Knoppix, Ubuntu, Kubuntu, Xubuntu, Edubuntu.

Second way is an installation of GNU/Linux distribution on separate partition of hard disk. It is recommended to backup all your data before manipulation with partitions. Installation media can be obtained by downloading from internet as ISO image or ordered from website of distribution. GNU/Linux distribution can be installed following instructions from screen or user guide.

Thirdly it is installation to Windows partition through Wubi installer. Wubi is an officially supported Ubuntu installer for Windows users that can bring you to the GNU/Linux world with a single click. Wubi allows you to install and uninstall Ubuntu as any other Windows application, in a simple and safe way. [5]

The last way of installation is installation to Virtual machine or Network installation.

3 GNU/Linux Desktop and application

Nowadays computer with installed GNU/Linux distribution is an sufficient alternative or replacement for Microsoft Windows based computer. There are a lot of applications which can substitute proprietary application used with Windows operating system. Possible replacements of Windows software for GNU/Linux can be found on web site linuxrsp.ru [6] or osalt.com [7] websites.

When FOSS replacements are used, keep in mind these differences: applications have another GUI; don't contain some functions, which applications running in Windows contain; contain some functions, which application running in Windows don't contain. For same result, use of different consecution is needed, some consecutions can be learned from scratch.

In case of Windows applications that can be ran under GNU/Linux, Wine can be used. Wine is an Open Source implementation of the Microsoft Windows API. [8]

4 GNU/Linux success stories in education

There are many success stories of using GNU/Linux and FOSS in schools. Here are some choosen examples. It seems that motion to GNU/Linux and FOSS in good direction in using computer software.

Some 16,000 Italian students in the mountainous South Tyrol province of Bolzano in northern Italy found 2,460 classroom computers upgraded from Windows XP to GNU/Linux in school in September 2005. [9]

In October 2007 BBC announced that all computers in Russian schools are to be run on Linux in 2009 - which means they will not have to pay for a license for software, such as Microsoft's Windows. [10]

In April 2008 Switzerlands Department of Public Instruction informed about plan to run 9000 computers that will run only Ubuntu and free and open source software beginning of September 2008. Switzerlands Department of Public Instruction motto is Long Live Free Software. [11]

5 Educational applications

Many applications are there for different grades of schools. Free and open source educational software covers applications used from schools to universities

Here are some applications for a children aged from 2 to 12 years.

GCompris is an educational software suite comprising of 100 activities and more are being developed. It for children aged from 2 to 10 years. [12]

Tux Paint is a free drawing program for children 3 to 12 years of age. It contains an easy-to-use interface, an encouraging cartoon mascot who guides children as they use the program and activities that are accompanied by fun sound effects. [13]

Tux Typing and Tux Of Math Command is application tux4kids project framework. Tux Typing is not an typing tutor for children. Tux Typing provides an action-filled experience for kids who want to learn how to type. This program is is a fun way to let kids practice finding the keys on the keyboard as fast as possible! Tux, of Math Command is a math drill game starring Tux, the Linux Penguin. Lessons are included from simple number typing through addition, subtraction, multiplication, and division of positive and negative numbers. It is intended for kids from 4 to 10 years. [14]

6 Document preparation systems

An electronic and printed papers and documents can be created in many different ways. There are two basic approaches to make documents. First is WYSIWYG (Whay You See Is What You Get) approach, then the WYSIWYM (Whay You See

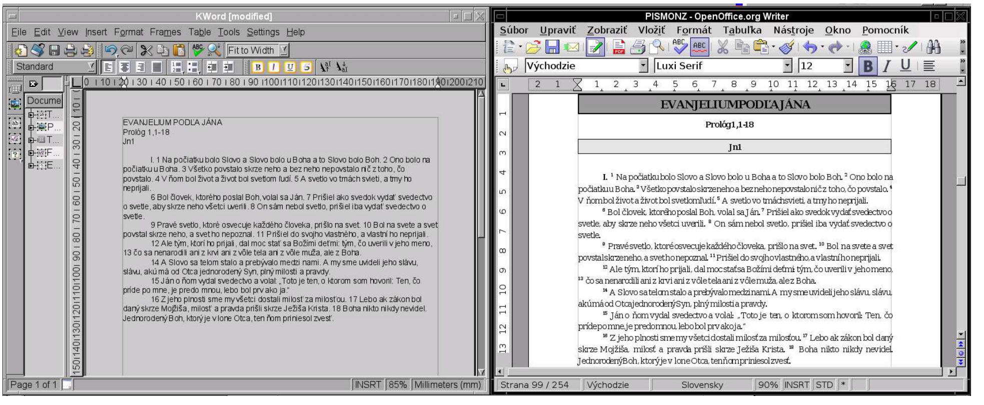


Figure 1: Kword and OpenOffice.org Writer

	OpenOffice.org	*KOffice*	*Microsoft Office*
Word processor	OO Writer	KWord	Word
Spreadsheet application	OO Calc	KSpread	Excel
Presentation program	OO Impress	KPresenter	Powerpoint
Database application	OOBase	Kexi	Access
Cost	$0	$0	$149 - $679

Table 1: Major office suite applications

Is What You Mean) approach, and last approach is combination of previous two approaches.

Applications from WYSIWYG group are Office Suites and DTP application. Applications from WYSIWYM group are computer document markup and formating languages and systems such as Docbook, TeX, LaTeX.

Office suite is commonly used application, usually installed on the most of computers. OpenOffice.org and Koffice suites are two major best known multiplatform FOSS office suites. Use of office suite is the most common way of electronic and printed documents creation.

OpenOffice.org suite consists of OpenOffice.org Writer - word processor, OpenOffice.org Calc - spreadsheet application, OpenOffice.org Impress - presentation program, OpenOffice.org Base - database management system, OpenOffice.org Draw - powerful graphics package and OpenOffice.org Math - component for mathematical equations. [15]

KOffice suite consists of productivity, creativity, management and supporting applications. Productivity applications are KWord - word processor; KSpread - spreadsheet application, KPresenter - presentation program and Kexi - an integrated environment for creating databases and database applications. Creativity applications are Kivio - a Visio-style flowcharting application, Karbon14 - a vector drawing application, Krita - a layered pixel image manipulation application. Management application is KPlato - an integrated project management and planning tool. Supporting applications are KChart - an integrated graph and chart drawing tool, KFormula - a powerful formula editor, Kugar - a tool for generating business quality reports. [16]

Second choice how electronic document can be prepared is to use DTP application. Scribus is an open-source DTP program that brings professional page layout to Linux/Unix, MacOS X, OS/2 and Windows desktops with a combination of ”press-ready” output and new approaches to page layout. [17]

Third choice to prepare electronic document is to use a typesetting system. LaTeX is a high-quality typesetting system; it includes features designed for the production of technical and scientific documentation. LaTeX is the de facto standard for the communication and publication of scientific documents. LaTeX is not a word processor. Instead, LaTeX encourages authors not to worry too much about the appearance of their documents but to concentrate on getting the right content. [18]

From all of the above mentioned applications can be produced PDF output, which is de facto standard for electronic documents sharing over the internet.

7 Graphics editors

There are a lot of open source graphic editors available. The bitmap graphic editors are GIMP, Krita, Cinepaint. The vector graphic editors are Inkscape, OpenOffice.org Draw.

GIMP is best known as an open source versatile bitmap graphics manipulation editor. It can be used for photo enhancement, digital retouching, etc. [19]

Krita as part of KOffice is a painting and image editing application. Krita contains both ease-of-use and fun features like guided painting and high-end features like support for 16 bit images. [20]

CinePaint is a deep paint image retouching tool that supports higher color fidelity than ordinary painting tools. It is recognized as a tool for motion picture artists. It is a more professional graphical editing application aimed at the movie industry. One of the major differences from Gimp is that CinePaint support up to 32-bit colors. [21]

Inkscape is an Open Source vector graphics editor, using the W3C standard Scalable Vector Graphics (SVG) file format. [22]

OpenOffice.org Draw is the powerful graphics package. From a quick sketch to a complex plan can be drawn. This one gives you the tools to communicate with graphics and diagrams. The maximum page size is 300cm by 300cm. This is powerful tool for technical or general posters, etc. [23]

8 Sound editors

Audacity is open-source cross-platform sound editor for Windows, Linux, Mac OS and others. It is available for free together with source code. It is released under open source license GNU GPL. It can be used commercially and non-commercially, without paying license fees. [24]

Audacity can open wav, aiff, au, mp2, mpa, mpg, mpeg, ogg, flac and mp3 files. To work with mp3 format, it needs LAME library to be installed. Audacity does not contain LAME itself. LAME can not be distributed by Audacity, because mp3 has license restrictions. It is localized to several languages, including Czech and Slovak languages.

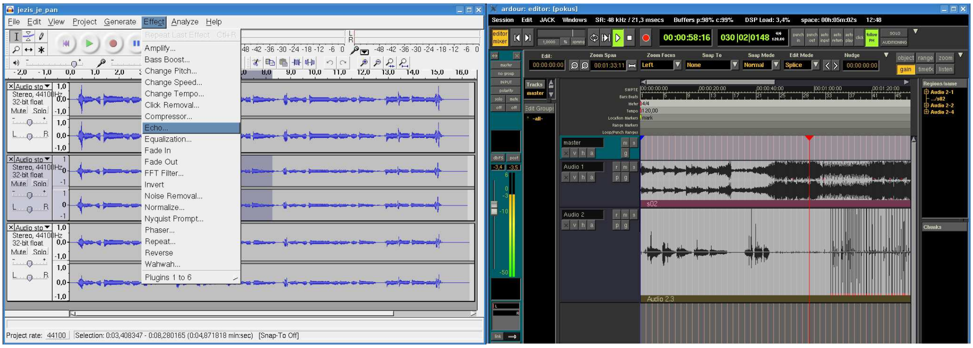


Figure 2: Audacity and Ardour

There are others sound editor Kwav, Krec, Krecord, Sound-recorder (command line), Rezound, Snd, Sweep, GNUsound, etc.

Ardour is a digital audio workstation (DAW). You can use it to record, edit and mix multi-track audio. You can produce your own CDs, mix video soundtracks, or just experiment with new ideas about music and sound.

Ardour capabilities include: multichannel recording, non-destructive editing with unlimited undo/redo, full automation support, a powerful mixer, unlimited tracks/bu sses/plug-ins, timecode synchronization, and hardware control from surfaces. Above all, Ardour strives to meet the needs of professional users. Ardour supports a wide range of audio-for-video features such as video-synced playback and pullup/pulldown sample rates. [25]

Another DAW is Traverso DAW. Traverso DAW is a complete solution from recording to CD mastering. This considerably lowers the learning curve, letting you get your audio processing work done faster. [26]

9 Applications for working with music score and midi

Rosegarden is a well-rounded audio and MIDI sequencer, score editor, and general-purpose music composition and editing environment. Rosegarden is an easy-to-learn, attractive application that runs on Linux, ideal for composers, musicians, music students, and small studio or home recording environments. [27]

MuseScore is a free cross platform WYSIWYG music notation program, licensed under GNU GPL. It allows easy and fast note entry with mouse, keyboard or MIDI. [28]

There are other midi and note score editors such as Noteedit, Canorus and Denemo.

10 Video editors

Kino is a non-linear DV editor for GNU/Linux. It features excellent integration with IEEE-1394 for capture, VTR control, and recording back to the camera. It captures video to disk in Raw DV and AVI format, in both type-1 DV and type-2 DV (separate audio stream) encodings. [29]

Avidemux is a free video editor designed for simple cutting, filtering and encoding tasks. It supports many file types, including AVI, DVD compatible MPEG files,

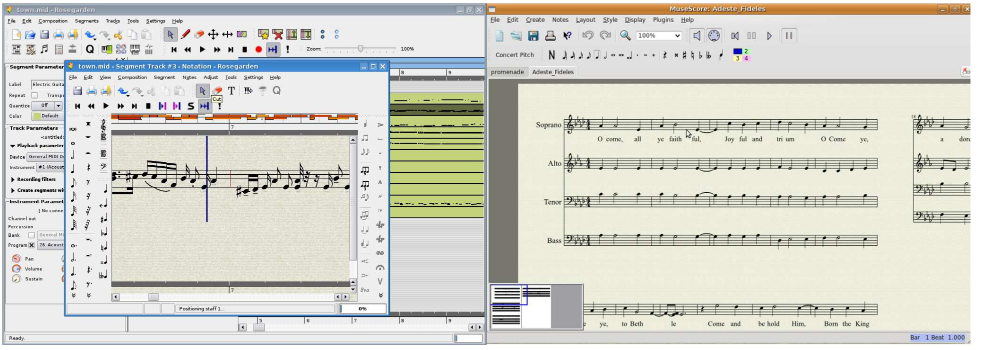


Figure 3: Rosegarden and MuseScore

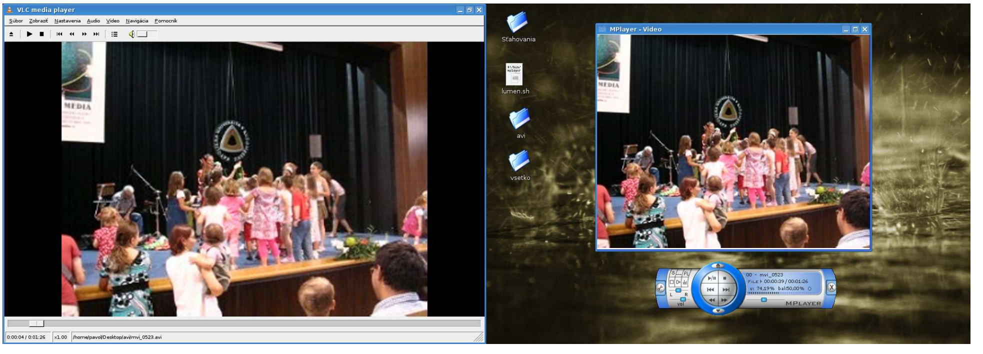


Figure 4: VLC and Mplayer

MP4 and ASF, using a variety of codecs. Tasks can be automated using projects, job queue and powerful scripting capabilities.

Avidemux is available for Linux, BSD, Mac OS X and Microsoft Windows under the GNU GPL license. The program was written from scratch by Mean, but code from other people and projects has been used as well. [30]

Cinelerra is the most advanced non-linear video editor and compositor for Linux. [31]

11 Movie players

VLC player is highly portable multimedia player. It can play various audio and video formats and streaming protocols. [32]

MPlayer is a movie player which runs on many systems and can play most of the movies with supported codecs. One of the great features of MPlayer is the wide range of supported output drivers and direct VESA displaying. Both players have subtitles support. [33]

Other movie players worth mentioning are Xine and Totem.

12 Open source multimedia formats ans tools

There are many audio and video formats. Some of these formats are proprietary and some are open source. Open source audio formats are Ogg Vorbis, Flac, Speex. Open source video format is Ogg Theora.

When working with audio or video you need conversion tools. Audio conversion tools are Lame, Sox, flac, Oggenc. Video conversion tools are ffmpeg and transcode.

SoX is the cross-platform command line Swiss Army knife of sound processing programs. SoX can convert various formats of computer audio files in to other formats. It can also apply various effects to these sound files. SoX can play and record audio files on many major platforms.

13 Streaming and podcasting

For streaming of audio you can use Icecast or for audio and video streaming can used VLC or VLS solution from VideoLAN project, which produces free and open source software for video, released under the GNU General Public License.

There are two possibilities for podcasting in Moodle. Starting from Moodle 1.6 discussion forum tool can be used. Discussion forum will be created and activated RSS feeds needs to be activated for the forum. Messages with media files as attachments will be posted. These will be delivered as podcasts in the RSS feed.

Ipodcast optional module can be used, which creates a specific podcasting activity type in Moodle. The advantage of this method is that it includes extra metadata designed to work well with Apple's iTunes software.

To hear or to see podcasts, software for receiving podcasts is needed. These FOOS applications can be used - Songbird, Rhythmbox, Amarok, Juice or Apple's iTunes. [34]

14 Other applications used in education

There are many other FOSS applications which can be used in education. Integrated Library Systems are Koha and Emilda. Content Management Systems like Joomla or Drupal are used to build portals or web sites. These applications allow to keep track of web site content such as text, photos, music, video or anything else. A major advantage of using CMS is that minimal programming skills are required. [35, 36]

Learning Management System (LMS) also known as Virtual Learning Environment (VLE), is web application that educators can use to create effective online learning sites. Moodle or ILIAS can be used to easily manage learning resources in an integrated system. [37, 38]

Conclusion

As you can see all over the world exists big amount of quality FOSS, which can by used from schools to universities. These programs are often multiplatform applications and can be used in Windows, GNU/Linux, Mac OS or another OS.

When FOSS is used, big amount of money can be saved. This money can be used to buy and maintain another resources in school. There is big community of users and developers, which can share information about using and installing open source application with you. Documentation are freely available on website of the individual projects. On other hand, there are also organisations, groups, which can grant commercial support for fee.

FOSS is more than a software. FOSS influences also philosophy, culture and life style. FOSS is a community driven software. Using of FOSS builds communities of users and developers and support cooperation, inspires people to be more open, tolerant and more pro community oriented. Finally, all of these applications can be used at no cost.

References

[1] Various Licenses and Comments about Them, GNU Project, Free Software Foundation (FSF), http://www.gnu.org/licenses/license-list.html
[2] Licenses by Name, Open Source Initiative, http://www.opensource.org/licenses/alphabetical
[3] Distrowatch, http://distrowatch.com
[4] Lwn.net, http://lwn.net/Distributions
[5] Wubi - Ubuntu Installer for Windows, http://wubi-installer.org
[6] Table of Equivalents/Replacements of windows programs, http://www.linuxrsp.ru/win-lin-soft/table-eng.html
[7] Open Source Alternative, http://www.osalt.com
[8] WineHQ - Run Windows applications on Linux, BSD and Mac OS X, http://www.winehq.org
[9] Linux displaces 2,460 Windows XP desktops in rural Italian schools, http://www.desktoplinux.com/news/NS2079183879.html
[10] Russian schools move to Linux, http://news.bbc.co.uk/1/hi/technology/7034828.stm
[11] 9000 PC in Swiss schools going Linux only, http://education.zdnet.com/?p=1615
[12] Free Education Software Gcompris, http://gcompris.net
[13] Tux Paint, http://www.tuxpaint.org
[14] Tux 4 Kids project, http://tux4kids.alioth.debian.org
[15] OpenOffice.org - The Free and Open Productivity Suite, http://www.openoffice.org
[16] The KOffice Project, http://www.koffice.org
[17] Scribus Open Source Desktop Publishing, http://www.scribus.net
[18] LaTeX project: LaTeX - A document preparation system , http://www.latex-project.org
[19] Gimp, http://www.gimp.org
[20] Krita, http://www.koffice.org/krita
[21] Cinepaint, http://www.cinepaint.org
[22] Inkscape, http://www.inkscape.org
[23] OpenOffice.org Draw - http://www.openoffice.org/product/draw.html
[24] Audacity, http://audacity.sourceforge.net
[25] Ardour, http://ardour.org
[26] Traverso DAW, http://traverso-daw.org
[27] Rosegarden: music software for Linux, http://www.rosegardenmusic.com
[28] Free music composition & notation software, http://www.musescore.org
[29] Kino: Linux Digital Video, http://www.kinodv.org
[30] Avidemux, http://fixounet.free.fr/avidemux
[31] Cinelerra CV, http://cinelerra.org
[32] VLC media player Overview, http://www.videolan.org/vlc
[33] MPlayer - The Movie Player, http://www.mplayerhq.hu

[34] Ipodcast module, MoodleDocs, http://docs.moodle.org/en/Ipodcast_module
[35] Joomla, http://www.joomla.org
[36] Drupal, http://drupal.org
[37] Moodle.org: open-source community-based tools for learning, http://www.moodle.org
[38] ILIAS Open Source LMS, http://www.ilias.de
[39] L. Piegl, W. Tiller: *The NURBS Book,* Monographs in Visual Communications. Springer, Berlin, 1997

The attitudes of participants in the process of solving mathematical tasks with insufficient data

Joanna Major

Institute of Mathematics
Pedagogical University in Kraków
Podchorążych 2, 30-084 Kraków
e-mail: jmajor@up.krakow.pl

Abstract. The subject of this paper are selected issues concerning the pupils' work on non-standard mathematical tasks. In particular, special observations were made on the pupils' attitudes in response to mathematical tasks.

Keywords: Didactic of mathematics, mathematical tasks with insufficient data

Classification: Primary 97D50; Secondary 97C30.

Mathematical knowledge and skills have been playing a more and more important role in our daily lives. At the same time, solving tasks has been the backbone of teaching mathematics at every level. Having in mind the goal of preparing students, in the course of the educational process, to living in the surrounding reality, one should emphasize the tasks which allow the pursuit of general objectives of mathematical instruction, i.e. those developing skills and attitudes necessary to a modern person, regardless of his or her field of activity. This goal may be reached by assigning non-standard problems to students[1].

Solving this type of tasks develops, among others, intellectual attitudes evidenced by logical, creative, and independent thinking as well as by overcoming difficulties, and can improve the ability to analyze the content of the task and understanding of the global structure of the task. Moreover, as Polya notes, in mathematics itself, *skills are more important than knowledge. What is skill in mathematics? It is the ability to solve problems, and not just typical tasks but those that require independent judgment, judgment ability, originality, creativity* (Polya, 1975).

Returning to the discussion related to our existence, it should be noted that in everyday situations we encounter a broad range of tasks, which are usually so to say ill-posed (have too much or too little information, or the data are contradictory). Mathematized, these tasks become the non-standard tasks. Thus, solving problems in classroom with either redundant or insufficient data provides a great instructional value. In the era of modern technology, obtaining information is not a problem. The main problem is to realize that in a given situation there is either excess or deficiency of information. Therefore, among the valuable skills are the ability to select information, verifying if the data are not contradictory, and, in case of tasks with insufficient information, considering various solutions or justifying that the data are insufficient to solve the task.

[1]By **standard tasks** we mean those that meet the following criteria:

1. there is sufficient information to obtain an unambiguous solution and at the same time there are no redundant data,
2. the content of the task does not lead to contradiction,
3. the content of the task is adequate, by which it is meant that the questions are closely connected to the data, the task relates to the real life, its conditions are sufficiently precise, and the task may be subject to arithmetic mathematization.

Negating any of these features leads to a **non-standard task** (see Gleichgewicht, 1988).

The guiding concept of this study was to answer the question about the impact of excess or deficiency of data on students solving word problems. The study aimed at finding answers, even partial, to the following questions:

- Are the students able to see the deficiency of data in the content of the task and state that the task has an ambiguous solution or indeed cannot be solved at all?
- Will the students use all the data or will they eliminate the redundant ones and use only the necessary information to solve the task?

The study has also attempted to answer additional questions:

- What are the difficulties encountered and errors made by the students when solving non-standard tasks?

In addition, the study attempted to determine the attitudes[2] the participants presented when solving tasks. It aimed also at determining the subjective feelings as to, among others, the easiness of the tasks or participants' attitudes to solving word problems. Partial answers to some of these questions can be found in the paper (Major, 2011).

The tasks used for the study (as research tools) were selected so as to be either realistic or purely mathematical, i.e. such as the students encounter the most frequently in classroom.

Due to the multitude of relevant threads related to the students' work on non-standard tasks, I will only present a few selected comments on the attitudes displayed by the students when solving tasks with insufficient information (deficient data) and having an ambiguous solution. The observations will be illustrated by the results of the research related to the work of students on purely mathematical tasks which, in terms of the mathematical level, do not go beyond the primary education (13-year-old students).

In this paper, I will discuss the results obtained by 35 third-grade high school students (16-year-olds), who in the opinion of mathematics teachers display high levels of knowledge and skills in mathematics.

Here is one of the five tasks solved by the participants.

1. Determine the perimeter of the rectangle, knowing that its area is equal to 18 cm^2 and the side lengths are natural numbers. Provide the calculation with the comment and give answers.

The analysis of the students' work, their subsequent comments to tasks and solutions, as well as interviews conducted with the students after they completed the questionnaire's tasks, brought me to distinguishing several types of students' attitudes related to the work on non-standard mathematical tasks. They are the following:

1. **Solving the problem on the example** i.e. the solving the problem using side lengths indicated by the participant

Students presenting this attitude (7 participants) indicated specific side lengths of the rectangle (one pair of natural numbers), determined the perimeter of the rectangle for the indicated side lengths, provided answers to the task, and thus

[2]The term attitude refers here to the approach of a person to life or to certain phenomena which expresses his or her opinions, also: the manner of conduct or behavior in relation to certain phenomena, events, or in relation to people (Słownik Języka Polskiego, 2000).

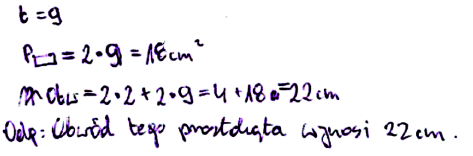


Figure 1: Solution 1

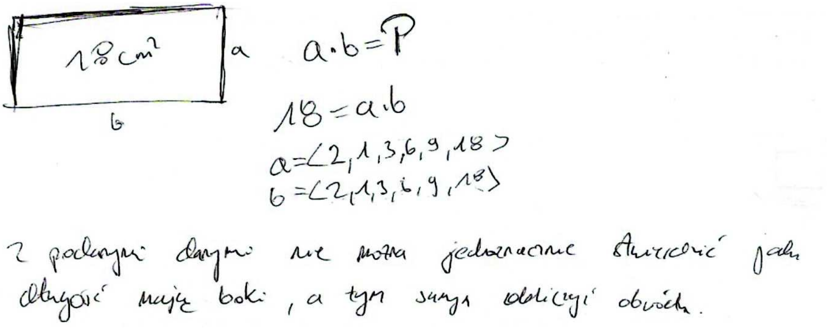


Figure 2: Solution 2

finished the work. The students felt no need to seek other pairs of numbers that would meet the conditions of the problem nor to argue that this was the only solution to the problem (see solution 1).

The analysis of the participants' comments given after the completion of the task allows a reasonable supposition that the participants found empirically (guessed) one of the possible solutions, whereas their conviction that there was a single solution became a barrier in further search for other possible solutions. This type of attitude results from the participants' experience acquired when solving standard mathematical tasks.

2. **Abandoning the task when stating the ambiguity**

Participants whose works were classified here (6 people) indicated a pair of numbers which satisfy the conditions of the task (i.e., found the side lengths of a rectangle with the area equal to 18 cm^2). Then they stopped working on the task without calculating the perimeter of the rectangle. It can be assumed that the reason for abandoning the task was the participants' conviction that they had made a mistake or misunderstood the task, or indeed they concluded that the task was ill-posed. This may imply that students are not familiar with the situation of considering several alternative cases, which would be contrary to the rules of conduct applied by participants when solving tasks. The participants stated that they *get different lengths of the sides, and it cannot be that way.* Some of them remarked e.g.: *I cannot unambiguously determine the lengths of the sides so I cannot calculate the perimeter of the rectangle.* The participants indicated also (see the solution 2) that the data are insufficient to obtain a clear solution.

3. **Solving the problem without providing an answer**

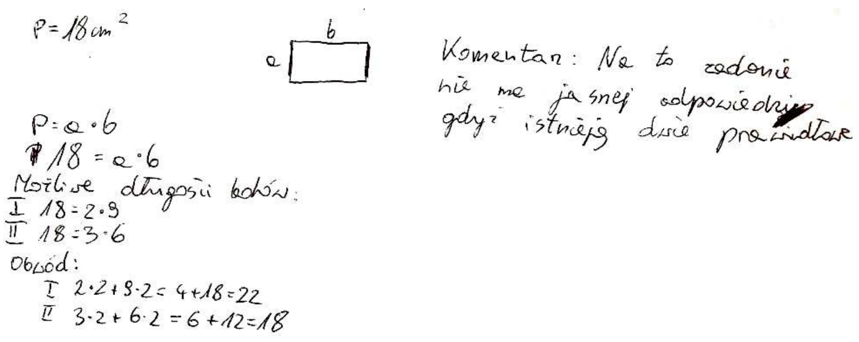


Figure 3: Solution 3

The participants displaying this attitude (12 people) gave different possible lengths of the rectangle's sides satisfying the conditions of the task. For the indicated lengths of the sides they calculated the perimeter of the rectangle. Then they gave no answer to the question of the task, stating only: *I don't know what to do, I received two solutions,* or (as in the solution 3) *For this task there is no clear answer, because there are two.*

4. **Realizing the ambiguity of the problem's solution and indicating a sample solution of the problem**

Participants displaying this attitude (6 people) gave different possible lengths of the rectangle's sides satisfying the conditions of the task. For the indicated lengths they calculated the perimeter the rectangle. Then they failed to provide a global answer to the task, and remarked only e.g.: *For the side lengths of 3 and 6 cm, the perimeter is 18 cm.* Thus, the answer consisted of a sample pair of numbers and the perimeter of the rectangle. Apparently, the students did not care to show all the solutions of this problem. They limited their answers to one sample solution. Participants' comments after solving the task indicate that they did not see anything worrying in the fact that, responding to the task, they provided only a sample answer neglecting all the others. At the same time the students did not provide any arguments for choosing the particular option (answer); hene, one can assume that the selection was random.

5. **Realizing the ambiguity of the problem's solution and attempting to give all (three) solutions**

The participants showing this attitude (4 students) gave different possible lengths of the rectangle's sides. For the indicated lengths they calculated the perimeter of the rectangle. The participants tried to cover all cases in their answers, i.e., different values of the perimeter for different lengths of the sides (see solutions 4 and 5).

It should be noted that three of the students provided comments to the solution. They wrote that *for different pairs of numbers different perimeters are obtained.* One of them also wrote that *the larger the difference between the lengths of the sides, the larger the perimeter.*

For the purpose of this study, it is important to present comments to the problems

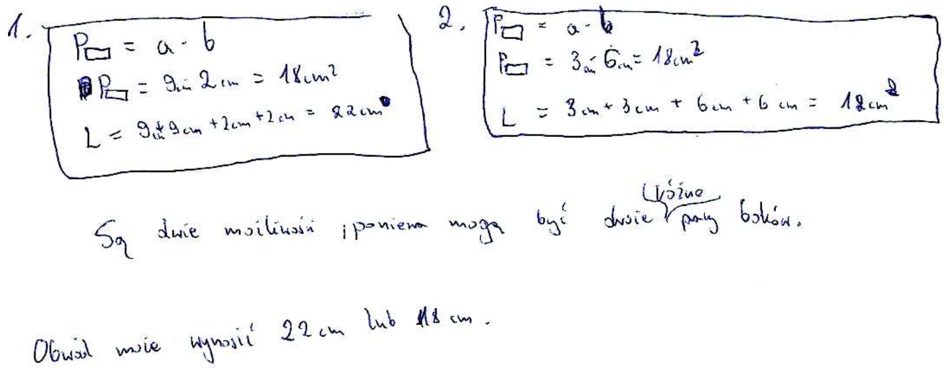


Figure 4: Solution 4

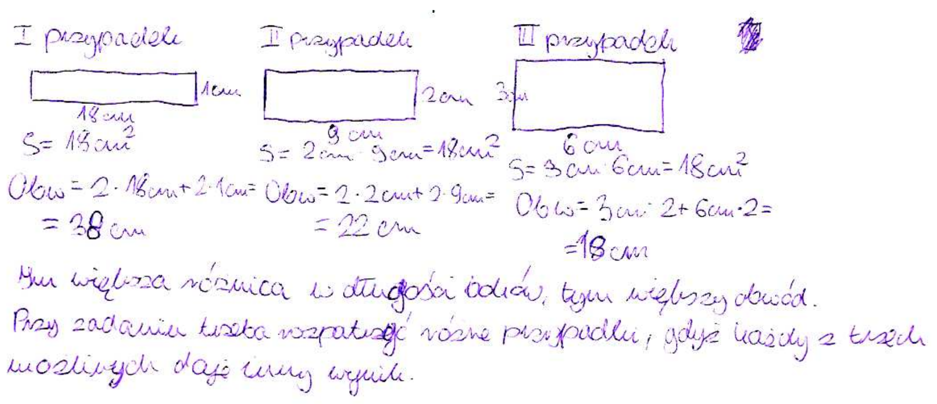


Figure 5: Solution 5

and their solutions recorded in the questionnaire as well as oral remarks provided by the participants displaying attitudes 3 and 4. These students, having indicated some solutions, prove unable to draw conclusions from their reasoning. The comments made by the participants should be noted:

There are various possible solutions and it's hard to determine which one is correct;

They all fit and they are always natural numbers and the area is 18 cm^2;

I got a few possibilities and I don't know what to do next;

This question has no clear answer because there are two correct answers;

I don't know what the answer is;

I don't know what to do, which answer is good;

I don't know which of the answers is good;

There are two solutions, and it cannot be that way;

This problem is impossible to solve, all the answers fit, they are natural numbers, and the problems must always have answers.

Some of the above statements indicate that the participants consider only a simple, affirmative sentence as the correct answer to the problem. The interviews clearly showed that the participants do not accept situations when several solutions to the task are possible and they do not possess tools allowing them to indicate (select) the "right" solution.

The presented attitudes of the students (who were perceived by teachers as mathematically talented) are disturbing for several reasons. A worryingly large number of young people attempt to solve the task automatically, schematically, almost without thinking, which would probably not be the case without teachers' contribution. It can be assumed that in the course of their mathematical instruction, the students were hardly ever given non-standard tasks. Moreover, in the opinion of teachers, students' skills are proved by solving tasks smoothly and by applying the procedures learnt. The results seem to indicate that the students' ability to solve mathematical problems has "broken down" into two partial skills:

- the attempt to identify the type of the task, and

- the attempt to recall the appropriate procedure to solve it.

The participants tried to solve the task using the "recalling strategy" (replicating a well-known scheme), and after it failed they remained helpless.

The results of the students' work on the tasks with insufficient data demonstrate that discovering a solution to the task often results in giving up the work. In many responses the students, having found some solutions to the tasks, did not formulate answers. It should be noted that most word problems presented in textbooks are convergent problems, which may create a limitation involving a false suggestion that there is always but one correct solution to the problem or task. Attitudes of this type result from the participants' beliefs regarding the characteristics of mathematical problems. Therefore, due the experience acquired in the course of their education, the participants developed wrong perceptions regarding the forms of solutions to the problems.

The participants displayed no need to develop different solutions and compare them. This may also result from a "cold" emotional attitude to the tasks solved in classroom and specifically to the task used in the study.

While solving the problem, the majority of the students failed to transfer everyday situations to purely mathematical ones. This may result from the specialist language used in classroom, which is different from that of everyday life.

It can also be assumed that the students treat mathematics statically, as a finite set of facts, algorithms, etc. In case of a cognitive conflict, the participants are not ready to look for (develop) subjectively new solutions (dynamic approach to mathematics).

This brings us to several questions. Why mathematically talented students, having mathematical knowledge and sufficient skills to solve rather difficult, but standard tasks, fail to solve a task of a moderate difficulty, although requiring a creative approach? Why do they fail to use their intellectual capacity? In the light of the presented study results, yet another question seems essential, i.e. how to change the students' attitudes?

References

[1] B. Gleichgewicht, *Arytmetyczne zadania tekstowe dla nauczycieli klas 1-4,* WSiP, Warszawa 1988

[2] J. Major, *Remarks concerning solving non-standard task by pupils,* Acta Mathematica 13, Nitra 2011, 143-148

[3] T. Madrzycki, *Psychologiczne prawidłowości kształtowania się postaw,* PZWS, Warszawa, 1977

[4] G. Polya, *Odkrycie matematyczne,* WNT, Warszawa, 1975

[5] *Słownik Języka Polskiego*, PWN, Warszawa, 2000

Once again on continuous random variable and geometrical probability space

Maciej Major

Institute of Mathematics
Pedagogical University in Kraków
Podchorążych 2, 30-084 Kraków
e-mail: mmajor@up.krakow.pl

Abstract. A random varible is one of important notions of the probability theory. In this work we will suggest a method of provingchosen theorems concerning continous random variables, with the use of the geometrical probability space.

Keywords: Didactic of mathematics, continous random variable, geometrical probability space

Classification: Primary 97B40; Secondary 60D05, 60G50.

1 Introduction

One of more important concepts of the probability calculus is a random variable. This notion is so important because it finds applications in many fields like mathematical statistics, economics, theory of insurance, physics and even psychology. Particularly important are so-called continuous random variables.

Mathematical apparatus applied for examining these random variables is rather complicated (the characteristic function, the Riemanna-Stieltjesa integral, the notion of the functional convolution).

In the works [2] [3] and [4] it is presented how possible it is to use geometrical probability space for examining continuous random variables. In this work we will suggest a method of proving chosen theorems concerning continuous random variables, with the use of the geometrical probability space.

2 Basic definitions

To begin with, we will quote definitions and theorems which are essential for the more distant part of this work.

Definition 1. Let $(\Omega, \mathcal{Z}, P)$ be any probability space. A random variable in this probability space is defined as any function X from the set Ω in $\mathbb{R}$, that satisfies the condition:

$$\{\omega \in \Omega : X(\omega) < t\} \in \mathcal{Z} \text{ for any } t \in \mathbb{R}. \tag{1}$$

Theorem 1. If X is a random variable in probability space $(\Omega, \mathcal{Z}, P)$ and $\mathcal{B}$ is the set of Borel subsets of a straight line and P_X is function defined by formula:

$$P_X(A) = P(\{\omega \in \Omega : X(\omega) \in A\}) \text{ for any } A \in \mathcal{B}, \tag{2}$$

then the triple $(\mathbb{R}, \mathcal{B}, P_X)$ is also the probability space.

Definition 2. Let X be a random variable in probability space $(\Omega, \mathcal{Z}, P)$. The function P_X defined by formula (2) on the set of Borel subsets of a straight line is called the probability generated on straight line by the random variable X or the distribution of the random variable X and the triple $(\mathbb{R}, \mathcal{B}, P_X)$ is called the probability space generated on straight line by the random variable X.

Definition 3. Function F_X defined on $\mathbb{R}$ by the formula

$$F_X(t) = P_X((-\infty, t)) \text{ for any } t \in \mathbb{R},$$

is called the cumulative distribution function also cumulative density function or briefly distribution function of random variable X.

Definition 4. Random variable X, for which exists such nonnegative and integralable function f_x defined on $\mathbb{R}$ that:

$$F_X(t) = \int_{-\infty}^{t} f_X(v)dv,$$

is called continuous and its distribution P_X is called continuous distribution. The function f_X is called the density of random variable X or the density of distribution P_X.

Definition 5. Let Ω be a subset of k-dimensional Euclidean space ($k = 1, 2, 3, \ldots$) having positive k-dimensional Lebesgue measure, let $\mathcal{Z}$ be a set of subsets of the Ω set having Lebesgue measure and let P be a function defined on $\mathcal{Z}$ by the formula:

$$P(A) = \frac{m_l(A)}{m_l(\Omega)}, \text{ where } m_l \text{ denotes the Lebesgue measure.} \tag{3}$$

The triple $(\Omega, \mathcal{Z}, P)$ is called the geometric probability space, and P is called geometric probability.

3 Main theorem

Theorem 2. Let X and Y be independent random variables of the density functions f_X and f_Y and let

$$\Omega^{XY} = \left\{(x, y, z) \in \mathbb{R}^3 \colon 0 \leq z \leq f_X(x)f_Y(y)\right\}.$$

The triple $(\Omega^{XY}, \mathcal{Z}, P)$ where $P(A) = m_l(A)$ and Z is a family of subsets of the set Ω having the Lebesque measure is a geometrical probability space. Functions $X(x, y, z) = x$ and $Y(x, y, z) = y$ are the continuous random variables of the density function f_X and f_Y.

Figure 1 presents the idea of the proof.

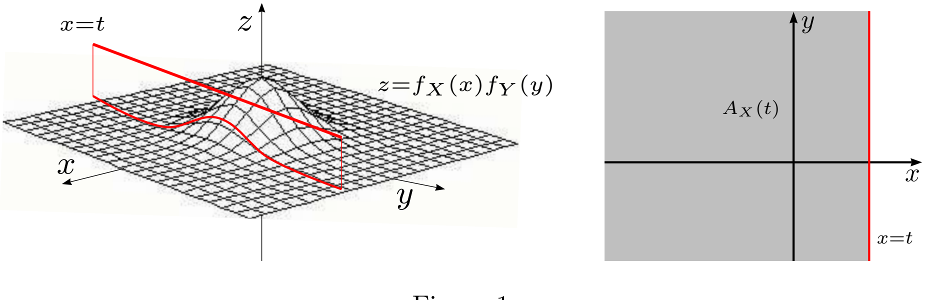


Figure 1

Proof.

Let

$$A_X(t) = \{(x,y,z) \in \Omega^{XY} \colon X(x,y,z) < t\} = \{(x,y,z) \in \Omega^{XY} \colon x < t\}$$

for $t \in \mathbb{R}$.

Let us notice, that

$$\begin{aligned} F_X(t) &= P(X < t) \\ &= P(\{(x,y,z) \in \Omega^{XY} \colon x < t\}) \\ &= m_l(A_X(t)) \\ &= \iint_{(-\infty,t)\times(-\infty\times\infty)} f_X(x) f_Y(y) dx dy \\ &= \int_{-\infty}^{t} \int_{-\infty}^{\infty} f_X(x) f_Y(y) dy dx \\ &= \int_{-\infty}^{t} f_X(x) \int_{-\infty}^{\infty} f_Y(y) dy dx \\ &= \int_{-\infty}^{t} f_X(x) dx. \end{aligned}$$

It results that the density function of random variable X is given by the function $f_X(x)$.

Acting by analogy it is possible to state, that function $Y(x,y,z) = y$ is the continuous random variable of the density function f_Y.

Theorem 3. Let X and Y be independent random variables of normal distributions and of parameters μ_X, σ_X and μ_Y, σ_Y and let $Z = X + Y$. Then Z is a random variable of normal distribution and of parameters and $\mu_Z = \mu_X + \mu_Y$ and $\sigma_Z = \sqrt{\sigma_X^2 + \sigma_Y^2}$.

The paper [4] proved this theorem when $\mu = 0$ and $\sigma = 1$. In this paper we present the proof in the general case.

Proof.

Let

$$\Omega^{XY} = \{(x,y,z) \in \mathbb{R}^3 \colon 0 \le z \le f_X(x) \cdot f_Y(y)\},$$

where

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\frac{(x-\mu_X)^2}{\sigma_X^2}}, \quad f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{1}{2}\frac{(y-\mu_Y)^2}{\sigma_Y^2}}.$$

Let functions $X(x,y,z) = x$ and $Y(x,y,z) = y$. They are the continuous random variables on the geometrical probability spaces $(\Omega^{XY}, \mathcal{Z}, P)$.

From theorem 2 we have

$$X \colon N(\mu_X, \sigma_X), \; Y \colon N(\mu_Y, \sigma_Y).$$

Let $Z := X + Y$. It is easy to see that $Z(x,y,z) = x + y$.

Let

$$A_Z(t) = \{\omega \in \Omega^{XY} : Z(\omega) < t\} = \{(x, y, z) \in \Omega^{XY} : y < -x + t\}.$$

We have (see Figure 2)

$$\begin{aligned} F_Z(t) &= P(Z < t) \\ &= m_L(A_Z(t)) \\ &= \iint_{\{(x,y)\in\mathbb{R}^2:\ x+y<t\}} \frac{1}{2\pi\sigma_X\sigma_Y} e^{-\frac{1}{2}\left(\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right)} dxdy. \end{aligned}$$

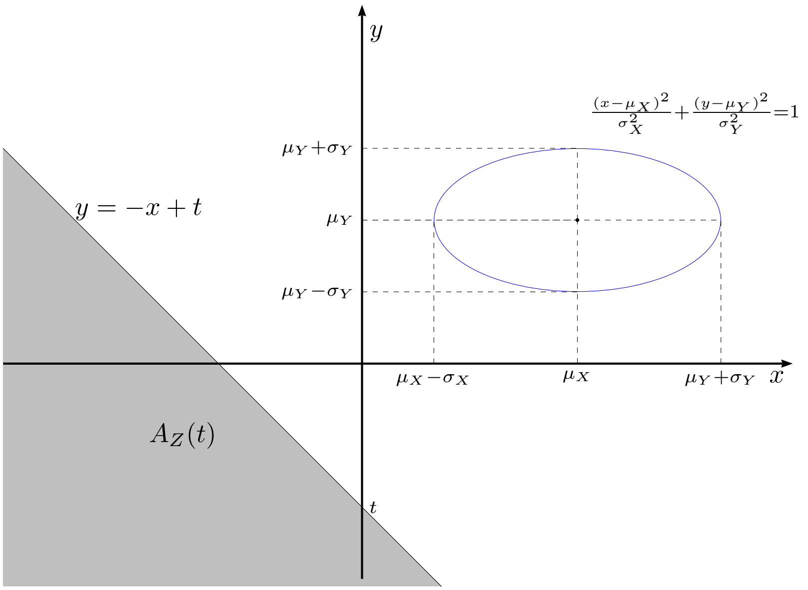


Figure 2

If

$$u = \frac{x-\mu_X}{\sigma_X}, \quad v = \frac{y-\mu_Y}{\sigma_Y},$$

then

$$x = \sigma_X u + \mu_X, \quad y = \sigma_Y v + \mu_Y \quad \text{and clearly} \quad J = \sigma_X\sigma_Y.$$

We have (see Figure 3)

$$\begin{aligned} F_Z(t) &= \iint_{\{(u,v)\in\mathbb{R}^2:\ v<\frac{\sigma_X}{\sigma_Y}u+\frac{t-\mu_X-\mu_y}{\sigma_Y}\}} \frac{1}{2\pi\sigma_X\sigma_Y} e^{-\frac{1}{2}\left(u^2+v^2\right)}\sigma_X\sigma_Y dudv \\ &= \iint_{\{(u,v)\in\mathbb{R}^2:\ v<\frac{\sigma_X}{\sigma_Y}u+\frac{t-\mu_X-\mu_y}{\sigma_Y}\}} \frac{1}{2\pi} e^{-\frac{1}{2}\left(u^2+v^2\right)} dudv. \end{aligned}$$

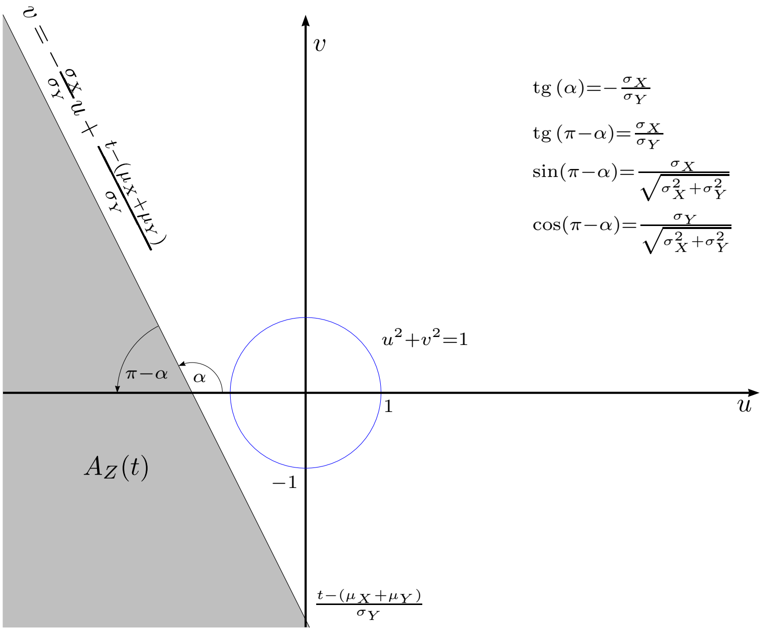


Figure 3

If

$$\beta=\pi-\alpha \quad \text{and} \quad p=u\cos\beta-v\sin\beta, \quad \text{and} \quad q=u\sin\beta+v\cos\beta,$$

then

$$u=p\cos\beta+q\sin\beta, \quad v=q\cos\beta-p\sin\beta \quad \text{and} \quad J=1.$$

We have (see Figure 4)

$$\begin{aligned}
F_Z(t) &= \iint_{\left\{(p,q)\in\mathbb{R}^2:\ q<\frac{t-(\mu_X+\mu_Y)}{\sqrt{\sigma_X^2+\sigma_Y^2}}\right\}} \frac{1}{2\pi} e^{-\frac{1}{2}\left(p^2+q^2\right)} dp dq \\
&= \int_{-\infty}^{\frac{t-(\mu_X+\mu_Y)}{\sqrt{\sigma_X^2+\sigma_Y^2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{q^2}{2}} \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{p^2}{2}} dp\right) dq \\
&= \int_{-\infty}^{\frac{t-(\mu_X+\mu_Y)}{\sqrt{\sigma_X^2+\sigma_Y^2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{q^2}{2}} dq \\
&= \Phi\left(\frac{t-(\mu_X+\mu_Y)}{\sqrt{\sigma_X^2+\sigma_Y^2}}\right).
\end{aligned}$$

Let us notice, that

$$f_z(t)=F_Z'(t)=\Phi'\left(\frac{t-(\mu_X+\mu_Y)}{\sqrt{\sigma_X^2+\sigma_Y^2}}\right)=\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\sigma_X^2+\sigma_Y^2}}e^{-\frac{(t-(\mu_X+\mu_Y))^2}{2(\sigma_X^2+\sigma_Y^2)}}.$$

Finally we have

$$Z\colon N\left(\mu_x+\mu_y,\sqrt{\sigma_X^2+\sigma_Y^2}\right).$$

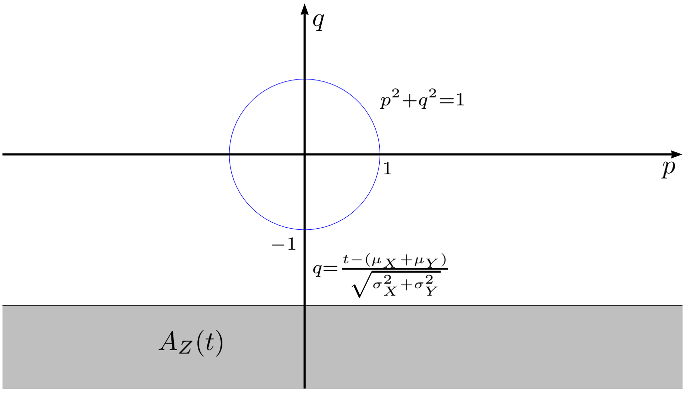


Figure 4

4 Conclusion

It is worthwhile to solve the problems presented above with the students of math teachers' training majors. Proving the theorems with elementary methods with the use of mathematical analysis and geometrical methods, allows to consider the elements of probability calculus in a different (than traditional) way. Quite elementary tools make the presented problems simple to understand and to operative use.

References

[1] Mirosaw Krzyśko, *Wykady z teorii prawdopodobieństwa*, Wydawnictwo Naukowo-Techniczne, Warszawa 2000.

[2] Maciej Major, *Geometric probability space as the tool of examining continuous random variables*, Acta Mathematica 11, Nitra 2008, 147-152.

[3] Maciej Major, *Continuous random variable*, Mathematica II, Ružomberok 2009, 41-49.

[4] Maciej Major, *Continuous random variable and geometric probability space*, Acta Mathematica 12, Nitra 2009, 165-170.

[5] Adam Płocki, *Stochastyka dla nauczyciela, Rachunek prawdopodobieństwa, kombinatoryka i statystyka matematyczna jako nauka 'in statu nascendi'*, Płock 2002.

The educational proposal for a pupil with the special ability for mathematics

Dagmar Malinová

Department of Mathematics and ICT Education, Faculty of Education,
J. E. Purkyně University in Ústí nad Labem
Hoření 13, 400 96, Ústí nad Labem
dagmar.malinova@ujep.cz

Abstract. In Czech schools, there is the predomination of the integrated form of education for intellect gifted pupils. Teachers are trying to find the right and suitable content and form of the educational proposal. Characters of mathematical tasks, especially its difficulty, are seemed to be very important in this matter. The article brings findings from the empirical probe from the work with the pupil of a primary school, who shows the noticeable talent just in the mathematical sphere.

Keywords: gifted pupil, mathematics, mathematical giftedness, primary school, mathematical task.

Classification: D12.

1 Introduction

With cognitively gifted, who have the potential for the outstanding results, but they are not reaching such results so far, we meet especially among preschool and young-age school children. In this case, we speak about so called latent talent. With talented ones, who already reach outstanding results and they demonstrate the unique talent we often meet at the upper primary school by the children of elder school age. [3]

In the article, there are some experiences of the work with an abnormally gifted pupil, who does the high performance in the field of mathematics on the first grade of the basic school, mentioned. It is the pupil of the fifth grade of lower primary school who was diagnosed by experts as an abnormally talented for mathematics. The pupil proves the above-average results in the field of the music as well. In other fields and subjects his performances are average, comparable with children of the same age.

In contrast to bright children has the above-average gifted pupil much more needs for getting the knowledge. As was found out during the semi-annual research, this pupil is able to solve more difficult exercises than expected.

2 The thematic content of the tasks

In Czech schools integrated form of education of intellectually gifted dominates. Teachers are looking for them the appropriate content and form of the educational choice; they often express the fear that thanks to the special care, gifted kids will be "too forward" compared to peers, they will know the curriculum of higher grades, in higher grades they will be bored and the teacher does not know what to teach them. Teachers of integrated gifted pupils require the emphasis on education enrichment versus acceleration. It is therefore appropriate to select topics for this age group that are not at the focus of school mathematics. For gifted children in primary school in mathematics we proved the following topics:

- Number systems,

- combinatorics,
- logical reasoning, tasks type zebra,
- coding, working with symbols, variable relationships between variables,
- Diofantine equations,
- Dirichlet's principle,
- the division of the geometric shape on parts,
- modeling – the construction of polyhedras.

3 The difficulty of tasks and exercises

Too easy or too difficult tasks have their risks. If tasks are too easy, or if too many similar tasks are given to a talented pupil, the pupil quickly loses the interest in the work or responds negatively. If the task is too difficult for the pupil, the teacher is expected to help. Learn more about how to balance the individual work of the pupil and support of the teacher in the book *How to solve it* by Polya (1945).

How does the teacher regard the difficulty? Teachers often regard the difficulty of tasks for the pupil by themselves. If they evaluate it themselves as tough, they expect that it will be difficult for a gifted pupil too. For example Diofant's tasks that teachers and students of teaching described as challenging enough were very easy for the gifted pupil. He solved them by insight with no need to record any calculation. Teachers in an effort to provide the pupil a more challenging task often choose common tasks from textbooks for the higher grade.

What is the difficulty of tasks for a pupil? When a gifted pupil is educated, when acceleration form of education is applied, he learns mathematics at higher grade or visits only certain lessons of mathematics in a higher grade, or his teacher in the base classroom gives him tasks from textbooks for higher grades. It happens that the pupil does not know the necessary mathematical concepts, relationships and symbolic writings, because he has never met them before. Although a gifted pupil acquires new knowledge very quickly, in this form of education happens that he does not have the necessary knowledge at the moment and if a teacher does not provide it to him immediately, the task becomes unsolvable for a pupil. It was observed that in such situations, a pupil is experiencing a psychological discomfort. For example the pupil visiting lessons of mathematics at the higher grade said that he does not like geometry. On closer inspection, it was found out that the only problem is that he does not understand symbolic writings of relationships between geometric shapes, he just cannot "translate" that.

Here is an assignation of a quite easy task which was classified (by the pupil) as insoluble in this version: *We divide cakes onto plates. If we give 6 cakes on a plate, two cakes are left. In case we will put 8 cakes on a plate, one plate will be empty. How many plates and cakes do we have?* The sample of an empirical probe with a gifted pupil in mathematics. [2]

The pupil drew a picture (Fig. 1) and he has helplessly read the assignation of the task repeatedly.

The task is not expressed precisely. Intellectually gifted pupils tend to perfectionism. [4], [5]

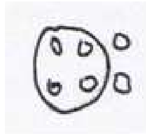


Figure 1

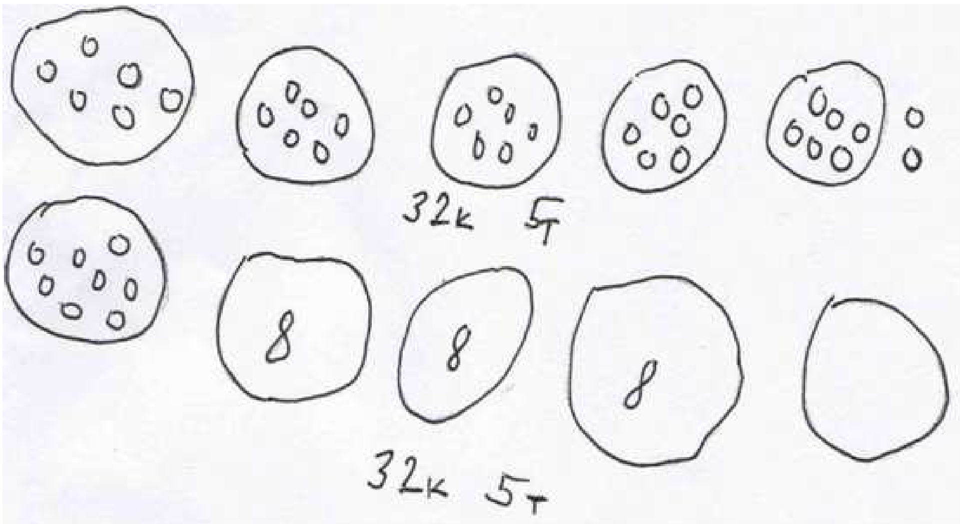


Figure 2

If a mathematical task is not formulated carefully it becomes unsolvable for gifted pupils, or it changes into "What did the teacher mean?", "What does the teacher want to hear?".

After the reformulation of an assignment the pupil solved the task with no difficulties, he helped himself by graphical illustration (Fig. 2).

A gifted student may perceive a task as a difficult in these cases, however, he does not perceive it negatively: to solve the task more difficult mental operations are needed, the task has a complex character, the task requires an interaction of various previous knowledge, algorithms in a new context or the task contains more new features.

4 Examples of tasks

In the thematic unit *Number systems* the pupil with no difficulties expressed natural numbers in different number systems, he transferred numbers from the decimal system to the z-adical system and vice versa, he added, multiplied and subtracted numbers expressed in z-adical system. There are examples of other tasks, solved by a gifted pupil:

Determine a base z, for which the following applies:

1. $123_z + 123_z = 312_z$ (Answer: $z = 4$)
2. $203_z + 111_z = 1021_z$ (Answer: no solution)
3. $9_z + 8\mathrm{B}_z = 1034_z$ (Answer: $z = 16$)
4. $12_z + 3_z = 15_z$ (Answer: $z > 5$, $z \in \{6, 7, 8, 9, 10\}$)

Tasks 1. – 3. were solved by the pupil quickly and with no difficulties. The task 4. was the one he was deeply interested in. First, he found out that z is greater than 5 and the task has infinitely many solutions. Reminded by the teacher he further dealt with the task and determined $z \in \{6, 7, 8, 9, 10\}$. To verbally justify, why the base cannot be greater than 10, was found as difficult for the pupil. The depth and way of pupil's thinking could be described by mathematical problems and questions formulated by himself. There are, for example, some questions that gifted pupil has asked while dealing with the non decimal number systems:

- How can I sign in another (non decimal) numeral system a rational, or an irrational number?
- What if we use as the basis the number π?

Two tasks with combinatorial elements:

4.1 Task – The Construction of cubes

John built the building made of cubes. On Fig. 3 you can see how does the building look from the front (frontal view) in Fig. 4 you can see, as it looks from above (plan view). [1]

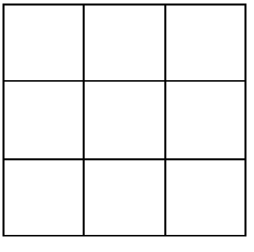


Figure 3

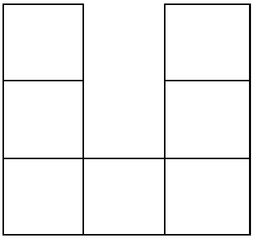


Figure 4

(a) *Draw a side view of the building (as it appears from the side).*

(b) *Determine the smallest and largest possible number of cubes for this construction.*

(c) *How many different structures can be built?*

While solving this task, the pupil has used cubes for building the constructions (Fig. 5) and he also has made some illustrations.

Sub-Task (b) has a convergent character, sub-tasks (a) and (c) have a divergent character and to find their solutions the pupil also used his knowledge of combinatorics. The pupil has solved task (b) first using cube as a model. When solving task (a) he modeled a side view separately, he modeled only some solutions; he has recorded all solutions in writing. He thought of unchangeable bottom layer of cubes (Fig. 6), the construction of other layers he has also understood as a variations of numbers of the cubes in particular columns. For example, the situation in Figure 6 was expressed in numbers (2, 0, 0). Task (c) was solved in a more difficult way. Here

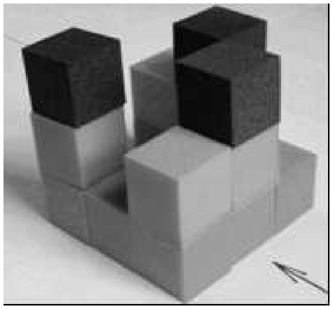


Figure 5

the pupil only some constructions has modeled, then he solved the task while using the plan view, where he wrote into the boxes the numbers expressing the number of cubes in the column (Fig. 7). During solving of the task he has used solution of the task (a) and his knowledge of how to solve the combinatorial tasks.

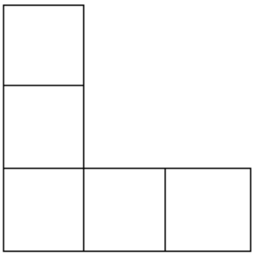


Figure 6 Side view

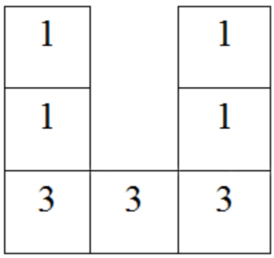


Figure 7 Plan view

4.2 Task – Polygons (using the Polydron set)

Preparatory task: *Put together all different polygons of four equilateral triangles.*

The pupil has recorded the solution (Fig. 8). After finding out that the task has infinitely many solutions, he finished.

Put together all different models of polygons using 4 parts at the shape of equilateral triangle. Find the number of options.

(a) *All parts are of the same colour (4 colours).*

(b) *Each part has a different colour.*

(c) *Parts can be any colour.*

For putting together pieces of the Polydron[1] set, were thought only possibilities where one whole part of triangle suits to the side of different triangle. Tasks (a) and (b) were not difficult for the pupil, he solved them himself without manipulation with models, without difficulties, using a solution of the previous task. The task (c) was much more difficult. The pupil used the coloured pieces of the Polydron set for

[1]More information about the Polydron construction set on www.polydron.co.uk.

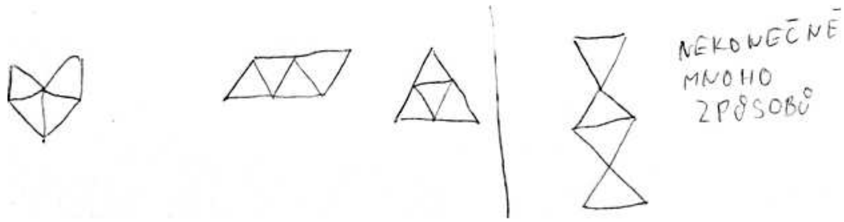


Figure 8

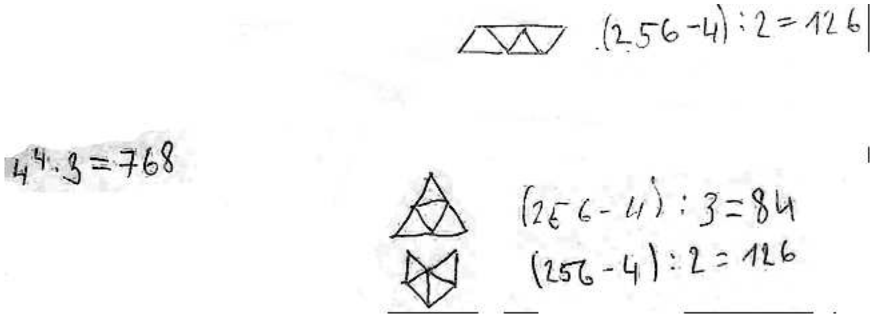


Figure 9 Sample of pupil's solution of the task (c)

modelling; after being notified that a solution in his first solutions some models are counted more than once, because he didn't consider the conformity of shapes after their rotation or overturning.

5 Conclusion

The pupil of the first degree of elementary school solved with passion and without significant difficulties a number of tasks, including some of the collection of tasks for leaving exam of high schools. A key role is represented by teacher in this case, he selects appropriate tasks and provide adequate support to a pupil to make him use his potential to gain as much experience from an independent work and to develop his talent. The pupil worked economically, he often wrote directly the result of the task. It was necessary to keep him to record a written solution, at least in a brief form. Also, verbal comments and explanations were requested. An interesting attitude was recorded while the pupil was manipulating with objects while solving some of those tasks. Because the pupil has mathematical talent, compared to peers and classmates, he is capable of high levels of abstraction, creating mental images and manipulating them, it was expected that the direct manipulation will not be used by the pupil, or it will rejected. The pupil used the possibility of direct manipulation in the case that the task was sufficiently demanding and of non-routine character, manipulating objects allowed insight into the sub-step solution. One of the aims of the article was also to highlight the fact that even primary school pupil, who has an extraordinary talent for mathematics, is able to solve much more difficult tasks than teachers expect.

References

[1] BECHEANU, M. Identifikace žáků nadaných na matematiku. In: *Objevování, motivace a podpora matematických talentů na evropských školách*. Praha: MATH.EU Project, 2006, p. 100 – 147. ISBN 9963-634-31-1.

[2] BLAŽKOVÁ, R. & VAŇUROVÁ, M. Charakteristika nadaného žáka s poruchou učení z hlediska matematických úloh. In: *Acta Universitatis Palackianae Olomucensis: Matematika 4*. Olomouc: Univerzita Palackého, 2010, p. 57 – 61. ISBN 978-80-244-2511-5.

[3] HŘÍBKOVÁ, L. *Mimořádně nadané děti ve škole a v rodině*. Ústí nad Labem: Univerzita J. E. Purkyně, 2010. ISBN 978-80-7414-319-9.

[4] LAZNIBATOVÁ, J. *Nadané diet'a: jeho vývin, vzdelávanie a podporovanie*. Bratislava: Iris, 2007. ISBN 80-890-1853-X.

[5] MACHŮ, E. Intellectually Gifted Pupils and Educational Problems Connected to Their Gift Development in Common Integrated Classroom. In: *Talented learner*. Brno: Masarykova univerzita, 2011, p. 96 – 106. ISBN 978-80-210-5701-2.

[6] POLYA, G. *How to solve it: a new aspect of mathematical method*. Bronx, NY: Ishi Press International, 2009. ISBN 48-718-7830-9.

Financial Literacy Standard at Elementary Schools in the Czech Republic

David Nocar

Palacký University in Olomouc, Faculty of Education, Department of Mathematics
Žižkovo náměstí 5, 771 40 Olomouc, Czech Republic
david.nocar@upol.cz

Abstract. The contribution presents a current problem of contemporary society, which is financial literacy and related skills. It presents the definition of financial literacy according to the document "National Strategy for Financial Education" and highlights current possibilities to implement financial literacy according to the "Framework Education Programme for Basic Education" and the standard of financial literacy for primary and secondary level of elementary schools defined in the document "Building financial literacy at elementary and secondary schools".

Keywords: financial literacy, financial literacy strategy documents in the Czech Republic

Classification: M30.

1 Introduction

First we should ask ourselves why we should implement financial literacy education at primary schools. The answer is very simple: the sooner, the better. Recent surveys (2007[1], 2010[2]) of financial literacy of Czech adult population show very unsatisfactory results. So the reason to implement financial and economic literacy into school education is the effort to increase the level of financial literacy of Czech population. The surveys show that we cannot rely on parents only. The parent generation of present pupils is missing the needed competences because they could not gain them during their own education since this field is completely new in the education process. A person gets basic competences and literacy needed for life in society mainly at school. Within the compulsory education the elementary schools represent a united starting level for all pupils. Therefore our pupils should get basic competences from all areas of life there, which means a certain level of financial literacy too. Pupils of elementary schools (6 - 15 years old) get in touch with economic aspects of life (parents go to work - they earn money; they go shopping with parents - they learn they cannot afford everything, expenses are limited by family/ home budget; they are influenced by sales in shops or by TV advertisements at home; they are aware of private ownership; they try to manage their own finances etc.).

2 Defining the term financial literacy, strategic documents

The Work group for financial education in accordance with the Ministry of Finance of the Czech Republic, Ministry of Industry and Trade of the Czech Republic, Ministry of education of the Czech Republic, Czech National Bank and consumers' and

[1]FINANČNÍ GRAMOTNOST – Kvantitativní výzkum pro MF ČR. [online]. STEM/MARK, July 2007. [cit. 2012-02-29]. Available on World Wide Web: http://www.mfcr.cz/cps/rde/xbcr/mfcr/Vyzkum_Financni_gramotnost_2007_STEM_pdf.pdf.

[2]FINANČNÍ GRAMOTNOST V ČR – Kvantitativní výzkum – Finanční gramotnost obyvatel ČR. [online]. STEM/MARK, May - October 2010. [cit. 2012-02-29]. Available on World Wide Web: http://www.mfcr.cz/cps/rde/xbcr/mfcr/EXT_-_PPT_zaverecna_zprava_mereni_FG_-_plne_zneni.pdf.

professional associations defined financial literacy in the Czech Republic in 2006. The definition is adopted from the document "National Strategy for Financial Education", which is the central document for financial education in the Czech Republic. [1]

"Financial literacy is a set of knowledge, skills, and attitudes of a citizen necessary for ensuring his/her own financial well-being and the financial well-being of his/her family within the present society, and for his/her active involvement in the market of financial products and services. A financially literate citizen is familiar with the issues of money and prices, and is able to manage his/her personal and/or family budget responsibly, including the management of financial assets and liabilities in consideration of changing life situations." [2]

At the beginning we mentioned economic not financial aspects of life. So we need to explain the difference between financial literacy and economic literacy.

Financial literacy is a specialized part of the more general economic literacy, which also includes e.g. The ability to secure an income, consider the consequences of personal decisions in respect of current and future income, knowledge of the labour market, ability to make decisions on expenditures etc. [1]

We have to consider whether all activities and tasks implemented into the educational process belong to the above defined financial literacy or whether they overlap into other parts of economic literacy. Most authors creating their study materials and tasks do not strictly follow the financial literacy but the join them into so called financial and economic literacy (e. g. a book published by M. Skořepa and E. Skořepová in cooperation with Czech national Bank[3]). The following part of our paper deals only with the issues of financial literacy.

Financial literacy is a broad set of competences, interconnected with many other types of literacy e.g. *numeric literacy - the ability to get, use, interpret and communicate mathematical information and ideas so that they can use them to cope with mathematical demands of many situations in their life* [3] *a interconnected numeracy - the ability to know and understand the role of mathematics in the world, make well supported decisions and get to the heart of mathematics so that it would fulfil their life needs as a creative, interested and thinking citizen* [4].

As more and more people get into debts nowadays, they get into troubles, they do not know how to solve them, what are the consequences, where to look for help and what should they avoid not getting into bigger trouble, it is important a certain level of *legal literacy - knowledge of legal system, rights and duties, and facilities where to look for help* [1]. And we could go on finding interconnection with other types of literacy. It implies that the system of financial literacy is complicated but also important for a responsible attitude to life.

Financial literacy is sometimes described as a set of competences needed for an effective management of personal and family finances. From this viewpoint it contains three components, sometimes called partial specific literacy:

- *money literacy* - capabilities necessary for the management of cash, noncash resources, financial transactions, instruments intended for this purpose (man-

[3]SKOŘEPA, M. - SKOŘEPOVÁ, E. Finanční a ekonomická gramotnost pro základní školy a víceletá gymnázia - manuál pro učitele. Praha : Nakladatelství Scientia, 2008. ISBN 978-80-86960-40-1.

agement of current and savings account, use of cash point, internet and phone banking, debit and credit cards, or so called sealed tender)

- *price literacy* - capabilities necessary to understand price mechanisms (price of goods, taxes, sales, price of financial instruments and services including fees and interest rates)
- *budget literacy* - capabilities necessary to manage personal/family budget (knowledge of personal/family incomes and expenses, setting goals and priorities according to the personal/ family budget) and capability to manage different life situations from the financial perspective. Budget literacy also includes management of financial assets (deposits, investments, insurance) a liabilities (loans, mortgage, leasing). See [1]

The definition of financial literacy introduced in this paper represents the basis of a document of the Ministry of Finance of the Czech Republic, the Ministry of Education of the Czech Republic, the Ministry of Industry and Trade of the Czech Republic *"Building financial literacy at elementary and secondary schools"* created on the basis of government resolution no. 1594 from 7^{th} December 2005, which set standards of financial literacy and defined target level of financial literacy for elementary and secondary education.

3 Financial literacy and Framework Education Programme for Elementary Education.

We cannot find the term financial literacy in the Framework Education Programme. Nevertheless this strategic document offers enough space and potential to implement and develop this issue. The introduction of this document says:

"Elementary education should help pupils form and gradually develop their key competencies and provide them with the firm foundations of general education focusing mainly on situations close to real life and on practical conduct." [5]

It is evident that competencies arising from financial literacy can be considered a part of general knowledge focusing mainly on situations close to real life and on practical conduct. How are the key competencies defined in the Framework Education Programme for Elementary Education?

"Key competencies are a set of knowledge, skills, abilities, attitudes and values which are important for the personal development of an individual and for the individual's participation in society. Their selection and conception are based on values generally accepted in society as well as commonly held ideas on which competencies of the individual contribute to his/her education, contented and successful life and to strengthening the functions of civil society. The purpose and aim of education are to equip all pupils with a set of key competencies on the level which is attainable for them and thus to prepare them for their further education and their participation in society. The acquisition of key competencies is a long-term and complicated process, which begins with preschool education, continues during elementary and secondary education and is gradually refined in subsequent life. While the level of key competencies acquired by the pupils by the end of their elementary education cannot yet be regarded as final, the key competencies acquired form a nonnegligible basis for the pupils' lifelong learning and their start in life and in the work process.

Key competencies are not isolated phenomena; they are variously interconnected, multifunctional, have an interdisciplinary nature and can always be acquired as a result of the overall educational process. The entire educational content and all of the activities taking place at school must therefore be aimed at and contribute to forming and developing these competencies.

The educational content of the FEP EE conceives the subject matter as a means of mastering the expected activity-based outcomes, which gradually link and create preconditions for an effective and complex utilisation of the acquired abilities and skills on the level of key competencies.

At the elementary stage of education, the following are considered as key competencies: learning competency, problem-solving competency, communication competency, social and personal competency, civic competency, professional competency.

The following are descriptions of what a pupil should be able to do in terms of the competency in question by the end of his/her elementary education." [5]

If we focus on competencies arising from financial literacy we can find them in every single sentence cited above. We cannot imagine personal development of an individual and individual's participation in society without financial literacy. Generally accepted values of our society include the competencies of this literacy contributing to satisfactory and successful life and to strengthening functions of our society. Financial literacy just as all other competencies cannot be considered finished at elementary schools, but elementary education has to form the basis for further education levels, entry in the real life and working process. We do not have to discuss blending and multifunction effect of financial literacy with other competencies. Connection with other types of literacy and their competencies was mentioned in previous chapter of this paper. The aim of our comparison of the idea of financial literacy was to show how often it can be found in such a strategic document of elementary education. So the question arises whether the document should not be updated and completed with topics of financial literacy as soon as possible, as it is in the Framework Education Programme for Secondary General Education[4] and in the Framework Education Programme for Secondary Technical and Vocational Schools e.g. branch of study $75-31-J/01$ Teaching for educational assistants[5] etc.

Actualization of Framework Education Programme for Elementary Education can be expected because at page 10 we can read the following:

"The FEP EE is an open document, which will be innovated at certain intervals based on the changing needs of society, the teachers' experience with the SEP as well as the changing needs and interests of the pupils." [5]

How to implement materials supporting financial literacy into School Education Programme in compliance with Framework Education programme for Elementary Education.

[4]Rámcový vzdělávací program pro gymnázia. [online]. Praha: Výzkumný ústav pedagogický v Praze, 2007. [cit. 2012-02-29]. Available on http://www.vuppraha.cz/wp-content/uploads/2009/12/RVPG–2007–07_final.pdf. ISBN 978-80-87000-11-3.

[5]Rámcový vzdělávací program pro obor $75-31-J/01$ Pedagogika pro asistenty ve školství. [online]. Praha: Národní ústav odborného vzdělávání, 2009. [cit. 2012-02-29]. Available on: http://zpd.nuov.cz/RVP_3_vlna/RVP%207531J01%20Pedagogika%20pro%20asistenty%20ve˜%20 skolstvi.pdf.

Even though we have already stated that there have not been defined specific areas for developing financial literacy in the Framework Education Programme, it does not mean the elementary schools cannot include this topic into their School Education Programmes. The area of financial literacy is closely interconnected with other types of literacy and it can be incorporated in different educational areas as they are defined in the Framework Education Programme: Mathematics and its applications, Information and communication Technologies, Man and his World, Man and Society, Man and World of Work or Man and Health. We can find some space for financial literacy almost in every area and we can proceed the same way through the cross-curricular subjects.

"Framework Education Programme opens possibilities to work with financial literacy mainly in parts which discuss practical life, critical approach to information, using logic, mathematical and empirical methods in solving problems, responsibility for their decisions, activities needed to put a business into practice. The competence to solve problems is considered to be the most predominant for developing financial literacy. On the general level the key competences include everything that arises from the definition of financial literacy." [1]

Teachers or rather schools have already implemented relevant topics in their School Education Programmes and so they fulfil the standards of financial literacy defined in the document *Financial Education System for primary and secondary schools*. So far it is a voluntary activity it will be included in the Framework Education Programme during next revision. However teachers and schools are aware of the importance of fulfilling the standards and they were challenged to do so by Martin Krejza - section manager for education (Ministry of Education of the Czech Republic) at the beginning of 2011. In the letter Notification to headmasters of primary and secondary schools about implementing financial literacy into PISA 2012 and further recommendations he asks the headmasters to implement the topic of financial literacy into instruction as much as possible because the Ministry of Education perceives financial education as an important part of primary education of pupils.

At the beginning he reminds them of the international survey PISA 2012 and upcoming testing of 15 year old pupils. This survey is prevailed with mathematics and one of the alternative components - financial literacy.

4 Financial literacy standard for elementary schools

Financial literacy standards are defined in the document *Financial Education System for primary and secondary schools*. This document states:

"Specific financial literacy standards proceed from the definition of financial literacy. They define the ideal level of financial literacy for different target groups or the target state of financial education of different levels of education. These standards are then implemented into the Framework Education Programme (for Elementary and Secondary Schools) or they are the ground for creating individual education programmes and activities aiming to development and improvement of financial literacy of pupils and adults (in further education), or specific target groups." [6]

To fulfil financial literacy standards at elementary schools, to grasp the topic correctly and to introduce it to pupils suitably teachers need to gain some experience, recommendations, materials and examples from this are. The Ministry of

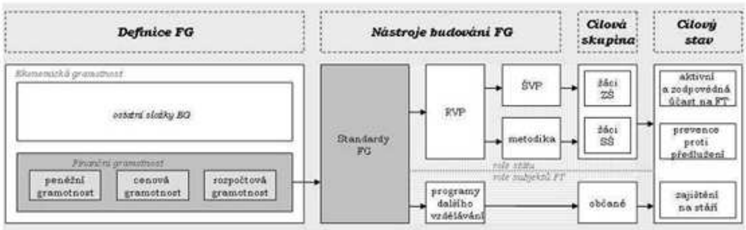


Figure 1: Role of financial literacy standards [6]

Education, the Research Institute of Education, the National Institution of Technical and Vocational Education and the Ministry of Finance agreed to participate on the preparation of the sources. We can expect other entities to prepare and publish other materials, which can become an inspiration for schools in writing their School Education Programme. *Financial literacy comprehensibly and without obstacles* by Jiří Brabec is an example of such materials[6].

Financial literacy standards for primary schools	
Money	
Content	Results
▪ *cash and cashless forms of money* ▪ *payment methods* ▪ *banks as administrators of money*	▪ *uses money in everyday situations* ▪ *estimates and checks price of shopping and money returned*
Household budget management	
Content	Results
▪ *budget, household income and expenses* ▪ *entitlement to claim a refund*	▪ *using examples shows why it is not possible to make all wanted purchases*
Financial products	
Content	Results
▪ *savings* ▪ *loans*	▪ *explains why one should save, when to obtain a loan, and how to repay debt*

Figure 2: Financial literacy standards for primary schools, See [6]

Till the revision of Framework Education Programme for Elementary Schools the document Financial Education System for primary and secondary schools recommends schools to implement financial literacy standards in the instruction voluntarily. Following figure introduces these standards for individual education stages.

There is no recommendation which cross-curricular subject does this standard belong to. The reason is simple the topic of financial literacy interconnects almost

[6]BRABEC, J. Finanční gramotnost srozumitelně a bez překážek. Plzeň: ABC Finančního vzdělávání, o.p.s., 2011. ISBN 978-80-905057-0-4.

Financial literacy standards for secondary schools	
Money	
Content	Results
▪ *making payments* ▪ *price creation* ▪ *inflation*	▪ *uses the most common payment instruments, exchanges money using a foreign exchange list* ▪ *determines price as the sum of costs, profit, and VAT* ▪ *explains how price differs according to customer, location, period ...* ▪ *explains the fundamentals of inflation and impact on incomes*
Household budget management	
Content	Results
▪ *household budget* ▪ *basic rights of customers*	▪ *differentiates between regular and irregular income and expenses and based on this makes a household budget* ▪ *explains the principle of a balanced, deficit and surplus budget* ▪ *using an example explains how to exercise consumer rights (when buying goods and services, including financial market products)*
Financial products	
Content	Results
▪ *bank services, active and passive operations* ▪ *financial products for investments* ▪ *insurance* ▪ *interest*	▪ *explains the use of debit and credit cards and their limits* ▪ *compares the most usual ways of investing free money (consumption, savings, investments)* ▪ *proposes how to resolve a budget deficit (loans, instalment sale, leasing)* ▪ *explains the significance of interest paid and excepted* ▪ *introduces the most common types of insurance and suggests when to use them*

Figure 3: Financial literacy standards for secondary schools, See [6]

all cross-curricular subjects defined in the Framework Education Programme for Elementary Education.

5 Conclusion

Financial literacy is a very topical and discussed issue nowadays. Everyone is influenced by a lot of factors (loans, different insurance, advertisements etc.) every day they cannot cope with properly because they are not oriented in the problematic adequately. They miss necessary competences not to fall into debts. Surveys from 2007 and 2010 show low level of financial literacy of Czech adults. To reduce the deficit of these competences we need to develop financial literacy starting in the school age. Although the area has not been implemented in the Framework so far there is plenty of space to do so. The ground for developing financial literacy of pupils forms the document Financial Education system for primary and secondary schools, which defines financial literacy standards for primary schools. We can expect this version of standard is not final because only reality shows us which areas are suitable or not, which need to be added. Competences arising from financial literacy are essential to life and it is needed to develop them on every level of education system. It is said “money is not everything” and it is important that money is not the most important thing in life but if people cannot manage their money they can get into trouble which can negatively influence the life quality.

Acknowledgements

The author has been supported by the project FRVŠ no. 274/2012 Innovation of the subject Economic and financial mathematics, Department of Mathematics, Faculty of Education, Palacky University in Olomouc, main researcher: David Nocar.

References

[1] J. Altmanová: *Gramotnosti ve vzdělávání - příručka pro učitele.* Výzkumný ústav pedagogický v Praze, Praha 2010. 978-80-87000-41-0

[2] *Národní strategie finančního vzdělávání – 2010. MF ČR.* http://www.mfcr.cz /cps/rde/xbcr/mfcr/Narodni_strategie_Financniho_vzdelavani_MF2010.pdf, 2010.

[3] *Numerická gramotnost dospělých: Konceptuální rámec pro PIAAC. PIAAC – Mezinárodní výzkum dospělých.* http://www.piaac. cz/attach/matematicka_ gramotnost.pdf, 2009

[4] *Koncepce matematické gramotnosti ve výzkumu PISA 2003. Ústav pro informace ve vzdělávání.* http://www.uiv.cz//soubor/673, 2004

[5] *Rámcový vzdělávací program pro základní vzdělávání. Metodický portál RVP.* http://www.vuppraha.cz/wp–content/uploads/2009/12/RVPZV–pomucka–ucitelum.pdf, 2010

[6] *Systém budování finanční gramotnosti na základních a středních školách.* http://www.msmt.cz/uploads/soubory/zakladni/SP_SBFG_2007_web.pdf, 2007

A method of deciphering the complete date of a given event

Grażyna Rygał, Arkadiusz Bryll

Pedagogical Department, Jan Długosz University in Częstochowa
Waszyngtona 4/8, 42-200 Częstochowa, Poland
g.rygal@op.pl

The Faculty of Management, Częstochowa University of Technology
Dąbrowskiego 69, 42-200 Częstochowa, Poland

Abstract. In the paper a method of calculating the day and the month of an event is presented. This method is also generalized to calculate the year of this event based on Diophantine equations.

Keywords: Diophantine equation, integers, successive substitutions.

Classification: 00A35.

1 Introduction

The known method of determining the day and the month of a given event based on Diophantine equations [2][1] can be generalized to take into account also the year of this event (a historical fact, birthday, marriage, death, etc.).

The day and the month of a given event can be determined as follows: we ask somebody to multiply the day of the event by 12 and the month of the event by 31 and to add the obtained products. The result of this addition is sufficient to determine the date of the event (without the year).

2 Deciphering the day and the month of an event

Let x be the day of the event and y be the month of the event. Let the sum of the products $12x$ and $31y$ be equal to c. Then we should solve the Diophantine equation

$$12x + 31y = c \tag{1}$$

under the assumption that

$$1 \le x \le 31, \quad 1 \le y \le 12. \tag{2}$$

Equation (1) can be obtained as follows. If (x_0, y_0) is the date of the event, then the condition

$$\frac{x}{31} + \frac{y}{12} = \frac{x_0}{31} + \frac{y_0}{12} \quad \text{or} \quad 12x + 31y = 12x_0 + 31y_0$$

should be fulfilled. Assuming $12x_0 + 31y_0 = c$ leads to equation (1).

Solving the Diophantine equations one can use the following theorems [1]:

Theorem 1. The equation

$$ax + by = c, \tag{3}$$

[1]Diophantus was a Greek mathematician who lived in the 3rd century BC.
A Diophantine equation (or indefinite equation) is called a linear equation with integer coefficients including at least two unknows. Solutions of the Diophantine equation are expressed in terms of integers.

where a, b, c are integers, has the solution in terms of integers if and only if the numbers a and b are coprime.

Theorem 2. If a pair of integers (x_0, y_0) is a solution of equation (3), then all the solutions of this equation can be obtained as follows:

$$\begin{cases} x = x_0 - bt \\ y = y_0 + at \end{cases} \quad (t \text{ is an arbitrary integer}). \tag{4}$$

As the numbers 12 and 31 are coprime, equation (1) has the following solutions in terms of integers

$$\begin{cases} x = x_0 - 31t \\ y = y_0 + 12t \end{cases} \quad (t \text{ is an arbitrary integer}), \tag{5}$$

and only one solution fulfills the condition (2).

Solutions of equation (1) can be determined by the method of successive substitutions and separation of the integer parts [1].

Example 1. It is known that if the day of the birth of the given person is multiplied by 12 and the month of the birth is multiplied by 31, then the sum of the products is equal to 410. Determine the date of the birth. Was it a leap-year?
The following equation

$$12x + 31y = 410 \tag{6}$$

should be solved under condition (2).

From equation (6) we determine that unknown which has the least coefficient and separate the integer parts:

$$x = \frac{1}{12}(410 - 31y) = 34 - 2y + \frac{1}{12}(2 - 7y).$$

Next we use the successive substitutions:

$$\frac{1}{12}(2 - 7y) = u; \quad \text{then} \quad y = \frac{1}{7}(2 - 12u) = -u + \frac{1}{7}(2 - 5u)$$

$$\frac{1}{7}(2 - 5u) = v; \quad \text{then} \quad u = \frac{1}{5}(2 - 7v) = -v + \frac{1}{5}(2 - 2v)$$

$$\frac{1}{5}(2 - 2v) = w; \quad \text{then} \quad v = \frac{1}{2}(2 - 5w) = 1 - 2w - \frac{1}{2}w$$

$$-\frac{1}{2}w = t; \quad \text{then} \quad w = -2t.$$

Therefore, we have:

$w = -2t,$

$v = 1 - 2w + t = 1 + 5t,$

$u = -v + w = -1 - 7t,$

$y = -u + v = 2 + 12t,$

$x = 34 - 2y + u = 29 - 31t.$

All the integer solutions of equation (6) have the following form:

$$\begin{cases} x = 29 - 31t \\ y = 2 + 12t \end{cases} \quad (t \text{ is an arbitrary integer}).$$

The solution fulfilling the condition (2) is $x = 29,\ y = 2$. Hence the birthday was on February 29 (a leap-year).

3 Deciphering the complete date

The complete date of the birth (including a year) can be found as follows.

Assume that the event was in the XX century. Ask a person to multiply the event day by $12 \cdot 2000$, the event month by $31 \cdot 2000$, and the event year by $31 \cdot 12$ and to present the sum of these products. If the sum of the products is equal to c, then we obtain the Diophantine equation

$$12 \cdot 2000 \cdot x + 31 \cdot 2000 \cdot y + 31 \cdot 12 \cdot z = c \tag{7}$$

under the assumption that x and y fulfill the condition (2), whereas

$$1900 \le z \le 2000. \tag{8}$$

Equation (7) is obtained in the following way: if (x_0, y_0, z_0) is the complete date of the birth, then

$$\frac{x}{31} + \frac{y}{12} + \frac{z}{2000} = \frac{x_0}{31} + \frac{y_0}{12} + \frac{z_0}{2000}.$$

Assuming that $12 \cdot 2000 \cdot x_0 + 31 \cdot 2000 \cdot y_0 + 31 \cdot 12 \cdot z_0 = c$, we get equation (7).

If the event was in the XIX century, then for determining the complete date of the event the following equation

$$12 \cdot 1900 \cdot x + 31 \cdot 1900 \cdot y + 31 \cdot 12 \cdot z = c$$

should be solved under conditions (2) and $1800 \le z \le 1900$.

Example 2. If the day of surrender of Japan is multiplied by $12 \cdot 2000$, the month of surrender is multiplied by $31 \cdot 2000$, and the year of surrender is multiplied by $31 \cdot 12$, the sum of the products is equal to 1329540. Determine the date of surrender of Japan.

According to the method given above, the solution of the equation

$$12 \cdot 2000 \cdot x + 31 \cdot 2000 \cdot y + 31 \cdot 12 \cdot z = 1329540 \tag{9}$$

should be found under conditions (2) and (8).

From equation (9) we determine the unknown which has the least coefficient:

$$z = \frac{1}{31 \cdot 12}(1329540 - 12 \cdot 2000 \cdot x - 31 \cdot 2000 \cdot y) = 3574 - 64x - 166y + \frac{1}{31} - \frac{16}{31}x + \frac{2}{3}y.$$

Using the successive substitutions and separating the integer parts we obtain:

$$\frac{1}{31} - \frac{16}{31}x - \frac{2}{3}y = t_1;$$

then

$$x = \frac{1}{48}(3 - 93t_1 - 62y) = -2t_1 - y + \frac{1}{48}(3 + 3t_1 - 14y).$$

$$\frac{1}{48}(3 + 3t_1 - 14y) = t_2;$$

then

$$t_1 = \frac{1}{3}(48t_2 + 14y - 3) = 16t_2 + 4y - 1 + \frac{2}{3}y.$$

$$\frac{2}{3}y = t_3; \quad \text{then} \quad y = \frac{2}{3}t_3 = t_3 + \frac{1}{2}t_3.$$

$$\frac{1}{2}t_3 = u; \quad \text{then} \quad t_3 = 2u.$$

Therefore, we have

$$\begin{aligned}
t_3 &= 2u, \\
y &= t_3 + u = 3u, \\
t_1 &= 16t_2 + 4y - 1 + t_3 = 16t_2 + 14u - 1, \\
x &= -2t_1 - y + t_2 = 2 - 31(t_2 + u), \\
z &= 3574 - 64x - 166y + t_1 = 3445 + 2000t_2 + 1500u.
\end{aligned}$$

Equation (9) has the following integer solutions

$$\begin{cases} x = & 2 - 31(t_2 + u), \\ y = & 3u, \\ z = & 3445 + 2000t_2 + 1500u, \end{cases}$$

where t_2 and u are arbitrary integers.

The solution fulfilling conditions (2) and (8) is obtained for $t_2 = -3, u = 3$. Then $x = 2, y = 9, z = 1945$. Hence the surrender of Japan was on September 2, 1945.

Example 3. Give the exact date of discovery of America by Christopher Columbus on the basis of the following information. If the day of this event is multiplied by $31 \cdot 500$, the month of the event is multiplied by $12 \cdot 500$, and the year of the event is multiplied by $31 \cdot 12$, then the sum of the products is equal to 1236024.

One should solve the Diophantine equation

$$12 \cdot 500 \cdot x + 31 \cdot 500 \cdot y + 31 \cdot 12 \cdot z = 1236024, \tag{10}$$

assuming that x and y fulfill the inequalities (2) and z fulfills the inequality

$$400 \leq z \leq 500. \tag{11}$$

From equation (10) we determine the unknown which has the least coefficient and separate the integer parts

$$z = \frac{1}{31 \cdot 12}(1236024 - 12 \cdot 500 \cdot x - 31 \cdot 500 \cdot y) = 3322 - 48x - 124y + \frac{1}{31}(20 - 12x - 31y).$$

Next the successive substitutions are applied:

$$\frac{1}{31}(20-12x-31y) = u; \text{ then } x = \frac{1}{12}(20-31y-31u) = 1-2y-2u+\frac{1}{12}(8-7y-7u).$$

$$\frac{1}{12}(8-7y-7u) = v; \quad \text{then} \quad y = \frac{1}{7}(8-7u-12v) = 1-u-2v+\frac{1}{7}(1+2v).$$

$$\frac{1}{7}(1+2v) = w; \quad \text{then} \quad v = \frac{1}{2}(7w-1) = 3w + \frac{1}{2}(w-1).$$

$$\frac{1}{2}(w-1) = t; \quad \text{then} \quad w = 2t+1.$$

Therefore, we have

$$\begin{aligned}
&w = 2t+1,\\
&v = 3w + t = 7t + 3,\\
&y = 1 - u - 2v + w = -4 - u - 12t,\\
&x = 1 - 2y - 2u + v = 12 + 31t,\\
&z = 3322 - 48x - 124y + u = 3242 + 125u.
\end{aligned}$$

Equation (10) has the following solutions

$$\begin{cases} x = & 12 + 31t,\\ y = & -4 - u - 12t,\\ z = & 3242 + 125u, \end{cases}$$

where t and u are arbitrary integers.

The solution fulfilling conditions (2) and (11) is obtained for $t = 0$ and $u = -14$. Hence Columbus discovered America on October 12, 1492.

We propose the Reader to solve the following problems.

1. It is known that if the day of the birth of some person is multiplied by 12 and the month of the birth is multiplied by 31, then the sum of the products is equal to 55. Determine the date of the birth. (Answer: January 2).

2. The event was in the XIX century. If the day of the event is multiplied by $12 \cdot 1900$, the month of the event is multiplied by $31 \cdot 1900$, and the year of the event is multiplied by $31 \cdot 12$, then the sum of the products is equal to 1523364. Which event was then? (Answer: September 14, 1812 – capture of Moscow by Napoleon's army)
3. Determine the exact date of the death of Mieszko I (the founder of the Polish state) on the basis of the following information. If the day of the death is multiplied by $12 \cdot 1000$, the month of the death is multiplied by $31 \cdot 1000$, the year of the death is multiplied by $31 \cdot 12$, then the sum of the products is equal to 824024. (Answer: May 25, 992).

References

[1] A. J. Banarski: *Równania nieoznaczone. Ułamki łańcuchowe. Kombinatoryka. Dwumian Newtona,* Biblioteczka Matematyczna Nr 3, PZWS, Warszawa, 1961
[2] J. I. Perelman: *Zanimatiel'naja algebra.* Moskwa–Leningrad, 1950

A contribution to functional thinking

Leona Salvetová, Jitka Laitochová

Palacký University in Olomouc, Faculty of Education
Žižkovo náměstí 5, 771 40 Olomouc, Czech Republic
lsalvetova@seznam.cz, jitka.laitochova@upol.cz

Abstract. The contribution is a portion of a dissertation thesis dealing with functional thinking of first year undergraduate mathematics students at Educational Faculties in the Czech Republic. The paper describes the research objectives, hypotheses and methodology used for processing data. Presented are test results obtained from the pilot study

Keywords: didactic test, functional thinking, TIMMS research, Referential Framework for Education.

Classification: Primary 97C99; Secondary 97D99.

1 Introduction

In everyday life, we are facing problems that cannot be solved on the basis of direct perception of our environment and the phenomena within it. The search for solutions based on inferences (i.e. deduction from the knowledge obtained so far) characterizes the process of thinking. Naturally, thinking has to be learned to be mastered. To think means to solve a problem using ones knowledge to infer conclusions of the situation given.

The paper is based on a proposed dissertation thesis topic: *The Level of Functional Thinking Achieved by Students Enrolled in Mathematics Education*. The concept of functional thinking is explained and the current state of the topic is tested within the group of first year students of Mathematics Education recruited from several pedagogical faculties within the Czech Republic. The paper also analyzes initial results of the research conducted at elementary schools in the region of the city of Zln. The elementary school student test scores were compared with university student scores.

2 Current state

The concept of a function belongs to the core terms of mathematics. The knowledge necessary for learning, irregardless of proficiency, of the concept and the training of functional thinking are an integral part of the whole course of compulsory school education. The necessity to develop functional thinking is beyond any doubt nowadays. This is evidenced by the increasing attention paid to functional thinking by many renowned theoreticians in mathematics education. Inductive methods and simple real-situation functional models seem to be undervalued in courses of study for future teachers.

Faculties specializing in training future mathematics teachers have recently experienced a decline in interest for the study of mathematics. The enrolled students show poor knowledge of mathematic foundations, as a consequence of which they struggle to meet the university standard level. The problem is caused by the fact that students lack proper knowledge of basic concepts upon which they should have been able to build during their university studies. As a result, university teachers have descended from their original standards, resulting in a lowering of the level of standards required for graduating teachers of mathematics. Functional thinking is incorporated within the *Referential Framework of Elementary and Secondary Education* in the educational area of mathematics and its applications.

3 The development of functional thinking

There are two methods to develop functional thinking. The first one is characterized by an effort to gradually master the theory through correct usage of concepts. Testing student ability in problem solving is regarded as the most precise way to detect and assess their knowledge and the level of thinking they have achieved. Problem solving is based on the usage of theoretical knowledge in order to find the solution to a problem. Problem solving is also useful for reinforcing theoretical knowledge and its further development (Sedláčková, 1993, page 7). Problem solving which makes use of functional thinking aims to identify the varying quantities, label the variables and then to find the mathematical relations between these quantities; all this enables the student to solve the mathematical relations and to interpret the obtained data (Stehlíková, 2007, page 260). The problems designed to develop functional thinking can be subdivided into the following groups:

1. Natural phenomena observation: comprised of problems designed to make a student observe and measure natural phenomena, such as temperature, and to form a table from the data and sketch a graph.
2. Trip planning: the problem provides the students with data to be interpreted.
3. Real life situation: problems are designed upon real life phenomena and/or graphs of real life functions.
4. Reviewing topics: provides a set of model problems suitable for reviewing topics and assessment of student knowledge.

Functional thinking can be developed through problems aiming at student achievement of subsidiary educational goals in mathematics while gradually mastering the theory and establishing a routine which results in the successful solution of practical problems. The subsidiary goals as individual aspects of functional thinking include the following capabilities (Blažková, 2010, page 54):

- To determine changes in the result of a mathematical operation caused by changes in the input variables.
- To determine the changes in geometry constructions in relation to changes in position and size of the objects involved.
- To determine whether a relation between two phenomena can be described quantitatively.
- The ability to characterize functions by tables, equations and graphs.
- The ability to understand domains and ranges of functions.
- To interpret properties of given functions.
- To know how to read graphs.
- The ability to generalize quantitative relations.

4 The goals of the dissertation thesis

The above mentioned facts were motivation to reflect on the level of functional thinking achieved by students of pedagogical faculties and to analyze the topic in the dissertation thesis. The main goal of the dissertation thesis is to report and assess the level of functional thinking achieved by first year university mathematics education students.

The subsidiary goals of the thesis involve:

- Measuring the success the students have achieved while solving different types of problems.
- Stating whether test scores of students are influenced by factors related to individual training, especially to their previous study at secondary schools.
- Assessing student capability to use the two aspects of functional thinking individually and independently of one another. (The two aspects involve the ability to obtain information from a graph, and the ability to use a graph for an adequate description of a mathematical relationship.)

4.1 Research working hypotheses

First year university mathematics education students have taken achievement tests at the end of their high school studies, called **maturita** in Czech schools.

The following working hypotheses are to be subject to question within the proposed experimental part of the dissertation thesis:

H1: Students who took the higher level of achievement tests score higher at functional thinking development tests in comparison with students who took the basic level of achievement tests or students who took no achievement tests.

H2: The more mathematics lessons that were incorporated into the weekly secondary school schedule, the better the level of functional thinking achieved by its graduates.

H3: The level of functional thinking achieved by first year university mathematics education students who had previous university or college studies is higher than the level of functional thinking achieved by students who had no university or college studies.

H4: Boys of age 12 to 15 do not have a higher level of functional thinking than girls of the same age.

H5: Pupils of age 12 to 15 have equal success at solving problems of graph interpretation compared to solving problems of graph creation from a functional relation.

H6: First year university mathematics education students have equal success at solving problems of graph interpretation compared to solving problems of graph creation from a functional relation.

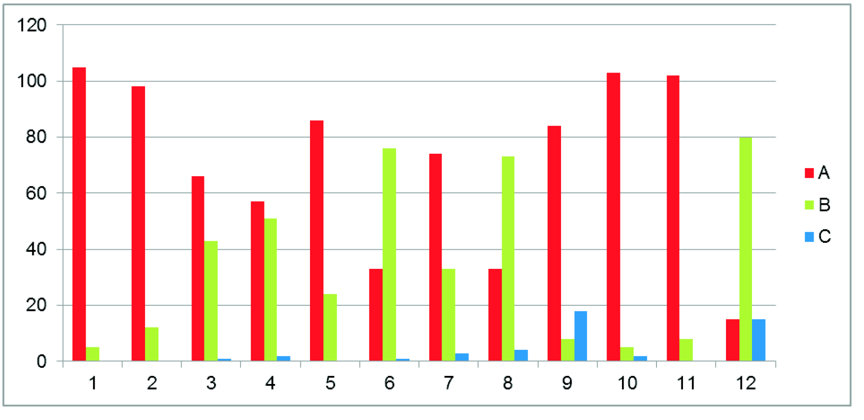
4.2 Methods of data collection and the research participants

The non-standardized didactic test used as the means of measurement of the participant's level of functional thinking consisted of twelve problems, some with multiple choice answers (closed problem), some without (open problem). Closed problems always had only one correct answer to each question. The problems were designed to test either the ability to draw a graph of a functional relation or to interpret a given graph. The problems were inspired by TIMMS (Trends in International Mathematics and Science Study), which suggested problems to compare the students educational results. Inspiration was also gained from associate professor Petr Eisenmann, who devoted his research career to studying functional thinking of students and pupils.

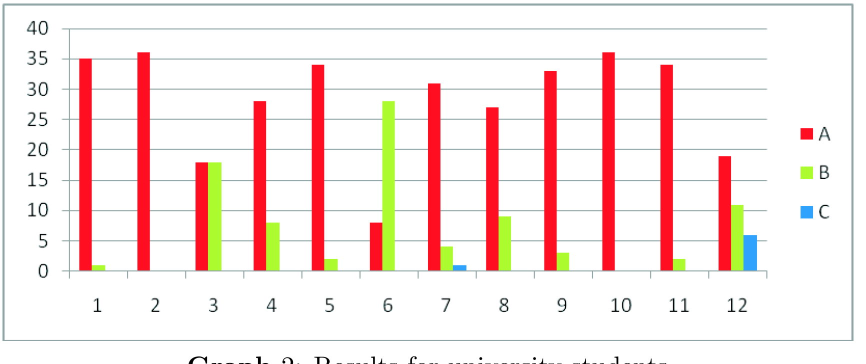
5 Results of the pilot research project

Thanks to financial support of the student grant competition 2012, a research project was awarded, named *The level of functional thinking achieved by the students of mathematics in 8th and 9th grades of elementary school.* The pilot research was conducted at two universities and two elementary schools. The target groups were 36 first year university mathematics education students at the Faculty of Education at Palacký University in the city of Olomouc and at the Faculty of Education at Jan Evangelista Purkyně University in the city of Ústí nad Labem. The second target group included 110 pupils age 14 to 15 at schools in the region of the city of Zlín. Graphs 1 and 2 show the results of both groups based on the following assessment scale.

A – completely correct solution
B – wrong solution
C – problem left unsolved



Graph 1: Results for pupils age 14 to 15



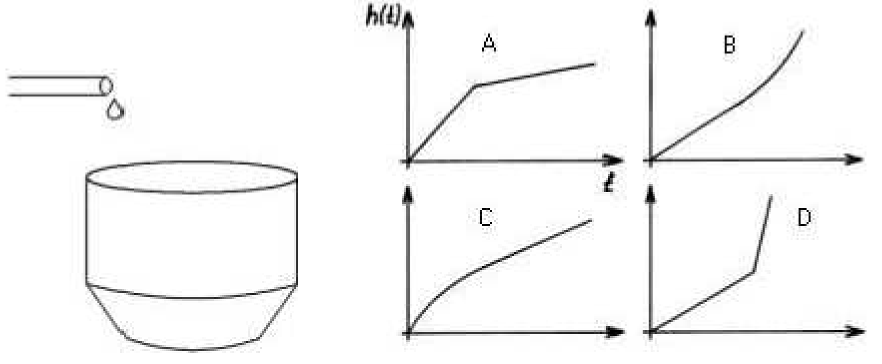
Graph 2: Results for university students

5.1 Wrong answer analysis

The graphs show that problems 6, 8 and 12 were the most difficult for pupils ages 14 and 15 (Czech grades 8 and 9). Problem 6 proved to be the most difficult for university students. These three problems will be discussed in detail.

Problem 6

The empty wooden barrel in the figure below starts filling with a constant flow of water at the time $t = 0$. Choose from the four graphs below the one which describes the relation between the height $h(t)$ of the water level in the barrel (vertical axis) and time t (horizontal axis).



The correct answer is graphic C. After the barrel is filled to where it is cylindrical, the water level increases linearly. Before that water level, the graph is not linear. Table 1 shows the results obtained from pupils age 14 to 15 and university students. The figures refer to the frequency of individual answers. The most frequent wrong answer for both groups was answer A.

Wrong solution	A	B	D
University students	19	7	2
Elementary school pupils	28	22	26

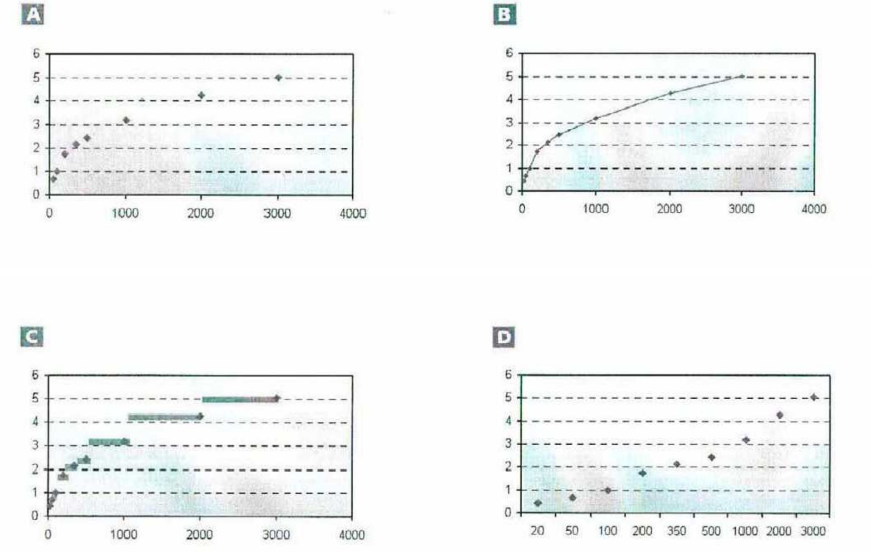
Table 1: The most frequent wrong answer of the university students and elementary school pupils given in problem 6

At elementary school the most frequent answer was A. The university students also chose A as the most frequent wrong answer.

Problem 8

Postal fees in New Zealand depend on the weight of the parcel. The prices are given in the table below. Each of the four graphs shows the relation between the postal fees in New Zealand dollars (NZD; vertical axis) for the weight of the parcel in grams (horizontal axis). Which graph (A,B,C,D) reflects the postal fees in New Zealand?

Parcel weight in grams	New Zealand postal fee in NZD
under 20 g	0.46
21 g – 50 g	0.69
51 g – 100 g	1.02
101 g – 200 g	1.75
201 g – 350 g	2.13
351 g – 500 g	2.44
501 g – 1 000 g	3.20
1 001 g – 2 000 g	4.27
2 001 g – 3 000 g	5.03



The correct answer is graphic C. Table 2 summarizes the wrong solution counts for elementary school pupils.

Wrong solution	A	B	D
Elementary pupils	8	25	40

Table 2: The most frequent wrong answers given by pupils solving problem 8

The most frequent wrong answer was D, which is understandable at elementary school level. The pupils rarely interpret real-life functional relations. Most pupils do not know about such graphs and they do not know how to draw them, so they chose the answer they regard as similar to a linear function, the only function they know. The ability of a pupil to choose or draw the correct graph therefore depends on pupil experience with real-life problems.

Problem 12

Locations A and B are 120 km apart. What time does it take for various vehicles to cover the distance, provided that the vehicles move at speed given in the table? Fill in the table and draw a graph of the relation between the time needed and the velocity of each vehicle.

Velocity in km per hour	10	30	60	100	120
Time in hours					

The correct solution should include a graph of an inverse proportion relation between the velocity and the time necessary to cover the distance, by various vehicles going at different constant speeds. Table 3 displays the most frequent wrong answers and their exact count. Partial solutions were regarded as wrong. For instance, the graph of the relation was often missing, and the solution was regarded as wrong. The problem proved the pupils to have an incomplete knowledge of inverse proportion in a Cartesian graph of finitely many discrete points. The scores achieved by the age 14 and 15 pupils are alarming, because direct and inverse proportions are taught as early as age 13.

Wrong solution	Table correct, but no graph	Wrong table	Table correct, but a directly proportional relation
Elementary school pupils	33	27	20

Table 3: The most frequent wrong solutions of pupils solving problem 12

6 Conclusions

The paper is devoted to the topic of functional thinking achieved by elementary school pupils and university students hoping to be future teachers of mathematics. Results are described for a pilot research program which aims to determine the level of success of pupils age 14 to 15 and first year university mathematics education students in solving problems that involve functional thinking. The wrong answers are analyzed and the research shows that both of these groups experienced greater difficulty solving problems that required them to draw a graph of a functional relation in comparison with problems intended to test their ability to interpret graphs of functional relations.

Central to the development of functional thinking is the concept of a mathematical function, which we propose to be developed through concrete examples of functions. The history of mathematics confirms this statement. The idea of a function is usually learned through real-life objects appearing in various problems from concrete life situations. The concept of a function materializes only after sufficiently many such concrete examples.

The dissertation thesis results could serve well in an argument for the importance of mathematics education for the general public. An effort to increase functional thinking may also increase student interest in mathematics. The effect of early educational efforts to increase functional thinking might be a starting point which leads to a decrease in the number of unsuccessful university mathematics students. The research results may also lead to innovations in programs of study and to new subjects relevant for the training of future teachers of mathematics.

Acknowledgements

The contribution was written with financial support of the Student Grant Competition, for the topic *The Level of Functional Thinking of the Students of Mathematics for Educational Purposes at Senior Elementary School* PdF_2012_020.

References

[1] Blažková, R. (2010). *Dyskalkulie II.* Brno: Masarykova univerzita.
[2] Eisenmann, P., Kopáčková, A. (2006). *Rozvoj funkčního myšlení ve výuce matematiky na základní škole.* Praha: Společnost učitelů matematiky JČMF.
[3] Sedláčková, J. (1993) *Rozvíjení myšlení žáků ve vyučování matematice*: 1. vydání. Olomouc: Univerzita Palackého.
[4] Stehlíková, N. (2007) *Náměty na podnětné vyučování v matematice.* 1. vydání. Univerzita Karlova v Praze.

Scientific Issues, MATHEMATICA IV, Ružomberok 2012

The use of multiple products of vectors in spherical trigonometry

Robert Sochacki, Grzegorz Bryll

University of Opole, Institute of Philosophy
Katowicka 89, 45-061 Opole, Poland
rsochacki@uni.opole.pl

Abstract. In this paper we present a new method of proving certain theorems in spherical geometry.

Keywords: spherical geometry, the law of sines, the law of cosines, Neper's formulas.

Classification: 97B40; 51N99.

On the basis of the properties of the multiple product of vectors one can easily prove the law of sines and the law of cosines for spherical triangles (see [5]).[1] On the basis of these theorems there may be obtained the so-called Neper's formulas. The scalar triple product of vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ will be denoted as follows:

$$\vec{a} \circ (\vec{b} \times \vec{c}) = (\vec{a}, \vec{b}, \vec{c}),$$

where " $\circ$ " is the dot product, and " $\times$ " is the cross product. Of course, we have the relationship of the form:

$$\vec{a} \circ (\vec{b} \times \vec{c}) = \vec{b} \circ (\vec{c} \times \vec{a}) = \vec{c} \circ (\vec{a} \times \vec{b}). \tag{1}$$

The vector triple product $\vec{a} \times (\vec{b} \times \vec{c})$ is expressed by vectors $\vec{b}$ and $\vec{c}$ as follows (see [3, 4]):

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \circ \vec{c})\vec{b} - (\vec{a} \circ \vec{b})\vec{c}. \tag{2}$$

In the further considerations we use properties of the products $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ and $(\vec{a} \times \vec{b}) \circ (\vec{c} \times \vec{d})$. On the basis of (2) the product $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ can be expressed as follows:

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = ((\vec{a} \times \vec{b}) \circ \vec{d})\vec{c} - ((\vec{a} \times \vec{b}) \circ \vec{c})\vec{d} = (\vec{a}, \vec{b}, \vec{d})\vec{c} - (\vec{a}, \vec{b}, \vec{c})\vec{d}. \tag{3}$$

Using formula (2) for the product $(\vec{a} \times \vec{b}) \circ (\vec{c} \times \vec{d})$, we obtain the so-called Laplace's formula: $(\vec{a} \times \vec{b}) \circ (\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}, \vec{c}, \vec{d}) = (\vec{c}, \vec{d}, \vec{a} \times \vec{b}) = \vec{c} \circ (\vec{d} \times (\vec{a} \times \vec{b})) = \vec{c} \circ [(\vec{d} \circ \vec{b})\vec{a} - (\vec{d} \circ \vec{a})\vec{b}]$, i.e. the formula of the form:

$$(\vec{a} \times \vec{b}) \circ (\vec{c} \times \vec{d}) = (\vec{a} \circ \vec{c})(\vec{d} \circ \vec{b}) - (\vec{a} \circ \vec{d})(\vec{b} \circ \vec{c}). \tag{4}$$

On the basis of (3) we have:

$$\vec{u} = (\vec{r}_1 \times \vec{r}_2) \times (\vec{r}_1 \times \vec{r}_3) = (\vec{r}_1, \vec{r}_2, \vec{r}_3)\vec{r}_1 - (\vec{r}_1, \vec{r}_2, \vec{r}_1)\vec{r}_3 = (\vec{r}_1, \vec{r}_2, \vec{r}_3)\vec{r}_1. \tag{5}$$

By (4) we obtain:

$$(\vec{r}_1 \times \vec{r}_2) \circ (\vec{r}_1 \times \vec{r}_3) = (\vec{r}_1 \circ \vec{r}_1)(\vec{r}_2 \circ \vec{r}_3) - (\vec{r}_1 \circ \vec{r}_3)(\vec{r}_1 \circ \vec{r}_2). \tag{6}$$

[1]Elementary proofs of these theorems can be found in [1, 2].

On a sphere of the radius $r = 1$, we consider a spherical triangle with vertices ABC formed from arcs of great circles. The unit vectors corresponding to the points A, B, C are $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, respectively. Moreover we assume that the vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ form a right-handed coordinate system (Fig. 1).

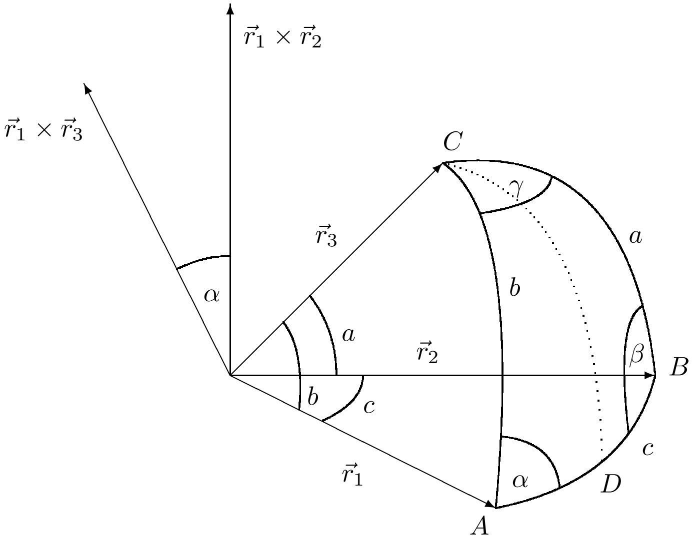


Figure 1

Let the interior angles of the triangle with the vertices A, B, C be α, β, γ, respectively. Suppose also the length of the sides of the triangle are a, b, c, respectively. The law of sines can be proved as follows. Since $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ are the unit vectors, so

$$|\vec{r}_1 \times \vec{r}_2| = |\vec{r}_1||\vec{r}_2| \sin(\vec{r}_1, \vec{r}_2) = \sin c. \tag{7}$$

From (5) we have:

$$|\vec{u}| = (\vec{r}_1, \vec{r}_2, \vec{r}_3), \quad \text{since} \ \ (\vec{r}_1, \vec{r}_2, \vec{r}_1) = 0. \tag{8}$$

Furthermore:

$$|\vec{r}_1 \times \vec{r}_3| = |\vec{r}_1||\vec{r}_3| \sin(\vec{r}_1, \vec{r}_3) = \sin(\vec{r}_1, \vec{r}_3) = \sin b,$$

then

$$\sin(\vec{r}_1, \vec{r}_3) = \sin b. \tag{9}$$

The vectors $\vec{r}_1 \times \vec{r}_2$ and $\vec{r}_1 \times \vec{r}_3$ form the angle α. Indeed: $\vec{r}_1 \times \vec{r}_2 \perp \vec{r}_1$ and $\vec{r}_1 \times \vec{r}_2 \perp \vec{r}_2$. So $\vec{r}_1 \times \vec{r}_2 \perp \widehat{AB}$. We have also $\vec{r}_1 \times \vec{r}_3 \perp \vec{r}_1$ and $\vec{r}_1 \times \vec{r}_3 \perp \vec{r}_3$, hence $\vec{r}_1 \times \vec{r}_3 \perp \widehat{AC}$. From this we obtain: $|\angle(\vec{r}_1 \times \vec{r}_2, \vec{r}_1 \times \vec{r}_3)| = |\angle(\widehat{AB}, \widehat{AC})| = \alpha$. By (5), (7) and (9) we obtain:

$$|\vec{u}| = |(\vec{r}_1 \times \vec{r}_2) \times (\vec{r}_1 \times \vec{r}_3)| = |\vec{r}_1 \times \vec{r}_2||\vec{r}_1 \times \vec{r}_3| \sin\alpha = \sin c \sin b \sin\alpha.$$

Hence and from (8) it follows that:

$$(\vec{r}_1, \vec{r}_2, \vec{r}_3) = \sin c \sin b \sin \alpha. \tag{10}$$

Therefore $\sin \alpha = \frac{(\vec{r}_1, \vec{r}_2, \vec{r}_3)}{\sin b \sin c}$, that is

$$\frac{\sin \alpha}{\sin a} = \frac{(\vec{r}_1, \vec{r}_2, \vec{r}_3)}{\sin a \sin b \sin c}.$$

Similarly we obtain:

$$\frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c} = \frac{(\vec{r}_1, \vec{r}_2, \vec{r}_3)}{\sin a \sin b \sin c}.$$

Thus we have the following law of sines:

$$\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c}. \tag{11}$$

This theorem may be formulated as follows:

Theorem 3. *In any spherical triangle the sines of the sides are proportional to the sines of the opposite angles.*

The law of cosines for the sides may be proved as follows. Since

$$(\vec{r}_1 \times \vec{r}_2) \circ (\vec{r}_1 \times \vec{r}_3) = |\vec{r}_1 \times \vec{r}_2||\vec{r}_1 \times \vec{r}_3| \cos \alpha = \sin c \sin b \cos \alpha,$$

and on the basis of (6) and given that $\vec{r}_1 \circ \vec{r}_1 = 1$ we have:

$$(\vec{r}_1 \times \vec{r}_2) \circ (\vec{r}_1 \times \vec{r}_3) = (\vec{r}_2 \circ \vec{r}_3) - (\vec{r}_1 \circ \vec{r}_3)(\vec{r}_1 \circ \vec{r}_2) = \cos a - \cos b \cos c.$$

Hence, we obtain:

$$\sin c \sin b \cos \alpha = \cos a - \cos b \cos c.$$

From these we obtain the following law of cosines:

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha. \tag{12}$$

For the other sides we have:

$$\cos b = \cos a \cos c + \sin a \sin c \cos \beta. \tag{13}$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma. \tag{14}$$

The law of cosines may be formulated in the following way:

Theorem 4. *In a spherical triangle the cosine of any side is equal to the product of the cosines of the other two sides plus the product of the sines of these two sides and the cosine of their included angle.*

On the basis of the laws of sines and cosines the so-called Neper's formulas can be deduced. For this purpose we assume in the spherical triangle ABC (see Fig. 1) that the side c is bisected by the point D and we draw an arc of the great circle from C to D. Using the law of cosine for the spherical triangles ACD and DCB we obtain:

$$\cos CD = \cos b \cos \frac{c}{2} + \sin b \sin \frac{c}{2} \cos \alpha, \quad (\Delta ACD). \tag{15}$$

$$\cos CD = \cos a \cos \frac{c}{2} + \sin a \sin \frac{c}{2} \cos \beta, \quad (\Delta DCB). \tag{16}$$

Comparing the formulas (15) and (16) we obtain:

$$\cos b + \sin b \tan \frac{c}{2} \cos \alpha = \cos a + \sin a \tan \frac{c}{2} \cos \beta,$$

that is:

$$\tan \frac{c}{2} (\sin a \cos \beta - \sin b \cos \alpha) = 2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}. \tag{17}$$

From the law of sines it follows that:

$$\sin b = \frac{\sin a}{\sin \alpha} \sin \beta. \tag{18}$$

Hence, on substituting in (17):

$$\tan \frac{c}{2} \left(\sin a \cos \beta - \frac{\sin a}{\sin \alpha} \sin \beta \cos \alpha \right) =$$

$$= \frac{\sin a}{\sin \alpha} \tan \frac{c}{2} \sin (\alpha - \beta) = 2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}. \tag{19}$$

From (18) we obtain:

$$\frac{\sin a}{\sin b} - 1 = \frac{\sin \alpha}{\sin \beta} - 1, \quad \text{whence} \quad \frac{\sin a - \sin b}{\sin b} = \frac{\sin \alpha - \sin \beta}{\sin \beta}.$$

Thus,

$$\frac{2 \cos \frac{a+b}{2} \sin \frac{a-b}{2}}{\sin b} = \frac{2 \cos \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2}}{\sin \beta}.$$

From this it follows that:

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\cos \frac{a+b}{2} \sin \frac{a-b}{2}}{\cos \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2}}. \tag{20}$$

Using (20) to (19) we have:

$$\tan \frac{c}{2} \cdot \frac{\cos \frac{a+b}{2} \sin \frac{a-b}{2}}{\cos \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2}} \cdot 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha - \beta}{2} = 2 \sin \frac{a+b}{2} \sin \frac{a-b}{2},$$

whence

$$\tan \frac{c}{2} \cdot \cos \frac{a+b}{2} \cdot \frac{\cos \frac{\alpha-\beta}{2}}{\cos \frac{\alpha+\beta}{2}} = \sin \frac{a+b}{2}.$$

So, finally we have the so-called Neper's formula for the side c:

$$\tan \frac{c}{2} = \tan \frac{a+b}{2} \cdot \frac{\cos \frac{\alpha+\beta}{2}}{\cos \frac{\alpha-\beta}{2}}. \tag{21}$$

In a similar manner, the other Neper's formulas can be shown to be true for any spherical triangle.

References

[1] H. Athen: *Vectorielle Begründung der Trigonometrie,* Mathematisch-Physikalische Semeslerberichte (Göttingen). 8(1) (1961), 82-94.

[2] W. Janowski: *Planar trygonometry and elements of spherical trygonometry* (in Polish), PZWS. Warszawa, 1959.

[3] V.P. Minorskij, V.P. Ul'anovskij: *Vektornaja al'gebra* (izd. 2), Gosudarstvennoe Izd-vo Tehniko-Teoretieskoj Literatury. Moskva-Leningrad, 1951.

[4] T. Trajdos-Wróbel: *Wstęp do analizy wektorowej,* PWN. Warszawa, 1959.

[5] F.A.Willers: *Elementar-Mathematik,* (9 Aufl.), Verlag von Theodor Steinkopff. Dresden-Leipzig, 1958.

Supporting Inquiry Skills through Mathematical Competition B-Day

Ján Šunderlík, Miroslava Sovičová, Soňa Čeretková, Janka Melušová

Department of Mathematics, Constantine the Philosopher University in Nitra
Tr. A. Hlinku 1, 949 74 Nitra, Slovakia
jsunderlik@ukf.sk

Abstract. The article describes the possibilities of development of inquiry skills of secondary school students during preparing and participating in mathematical competition B-Day. During the competition teams of three or four secondary school students work on open-ended tasks that require students' investigation and original approach to the problem. The competition is not only about finding the winner, but it is suitable for identifying the level of students' inquiry skills.

Keywords: Inquiry based learning, mathematics competition, inquiry processes

Classification: D54

1 Introduction

Secondary mathematics education continuously goes through several organisational changes that influence for example the reduction of mathematical content, the implementation of new teaching methods. The changes should be understood within the aims of mathematics education. They can be simplify to: mastery in basic skills in mathematical strands, to know how to apply this knowledge and to be able to approach new open problems in mathematics inquiry. All of them are important, but especially the third aim is crucial for many technical or applied mathematical professions. The first and partly the second aim can be evaluated during the standardised test. We agreed with the opinion as mentioned in [1] that it is hard to evaluate the inquiry skills within the two-hour standardised test.

In our contribution, we describe the pilot study of a successful mathematics competition B-DAY in Slovak settings. We consider the competition as a good example where pupils can experience mathematical investigation and use several inquiry processes. In the competition, pupils are supposed to work on complex tasks where they need to investigate unknown mathematical areas to find the solution. On the other side, the results of this kind of competition offer us good environment for monitoring pupils' inquiry processes.

2 Background of the competition

The Mathematics B-DAY is a mathematics competition for teams of secondary-school pupils at the age of 15-19. Although the competition was organized in Slovakia in the school year 2011/2012 for the first time, in the Netherlands, Germany and Belgium it has its tradition for several years. The main organiser of the competition is the Freudenthal Institute for Science and Mathematics Education of Utrecht University in Netherlands. In Netherlands, "Mathematics B is the mathematics needed for technical studies and studies in science and mathematics at university level, its core component is calculus. The assignment is related to the Mathematics A-lympiad. As in the A-lympiad the real world character of the task is essential, for the Mathematics B-day the emphasis is on mathematical problem solving, conjecturing and proving."

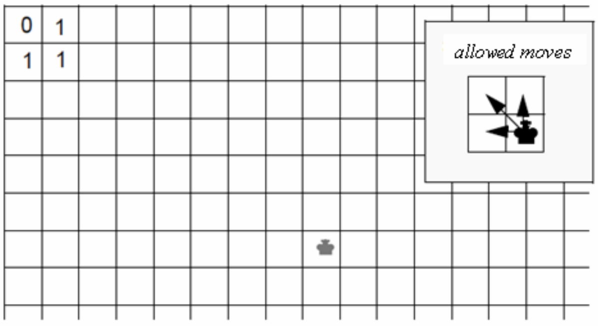


Figure 1: Possible moves of the king

3 B-DAY in Slovakia

In school year 2011/2012, the Mathematics B-DAY competition was introduced for the first time in Slovakia within the international project 7FP PRIMAS (www.primas-project.eu). One of the main aims of the project is to create the environment and realize the activities suitable for the implementation of inquiry-based learning in mathematics education. The competition was organized by partners of the project PRIMAS from the Department of Mathematics at Faculty of Natural Sciences, Constantine the Philosopher University in Nitra in collaboration with the partners of the project from the Freudenthal Institute of Utrecht University, Netherlands.

4 Assignment from 2011

The theme of the assignment and the context of the problems differ every year. In 2011, the title of the assignment for the Mathematics B-DAY competition was "The final move". The assignment was formally divided into four parts. The first part presented four combinatorial games which were intended to familiarize the pupils with looking for a winning strategy. The second part contained a theoretical background applicable to the games from the first part of the assignment. The third part included a detailed analysis of one game. The last part was a final assignment which was the main part of pupils' mathematical inquiry.The assignment respects the principle of lower level of knowledge to enter but quite high mathematical outcome of the possible results.

As an example of the scaffolding character of the tasks we present the task about the problems on the chessboard. This problem is presented in every part of the assignment. The king can move only up, left or diagonal up - left (See Figure 1). What pupils had to do was to indicate precisely which starting points will allow the first player to win, no matter how well the opponent plays. Other starting points mean the second player wins, no matter how well the first player plays. After playing the game pupils get the understanding of the solution.

- Try to find the winning and losing starting points.

In the second part, pupils were asked to deepen their understanding of the concept as well as to develop the appropriate terminology that was necessary for the final assignment. Pupils should focus more on the exploring, formulating hypothesis

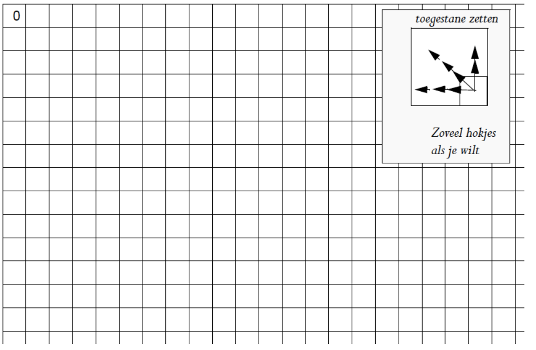


Figure 2: Possible moves of the queen

and finding conjectures. The tasks go different directions as listed in the first part but have common ways of solution(See Figure 2). The task follows as: :

1. So why do you have to put a 0 in square (0, 2)?
2. Now you can continue with square (1, 2). Why must that be a 1?
3. Fill in squares until you can see a pattern. Try to explain the pattern.Thus far the preparations. It's now your turn with the limited king in square (6, 10).
4. Can you win from that position? If yes, what is your first move?
5. Now put the king in square (1020, 389785). Can you win from that position? If yes, what is your first move?

In the third part the difficulty of tasks increased. Evidence of that can be identified from pupils' solutions. Only two out of five groups solved this problem, one group got correct solution. In the fourth part, pupils can decide between two final tasks. Both of them were complex tasks where pupils should use discovered strategies, understood terminology and systematic investigation to find and explain their solution. Second final task was to determine the winning and losing positions of the queen for the whole field. The allowed moves are in the Picture. The task was more complicated than the previous one. Pupils may find a characteristic description of the losing fields, what was excellent!

- It would also be nice if you can indicate the winning move for the following starting squares: (15, 31); (20, 21); (100, 200).

5 Data collection

The competition was held in Nitra, at the Department of Mathematics FNS CPU with the attendance of 22 pupils from four secondary-grammar schools. They formed six groups. Later on, 18 pupils in five groups from two secondary-grammar schools participated in the competition at the Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava. The teams of three or four pupils worked on an assignment for one whole day. The assignment comprises of one or more open problems with a common context. Pupils can use every accessible method, tool or aid, including the computer and the internet. The result of their work is the solution in the form of coherent mathematical text understandable for every reader, even for someone who does not know already in advance the topic and terms of the assignment. The participants should also include the description of the solutions, the answers to the questions asked, their logical thinking and reasoning into the report. The solution must be handed in in the electronic version during the competition day, at least seven hours from the beginning of the competition. Pupils worked on the tasks during one day from nine a.m. to five p.m. included short introduction at the beginning, lunch break and writing of the final report.

6 Research question

Within the pilot study, different perspectives were taken. In this article, we focused on the research questions to find out To what extend participated secondary school pupils are able to use inquiry skills for solving open problems from unknown mathematical context.

7 Theoretical framework

Developing of the inquiry processes is the main aim of the Inquiry based learning (IBL). This approach to learning recently spread within Europe through several projects and organisations. Our pilot study is framed within the 7FP project PRIMAS that characterises IBL as an innovative way of learning and teaching [2]. It aims to develop 'inquiring minds and attitudes' of young people which will be required in an uncertain future. Carrying out an experiment in which one tests a scientific hypothesis, developed by drawing on previous knowledge, generally enables understanding of cause-and-effect relationships. This leads to activity in which deliberately defined or selected variables are modified, controlled, monitored, measured, analysed and interpreted. These processes form the inquiry skills that we are about to observe within the results of the competition B-Day.

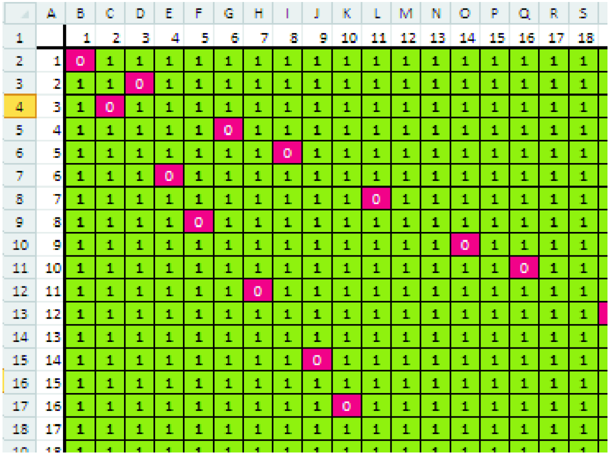
8 Methods

The main sources of analysis were pupils' final reports, filled questionnaires and semi-structured interviews with participating pupils after the competition. There was quite high percentage of unanswered questionnaires that is why we used them only as a supportive data. We are not presenting a deep description of the pupils' results as well as it was not our aim to monitor all pupils' thoughts and discussions during the competition that may have influenced the results. We presented on few selected examples, the general overview of the spirit of the competition, as well as sufficient evidence that will satisfy our research question.

9 Findings

According to the analysis of the pupils' solutions and information from a questionnaire or interview, we could take the mathematical content as unfamiliar for most of the pupils. The way of working was also new for pupils. On the question whether they did something like investigation they answered: *"It is totally different during mathematics lesson, it is everything so strict."*

Considering the secondary school pupils, all participated groups solved part one. Some of the groups didn't include results from the first and second part into the final report, but there is an evidence in supported materials that they solved the tasks. Solutions from the first part were mostly correct and pupils got good understanding of processes as experimenting, formulating hypothesis, finding conclusion. Another



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
1		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	2	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	3	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	4	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1
6	5	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1
7	6	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
8	7	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1
9	8	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1
10	9	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1
11	10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1
12	11	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1
13	12	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
14	13	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
15	14	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1
16	15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
17	16	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
18	17	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Figure 3: Students' solution

very important aim of the first part was to develop the inquiry processes. The whole competition had several possible directions that the pupils may go through.

In Bratislava none of the five groups chose the first final assignment. In Nitra, there were two out of six groups that solved the first final assignment. None of the groups solved the whole final problem, but most of the groups that solved this task correctly applied the developed backtracking strategy to solve the problem. Most difficult thing was to find a common conjecture that will help to identify general winning strategy.

There have been different kinds of description of the solution that pupils have written in their reports. They for example used excel (See Figure 3). Some of them included detailed description and explanation and some of them included only precisely formulated solution. During the interview, the groups of pupils reflected on the process of how they came up with the solution. We asked them whether they used some strategies that they had known before. They answered: *"We created them today, because we came here without knowing anything about this."*

Eight out of eleven groups have written some solution of the final assignment. They were able to used what they found out during the day, used the terminology and reason mathematically to support they solution. Even though pupils did not work this way based on the day activities, they were able to, in different levels, develop appropriate strategies to solve the problem. We also observe the big advantage of previous knowledge. Most successful pupils were pupils from classes specialized in informatics and mathematics. They differ from other groups in the character of solution. Pupils from informatics class were able to express observed relationship in a recurrent way. The formulas are for one line:

$$X : X(n+1) = x(n) + 2 + a(n)$$

$$Y : Y(n+1) = y(n) + 1 + a(n)$$

They also describe what $a(n)$ stands for. This knowledge was not mentioned in the assignment and also is not common in regular class. That gave us the conjecture that previous knowledge played also some role in the highest level of the task. There were two groups from both cities that did perform quite poor. They were able to solve some tasks from the first part but didn't manage to go further. The incorrect reasoning was present also in the basics tasks. One group did solve the initial task bud didn't continue with the other parts of the assignment. To clearly understand this, we need more information. Six groups did recognised patterns and gave appropriate reasoning of them. We consider this group as the group of good mathematics pupils that are keen on learning something new not only from textbooks or teacher but through their own investigation. Some problems which pupils were not able to solve by themselves were solved within the peers during a group work. The level of theoretical knowledge wasn't the most important thing. What mattered was the level of inquiry processes and the ability to satisfy their curiosity with investigation. As one pupil expressed during the interview: *"The important thing is that we did not need some..., how to say it.., some deep previous knowledge. Good sense is sufficient and correct judgement. Simply, I have some idea and I try to bring it to conclusion."*

10 Conclusions

The Mathematics B-DAY achieved a significant success among the pupils from both Slovak towns. They considered the problems from the assignment interesting and many of the pupils surprised the organisers by the level of their mathematical skills, by the level of the ability to express their ideas, logical reasoning and also by the results achieved in the final report. Connecting the pupils' solutions with expressed opinions, we see the potential of mathematical competitions like this in Slovakia. There is also a significant evidence of adapting the backtracking strategy in the problem solving process. Even though pupils haven't been used to do similar tasks in the school they were able to work within the inquiry. As we observe all of the pupils were interested in mathematics. That gives us the optimisms that with further preparation and more experience in inquiry processes, the results can be even better. The last group of pupils that didn't manage to overcome the initial problems gave us the information that the competition has its difficulty and not all pupils are able to succeed without the preparation.

Acknowledgements

This work has been supported by the project PRIMAS - Promoting Inquiry in Mathematics and Science across Europe (www.primas-project.eu) that has received funding from the European Union Seventh Framework Programme (7RP/2007-2013) under grant agreement 244380.

References

[1] H. van der Kooij: *A round mathematics,* [online]. [2012-05-05] http://www.primas-project.eu/artikel/en/1137/Mathematics+B-day/view.do ?lang=en

[2] M. Euler: *Report about the survey on inquiry-based learning and teaching in the European partner contries* [online]. [2012-05-05] http://www.primas-project.eu/servlet/supportBinaryFiles?referenceId=8&supportId=1247

Multi-target tracking algorithms in 3D

Rastislav Telgársky
Central New Mexico Community College
525 Buena Vista Dr. SE, Albuquerque, NM 87106, U.S.A.
rtelgarsky@cnm.edu; rastislav@telgarsky.com

Abstract. Ladars provide a unique capability for identification of objects and motions in scenes with fixed 3D field of view (FOV). This paper describes algorithms for multi-target tracking in 3D scenes including the preprocessing (mathematical morphology and Parzen windows), labeling of connected components, sorting of targets by selectable attributes (size, length of track, velocity), and handling of target states (acquired, coasting, re-acquired and tracked) in order to assemble the target trajectories. This paper is derived from working algorithms coded in Matlab, which were tested and reviewed by others, and does not speculate about usage of general formulas or frameworks.

Keywords: algorithms, target tracking, image processing, 3D

Classification: M10; P50.

1 Data generated by Ladar

The photon counting Ladar (laser radar) consists of the illumination laser (transmitter) and the sensor (receiver) at the same location (the monostatic setup). The focus of this paper is on algorithms for processing the images resulting from a prototype of a photon-counting Geiger-mode sensor with an array resolution of 32×32. The sensor has FOV of 50 by 50 meters at 10 km, and a frame rate of 12 kHz. Each frame contains the energy returned from one laser pulse. We construct a 3D image from 200 consecutive frames by calculating the empirical PDF of the range (depth) values; actually, the voxel intensities are defined by the 3D histogram values.

The record of Ladar activity is a file stored on a hard disk. One second of data contains 12,000 2D images, each of which has size 32×32, and each pixel stores 16 bits of data. We use 200 2D images to create one 3D frame. Thus, the frame speed reduces from 12 kHz down to 60 Hz. The file with 300 cumulative frames represents $300 / 60 = 5$ minutes of real running time and contains $300 \times 200 = 60{,}000$ of raw 2D images. The size of this file is about 125 MB.

The groups of 200 raw images correspond to one train of 200 pulses of the laser; each of the 32×32 images is the return from a single laser pulse. The pixel values in this image either correspond to a relative time between pulse emission by the laser and the pulse detection by this sensor pixel, or they represent the ceiling (when we stop waiting for return).

The integration time for one 32x32 image is 100 microseconds. If during this time interval of 100 microseconds a pixel is not hit by a photon, then its value is set to the ceiling value. We interpret the non-ceiling values of pixels as the range bins.

Due to very weak signal (laser is eye-safe), and thus very small number of photons per each 32×32 image (resulting from one pulse), we get very little information about the scene. Moreover, the pattern of each image is heavily distorted by a noise which is generated mostly by the sensor.

Each 3D image is created from 200 32×32 images as follows: for each pixel (x, y) we calculate the histogram of occurrences of range bins between an offset (say, 10) and ceiling − offset. Now, $g(x, y, z)$ is the count of the photons detected by the sensor at (x, y) and reflected from the distance z. There are no other transmitters in the area having the same frequency as the illuminating laser. It turns out that this new 3D image has size $32 \times 32 \times 600$.

2 Processing of 3D images

The raw 3D image is populated with too many non-zero voxels, some of which represent the detection noise. Since voxels representing noise have low values (typically, 1 or 2 photons), we could remove the noise by thresholding. However, this will not remove many of the outliers which correspond to small objects in the scene. There is a delicate balance between removing outliers and losing the shapes of objects of interest.

The following noise-reducing functions are user-selectable:

1. Thresholding
2. Thresholding and majority rule
3. Parzen Windows and thresholding

The selectable thresholding algorithms are: a fixed threshold, a fixed percentage of peak voxel value, a fixed convex combination of fixed percentage of peak voxel value and the past value (the moving average), or another hybrid threshold.

The majority rule is taken from mathematical morphology, which uses local processing to establish new image values. One of the schemes is the following. We consider an internal voxel and its neighborhood $3 \times 3 \times 3$. We consider the number of nonzero voxels in this neighborhood. This number is between 0 and 27. We make the following decision rule. We set the new value of the initial voxel to 1 or 0 depending on whether the number of nonzero voxels is greater than 2 or not, respectively. Actually, the majority rule (according to the original definition) operates on binary images, so we use first the thresholding to indicate whether the voxel value is significant or negligible, and then we use the majority rule. The majority rule is a voting scheme applied to voxel neighborhoods in order to set their new voxel values.

The Parzen Windows Probability Density Function Estimation, or Parzen Windows - for short, is an approach for an estimation of the "true" probability density function by approximating the local density points. One way of applying Parzen Windows is to take the convolution of $g(x, y, z)$ with a suitable 3D Gaussian density function. This density function depends on three scale parameters: the standard deviations in each of three dimensions. It seems from examples processed so far that Parzen Windows provide more details of target shape while also remove noise from the scene.

3 Connectivity and labeling

The processing creates a binary mask for the 3D image indicating the potential target locations. Now we find the connected components in the scene with 3D morphological connectivity operators available in Matlab. The structural elements can be selected with 6, 18, or 26 neighbors when based on $3 \times 3 \times 3$ neighborhood. We declare the list of these connected components as the acquired targets. Targets are numbered from 1 to N, and k-th target consists of voxels with value k. If N is greater than the selected maximal number of targets tracked, T_{max}, then we reduce the list as follows. We sort the list by means of an importance criterion in the decreasing order, so that the most important target appears as the first on the sorted list. Then we truncate the list to the length T_{max}. The importance criterion can be a combination of the following: target volume (number of voxels), speed, direction of motion, variance of motion, etc.

4 Target associations

After the first step in acquisition we have only $T_1 \leq T_{max}$ of newly acquired targets. However, in step n > 1, we have $T_{n-1} \leq T_{max}$ targets from previous step $n-1$ and also $T_n \leq T_{max}$ of the newly acquired targets at this step n. Both lists are sorted by decreasing importance criterion. We must find the best matches between the old targets and new targets, that is, which instances of targets in the current step n are extensions of target in the accumulated list indexed with T_{n-1}. If the association criteria are too strong, we may end up with no association, and therefore we must accept $T_{n-1} + T_n$ targets for further processing, in particular, for sorting by the importance criterion, and then limiting their total number by T_{max}.

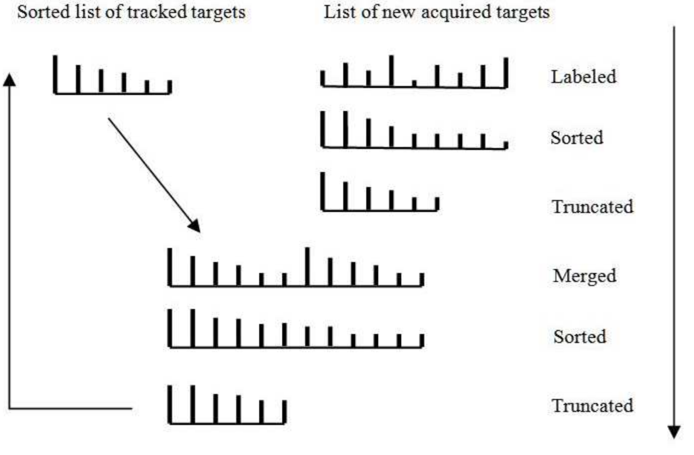


Figure 1: Merging of target lists

There are several methods of associating (matching) targets indexed by T_{n-1} to targets indexed by T_n. Here we form the target association matrix M with T_{n-1} rows and T_n columns, and define each entry as the numerical value of the relationship strength between an old and a new target. In the extreme, each old target would be related to (associated with) a new target, so all entries of the matrix M would be positive numbers. This would lead to an unmanageable complexity of the tracking program. Each entry of the matrix M can be considered as a weighted combination of correlations between attributes of old target and attributes of new target.

As we sort the target lists by a single criterion or a combination of criteria, we generate the entries of the association matrix M. Since the most undesirable feature in target tracking is a target jumping from a location to a location violating the continuity of motion, we use the target location as the main criterion for association. However, there are different ways to characterize the target location. The most popular is the target centroid, but we use here the target location characterized by the bounding box, which is also used in many popular program such as AutoCAD. Thus the target location is characterized by 6 numbers: minima and maxima of x, y, z coordinates of the target voxels.

We define a deterministic matching (association) criterion via bounding boxes. We shall say that an old target matches a new target if the bounding box of the old target is contained in an expansion of the bounding box of old target, and vice versa. The expansion of a bounding box is defined by decreasing the minima and increasing the maxima in each dimension by a fixed number (positive integer). This number is specified before running the tracking program.

We say that the new target is a valid extension of an old target, if the above symmetric relation holds. The next refinement of this notion is using the Kalman filter to predict the six values of the old targets bounding box and try to match the bounding box of new target to these predicted values. In this case each old target must have its own Kalman filter stored in the history array. As an alternative, we use the Kalman filter predictor for centroid, so we match a new target to an old target if its centroid is close enough to the predicted value of the centroid of the old target (from step $n-1$ to the step n).

5 Target states

After performing the matching procedure we end up with targets in 4 different states. To old targets which were not matched to new targets we give some chances for a short period of time, typically for 3 time frames; this number is selectable from the menu. This chance to be detected and matched later but not too late is called coasting. Here is the description of target states.

State 1 (yellow). It is the coasting state. The target was either detected in step $n-1$ but not matched (so, it is going into coasting state with bad count $= 1$) or target was in coasting state in step $n-1$, it is not matched to any new target in the current step n, and the number of bad counts in the consecutive coasting states did not exceed 3. If the number of consecutive coasting states is 4 and the target is not matched to any new target, then the target is not recorded in the tracking history array. We say, that after 3 unsuccessful trials, the target is declared to be lost.

State 2 (green). The target was detected (in the current step n), but it was not matched to any target from the previous step $n-1$. So, it is considered to be a new target.

State 3 (red). The target was detected in the current step n and matched to a coasting target in the step $n-1$, while the count of consecutive coasting states did not exceed 3. In another words, the previous target state was 1 (yellow) and the number of these coasting states did not exceed 3. We say that this target was re-acquired.

State 4 (magenta). The target is detected in the current step n and is matched to a valid (non-coasting) target in step $n-1$.

The target states are recorded in the tracking history array. This array is circular, maintained in each tracking step, and its length is 10. This number is larger than 7, which is the maximal selectable number of coasting states.

The total number of these 4 kinds of targets does not exceed $2 \times T_{max}$. But, if it is greater than T_{max}, we shall sort the target list again by the same criteria as was sorted the list of newly detected (and labeled) targets. Note that the optimal sort here is the merge sort, because both lists are already sorted according to the same criterion. Then we truncate the list to T_{max}, thus keeping the most important targets only. This operation, however, is pretty drastic, because we are removing the relationships (matches, associations) of the removed targets to the remaining targets. Of course, we must work with the list of length $2 \times T_{max}$ in order to sort it and then reduce it to the length T_{max}.

6 Target trajectories

We developed an alternative scheme to the association matrix M. Our scheme consists of two arrays which record forward and backward links of targets. Both of them extend target trajectories by one step if a match happens at time n. First, we should note that the matching relationship does not have to be symmetric. If the matching relation is not symmetric, then we end up with two association matrices M_1 and M_2, where M_1 records the matching of an old target to a new one and M_2 records the matching of a new target to an old one. We use the index-valued matrices, FWlink and BWlink, to store the match relations. Along with these deterministic relationships we could also store additional values assigned to these relationships, but it would require to define and maintain two more arrays. The relation

$$FWlink_{n-1,mt} = nt$$

means that the target with index mt in step $n-1$ (an old target) is linked forward to step n with the (new) target having the index nt. Similarly, the relation

$$BWlink_{n,nt} = mt$$

means that the target having index nt at the current step n is linked backward to the target with the index mt at the step $n-1$.

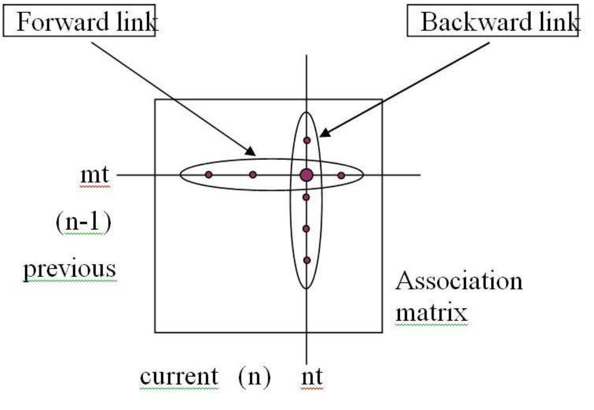


Figure 2: Target associations

Of course, recording one type of link is sufficient, especially in the situation when the matching relation is symmetric. However, keeping two links, the forward link and the backward link, allows for two different reconstructions of target trajectories.

7 Feature extraction, display and output

The purpose of feature extraction is usually two-fold. Some features are used for tracking the targets and some for target discrimination. For each target we record 23 features. Past targets with their features are remembered up to 10 frames back (in the history array). For matching (association) we could use predictors depending on up to 10 steps back, however, the stationary approach was working well with experimental data provided.

The tracking algorithms can distinguish between moving targets and static targets. Targets can split or merge any time. A target can enter the FOV any time as well as exit it. There are no prior assuptions about photon distributions, target sizes or target speeds. For all targets the important features are: the size (volume, number of voxels), the location, and the orientation. For moving targets we have also speed and acceleration. Those are the instant features. With target we can also associate the trajectories in x, y and z, or the time series of other attributes. Finally, each target lasts during entire tracking sequence, or is short-lived (ephemeral), or something between these extremes.

In general, an object can be tracked if it has a location and size, and we are able to estimate them.

We can plot trajectories of each target as three separate plots of x, y, and z as functions of time, or as one (parametric) 3D plot. The frames (edges) of boxes are displayed around targets tracked in 3D. At the end of tracking sequence, a video in AVI format can be saved.

8 Conclusion

Thousands of publications and internal research reports contain many diverse algorithms for target tracking which depend on initial specifications, hardware available and/or hardware build for the project, and software written for the projects. This paper describes a very special case of multiple target tracking with Ladar, but also general purpose algorithms having mathematical, engineering and educational values.

Acknowledgement

My thanks are due to Boeing-SVS engineering teams, especially, to Michael Salisbury, Matthew Heino and Elizabeth Rejman.

References

[1] Brian Aull: *3D Imaging with Geiger-mode Avalanche Photodiodes,* Optics and Photonics News, May 2005, pp. 42-46.

[2] Yaakov Bar-Shalom and Xiao-Rong Li: *Estimation and Tracking:* Principles, Techniques, and Software, 1998.

[3] Samuel Blackman and Robert Popoli: *Design and Analysis of Modern Tracking Systems,* Artech House, Boston 1999.

[4] Ronald P. S. Mahler: *Statistical Multisource-Multitarget Information Fusion,* Artech House 2007.

[5] Ralph Pringle: *Performance of Single Photon Counting Laser Radars,* Boeing-SVS, 9/12/2003, revised 2/18/2004.

[6] John N. Sanders-Reed: *Detection, Tracking and Discrimination,* Boeing-SVS, November 2001.

[7] John N. Sanders-Reed: *Multi-Target, Multi-Sensor, Closed Loop Tracking,* Proc. SPIE Vol. 5430 (2004), 94-112.

[8] Rastislav Telgarsky and Elizabeth Rejman: *Discrimination Algorithms IR&D, Modeling and Simulation,* Final Report, Boeing-SVS, September 2004.

[9] Rastislav Telgarsky, Elizabeth Rejman and Fredrick Thomas: *Feature Extraction in Photon Counting Ladar,* D&T Noon Seminar, Boeing-SVS, August 25, 2004.

Generalisation of Collector's Problem

Pavel Tlustý, Radka Štěpánková

University of South Bohemia
Studentská 13, České Budějovice 370 05
tlusty@pf.jcu.cz

University of South Bohemia
Jeronýmova 10, České Budějovice 371 15
radka.step@centrum.cz

Abstract. The paper deals with generalisation of so called *Collector's Problem*, a known problem from the theory of probability, when we are interested either in:

- how many cards we should buy in order that an event with a given probability happens,
- how many cards we have to buy *on average* to make a collection complete.

The generalisation is based on a more realistic background when we do not buy cards separately but in sets of several pieces.

Keywords: probability, Collector's Problem, stochastic graph

Classification: 97K50; 97K30.

1 Introduction

Let's deal with a problem which can be encounter in various forms in every-day situations.

A beer producer would like to increase the sale using the following promotion: One of n possible symbols is placed under the lid of every beer bottle. Customers buy beer bottles and collect these symbols. If they collect a whole collection of symbols, i.e. all n symbols, they will be given a reward. From the customer' point of view, the raised question is: *How many bottles of beer do we have to buy "on average" to be given the reward?*

2 Classic problem

Let's solve the above mentioned problem. We will assume that each of n symbols under the lid appears with the same probability (even distribution) and let's to deal with the following situation for $n = 6$. This assumption is not essential, but it enables us to use a dice to estimate the solution (see solution 3).

2.1 Solution 1

We repeat the trial (bottle purchase) as long as each of the six symbols of the collection appears. The whole process may be divided in six steps - the waiting time for obtaining the first symbol - random variable T_1, the waiting time for the second symbol - random variable T_2, ..., the waiting time for the sixth symbol - random variable T_6. The mean of the waiting time T for obtaining the whole collection is the sum of waiting times for particular symbols, i.e.

$$ET = ET_1 + ET_2 + ET_3 + ET_4 + ET_5 + ET_6$$

Taking into account the fact that the waiting time for $i-$th symbol represents the waiting time for the first success in Bernoulli attempt with success probability p_i is

$ET_i = \frac{1}{p_i}$. Then we get

$$ET = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_6} = \frac{1}{1} + \frac{1}{\frac{5}{6}} + \frac{1}{\frac{4}{6}} + \frac{1}{\frac{3}{6}} + \frac{1}{\frac{2}{6}} + \frac{1}{\frac{1}{6}} = 14.7$$

2.2 Solution 2

To solve the problem, we can use the stochastic graph in Fig. 1. The graph vertices represents collection states, i.e. how many symbols we already have, the graph edges represent probabilities of the transition from one state to the other. The calculation algorithm of the time wandering within the stochastic graph is described in detail in [1] and with its use, we get the following system of linear equations:

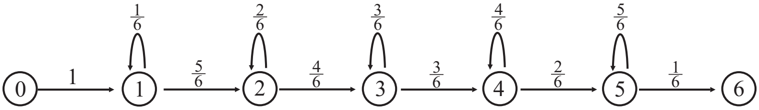


Figure 1: Stochastic graph of Collector's problem

$$\begin{aligned}
t_0 &= 1 + t_1\\
t_1 &= 1 + \tfrac{5}{6}t_2 + \tfrac{1}{6}t_1\\
t_2 &= 1 + \tfrac{4}{6}t_3 + \tfrac{2}{6}t_2\\
t_3 &= 1 + \tfrac{3}{6}t_4 + \tfrac{3}{6}t_3\\
t_4 &= 1 + \tfrac{2}{6}t_5 + \tfrac{4}{6}t_4\\
t_5 &= 1 + \tfrac{1}{6}t_6 + \tfrac{5}{6}t_5, \qquad t_6 = 0
\end{aligned}$$

Solving the given system of equations, we get $ET = t_0 = \frac{147}{10} = 14.7$.

2.3 Solution 3

To obtain an approximate solution (estimation), simulation methods may be used. Students can simulate the whole process using dice casting. They cast as long as they cast each of the numbers at least once and they count the total number of casts. If they repeat this casting attempt many times, they will get an assumption of the number of bought beer bottles necessary to complete the whole collection. To obtain a precise estimation, it is necessary to make so many attempts that the use of a computer is inevitable.

3 Generalised problem

The classic problem can be generalised in several ways:

1. Bottles are not bought one by one but in packages by k pieces.
2. How many bottles do we have to buy "on average" to obtain m rewards.
3. The symbols under the lids do not have the same probability of appearance.

Let's deal with the first situation, i.e. bottles are packaged by k pieces and let's choose the simplest version when $k = 2$.

It is clear that the wanted number of a double-pack is not a half of the expected number of bottles of the classic problem, but that the solution will be more complex. It is also clear that not each of the three solution of the classic problem may be easily used for the more complex version. The simplest it is in the case of simulations, i.e. this method is very universal, however, it does not provide a precise result, but only its estimation.

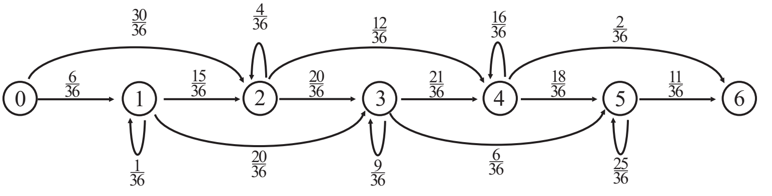


Figure 2: Stochastic graph of Collector's problem for $n = 6$ and $k = 2$

Let's show the solution using a stochastic graph. A stochastic graph representing the given problem is in Fig. 2. we can derive from it the following system of equations:

$$\begin{aligned}
t_0 &= 1 + \tfrac{6}{36}t_1 + \tfrac{30}{36}t_2 \\
t_1 &= 1 + \tfrac{1}{36}t_1 + \tfrac{15}{36}t_2 + \tfrac{20}{36}t_3 \\
t_2 &= 1 + \tfrac{4}{36}t_2 + \tfrac{20}{36}t_3 + \tfrac{12}{36}t_4 \\
t_3 &= 1 + \tfrac{9}{36}t_3 + \tfrac{21}{36}t_4 + \tfrac{6}{36}t_5 \\
t_4 &= 1 + \tfrac{16}{36}t_4 + \tfrac{18}{36}t_5 + \tfrac{2}{36}t_6 \\
t_5 &= 1 + \tfrac{25}{36}t_5 + \tfrac{11}{36}t_6, \qquad t_6 = 0
\end{aligned}$$

The solution is that $t_0 = \frac{70219}{9240} \doteq 7.599$.

4 Conclusion

The solution of the generalised Collector's problem shows that stochastic graph represents a strong tool for solving probabilistic problems. Their utilisation is rather wide and work with them may be manageable at upper-secondary school level.

References

[1] Krech, I., Tlustý P.: *Stochastické grafy a jejich aplikace,* Jihočeská univerzita, České Budějovice, 2012. (in czech)

[2] Płocki A., Tlustý P., *Pravděpodobnost a statistika pro začátečníky a mírně pokročilé*, Prometheus, Praha 2007. (in czech)

The generalization of the Lagrange functional equation

Iwona Tyrala

Jan Długosz University in Częstochowa
al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland
i.tyrala@ajd.czest.pl

Abstract. We will present solutions of a functional equation for mappings defined on a commutative unitary ring uniquely divisible by 2 and having values in the same ring. Inspired by Jean Dhombres' paper *Relations de dépendance entre les équations fonctionnelles de Cauchy* from 1988 we deal with the generalized Lagrange's equation.

Keywords: functional equation, alienation

Classification: 39B52, 39B72.

1 Introduction

Let X and Y be two rings and a map f be defined on X and having values in Y. We can ask: when does the system of two functional equations characterizing rings homomorphisms, e.i.

$$\begin{cases} f(x+y) = f(x) + f(y) \\ f(xy) = f(x)f(y) \end{cases} \quad \text{for all} \quad x, y \in X \tag{1}$$

may be replaced by a single functional equation:

$$f(x+y) + f(xy) = f(x) + f(y) + f(x)f(y), \tag{2}$$

or a more general equation:

$$af(x+y) + bf(xy) + c(f(x) + f(y)) + df(x)f(y) = 0,$$

for each $x, y \in X$. In particular in [1] J. Dhombres showed the following result.

Theorem 1 (Dhombres, 1988)**.** Let X and Y be two unitary rings and X be 2-divisible. Then each solution $f : X \to Y$ of equation (2) such that $f(0) = 0$ yields a solution of the system (1).

The above effect, such that each solution of a single equation satisfies the corresponding system of equations is called *the alienation phenomenon*. Generalizations of this result appeared in various directions (see e.g. [3, 4, 5, 8]). In 2010, W. Fechner [2] solved the problem of quadratic-multiplicative mappings. The system of Pexider type Cauchy equations:

$$\begin{cases} f(x+y) = f(x) + f(y) \\ g(x+y) = g(x)g(y) \end{cases} \tag{3}$$

was studied by R. Ger [6]. The aim of the present paper is to examine the alienation phenomenon for the generalization of the Lagrange functional equation. We shall start with solutions of this equations.

Theorem 2 (Cauchy, 1821, see [7])**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the additive Cauchy functional equation

$$f(x + y) = f(x) + f(y) \tag{4}$$

for all $x, y \in \mathbb{R}$. Then f is linear; that is, $f(x) = cx$, where c is an arbitrary constant.

In the next theorem, continuity of functions is not required.

Theorem 3 (Aczél, 1963, see [7])**.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions satisfying the Lagrange functional equation

$$g(x) - g(y) = (x - y)f\left(\frac{x + y}{2}\right) \tag{5}$$

for all $x, y \in \mathbb{R}$. Then there exist constants α, β, γ such that

$$\begin{cases} g(x) = \alpha x^2 + \beta x + \gamma \\ f(x) = 2\alpha x + \beta \end{cases} \tag{6}$$

for all $x \in \mathbb{R}$.

Remark 1. From the form (6) we see that between functions f and g there exists a dependence such that $g' = f$.

2 Main results

In view of the form of equation (5) we assume that functions f and g are defined on R, which is a commutative unitary ring uniquely divisible by 2 and this functions have values in R, too. Now we will consider the equations (4) and (5) as the system of equations.

Lemma 1. Let $(R, +, \cdot)$ be a commutative unitary ring uniquely divisible by 2 and let functions $f, g : R \to R$ satisfy the system of equations

$$\begin{cases} f(x + y) = f(x) + f(y) \\ g(x) - g(y) = (x - y)f\left(\frac{x+y}{2}\right) \end{cases} \tag{7}$$

for all $x, y \in R$. Then the function g has a form:

$$g(x) = \frac{1}{2}xf(x) + b \quad \text{for all} \quad x \in R, \tag{8}$$

where b is an arbitrary element in R and the function f is an additive.

Proof. Setting $y = x$ in (4), we get relation

$$f(2x) = 2f(x), \quad x \in R. \tag{9}$$

Putting $y = 0$ in (5) we obtain

$$g(x) - g(0) = xf(\frac{x}{2}), \quad x \in R. \tag{10}$$

From (9) and (10) we infer that

$$g(x) = \frac{1}{2}xf(x) + b, \quad x \in R,$$

where $b = g(0)$. □

Therefore, we have the following

Theorem 4. Let $(R,+,\cdot)$ be a nontrivial commutative unitary ring uniquely divisible by 2. Then functions $f,g:R\to R$ satisfy the system of equations (7) if and only if

$$\begin{cases} f(x)=ax, \\ g(x)=\frac{1}{2}ax^2+b \end{cases} \tag{11}$$

for each $x\in R$ and for a,b arbitrary elements in R.

Proof. It is easy to verify that (11) satisfes the system of equations (7). From Lemma 1 we obtain a form of function g. By (8) and (5) we conclude that

$$\frac{1}{2}xf(x)-\frac{1}{2}yf(y)=(x-y)f\left(\frac{x+y}{2}\right),\quad x,y\in R.$$

Jointly with (9) this implies that

$$\frac{1}{2}xf(x)-\frac{1}{2}yf(y)=\frac{1}{2}(x-y)f(x+y),\quad x,y\in R,$$

or, equivalently using (4) we get

$$xf(y)=yf(x),\quad x,y\in R. \tag{12}$$

Setting $y=1$ in (12) we obtain

$$f(x)=ax,\quad x,y\in R, \tag{13}$$

where $a:=f(1)$. Compare (13) with (8), this completes the proof. $\square$

Theorem 5. Let $(R,+,\cdot)$ be a commutative unitary ring uniquely divisible by 2. Then functions $f,g:R\to R$ satisfy the generalization of the Lagrange functional equation

$$f(x+y)+g(x)-g(y)=f(x)+f(y)+(x-y)f\left(\frac{x+y}{2}\right) \tag{14}$$

for every $x,y\in R$ if and only if

$$\begin{cases} f(x+y)=f(x)+f(y) \\ g(x)-g(y)=(x-y)f\left(\frac{x+y}{2}\right) \end{cases}$$

for each $x,y\in R$.

Proof. Interchanging the roles of x and y in (14) we obtain

$$f(x+y)+g(y)-g(x)=f(y)+f(x)+(y-x)f\left(\frac{x+y}{2}\right),\quad x,y\in R. \tag{15}$$

Adding (14) and (15) we get

$$2f(x+y)=2f(x)+2f(y)+f\left(\frac{x+y}{2}\right)(x-y+y-x),\quad x,y\in R.$$

In other words, function f is an additive. Moreover, subtracting (14) and (15) we infer that

$$2g(x) - 2g(y) = f\left(\frac{x+y}{2}\right)(x - y - y + x), \quad x, y \in R,$$

then functions f and g satisfy the Lagrange equation. $\square$

3 Conclusion

Examining proof of Theorem 5 we obtain the following

Corollary 1. Let $(R, +, \cdot)$ be a commutative unitary ring uniquely divisible by 2 while a, b, c, d are given elements of R. Then functions $f, g : R \to R$ satisfy the functional equation

$$af(x+y) + b(g(x) - g(y)) = c(f(x) + f(y)) + d(x-y)f\left(\frac{x+y}{2}\right), \quad x, y \in R$$

if and only if

$$\begin{cases} af(x+y) = c(f(x) + f(y)) \\ b(g(x) - g(y)) = d(x-y)f\left(\frac{x+y}{2}\right) \end{cases}$$

for each $x, y \in R$.

The functional equation (5) can be written as

$$g(x) - g(y) = (x-y)h(x+y), \quad x, y \in R$$

where $g(x) := h\left(\frac{x}{2}\right)$. Now, we do not assume that R is divisible by 2.

Corollary 2. Let $(R, +, \cdot)$ be a nontrivial commutative unitary ring. Then functions $f, g : R \to R$ satisfy the system of equations

$$\begin{cases} h(x+y) = h(x) + h(y) \\ g(x) - g(y) = (x-y)h(x+y) \end{cases} \quad \text{for all} \quad x, y \in R \tag{16}$$

if and only if

$$\begin{cases} h(x) = ax, \\ g(x) = ax^2 + b \end{cases} \tag{17}$$

for each $x \in R$ and for a, b arbitrary elements in R.

Remark 2. From the form (17) we see that between functions h and g there exists a dependence such that $g' = \frac{1}{2}h$.

References

[1] J. Dhombres: *Relations de dépendance entre les équations fonctionnelles de Cauchy,* Aequationes Math. **35** (1988), 186-212

[2] W. Fechner: *A characterization of quadratic-multiplicative mappings,* Monatsh Math. **164** (2010), 383-392

[3] R. Ger: *On an equation of ring homomorphisms,* Publ. Math. Debrecen **52** (1998), 397-417

[4] R. Ger: *Ring homomorphisms equation revisited,* Rocznik Nauk.-Dydakt. Prace Mat. **17** (2000), 101-115
[5] R. Ger, L. Reich: *A generalized ring homomorphisms equation,* Monatsh Math. (2010), 225-233
[6] R. Ger: *Additivity and exponentiality are alien to each other,* Aequat. Math. **80** (2010), 111-118
[7] P. K. Sahoo, T. Riedel: *Mean value theorems and functional equations,* World Scientific Publishing, USA, 1998
[8] I. Tyrala: *Solutions of the Dhombres-type trigonometric functional equation,* Sci. Issues Jan Długosz Univ. Częstochowa, Mathematics **XVI** (2011), 87-94

Fractal Geometry at Secondary Schools

Vladimír Vaněk

Department of Algebra and Geometry, Faculty of Science, Palacký University in Olomouc
17.listopadu 12, 77146 Olomouc, Czech Republic
vladimir.vanek@upol.cz

Abstract. The paper shows mathematical description of selected natural structures and phenomenon in nature from the view point of fractal geometry. Even though the fractal theory is rather complicated area, interfering into all natural sciences, we can introduce it to high-school students using only their present mathematical knowledge. We are going to mathematically describe fractal phenomenon which ruins usual principles of classical geometry in many cases and it can support the attractiveness of mathematics the queen of natural sciences.

Keywords: fractal, secondary school, geometry, didactics, mathematics

Classification: G14.

1 Introduction - problem of motivation

In this paper, we would like to introduce one of the interesting mathematical topics which can be used in math lessons at the secondary schools - *Fractals*. E.g. it can be applied while presenting terms like sequence, infinite series and its sum or in the theory of space dimensions.

It is always convenient to present an interesting task at the beginning so that it catches students' attention and involves them into a new problematic. The construction of the term fractal can be used as follows:

A chosen plane figure can be categorized in some of the categories according to the following characteristics:

1. *the figure has finite area and finite circumference,*
2. *the figure has infinite area and infinite circumference,*
3. *the figure has infinite area and finite circumference,*
4. *the figure has finite area and infinite circumference.*

Find suitable examples of individual categories.

Students do not usually have problems finding some examples of the first two categories. The third one is a little problematic and finding an example of the forth category is for most of the students an insolvable problem, although it is more common in nature and our surroundings than the examples of the other categories. We suggest an interesting plane figure as a simple example of the forth category - Koch snowflake.

First we describe the construction of a Koch snowflake and then we will prove that it is an example of the forth category.

2 Construction of the Koch snowflake

We start with an equilateral triangle with side k. We draw an equilateral triangle with side $k_1 = \frac{k}{3}$ above the center of each side of the triangle and the resulting shape is a 12-sided star. Above each side of the star we construct an equilateral triangle with side $k_2 = \frac{k_1}{3} = \frac{k}{9}$. The newly created figure has 48 sides. Again we construct an equilateral triangle above each side with side-length $k_3 = \frac{k_2}{3} = \frac{k}{27}$ and we continue this way (see Figure 1 and Figure 2). Every other figure will have four times more sides than the initial one and the side-length will be one third at the same time.

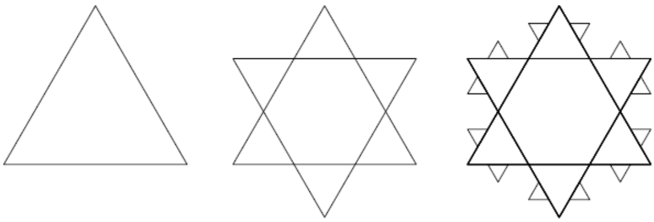


Figure 1: First steps of construction of the Koch snowflake

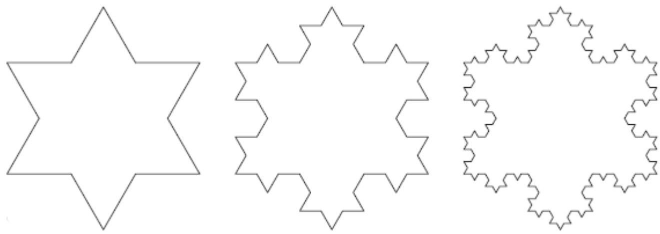


Figure 2: Koch snowflake

Now we can analyze the calculation of the circumference O of the Koch snowflake.

The circumference of the initial triangle was $3k$ then we added to every side two line segments $k/3$ and we took away one same line segment. So we added $3 \cdot k/3$ line segments to the initial triangle. During the second step we constructed triangles above the center of star sides. There are 12 of them and the length of the side of such a triangle is $k/9$ (see Figure 1). Following procedure is analogical and we get an expression for calculating the circumference of the Koch snowflake:

$$O = 3k + 3 \cdot \frac{k}{3} + 12 \cdot \frac{k}{9} + 48 \cdot \frac{k}{27} + 192 \cdot \frac{k}{81} + \ldots = 3k + k \cdot (1 + \frac{4}{3} + \frac{16}{9} + \frac{64}{27} + \ldots),$$

where $3k$ is the circumference of the initial triangle and following part shows us the enlargement of the circumference in every step we made. Elements in the brackets form a geometric series where $a_1 = 1$ and $q = \frac{4}{3}$. With regard to the value of the quotient $|q| > 1$ the geometric series diverges to infinity. As the construction of the Koch snowflake proceeds its circumference grows to infinity.

We do not have to do the calculation of its area. We only need to realize that we can circumscribe a circle about this figure. The area is apparently finite and therefore the area of the Koch snowflake is finite.

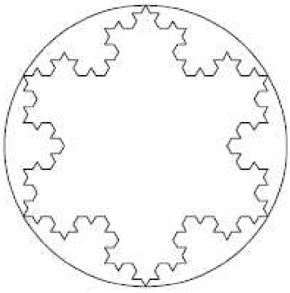


Figure 3: Proof that the area of the Koch snowflake is finite

Nevertheless it is not difficult to count the area of the Koch snowflake using mathematical apparatus which is already well known at the secondary school level.
Let the area of the initial triangle be S_0, $S_0 = \frac{k}{2} \cdot \frac{k\sqrt{3}}{2}$. After a lot of iterative steps we can express the area of the snowflake as follows:

$$S = S_0 + 3 \cdot \frac{S_0}{9} + 12 \cdot \frac{S_0}{9^2} + 48 \cdot \frac{S_0}{9^3} + \ldots = S_0 + \frac{S_0}{3} \cdot (1 + \frac{4}{9} + \frac{4^2}{9^2} + \ldots) = S_0 + S_0 \cdot \frac{1}{1 - \frac{4}{9}} = \frac{8}{5} S_0.$$

We proved that the Koch snowflake belongs to the fourth category since its circumference is infinite whereas its area is equal to $\frac{8}{5}$ of the area of the initial triangle.

3 Fractals

Previous example shows us one of the simplest fractals. The name "fractal" is of Latin origin, *fractus = broken.* Many personalities mainly of the 20^{th} century studied these figures. Let us remind some of them Georg Ferdinand Ludwig Phillip Cantor ($*$ 1845, † 1918), Niels Fabian Helge von Koch ($*$ 1870, † 1924), Waclaw Franciszek Sierpiński ($*$ 1882, † 1969), Gatson Maurice Julia ($*$ 1893, † 1978), Felix Hausdorff ($*$ 1868,† 1942), or Benoit Mandelbrot ($*$ 1924). Those who are interested can study the fractals in details in their works.

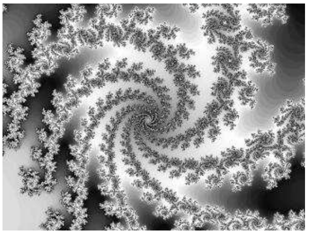


Figure 4: The beauty of fractals

Let us define several important terms connected with fractals which will be needed.

Definition 1:
Fractals are objects which are self-similar.

Definition 2:
Self-similarity (mathematically is this property called the invariance to scale change) means that the figure looks the same whether we look at it from far or from near.

Definition 3:
Fractal is a set whose Hausdorff-Besicovitch dimension is far greater than its topological dimension.

Let us describe this definition which can be found in all important works about fractals.

3.1 Hausdorff-Besicovitch dimension

The question is: *"What is the dimension of the Koch snowflake in the introduction of this paper?"*

The Koch snowflake is in fact a set of infinitely many infinitely short line segments which form a plane figure. So its topological dimension is equal to one and it should be independent on the choice of the scale. As it is an infinitely long curve (its length is dependent on the choice of the scale - it is called *Richardson effect*), it should occupy more space in a plane than a similar smooth curve. However it does not occupy the whole plane so its dimension is between 1 and 2. Therefore it is suitable to introduce a new "fractal" dimension (Hausdorff-Besicovitch) which is greater than 1 and fits our curve better.

Similar quality as in the Koch snowflake can be found in the length of the coast of maritime states, borders between two states, etc. Lewis Fry Richardson (an English mathematician and physicist) tried to measure the length of the border between Spain and Portugal. He came to a conclusion that using different scales the length of the border is between 987 km and 1214 km. During his measurements he derived an empirical expression

$$K = C\varepsilon^D,$$

where K is the length of approximation, C is a proportionality constant and ε is the length of the measuring instrument. The value of D was determined empirically and its mathematical meaning was not known to him. Only B.B. Mandelbrot presented the relation between the D value and Hausdorff-Besovitch dimension.

The expression to calculate Hausdorff-Besicovitch dimension can be easily deduced from the Richardson empirical expression and the fact that we cover the lengths of the curves, the surface areas, or the volume of the objects with line segments, squares, cubes with unit length, area, or volume.

$$N \cdot \varepsilon^D = 1$$

Calculating the logarithms we come to

$$\log N + D \log \varepsilon = 0$$

$$D = \frac{\log N}{\log \varepsilon}$$

N shows the minimal number of objects (unit line segments, squares, cubes) needed to cover the considered set and ε is the scale (the length of the instrument). Let us count the fractal dimension of the Koch snowflake.

The length of the side minimizes to 1/3 in every iterative step, so $\varepsilon = 1/3$ of its initial value. Concurrently the number of such line segments needed to cover the new object rises to $N = 4$. So the fractal dimension of the Koch snowflake is

$$D = \frac{\log N}{\log(1/\varepsilon)} = \frac{\log 4}{\log 3} \doteq 1,2618595.$$

4 Occurrence and application of fractals

Fractals[1] can be found almost everywhere around us. They are clouds (fractal dimension of the circumference of the projection of a cloud is 1.33), rocks (fractal dimension of the surface of uneroded rocks is 2.25), or the coast of maritime states (fractal dimension is 1.26), lightning strikes are believed to be examples of branch fractals. Human brain is also considered to be a fractal (fractal dimension of the brain surface is 2.76).

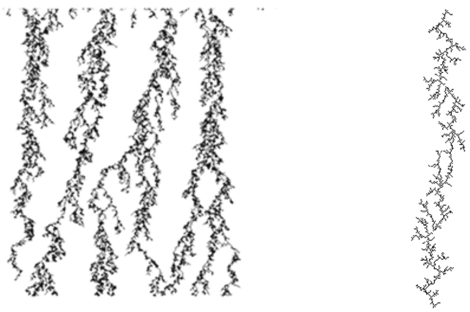


Figure 5: Lightning strikes display fractal characteristics

Some plants (Romanesque Broccoli) display fractal characteristics.

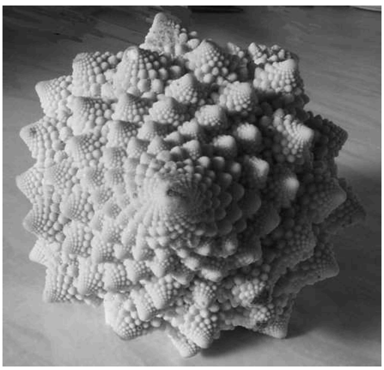


Figure 6: Romanesque Broccoli

Physical and chemical phenomena as e.g. diffusion (gradual delivery of particles of solid, liquid, or gaseous state in other solid, liquid, or gas - Brownian motion).

[1]The figures are adopted from [7]

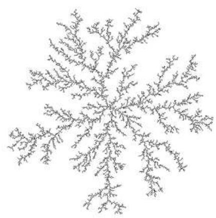


Figure 7: Fractal notation of Brownian motion

Change of magnetism of solids at Courier temperature displays fractal characteristics. At the critical temperature materials pass from magnetic to nonmagnetic state and vice versa. Closer look emerged that elementary magnets, which form magnetic material, do not convert into chaotic orientation suddenly all at once. Although the material seems not to be externally magnetic, some local fluctuations of elementary magnets emerge and they are organized (magnetic). So at the Courier temperature we cannot decide whether material is magnetic or not.

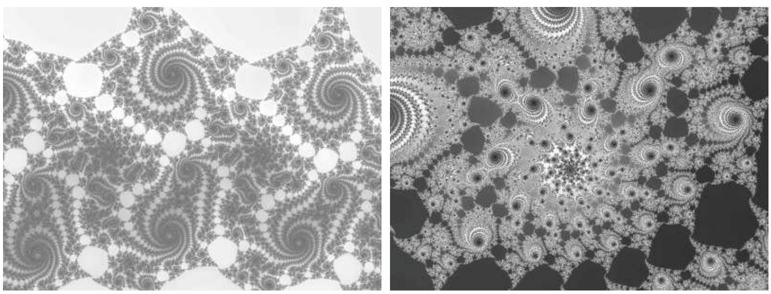


Figure 8: Local fluctuation of elementary magnets at Courier temperature

Simple fractal elements can be found even in the architecture, distribution of components in a printed circuit, or in the shape of a broad-spectrum antenna with minimal proportions.

Figure 9: Antenna of the shape of Koch curve

5 Conclusion

We tried to give the readers an idea of the beauty of nature under the magnifying glass of mathematics. We believe that the text or its parts has inspired the readers and they can use it in their classes or seminars. If it was your first meeting with the term fractal, or you are not a mathematics teacher, we hope that it was not the last meeting and you will look at nature and mathematics as two inseparable parts of our life.

Acknowledgements

The article has been written in the context of the European Social Fund project "*Broadening the accreditation of mathematics and descriptive geometry teaching at the Faculty of Science in Olomouc to include the combined mode of study*", reg. no. CZ.1.07/2.2.00/18.0013. This educational project is co-financed from the European Social Fund and the Czech state budget.

References

[1] B. B. Mandelbrot: *Fraktály: tvar, náhoda a dimenze,* Mladá fronta, Praha 2003
[2] I. Zelinka: *Aplikovaná informatika,* UTB Zlín, 2005
[3] P. Pauš: *Počítačové metody analýzy fraktálních množin,* ČVUT Praha, 2006
[4] M. Mašek: *Fraktálov kapacitory.* ČVUT Praha, 2009
[5] R. Wiesner: *Užití a zneužití fraktálů,* MU, Brno, 2006
[6] *http://www.fit.vutbr.cz/˜tisnovpa/fract/uvod.html,* 2000
[7] *http://www.root.cz/clanky/fraktaly-kolem-nas,* 2007
[8] *http://dagles.klenot.cz/rihova/koch.pdf,* 2007
[9] *http://martin.hinner.info/math/Fraktaly/jednoduche.php,* 1999
[10] *http://www.utopiansky.com/labratory/fractals/,* 2007

Students' investigation of Platonic solids - they are really only five

Kitti Vidermanová, Michaela Klepancová

Department of Mathematics, Faculty of Natural Sciences,
Constantine the Philosopher University in Nitra
Tr. A. Hlinku 1, 949 74 Nitra, Slovakia
kvidermanova@ukf.sk, michaela.klepancova@ukf.sk

Abstract. As George Polya said, the best way to learn something is to discover it himself. In mathematics education, it is very important that the teacher encourages the students to individual work to solve problems. We connect guided discovery learning with manipulatives as a teaching method for finding all regular convex-polyhedrons. During the activity we use a teaching aid - Polydron. In our article we inductive approach of learning about Platonic solids. Students are encouraged to investigate which regular polygons can form faces of a regular polyhedron. This leads to further investigation of inner angles of those polygons. Based on their findings, students are supposed to formulate hypothesis about number of Platonic solids and formally proof their hypothesis. Then we create all regular convex polyhedrons and show that they are only five.

Keywords: Guided discovery learning, Manipulatives, Solid geometry, Regular polyhedrons, Platonic solids.

Classification: D44; G44.

1 Introduction

In recent years, many studies have highlighted an alarming decline in young people's interest for key science and mathematics (Rocard et al., 2007). The European committee called the countries to change the education system. The Slovak government responsed for this call with a school reform in 2008. Although skills, key competencies and their achievement were defined in the new State Education Program, teachers still use traditional teaching methods. It is necessary to change teaching strategies to awake students' interest in science and mathematics studies.

In this article we describe how we can awake students' interest in solid geometry using discovery learning and manipulatives.

2 Discovery teaching methods

Discovery learning takes many different forms, ranging open-ended, minimally-guided investigation through to fairly tightly structured 'guided discovery' where the teacher still retains a fair degree of control. How much guidance the teacher uses during the discovery method depends on his/her decision based on his/her experience. If there is a lot of guidance, then it is the teacher who is doing the investigation. If there is not enough guidance, then the students may lose interest when they get stuck. (e.g. Lerman, 1989, Tanner, 1989 in Yeo, 2007)

Kirschner at al. (2006) claim that "although unguided or minimally guided instructional approaches are very popular and intuitively appealing, the point is made that these approaches ignore both the structures that constitute human cognitive architecture and evidence from empirical studies over the past half-century that consistently indicate that minimally guided instruction is less effective and less efficient than instructional approaches that place a strong emphasis on guidance of the student learning process."

Under the term *investigation approach* we understand a form of unguided discovery. The main limitation of this method is that some students never discover the concepts or principles themselves without teacher's guidance.

The Spectrum Institute for Teaching and Learning defined the ***Guided discovery*** teaching style as a series of logical and sequential designed questions or instructions that lead students to discover a predetermined concepts, principles, relationships or rules that was not previously known.

The structure of typical guided discovery learning session

The structure of every school lesson depends on many factors: number of students in class, their level of knowledge, teacher's enthusiasm, his/her attitude to new methods and strategies in education process, etc.

The common used procedure for doing guided discovery is: the teacher usually explains the lesson objectives to the students, provide initial input or explanation to help students begin the task efficiently. The teacher divides the studied problem into smaller problems or tasks termed goals. The solutions of this goals create a step-by-step procedure which could help to find out the target information or to solve the problem.

Westwood (2008) indicates the following format for a typical guided discovery learning session:

- a topic is identified or an issue is posed;
- teacher and students work together to brainstorm ideas for ways of investigating the topic;
- students work individually or in small groups to obtain and interpret data;
- inferences and tentative conclusions are drawn, shared across groups and modified if necessary;
- teacher clears up any misconceptions, summarises the findings and helps to draw conclusions.

The advantages of guided discovery teaching method were summarized by Westwood (2008):

- students are actively involved in the process of learning and the topics are usually intrinsically motivating;
- the activities used in discovery contexts are often more meaningful than the typical classroom exercises and textbook study;
- students acquire investigative and reflective skills that can be generalised and applied in other contexts;
- new skills and strategies are learned in context;
- the approach builds on students prior knowledge and experience;
- independence in learning is encouraged;
- it is claimed (but not proved conclusively) that students are more likely to remember concepts and information if they discover them on their own;
- group working skills are enhanced.

3 Manipulatives

Manipulatives are often defined as "physical objects that are used as teaching tools to engage students in the hands-on learning of mathematics" (Boggan et al., 2010).

Manipulatives are suitable for using in all areas of math instruction - teaching number and operations, algebra, geometry, measurement across all grade levels. They can be used to introduce, practice, or remediate a concept. Manipulatives can be almost anything blocks, shapes, spinners, penny or even paper that is cut or folded. They may be store-bought, brought from home and teacher- or student- made.

Manipulatives become popular, because their using encourage students' understanding. Pupils and students on some level of mental maturity are unable to understand the content of abstract concepts, respectively relations expressed only through words and mathematical symbols. It correspondences with Bruner's three stages of representation (McLeon, 2008): 1. Enactive representation (action-based), 2. Iconic representation (image-based), 3. Symbolic representation (language-based).

Heddens (1997) argue that using manipulative materials in teaching mathematics will help students learn:

- to relate real world situations to mathematics symbolism,
- to work together cooperatively in solving problems,
- to discuss mathematical ideas and concepts,
- to verbalize their mathematical thinking,
- to make presentations in front of a large group,
- that there are many different ways to solve problems,
- that mathematics problems can be symbolized in many different ways,
- that they can solve mathematics problems without just following teachers' directions.

Before start the teaching using manipulatives, Mink (2010) proposes to answer in mind the following question:

- What outcome are the students supposed to walk away with at the end of this activity?
- What size and how many of the manipulatives does each student/group of students need?
- How can the manipulatives be arranged to the easiest using in the activity?
- What is the size of the students workspace?
- What can be done to make sure that students do not take the manipulatives away from the activity or classroom?
- Does the organization method being utilized enable more positive learning opportunities to occur?

When we use manipulatives using group-work, it is often helpful to assign students roles within their groups. This will help each student feel important and give him or her responsibility for the success of the activity. It is important to rotate roles so that every student has a chance to have each role at least once during the year.

Here is an example from Mink (2010) of various roles that could be given during instruction with manipulatives:
Group Leader: This student encourages other students in group to participate in investigation, solution and discussion. He/she is the only student who may approach the teacher with a question from the group.

Timekeeper: This student keeps track of the time elapsing during the activity. He/she motivates group members to stay on task and use time wisely and notifies team members when it is time to clean up the activity.
Materials Officer: This student is in charge of getting the materials for one week. He/she makes sure each student has everything needed for the assigned task.
Materials Organizer: This student counts materials to make sure none were lost during the activity. He/she is in charge of making sure that all of the supplies are back in the correct containers or bags at the end of the activity.

4 The investigation of Platonic solids

The solid geometry is one of the most important parts of mathematics. We mean that teaching of solid geometry has the most influence on the development of students' spatial imagination. It is evident that this ability is very important for different professions. Therefore the development of students' spatial imagination is one of the main goals of geometry education. The Platonic solids are included in the curriculum for the higher secondary education in Slovakia. Generally, teachers only tell to students that there exist five regular polyhedrons and they do not give more attention to them. We suggest teaching the Platonic solids using manipulatives under guided discovery learning as a teaching method. In this activity we recommend teacher 's guidance consisting of 3 goals and some supporting questions (or notes) within goals. Within solutions of the goals students have time for their own investigation.

The assigned task for students: ***The regular polyhedron is a solid whose faces are identical regular polygons and at each vertex meet the same number of faces. How many regular polyhedron do there exist? How do they look like?***

Goal 1: Which regular polygons can form faces of a regular polyhedron?
Supporting questions (use only if the students need them):

1. How many faces meet at one vertex? (Answer: At least three.)
2. From what depends the number of faces meeting at one vertex? (Answer: The interior angles of all the polygons meeting at one vertex of a polyhedron add to less than 360°.)

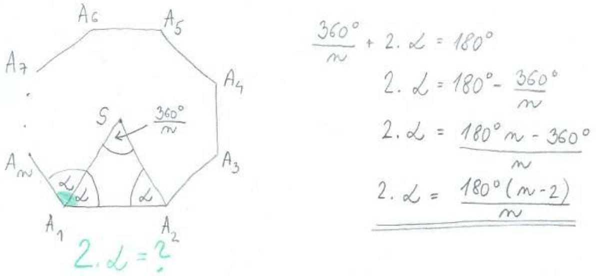


Figure 1: The interior angle of an n-sided regular polygon (Student's solution)

The students calculate the interior angle of a regular polygon. We can see one student's solution at Figure 1.

Using these supporting questions it was easy for students to find all possibilities.

Firstly they studied the 3-sided regular polygon - equilateral triangle (three is the smallest possible number for sides of polygon). The interior angle of an equilateral triangle is 60°. The students manipulated with Polydron and connect 3 triangles (Figure 2a), 4 triangles (Figure 2b), 5 triangles (Figure 2c) and 6 triangles (Figure 2d). In the fourth possibility the students discover that they can not create a regular polyhedron with 6 triangles meeting at vertex. The interior angles of six triangles add to exactly 360° and they cover the plane.

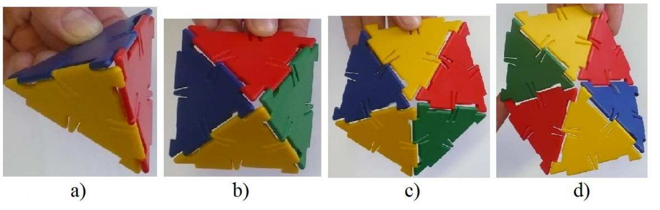


Figure 2: Equilateral triangles meeting at one vertex (Student's solution)

Secondly they studied the 4-sided regular polygon - square. The interior angle of a square is 90°. The students manipulated with Polydron and connected 3 squares (Figure 3a) and 4 squares (Figure 3b) at one vertex. With four squares is the situation analogous to previous care with 6 triangles.

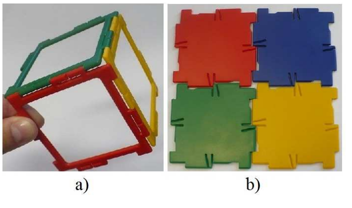


Figure 3: Squares meeting at one vertex (Student's solution)

Thirdly the students investigated the 5-sided regular polygon - regular pentagon. The interior angle of a pentagon is 108°. The students manipulated with Polydron and connected 3 pentagons (Figure 4a) and 4 pentagons (Figure 4b) at one vertex. They discovered that there is no possibility to use 4 pentagons to crate a convex "space angle".

Finally the 6-side regular polygon - hexagon. When the students connected three hexagons at one vertex, again was repeated the problem with covering of the plane (Figure 5). There does not exists regular polyhedron with hexagon faces.

Discussion of goal 1

There are five possible regular polyhedrons:

1. at each vertex meet 3 equilateral triangles,
2. at each vertex meet 4 equilateral triangles,
3. at each vertex meet 5 equilateral triangles,

a) b)

Figure 4: Regular pentagons meeting at one vertex (Student's solution)

Figure 5: Regular hexagons meeting at one vertex (Student's solution)

4. at each vertex meet 3 squares,
5. at each vertex meet 3 regular pentagons.

We can show the mathematical reasoning to students too: Let p the number of polygons meeting at a vertex, n the number of vertices of each polygon. Of course $p \geq 3, n \geq 3$. We must have

$$\begin{aligned} p.\frac{(n-2).180^\circ}{n} &< 360^\circ \\ p &< \frac{2.n}{n-2} \end{aligned}$$

The solutions of the inequality are:
If $n = 3$ then $p < 6$ we have three results: $p \in \{3;4;5\}$
If $n = 4$ then $p < 4$ we have three results: $p \in \{3\}$
If $n = 5$ then $p < \frac{10}{3}$ we have three results: $p \in \{3\}$

We can use Schläfli symbol $\{p, n\}$ for combinatorial description of the polyhedron. Our results we can write using this symbol as $\{3, 3\}$; $\{4, 3\}$; $\{5, 3\}$; $\{3, 4\}$ and $\{3, 5\}$.

Goal 2: How do the regular polyhedrons look like? How many faces are there?

Hint: Create the regular polygons from the results of Goal 1. Don't forget that at every vertex meet the same number of faces.

The students created all five regular polyhedrons and counted the number of the faces in each one. At the Figure 6 we can see the student's solution. We declaim to students the connections among the regular polyhedrons and theirs names. The hexahedron we know as cube. This solid is familiar for students trivially.

Goal 3: How many edges and vertices do the regular polyhedrons have? Fill in the table.

Supporting questions (only when the students need them):

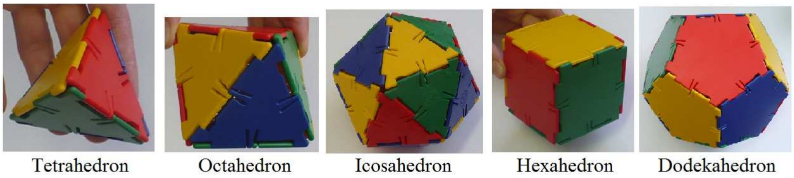


Figure 6: The five regular polyhedrons - Student's solution

1. How many sides and vertices have the polygons formed the faces of polyhedrons?
2. How many of them are connected in one edge or vertex? (For example the cube' Six squares have together 24 vertices and 24 sides - in cube connect two squares' sides to one cube's edge, we have 12 cube's edges; and 3 squares' vertices connect to one cube' vertex, we have 8 cube' vertices.)

Polyhedron	Number of faces	Number of vertices	Number of edges
Tetrahedron	4		
Hexahedron	6		
Octahedron	8		
Dodecahedron	12		
Icosahedron	20		

The students tried to count the edges and vertices on the created polyhedrons. With the tetrahedron, cube and octahedron they had not any difficulties. Studying the dodecahedron and the icosahedron they needed help. They helped themselves with nets of the polyhedrons.

5 Conclusion

Discovery learning is an active learning process where students develop higher-level skills to build a deep understanding of main concepts and relations. With active learning we better prepare students for adult world they will one day enter. It is important to teach students to systematic work and develop their reasoning skills.

We can agree with Mink (2010) that using manipulatives many students may have experienced greater success in understanding mathematical concepts. We can not imagine doing the described activity without manipulatives. Don't forget, that the using only manipulatives does not guarantee the success. It is not good use manipulatives in a rote manner. The findings of much research indicate that the using of manipulatives improves students' abilities, reasoniing and skills (e.g. Vidermanová, 2007). On the other side, the Moyer's research showed that teachers consider manipulatives for fun, but not necessary, for teaching and learning mathematics (Moyer, 2001). Actually in one study, students not using manipulatives outperformed students using manipulatives on a test of transfer (Fennema, 1972 in Durmus and Karakirik, 2006).

Finally we emphasize one importance by using manipulatives - when we first expose some manipulative material (games, boxes of blocks and bricks, etc.), it is necessary to allow students to play with them for a short time. Another advantage of manipulatives is that they are appropriate for pupils and students of all ages and ability levels.

Acknowledgements

The authors have been supported by grant KEGA 038UKF-4/2011 Solid Geometry in Arrangement of Math teachers with Accent of Factor Handling Activity and Application Tasks.

References

[1] M. Boggan, S. Harper, A. Whitmire: *Using manipulatives to teach elementary mathematics.* In: Journal of Instructional Pedagogies, Vol. 3, 2010. ISSN: 1941-3394. Retrieved June 26, 2012 from http://www.aabri.com/manuscripts/10451.pdf

[2] S. Durkus, E. Karakirik: *Virtual Manipulatives in Mathemtics Education: A Theoretical Framework.* In: The Turkish Online Journal of Educational technology. Vol. 5, Is. 2, 2006. pp. 117-123. ISSN 1303-6521

[3] J. W. Heddens: *Improving Mathematics Teaching by Using Manipulatives.* Retrieved June 26, 2012 from www.fed.cuhk.edu.hk/ fllee/mathfor/edumath/ 9706/13hedden.html

[4] P. A. Kirschner, J. Sweller, R. E. Clark: *Why Minimal Guidance During Instruction Does Not Work: An Analysis of the failure of Constructivistist, Discovery, Problem-Based, Experiential, and Inquiry/Based teaching.* In: Educational Psychologist, 41 (2), 2006. pp. 75-86. ISSN 1532-6985 (online)

[5] S. McLeod: *Jerome Bruner.* Retrieved June 26, 2012 from http://www.simply psychology.org/bruner.html

[6] D. V. Mink: *Strategies for Teaching Mathematics.* Shell Education, California, 2010. ISBN 978-1-4258-0249-3

[7] P. S. Moyer: *Are we having fun yet? How teachers use manipulatives to teach mathematics.* In: Educational Studies in mathematics. Vol. 47, No. 2, 2001. pp. 175-197. ISSN 1573-0816 (electronic version)

[8] M. Rocard et al.: *Science Education Now: A renewed Pedagogy for the Future of Europe.* European Communities, Belgium, 2007, 29 p. ISSN 1018-5593. Retrieved June 26, 2012 from http://ec.europa.eu/research/science-society/document_ library/pdf_ 06/report-rocard-on-science-education_ en.pdf

[9] Spectrum Institute for Teaching and Learning: *Guided Discovery.* Retrieved June 26, 2012 from http://www.spectrumofteachingstyles.org/styles/f

[10] K. Vidermanová: *Statistical Evaluation of an Experiment Using Polydron.* In: Acta mathematica 10. CPU, Nitra, 2007. pp. 231-236. ISBN 978-80-8094-181-9

[11] P. S. Westwood: *What teachers need to know about Teaching methods.* ACER Press, Australia, 2008. ISBN 978-0864319128

[12] J. B. W. Yeo: *Mathematical tasks: Clarification, classification and choice of suitable tasks for different types of learning and assessment.* Technical report ME2007-01, Mathematics and Mathematics Education, National Institute of Education. Retreived June 26, 2012 from http://math.nie.edu.sg/bwjyeo/publication/MMETechnicalReport2007_ MathematicalTasks_ ME200701.pdf

Catholic University in Ružomberok
Faculty of Education

Mathematica IV

SCIENTIFIC ISSUES

ISBN 978-80-8084-954-2