An algorithm for making magic cubes

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Magic squares fascinated people throughout centuries. A magic cube is a generalization of a magic square. Probably the first magic cube appeared in a letter of *Pierre de Fermat* from 1640.

A magic cube of order n is a cubical array

$$\mathbf{M}_n = [\mathbf{m}_n(i,j,k);\ 1 \leq i,j,k \leq n], \tag{1}$$

containing natural numbers $1, 2, \dots, n^3$ such that the sums of the numbers along each row (n-tuple of elements having the same coordinates in two places) and also along each of its four great diagonals are the same, e.i. $\frac{n(n^3+1)}{2}$.

Figure 1 shows the magic cube $\mathbf{M}_3$ which was constructed using the formula (1). The element $\mathbf{m}_3(1,1,1) = 8$ is in three rows containing the triples $\{8,15,19\}$, $\{8,24,10\}$, $\{8,12,22\}$. On the four diagonals there are the triples $\{8,14,20\}$, $\{19,14,9\}$, $\{10,14,18\}$ and $\{6,14,22\}$.

Here we gives an algorithm for making magic cubes of order $n \neq 2$. The proof of the correctness of our formulas follows from [3] and [4]. We use the following notation:

$$\begin{aligned}
x \pmod n \quad &:= \quad \text{the remainder after division of } x \text{ by } n \\
\overline{x} \quad &:= \quad n+1-x, \\
x^* \quad &:= \quad \min\{x, \overline{x}\} \\
\widetilde{x} \quad &:= \quad \begin{cases} 0 & \text{for} \quad 1 \leq x \leq \frac{n}{2} \\ 1 & \text{for} \quad \frac{n}{2} < x \leq n. \end{cases}
\end{aligned}$$

We construct a magic cube $\mathbf{M}_n = [\mathbf{m}_n(i,j,k)]$ of order n using the following three formulas:

1. If $n \equiv 1 \pmod 2$ then

$$\mathbf{m}_n(i,j,k) = \ a_{i,j,k}\ n^2 \ + \ b_{i,j,k}\ n \ + \ c_{i,j,k} \ + \ 1$$

with $a_{i,j,k} = (i-j+k-1) \pmod n, b_{i,j,k} = (i-j-k) \pmod n, c_{i,j,k} = (i+j+k-2) \pmod n$

2. If $n \equiv 0 \pmod 4$ then

$$\mathbf{m}_n(i,j,k) = \begin{cases} (i-1)\ n^2 + (j-1)\ n + k & \text{if} \quad \mathbb{F}(i,j,k) = 1 \\ (\overline{i}-1)\ n^2 + (\overline{j}-1)\ n + \overline{k} & \text{if} \quad \mathbb{F}(i,j,k) = 0 \end{cases}$$

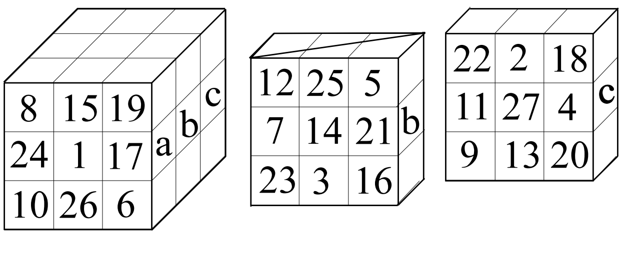

Figure 1:

where

$$\mathbb{F}(i,j,k) = (i + j + k + \widetilde{i} + \widetilde{j} + \widetilde{k}) \pmod 2$$

3. If $n \equiv 2 \pmod 4$ (in this case $t = \frac{n}{2}$ is odd) then

$$\mathbf{m}_n(i,j,k) = \mathbf{d}(u,v)t^3 + \mathbf{m}_t(i^*, j^*, k^*)$$

where

$u = (i^* - j^* + k^*) \pmod t + 1$,

$v = 4\widetilde{i} + 2\widetilde{j} + \widetilde{k} + 1$, and

$\mathbf{d}(u,v)$ for $1 \le u \le t, 1 \le v \le 8$ is defined by the table $(x = 1, 2, \ldots, \frac{n-6}{4})$

	$\mathbf{d}(u,1)$	$\mathbf{d}(u,2)$	$\mathbf{d}(u,3)$	$\mathbf{d}(u,4)$	$\mathbf{d}(u,5)$	$\mathbf{d}(u,6)$	$\mathbf{d}(u,7)$	$\mathbf{d}(u,8)$
$\mathbf{d}(1,v)$	7	3	6	2	5	1	4	0
$\mathbf{d}(2,v)$	3	7	2	6	1	5	0	4
$\mathbf{d}(3,v)$	0	1	3	2	5	4	6	7
$\mathbf{d}(2x+2,v)$	0	1	2	3	4	5	6	7
$\mathbf{d}(2x+3,v)$	7	6	5	4	3	2	1	0

Using analogous formulas it is easy to make a computer program which constructs a magic square for every $n \neq 2$.

References

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[3] M.Trenkler, *A construction of magic cubes*, The Mathematical Gazette, **84**, 36–41, March 2000

[4] M.Trenkler, *Magic p-dimensional cubes*, Acta Arithmetica **96**, 361–364, 1998

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